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ON THE SYNTACTIC CHARACTERIZATION OF SOME MODEL THEORETIC RELATIONS

By

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ABSTRACT

In this thesis we consider binary relations over the class of L - structures , for some fixed language L . Such a binary relation R , induces a binary relation R* between the class of theories in L ; in the following natural way . If T_1 and T_2 are theories in L then $T_1 R^* T_2$ iff $\exists A,B \; A \models T_1 , B \models T_2$ and A R B

We characterize syntactically those pairs of theories related by R* by introducing the concept of a notion of goodness for R . This consists of a set of ordered pairs of sentences in L , Δ , with the property that for theories T_1 and T_2

$$* \quad T_1 R^* T_2 \text{ iff for no } \langle \phi_1, \phi_2 \rangle \in \Delta \text{ do we have } T_1 \vdash \phi_1 \text{ and } T_2 \vdash \phi_2 .$$

Provided Δ is defined in a syntactically simple way , we find , by negating both sides of * and restricting the theories to sentences that the property * closely resembles an Interpolation Theorem for R . Actually , a notion of goodness is more complicated than this and our results are more general .

In the established approaches to find Interpolation Theorems , the weak point has been in the understanding of "syntactically simple " . We show , by considering certain relations which can be "described " by a theory in a particular language extending L , that a notion of goodness can often be found immediately from such a theory . Indeed we find a model theoretic

condition on R for which this is possible . It turns out to be a "union of chains " condition .

Using this approach we obtain many Interpolation Theorems by analysing the structure of the theories used to "describe " R . In particular the methods are used to prove a version of Feferman's Interpolation Theorem in a many-sorted language .

We give a characterization of those theories with the Amalgamation Property and the Strong Amalgamation Property . We conclude with a solution of an open problem of G. Grätzer .

To my wife :

but for whom this thesis would not
be in its present state .

Contents

CHAPTER 1

1.1 Notation

1.11 Set Theoretic

We use the standard notation for set theoretic concepts.

e.g. $\cap$ (intersection), $\cup$ (union) $-$ (difference), X (cross product) and ϕ for the empty set. We write $X \subset Y$ if X is a (not necessarily proper) subset of Y.

We use m, n etc. for integers and ω for the order type of the integers. Other ordinals will be denoted by μ, ν, κ etc. $\aleph_0$ is the cardinality of the integers, Card(ω).

A sequence of objects $a_1, \ldots, a_n$ will sometimes be thought of as the set $\{ a_1, \ldots, a_n \}$. The context will decide which case holds. The length of the above sequence, denoted by $lg(a_1, \ldots, a_n)$ is n.

1.12 Languages and Theories

We consider First Order Predicate Languages L, with equality, whose logical connectives are $\cap$, $\cup$, $\neg$, $\rightarrow$, $\leftrightarrow$ and quantifiers are $\forall$ and $\exists$. L may contain functions and individual constants as well as predicate letters. Terms, formulae, sub - formulae, sentences and other notions are defined as usual. (See e.g. [B.S] chapter 3.)

We use ϕ, ψ, θ, Φ etc. for formulae and write $\phi \in L$ if ϕ is a formula in L.

A set of sentences (in L) T is called a theory (in L) ; we write $T \vdash \phi$ if we can prove ϕ from T . If T is a theory we write L(T) for the language of T , and $L(\phi)$ for $L(\{\phi\})$.
If L is a language and $\hat{E}$ is a set of individual constant symbols , $L(\hat{E})$ is the language obtained from L by the addition of the individual constant symbols in $\hat{E}$. Const(L) is the set of individual constant symbols in L . Other notation used for extending languages will be defined or it will be obvious what is meant .

If ϕ is a formula in L and U is a unary predicate symbol, then ϕ^U is the relativization of ϕ to U . For a definition see e.g. [B.S] page 249.

We use t for some arbitrary true sentence and f will denote $\neg t$.

If $\phi(v_1,\dots,v_n)$ is a formula and for $1 \leqslant i \leqslant n$ $\bar{a}^i$ is a sequence of terms s.t. $lg(\bar{a}^i) = m$ (say) then

$$\wedge\!\!\wedge\phi(\bar{a}^1,\dots,\bar{a}^n) \text{ means } \bigwedge_{j<m}\phi(\bar{a}^1_j,\dots,\bar{a}^n_j) .$$

If $\bar{x}$ is a sequence of variables , $\bar{a}_x$ is a sequence of individual constants s.t. $lg(\bar{a}_x) = lg(\bar{x})$.
If the variables in ϕ (free or bound) include $\bar{x}$, then $\phi(\bar{a}_x)$ is obtained by replacing each variable $\bar{x}_j$ in ϕ by $(\bar{a}_x)_j$, and in case $\bar{x}_j$ was a bound variable in ϕ then the quantifiers of $\bar{x}_j$ are omitted , for $j \in lg(\bar{x})$.

e,g. $\exists x_1(x_1 < x_2)(\bar{a}_{x_1x_2})$ is $a_{x_1} < a_{x_2}$.

1.13 Models and L - structures

We use A,B,C,D and E as names for L - structures . The lamguage of A , L(A) is L . We assume the reader is familiar with the notion of satisfaction of formulae $\psi \in L$ in the L - structure A . In particular, for a theory T in L , $A \models T$ iff $A \models \psi$ for $\psi \in T$. Th(A) is the theory of A i.e.

$$\{ \psi : A \models \psi \quad \psi \text{ a sentence in } L(A) \}$$

Sometimes we do not distinguish between A and dom(A) , the domain of A . Thus ,for instance , $a \in A$ means $a \in \text{dom}(A)$. If A is an L - structure , A^+ is the L(dom(A)) - structure $(Aa)_{a \in A}$.

We write $A \simeq B$ if A is isomorphic to B ; $A \subset B$ if A is a substructure of B and $A \preccurlyeq B$ if A is an elementary substructure of B . If $X \subset \text{dom}(C)$ then $C \| X$ is the structure whose domain is the smallest subset of C extending X closed under the n-ary functions of C for $n \geqslant 1$ and for atomic formulae $\theta(\vec{v})$ in L(C) whose individual constants belong to Const($L(C\|X)$)

$$C\|X \models \theta(\vec{a}) \quad \text{iff} \quad C \models \theta(\vec{a})$$

where $\vec{a} \in \text{dom}(C\|X)$ and $L(C\|X)$ is the same as L(C) except that individual constants whose interpretations in C are not in $\text{dom}(C\|X)$ are omitted .
In particular if $\text{Const}(L(C))^C \subset X$ then $C\|X$ is the substructure of C generated by X .

$C|L$ is the L - structure obtained from C by omitting all the interpretations of symbols not occurring in L . (L(C) will always extend L when this notation is used).

If $\psi(v_1,\ldots,v_n)$ is a formula in L(C) then $\psi^C(v_1,\ldots,v_n)$ is the n-ary relation over C defined by $\langle c_1,\ldots,c_n\rangle \in \psi^C(v_1,\ldots,v_n)$ iff $C \models \psi[c_1,\ldots,c_n]$.

1.14 Canonical Structures .

A theory T is consistent if for no $\psi \in L(T)$ do we have $T \vdash \psi$ and $T \vdash \neg\psi$, it is s.t.b. (said to be) complete if for all sentences $\psi \in L(T)$ $T \vdash \psi$ or $T \vdash \neg\psi$.

We call a theory T a Henkin Theory iff (if and only if) for all $\psi(v_0, a_1,\ldots,a_n) \in L(T)$ there is a $c \in Const(L(T))$ s.t. (such that) the sentence

$$(\exists v_0 \psi(v_0 a_1,\ldots,a_n) \rightarrow \psi(c, a_1,\ldots,a_n)) \in T .$$

It is well known that every consistent theory T can be extended to a consistent Henkin Theory , T'. Where for some set of individual constants $\hat{E}$

$$L(T') = L(T)(\hat{E}) .$$

We call such a theory T' a Henkinization of T . If T is a consistent theory, there is a complete extension , T' s.t. L(T) = L(T') and if T is a Henkin Theory so is T'.

We call T' a H.C.C. extension of T if T' is a complete consistent extension of T which is a Henkin Theory . It suffices that T be consistent for such to exist .

If T' is a H.C.C. extention of T then T' is a conservative extension of T, i.e. for any sentence $\psi \in L(T)$ $T' \vdash \psi \Longrightarrow T \vdash \psi$

Let T be any consistent complete Henkin Theory, we define the canonical model $[T]$ of T to be the $L(T)$ structure whose domain is the set of closed terms in $L(T)$ factored by the equivalence relation $\sim$ defined by $\tau \sim \sigma$ iff $T \vdash \tau = \sigma$. For $c \in Const(T)$ $\hat{c}$ is the equivalence class under $\sim$ containing c. The relations and functions of $[T]$ are defined as usual, e.g.

if $R(\vec{v}) \in L(T)$ then

$$[T] \models R[\vec{\tau}] \text{ iff } T \vdash R(\vec{\tau}) .$$

It can easily be shown that $[T] \models T$.
For any L - structure A, $Th(A^+)$ is a Henkin Theory which is complete and consistent and

$$[Th(A^+)] \big| L \simeq A .$$

1.2 Acknowledgments

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CHAPTER 2

2.1 Introduction

Let L be a First Order Language and R be a binary relation between L-structures.

We say that R has a Preservation Theorem if we can define in some syntactically "simple" way a set of sentences Δ in L s.t.

for any sentence $\phi \in L$

$\forall A \forall B$ (A and B L-structures $A \models \phi$ and AR B imply $B \models \phi$)

iff ϕ is equivalent to some member of Δ

There are many generalizations of the above given in the literature, for example

a) We introduce a theory T in L and in the above add the further condition " $A \models T$ and $B \models T$" to the L.H.S. (left hand side) and replace " logically equivalent" by " equivalent under T " in the R.H.S.

b) We obtain an Interpolation Theorem for R if under the above conditions for Δ,

for any sentences ϕ , $\psi \in L$ we have

2.11 $\forall A \forall B$ (A , B L-structures $A \models \phi$ and A R B imply $B \models \psi$)

iff there is $\theta \in \Delta$ s.t. $\phi \vdash \theta$ and $\theta \vdash \psi$

In the above case θ is s.t.b. an interpolant for ϕ and ψ .

Many relations R have an interpolation theorem and hence a preservation theorem by substituting ϕ for ψ

For example if R is the relation between L-structures given by

A R B iff there is an embedding of A into B

then letting Δ be the set of existential sentences in L we obtain , as is well known , an interpolation theorem for this R.

Suppose we rewrite 2.11 by negating both sides to obtain

$\exists A \exists B$ (A , B L-structures s.t. $A \models \phi$ and A R B and $B \models \neg\psi$)

iff for no $\theta \in \Delta$ do we have $\phi \vdash \theta$ and $\neg\psi \vdash \neg\theta$

This reformulation makes sense if we

1) replace sentences ϕ , $\neg\psi$ by theories T_1 , T_2 in L

2) replace Δ by a set of ordered pairs of sentences (but still maintaining a similar condition on the simplicity of Δ)

to obtain a property that R might possess.

Namely

<u>2.12</u> For all theories T_1 T_2 in L

$\exists A \exists B$ (A,B L-structures s.t. A R B and $A \models T_1$ and $B \models T_2$)

iff for no $\langle\theta_1 , \theta_2\rangle \in \Delta$ do we have $T_1 \vdash \theta_1$ and $T_2 \vdash \theta_2$

This reformulation is now suitable for considering n-ary relations P , for by letting Δ be a set of n-tuples of sentences in L we obtain a meaningful property of P in the obvious way.

It is a further generalization of 2.12 that we shall consider in this thesis.

Since we have weakened the original condition on Δ, it might be supposed that for all relations R there is a Δ satisfying 2.12. The following Lemma suggests otherwise.

2.13 Lemma

There exists a binary relation R s.t. for no Δ does 2.12 hold.

Proof:

Let L be any language.

We define a binary relation R between L-structures as follows

$A \; R \; B$ iff A is finite.

Suppose a set of ordered pairs of sentences Δ exists s.t. 2.12 holds with this Δ.

We let $T_1 = \{\exists x_0 \ldots x_n \, (\bigwedge_{i \in n} \bigwedge_{\substack{j \in n \\ i \neq j}} x_i \neq x_j) \; n \in \omega\}$

$T_2 = \{\exists x \, (x = x) \}$

Since the L.H.S. of 2.12 cannot hold for this choice of T_1, T_2 there must be $<\theta_1, \theta_2> \in \Delta$ s.t.

$T_1 \vdash \theta_1$ and $T_2 \vdash \theta_2$

So there is a finite subset, say T_1' of T_1 s.t.
$T_1' \vdash \theta_1$

But clearly T_1' has a finite model.

It follows easily that we have a contradiction.

□

2.2 Simple Relations

Let L be any First Order Language.

For simplicity we consider binary relations in this section. The following definitions can be extended to include n-ary relations if required.

A relation R betwwen pairs of L-structures A, B often asserts the existence of a finite number of relations R_i : $i \in m$ s.t.

$R_i \subseteq A^{n_i} X B^{n_i}$ for some n_i $i \in m$

together with certain simple conditions on the R_i

For example

i) R_1 is a function from A^3 to B^3

ii) R_2 is an embedding of A into B

For such relations we can define a useful new language.

Let $\hat{B}$ be a set of individual constants s.t. $\hat{B} \cap \mathrm{Const}(L) = \phi$, for which there is a bijection

$1: \mathrm{Const}(L) \longrightarrow \hat{B}$

If L' is the language obtained from L by omitting all the individual constants in Const(L) then 1 induces in the natural way a bijection between the class of L-structures and the class of L'($\hat{B}$) structures.

When the context permits we shall not distinquish between L-structures and L'($\hat{B}$)-structures.

However, the reason for introducing the new set of individual constants will be seen from the next two definitions.

<u>Def</u>

We define $L^{n_1, n_2, \ldots, n_m}$ ($L^{\underline{n}}$) to be the language extending L by adding

the new individual constants $\hat{B}$

two new unary predicates U_1 U_2

m new predicates $R_1^{n_1},\dots,R_m^{n_m}$ where for

$1 \leqslant i \leqslant m$ $R_i^{n_i}$ is $2n_i$-ary .

For many binary relations R between L-structures there are $m \in \omega$ and $n_1,\dots,n_m \in \omega$ s.t. $L^{n_1,\dots,n_m}$ is a suitable language for discussing R .
The next definition defines the class of those relations we shall be interested in .

<u>2.21</u> <u>Def</u>

A binary relation R between L-structures is s.t.b. $(n_1,n_2,\dots,n_m)$-simple (or just $\underline{n}$ - simple) if there is a theory T in $L^{\underline{n}}$ s.t.

i) For any L-structures A and B

A R B iff $\exists$ C an $L^{\underline{n}}$-structure s.t.

a) $C \models T$ and U_1^C , U_2^C are closed under the functions in L .

b) $C\|U_1^C|L = A$ c) $C\|U_2^C|L'(\hat{B}) = B$.

For any $L^{\underline{n}}$-structure D we set $D_1 = D\|U_1^C|L$ and $D_2 = D\|U_2^C|L'(\hat{B})$.

ii) Whenever C and D are $L^{\underline{n}}$-structures s.t. $C \models T$ and U_1^C , U_2^C are closed under the functions in L , $f:C_1 \simeq D_1$ and $g:C_2 \simeq D_2$ and for $1 \leqslant i \leqslant m$

$\langle a_1,\dots,a_{n_i},b_1,\dots,b_{r_i}\rangle \in R_i^{n_i\,C} \cap ((U_1^C)^{n_i} X (U_2^C)^{n_i})$ iff

$\langle fa_1,\dots,fa_{n_i},gb_1,\dots,gb_{n_i}\rangle \in R_i^{n_i\,D} \cap ((U_1^D)^{n_i} X (U_2^D)^{n_i})$,

then $D \models T$.

Though the above definition is a little long, we claim that it is a natural definition to consider, indeed similar and related definitions can be found in [L] and [M], chapter 11.

Part 1) of the definition demands that if

$A \simeq A'$ and $B \simeq B'$ and $A\ R\ B$ then $A'\ R\ B'$.

Part ii) says that what is " going on " outside of C_1 and C_2 is irrelevant. Indeed we can make this more precise by demanding that in the definition T contains the set of sentences Σ which says

" U_i is nonempty and closed under the functions whose names occur in L. (for i=1,2)

U_1 contains the individual constants in L,

U_2 contains the individual constants in $\hat{B}$

and for $1 \leqslant i \leqslant m$

$R_i^{n_i} \subset ((U_1)^{n_i} \times (U_2)^{n_i})$ " in part i) of Def 2.21 .

Clearly Σ is a theory in L^n . Demanding that T contains Σ in i) has the effect of tidying up the definition (and our picture) without altering the concept of n-simple. In particular if an $L^{\underline{n}}$-structure C is s.t. $C \models \Sigma$ then the $\mathrm{dom}(C_i) = U_i^C$ for i = 1,2

If T is a theory in $L^{\underline{n}}$ and a binary relation R is n-simple by virtue of $T \cup \Sigma$ in Def. 2.21 then we call it T_R. (there may be more than one such T_R)

Conversely, if T is a theory in $L^{\underline{n}}$ and we define a binary relation R between L-structures A, B by

A R B iff $\exists$ an L^{n}- structure C s.t.

$C \models T \cup \Sigma$

$C_1 = A$

$C_2 = B$

then if R is $\underline{n}$-simple and some $T_R = T$ then we shall call R R_T.

Thus for $\underline{n}$ -simple relations we have defined theories T_R , and for certain theories T we have defined a binary relation R_T which is $\underline{n}$-simple.

Infact one can give a syntactic condition on T for which R_T is defined, see 3.4 .

For such T there is clearly a T_{R_T} s.t.

$$T = T_{R_T}$$

The relation of $\subset$ is not $\underline{n}$ -simple for any $\underline{n}$ but the relation of embedding is; as is isomorphism, homomorphism, end-extension (when suitably defined) and many other relations.

For the rest of this section let R be a fixed n-simple binary relation and choose some T_R.

<u>Def</u>

We write $A, R_1, \ldots, R_m, B \models T_R$ if for some L^{n}-structure C

$C \models T_R \cup \Sigma$

$C_1 = A$

$C_2 = B$

for $1 \leqslant i \leqslant m$

$R_i = R_i^{n_i C}$

It then follows from Def. 2.21 that A R B

Def

We say $T_1, R_1, \ldots, R_m, T_2$ is an n -sequence (in L) if for some set of indivual constants $\hat{E}$

T_1, T_2 are theories in $L(\hat{E})$

and for $1 \leqslant i \leqslant m$

$R_i \subset Const(L(T_1))^{n_i} \times Const(L(T_2))^{n_i}$

Def

If $T_1, R_1, \ldots, R_m, T_2$ is an n-sequence we write $T_1, R_1, \ldots, R_m, T_2 \models T_R$ iff

T_1 and T_2 are H.C.C. theories s.t.

$[T_1] \Vdash R_1', \ldots, R_m', [T_2] \Vdash \models T_R$

where for $1 \leqslant i \leqslant m$

$\bar{\bar{a}}\ R_i'\ \bar{\bar{b}}$ iff $\bar{a}\ R_i\ \bar{b}$.

If γ is an n-sequence $T_1, R_1, \ldots, R_m, T_2$ we write T_1^{γ} for T_1 R_1^{γ} for R_1 and so on.

Def

If γ and δ are n-sequences we write

$\gamma \subset \delta$ iff $T_i^{\gamma} \subset T_i^{\delta}$ $i = 1,2$

and $R_i^{\gamma} \subset R_i^{\delta}$ $1 \leqslant i \leqslant m$.

2.22 Def

We say an n-sequence γ is an approximation to T_R if there is an n-sequence δ s.t.

$\gamma \subset \delta$ and $\delta \models T_R$

If γ is an approximation to T_R and T is another possible choice of T_R then γ may not be an approximation to T. For consider the (1)-simple relation P for which $A\,P\,B$ iff A , B are L-structures . A suitable T_P is

$$T_1 = \{ \exists v_0 \in U_1\ \exists v_1 \in U_2\ (v_0\ P^1_1\ v_1\ ) \}$$

but so also is

$$T_2 = \{ \neg (\exists v_0 \in U_1\ \exists v_1 \in U_2\ (v_0\ P^1_1\ v_1\ )) \}$$

Let γ be $a = a$, $\{ \langle a , b \rangle \}$, $b = b$.

Then γ is an approximation to T_1 but not T_2 .

This problem does not arise , however , for those $\underline{n}$-sequences whose relations are empty. More precisely if γ is an $\underline{n}$-sequence of the form $T_1, \phi, \dots, \phi, T_2$ and G_1 and G_2 are two possible choices for T_R then γ is an approximation to G_1 iff γ is an approximation to G_2 . For if γ is an approximation to G_1 then there are $A \models T_1$ $B \models T_2$ and suitable $R_1, \dots, R_m$ s.t.

$$A, R_1, \dots, R_m, B \models G_1$$

So $A\,R\,B$

Hence there is $C \models G_2 \cup \Sigma$ s.t.

$$C_1 = A \quad \text{and} \quad C_2 = B ,$$

So $A, (R_1^{n_1})^C, \dots, (R_m^{n_m})^C, B \models G_2$

and so γ is an approximation to G_2

Symmetry gives us our result.

2.3 $\underline{n}$ - sets

We suppose that $L^{\underline{n}}$ has the variables of the form $v_{jik\rho}$ for

$o \leqslant j \leqslant 1$, $o \leqslant i \leqslant m$, $1 \leqslant k \leqslant n_i$ when $1 \leqslant i \leqslant m$

$k = o$ when $i = o$, $\rho \in \omega$.

For variables of the form $v_{0ik\rho}$ we write $x_{ik\rho}$ and refer to them as x-variables.

For variables of the form $v_{1ik\rho}$ we write $y_{ik\rho}$ and refer to them as y-variables.

$\vec{x}$ with or without subscripts denotes a sequence of x-variables. Similarly for $\vec{y}$.

We say $\vec{x}$ corresponds to $\vec{y}$ if they are of the same length and for $j < lg(\vec{x})$ if $x_j = v_{0ik\rho}$ then $y_j = v_{1ik\rho}$.

If we use $\vec{x},\vec{y}$ (with the same subscripts) in the same context then they will correspond.

We say $\vec{x}$ is a complete sequence if whenever $x_{it\rho} \in \vec{x}$ where $1 \leqslant i \leqslant m$ and $1 \leqslant t \leqslant n_i$ then $x_{is\rho} \in \vec{x}$ for $1 \leqslant s \leqslant n_i$

We say $\vec{x}$ is similar to $\vec{x}_1$ if

$$lg(\vec{x}) = lg(\vec{x}_1)$$

and for some function $f; \{1,\dots,m\} \times \omega \longrightarrow \omega$ for $j < lg(\vec{x})$, if the j^{th} element of $\vec{x}$ is $x_{it\rho}$ then the j^{th} element of $\vec{x}_1$ is $x_{it\,f(i\rho)}$.

The above definitions enable us to simplify later definitions.

If $\phi_1(\vec{x})$, $\phi_2(\vec{x}_1)$ are formulae in L then we write $\phi_1(\vec{x}) \sim \phi_2(\vec{x}_1)$ if $\vec{x}$ is similar to $\vec{x}_1$ and $\phi_2(\vec{x})$ is obtained from $\phi_1(\vec{x})$ by (possibly) changing bound variables, in such a way that no free variable becomes bound and no bound variable occurs among the $\vec{x}_1$. For further information on the notions involved here see [B.S] page 53.

A similar definition is assumed for formulae containing y-variables free, rather than x-variables.

We give now an important definition of a class of ordered pairs which represents the possible choices of those Δ occurring in 2.12 . The justification for this will be seen in Theorem 2.42 below.

2.31 Def

A set of ordered pairs of formulae (in L) Δ is called an n-set (in L) if

i) If $<\phi_1(\vec{v}), \phi_2(\vec{v}_1)> \in \Delta$ where $\vec{v}, \vec{v}_1$ are precisely the free variables occurring in ϕ_1, ϕ_2 respectively then $\vec{v}$ are x-variables and $\vec{v}_1$ arey-variables and $\vec{v}$ corresponds to $\vec{v}_1$

ii) If $<\phi_1(\vec{x}), \phi_2(\vec{y})> \in \Delta$ then $\vec{x}$ is complete and if $x_{i\,t\rho} \in \vec{x}$ then $1 \leqslant i \leqslant m$.

By i) $\vec{x}$ corresponds to $\vec{y}$, though this also follows from our convention.

iii) $\{ <t,f> , <f,t> \} \subset \Delta$

iv) If $<\phi_1,\phi_2> \in \Delta$ and $<\theta_1,\theta_2> \in \Delta$ then $<\phi_1 \cap \theta_1 , \phi_2 \cup \theta_2> \in \Delta$ and $<\phi_1 \cup \theta_1 , \phi_2 \cap \theta_2> \in \Delta$

v) If $<\phi_1(\vec{x}) , \phi_2(\vec{y})> \in \Delta$ and $\theta_1(\vec{x}_1) , \theta_2(\vec{y}_1)$ are s.t. $\phi_1(\vec{x}) \sim \theta_1(\vec{x}_1)$ and $\phi_2(\vec{y}) \sim \theta_2(\vec{y}_1)$ then $<\theta_1(\vec{x}_1) , \theta_2(\vec{y}_1)> \in \Delta$.

(By our convention i) still holds).

If such is the case we write $<\phi_1(\vec{x}),\phi_2(\vec{y})> \sim <\theta_1(\vec{x}_1),\theta_2(\vec{y}_1)>$

The following facts are easily proved.

a) The intersection of a set of n -sets (in L) is an n set (in L).

b) Any set of pairs of formulae satisfying i), ii) of the definition can be extended to a unique smallest n-set.

c) { <t,f> , <f,t> } is considered to be an n-set for any n.

2.4 Goodness

We now link n-sets even more closely with 2.12.

Def

If γ is an n-sequence and $\vec{x}$ a sequence of x-variables, then we say

$\vec{a}$ and $\vec{b}$ are γ consistent for $\vec{x}$ if

i) $\vec{a} \in \mathrm{Const}(L(T_1^\gamma))$ s.t. $\lg(\vec{a}) = \lg(\vec{x})$

ii) $\vec{b} \in \mathrm{Const}(L(T_2^\gamma))$ s.t. $\lg(\vec{b}) = \lg(\vec{x})$

iii) whenever $v_{j1},\ldots,v_{jn_i}$ contained in $\vec{x}$ isof the form $x_{i1\rho},\ldots,x_{in_i\rho}$ where $1 \leqslant i \leqslant m$ then

$$< a_{j1},\ldots,a_{jn_i}, b_{j1},\ldots,b_{jn_i} > \in R_i^\gamma$$

2.41 Def

If Δ is an n-set , an n-sequence γ is s.t.b. Δ good if whenever $<\phi_1(\vec{x}),\phi_2(\vec{y})> \in \Delta$ and

$\vec{a}$ and $\vec{b}$ are γ consistent for $\vec{x}$

we do not have

$$T_1^\gamma \vdash \phi_1\vec{a} \quad \text{and} \quad T_2^\gamma \vdash \varphi_2\vec{b} .$$

If γ is not Δ good we say γ is Δ bad.

The following Theorem collects up some of the facts following from the definitions .

2.42 Theorem

Let Δ be any $\underline{n}$-set (in L)

a) If γ and δ are $\underline{n}$-sequences s.t. $\gamma \subset \delta$ then if δ is Δ good, γ is Δ good

b) If the $\underline{n}$-sequence $\gamma = T_1, R_1, \ldots, R_m, T_2$ is Δ good then there is an extension of γ of the form $T_1^2, R_1, \ldots, R_m, T_2^2$ which is Δ good, where T_1^2 and T_2^2 are H.C.C. theories in some $L(\hat{E})$.

(See Chapter 1 for the definition of H.C.C)

c) If $\{\gamma_\alpha\}_{\alpha<\mu}$ is a sequence of $\underline{n}$-sequences s.t. $\gamma_\alpha \subset \gamma_\beta$ for $\alpha \leq \beta < \mu$ and γ_α is Δ good $\alpha<\mu$ then $\bigcup_{\alpha<\mu} \gamma_\alpha = \bigcup_{\alpha<\mu} T_1^{\gamma_\alpha}, \bigcup_{\alpha<\mu} R_1^{\gamma_\alpha}, \ldots, \bigcup_{\alpha<\mu} R_m^{\gamma_\alpha}, \bigcup_{\alpha<\mu} T_2^{\gamma_\alpha}$ is Δ good

Proof

Part a) follows from the definition 2.41

Part b); Since γ is Δ good T_1^γ, T_2^γ are both consistent (See 2.31 iii))

We can Henkinize T_1^γ, T_2^γ to obtain T_1^0, T_2^0 say.

It can easily be checked that $T_1^0, R_1, \ldots, R_m, T_2^0$ is Δ good, by the conservative property of Henkinization.

We can complete T_1^0 and T_2^0 resp. still remaining good by 2.31 iv)

For suppose $T_1^1, R_1, \ldots, R_m, T_2^1$ is Δ good and for some sentence $\phi \in L(T_1^1)$

$$T_1^1 \nvdash \phi \quad \text{and} \quad T_1^1 \nvdash \neg\phi$$

I claim that

$\gamma_1 = T_1^1 \cup \{\phi\}, R_1, \ldots, R_m, T_2^1$ is Δ good or

$\gamma_2 = T_1^1 \cup \{\neg\phi\}, R_1, \ldots, R_m, T_2^1$ is Δ good .

For if not there will be $\langle\phi_1(\vec{x}), \phi_2(\vec{y})\rangle \in \Delta$ and $\langle\sigma_1(\vec{x}_1), \theta_2(\vec{y}_1)\rangle \in \Delta$ and constants

$\vec{a}$, $\vec{b}$ γ_1 consistent for $\vec{x}$ and

$\vec{c}$, $\vec{d}$ γ_2 consistent for $\vec{x}_1$ s.t.

$$T_1^1 \cup \{\phi\} \vdash \phi_1\vec{a} \quad \text{and} \quad T_2^1 \vdash \phi_2\vec{b}$$

$$T_1^1 \cup \{\neg\phi\} \vdash \theta_1\vec{c} \quad \text{and} \quad T_2^1 \vdash \theta_2\vec{d}$$

and hence

$$T_1^1 \vdash \phi_1\vec{a} \cup \theta_1\vec{c} \quad \text{and} \quad T_2^1 \vdash \phi_2\vec{b} \cap \theta_2\vec{d}$$

w.l.o.g. we may suppose $\vec{x} \cap \vec{x}_1 = \phi$

it then follows easily that

$T_1^1, R_1, \ldots, R_m, T_2^1$ is not Δ good which contradicts our supposition, hence the claim follows .

Repeated application of the above proceedure for both T_1^1 and T_2^1 ensures our result.

Part c) again follows from the definition 2.41

□

<u>Remark</u>

I was tempted to say that the <u>n</u>-simple binary relation R was Syntactically Characterizable if there was a T_R and an <u>n</u>-set Δ defined in a syntactically simple way s.t. for any <u>n</u>-sequence γ

2.43 γ is an approximation to T_R iff γ is Δ good.

However, in the above , the word "simple " is very loose. That care must be exercised so as not to obtain a trivial result is shown by the following :

<u>2.44 Lemma</u>

If R is a (1)-simple binary relation, for any T_R there is a (1)-set Δ s.t. 2.43 holds.

<u>Proof</u>

Let Δ_1 be the set of pairs of formulae $\langle\phi_1(\vec{x}), \phi_2(\vec{y})\rangle$ (in L) where

$$T_R \vdash \forall \bar{x} \in U_1 \forall \bar{y} \in U_2 ((\phi_1 \bar{x})^{U_1} \cap \Lambda\Lambda\, \bar{x} R_1 \bar{y} \rightarrow (\neg \phi_2(\bar{y}))^{U_2})$$

where $\bar{x}$ corresponds to $\bar{y}$ (i.e. our convention holds even for bound variables in the same context) and each variable in $\bar{x}$ is of the form $x_{11\rho}$ for $\rho \in \omega$

With these restrictions Δ_1 satisfies 2.31 i) ii) and so can be extended to a unique smallest (1)-set Δ say. That 2.43 holds for this Δ follows by a simple compactness argument and definition 2.31 .

□

Fortunately it is possible to give a very precise definition of " simple " . Indeed the next section is devoted to this. It will be seen later that the definition is a nice extension of the usual vague ideas of "simple " .

2.5 Operators

Let IL be the language obtained from L by the addition of a set $\{ X_n : n \in \omega \}$ of propositional variables. We are not interested in these variables other than as markers . They behave just like atomic formulae in the formation of formulae

2.51 Def

An s - operator in L is an ordered s-tuple of formulae in IL. $<\Phi_1, \dots, \Phi_s>$

We shall be interested only in the cases when s = 1,2

2.52 Def

If s = 1,2 and Δ is a set of s-tuples, where in case s = 2 Δ is an n-set for some n and K is a set of s - operators in L then $K[\Delta]$

is defined to be the least set Δ' of s-tuples of formulae in L s.t.

i) $\Delta \subset \Delta'$

ii) If $<\Phi_1(X_1,\dots,X_p),\Phi_s(X_1,\dots,X_p)> \in K$ where for $1 \leqslant j \leqslant s$ $X_1,\dots,X_p$ include all the propositional variables in Φ and if $<\phi_1^i\ \phi_s^i> \in \Delta'$ for $1 \leqslant i \leqslant p$ then $<\Phi_1(\phi_1^1,\dots,\phi_1^p),\Phi_s(\phi_s^1,\dots,\phi_s^p)> \in \Delta'$

iii) If $<\phi_1,\phi_s> \in \Delta'$ and $<\phi_1,\phi_s> \sim <\theta_1,\theta_s>$ (where $<\theta_1,\theta_2>$ is an s-tuple) then $<\theta_1,\theta_2> \in \Delta'$

For the notion of $\sim$ in case s = 1, 2 see 2.31.

As an example, the set of existential formulae (in L can be described as

$$\{ <\exists vX>, <X_1 \cap X_2>, <X_1 \cup X_2> \}[Z]$$

where Z is the set of atomic and negated atomic formulae in L ,

In case s = 1 this definition extends notions introduced by Keisler in [K_1]

In case s = 2 this will enable us to describe new n-sets from certain theories and old n-sets. As the following suggests.

2.53 Π - Sentences

Suppose L is our fixed language and $L'(\hat{B})$ is as defined in 2.2 . Assume a L^n has been chosen for some n .We assume in what follows all the variables are in L^n (See 2.3)

A formula of type 1 is a formula of form

$$(\theta(\vec{x}))^{U_1} \text{ where } \theta(\vec{x}) \in L$$

A formula of type 2 is a formula of form

$$(\theta(\vec{y}))^{U_2} \text{ where } \theta(\vec{y}) \in L'(\hat{B})$$

Just as we make the convention that if $\vec{x}$ and $\vec{y}$ occur in the same context they correspond , so we make the convention that if the type 1 formula $(\theta(\vec{x}))^{U_1}$ and the type 2 formula $(\theta(\vec{y}))^{U_2}$ occur in the same context , then $\theta(\vec{y})$ is obtained from $\theta(\vec{x})$ by replacing each individual constant a in $\theta(\vec{x})$ by l(a) (defined in 2.2) and replacing $\vec{x}$ by the corresponding sequence $\vec{y}$.

A formula of type 3 is any finite conjunction of formulae of the form $R_i^{n_i}(x_{i1\rho},\dots,x_{in_i\rho},y_{i1\rho},\dots,y_{in_i\rho})$ for $1\leqslant i\leqslant m$. Thus if $\psi(\vec{x},\vec{y})$ is a formula of type 3 then $\vec{x}$ corresponds to $\vec{y}$ and $\vec{x}$ is complete.

Let W be the set of formulae of the form

$$\bigvee_{k\in t}(\exists \vec{x}_{k_1}\in U_1\exists \vec{y}_{k_2}\in U_2(\theta_{k_1}{}^{U_1}\cap\theta_{k_2}{}^{U_2}\cap\theta_{k_3}))$$

where $\vec{x}_{k_1},\vec{y}_{k_2}$ occur in θ_{k_3} for $k\in t$ and we assume through out that $\theta_{k_1}{}^{U_1}$ is a formula of type 1 $\theta_{k_2}{}^{U_2}$ is a formula of type 2 and θ_{k_3} is a formula of type 3 .

Let S_1 be the set of 1- operators of the form a) or b) viz:

a) $\bigvee_{k\in t}(\exists x_{k_1}\in U_1\exists y_{k_2}\in U_2(\ \theta_{k_1}{}^{U_1}\cap\ \theta_{k_2}{}^{U}{}_2\cap\theta_{k_3}\cap X_k\ ))$

where if a variable in $\vec{x}$,$\vec{y}$ of form $v_{jit\rho}$ for $i\geqslant 1$ occurs , then it occurs in θ_{k_3}

Note that a variable of form $v_{j00\rho}$ cannot occur in a formula of type 3 .

b) $\forall \vec{x}_1\in U_1\forall \vec{y}_2\in U_2\ (\theta_1{}^{U_1}\cap\theta_2{}^{U_2}\rightarrow X_1)$

where $\theta_1{}^{U_1}$ is a formula of type 1 and $\theta_2{}^{U_2}$ is a formula of type 2 .

Let S_2 be the set of 1 - operators of the form

$\forall \vec{x}_1\in U_1\forall \vec{y}_2\in U_2(\theta_3\rightarrow X_1)$ where θ_3 is of type 3 .

2.54 Def

A Π -sentence (in L) is a sentence in $S_2[S_1[W]]$ Φ s.t. whenever a variable of form $v_{jik\rho}$ where $1 \leqslant i \leqslant m$ occurs in such a sentence then it occurs in a sub-formula of type 3 ; where sub-formulae of form t^{U_1} ,t^{U_2} may be omitted, and if Φ is of the form $\forall \vec{x}_1 \in U_1 \forall \vec{y}_1 \in U_2 (\theta_3 \rightarrow \Theta)$ then every variable in θ_3 occurs free in Θ .

2.55 Remark

If T is any set of Π-sentences in L^{n} then R_T is defined , as can be seen from the fact that all quantifiers are bounded . For the definition of R_T see 2.21 .In such a case we write T for T_{R_T}
As an example we describe the relation of embedding between L-structures as a (1)-simple relation defined by:

$$\forall x_{111} \in U_1 \; \exists y_{111} \in U_2 \; (x_{111} R_1^1 y_{111})$$

$$\forall \vec{x} \in U_1 \forall \vec{y} \in U_2 \; (\Lambda\Lambda \; \vec{x} R_1^1 \vec{y} \longrightarrow (\theta \vec{x}^{U_1} \longrightarrow \theta \vec{y}^{U_2}))$$ where

$\Lambda\Lambda \; \vec{x} R_1^1 \vec{y}$ is a formula of type 3 and $\theta(\vec{x})$ is an atomic or negated atomic formula in L .
Ofcourse each of the above sentences are Π -sentences in $L^{(1)}$ if we allow ourselves to omit t^{U_1} and t^{U_2} from the formulae, which we do.
It is not at all clear as to why we have been so painstakingly precise with the variables. Part of the reason is so that from Π -sentences we can define 2- operators with which we define n-sets which in turn have a good chance of satisfying 2.59 below.

2.56 The OP Function

Let Φ be a formula in $S_2[S_1[W]]$; we define OP(Φ) to be a 2 - operator by induction on the complexity of Φ .

Case 1 Φ is of the form

$$\forall x_1 x_2 \in U_1 \ \forall y_3 y_4 \in U_2 (\theta_3 \rightarrow \Theta)$$

where $\Theta \in S_2[S_1[W]]$ θ_3 is a formula of type 3 and we suppose x_1, y_3 occur in θ_3 and x_2, y_4 do not.

Then $OP_1(\Phi) = \exists x_2(OP_1(\Theta))$

$OP_2(\Phi) = \exists y_4(OP_2(\Theta))$

$OP(\Phi) = \langle OP_1\Phi, OP_2\Phi \rangle$

Case 2 Φ is of the form

$$\bigvee_{k \in t} \exists x_{k_1} \in U_1 \ \exists y_{k_2} \in U_2 \ (\theta_{k_1}^{U_1} \cap \theta_{k_2}^{U_2} \cap \theta_{k_3} \cap \Theta_k)$$

where for $k \in t$ $\Theta_k \in S_1[W]$

Then $OP_1(\Phi) = \bigwedge_{k \in t} \forall x_{k_1} (\theta_{k_1} \rightarrow OP_1(\Theta_k))$

$OP_2(\Phi) = \bigwedge_{k \in t} \forall y_{k_2} (\theta_{k_2} \rightarrow OP_2(\Theta_k))$

$OP(\Phi) = \langle OP_1(\Phi), OP_2(\Phi) \rangle$

Case 3 Φ is of the form

$$\forall x_1 \in U_1 \ \forall y_2 \in U_2 (\theta_1^{U_1} \cap \theta_2^{U_2} \rightarrow \Theta)$$

where $\Theta \in S_1[W]$

$OP_1(\Phi) = \exists x_1(\theta_1 \cap OP_1(\Theta))$

$OP_2(\Phi) = \exists y_2(\theta_2 \cap OP_2(\Theta))$

$OP(\Phi) = \langle OP_1(\Phi), OP_2(\Phi) \rangle$

Case 4 Φ is of the form

$$\bigvee_{k \in t} \exists x_k \in U_1 \exists y_k \in U_2(\theta_{k_1}^{U_1} \cap \theta_{k_2}^{U_2} \cap \theta_3)$$

$OP_1(\Phi) = \bigwedge_{k \in t} \forall x_{k_1} (\theta_1 \rightarrow X_k)$

$OP_2(\Phi) = \bigwedge_{k \in t} \forall y_{k_2} (\theta_2 \rightarrow X_k)$

$OP(\Phi) = \langle OP_1(\Phi) \quad OP_2(\Phi) \rangle$

If formulae of type 1 or 2 of form t^{U_1}, t^{U_2} occur or have been omitted in Φ, then $OP_1\Phi$ and $OP_2\Phi$ omit the formulae and the logical connective immediately following. Thus for instance the 2 - operators obtained from the sentences in Remark 2.55 become :

$\langle \exists x_{111}(X_1), \forall y_{111}(X_1) \rangle$ and $\langle \theta x \cap X_1, \theta y \rightarrow X_1 \rangle$

As a more complicated example let Φ be the Π-sentence in $L^{(1)}$

$$\forall x_{111}\in U_1 \forall y_{111}\in U_2 (x_{111} R_1^1 y_{111} \rightarrow (\forall x_{112}\in U_1 ((x_{111}<x_{112})^{U_1} \rightarrow$$
$$\rightarrow \exists y_{112}\in U_2 ((y_{111}<y_{112})^{U_2} \cap x_{112} R_1^1 y_{112})))$$

then $OP(\Phi)$ is

$$< \exists x_{112}(x_{111}<x_{112} \cap X), \forall y_{112}(y_{111}<y_{112} \rightarrow X) >$$

A point to note is that if Φ is any Π-sentence then the free individual variables in $OP_1\Phi$ correspond to the free individual variables in $OP_2(\Phi)$ and each forms a complete sequence.

If T is a set of Π - sentences in $L^{\underline{n}}$ then

$$OP(T) = \{ OP(\Phi) : \Phi \in T \}$$

<u>2.57</u> <u>Lemma</u>

If T is a set of Π-sentences in $L^{\underline{n}}$ and Δ is an $\underline{n}$-set then

$$OP(T) \cup \{ <X_1 \cap X_2, X_1 \cup X_2 >, <X_1 \cup X_2, X_1 \cap X_2> \}[\Delta]$$

is also an $\underline{n}$-set.

We write $OP(T)[[\Delta]]$ for the above set.

<u>Proof</u>

We sketch the proof.

iii) of Def 2.31 follows since Δ is an $\underline{n}$-set

iv) and v) of def 2.31 follow trivially from Def 2.52.

i) and ii) follow from the following facts.

From Def 2.54 every variable of the form $v_{jik\rho}$ for $1 \leq i \leq m$ occurs in a formula of type 3. Since we are dealing with sentences in 2.54 it follows from the definition of formulae of type 3 that the variables of the form $v_{0ik\rho}$ for $1 \leq i \leq m$

form a complete sequence as do the variables of form $v_{\ddagger i k \rho}$, which clearly correspond . In view of the point made prior to the lemma the x-variables and y-variables which are quantified in $OP_1(\Phi)$ and $OP_2(\Phi)$ resp. form complete and corresponding sequences. Thus it follows that if $OP(\Phi)$ is applied to pairs of formulae satisfying i) ,ii) of 2.31 then so does the resulting pair of formulae. Induction will give the result.

□

2.58 SYNTACTIC CHARACTERIZATIONS DEF.

We say that an n- simple binary relation R between L -structures is Syntactically Characterizable (written S.C.) if there is a T_R which is a set of Π - sentences in L^n s.t. for any n-sequence γ

2.59 γ is an approximation to T_R iff
γ is $OP(T_R)[[\{<t,f>,<f,t>\}]]$ good.

If T is a set of Π - sentences in L^n then we write $[[OP(T)]]$ for the n -set $OP(T)[[\{<t,f>,<f,t>\}]]$

The reader may care to return to section 2.1 and the end of section 2.4 to compare the above with the notions developed there .

As an example, if R is the relation of embedding between L-structures , T_R is chosen as in 2.55 then $[[OP(T_R)]]$ becomes the set of pairs of formulae $<\theta_1\vec{x}\ \theta_2\vec{y}>$ where $\theta_1\vec{x}$ is existential and $\theta_2\vec{y}$ is the negation normal form of $\neg\theta_1\vec{x}$ (with suitable conditions on the variables) .

2.5 10 Remark

In order to show that every approximation to T_R is $[[OPT_R]]$ good it suffices to show that every n-seqnence γ s.t. $\gamma \models T_R$ is $[[OPT_R]]$ good by Theorem 2.42 a) .

If Δ is an n -set satisfying 2.59 for some T_R then Δ is called a notion of goodness for R .

The main problem for the rest of Chapter 2 and Chapter 3 is to characterize a large class of n -simple binary relations which are S.C.

In the usual proofs of Interpolation Theorems there is in proving the corresponding assertions to 2.59 an "easy" direction and a "hard" direction . This remains true in our case . The next section is devoted to proving a result about the "easy" direction.

2 .6 Theorem

If T is a set of Π- sentences , Δ is an n -set and γ is a T approximation which is Δ good then γ is $OP(T)[[\Delta]]$ good .

Proof

For an understanding of " T approximation " see Remark 2.55.

It suffices to show that if $\gamma \models T$ and γ is Δ good then γ is $OP(T)[[\Delta]]$ good .

Suppose γ is not $OP(T)[[\Delta]]$ good , then there will be $<\theta_1\mathfrak{x}, \theta_2\mathfrak{y}> \in OP(T)[[\Delta]]$ and constants $\mathfrak{a}$, $\mathfrak{b}$ s.t. $\mathfrak{a}$ and $\mathfrak{b}$ are γ consistent for $\mathfrak{x}$ where

$$T_1^{\gamma} \vdash \theta_1\mathfrak{a} \quad \text{and} \quad T_2^{\gamma} \vdash \theta_2\mathfrak{b} \quad .$$

It is easy to see that we can assume that $<\theta_1\mathfrak{x}, \theta_2\mathfrak{y}>$ is of the form :

$<OP_1\Phi(X_1,\ldots X_\rho)[\phi_{11},\ldots,\phi_{\rho 1}], OP_2\Phi(X_1,\ldots,X_\rho)[\phi_{12},\ldots,\phi_{\rho 2}]>$
for some $\Phi \in T$ where $<\phi_{i1}\ \phi_{i2}> \in OP(T)[[\Delta]]$
for $1 \leqslant i \leqslant p$. We shall show that for some $i, 1 \leqslant i \leqslant p$, constants $\vec{c}$,$\vec{d}$ γ consistant for some $\vec{x}_1$ can be found s.t. $T_1^\gamma \vdash \phi_{i1}(\vec{a},\vec{c})$ and $T_2^\gamma \vdash \phi_{i2}(\vec{b},\vec{d})$
where $<\phi_{i1},\phi_{i2}>$ is of the form $<\phi_{i1}(\vec{x},\vec{x}_1),\phi_{i2}(\vec{y},\vec{y}_1)>$
Thus reducing the complexity of $<\theta_1,\theta_2>$.
Having proved this it follows easily that γ is not Δ good contradicting the choice of γ .
The method we employ is to show that infact we can reduce the complexity of Φ.

We must in general deal with an arbitrary formula in $S_2[S_1[W]]$ (which will be a sub-formula of Φ)
Suppose For our induction hypothesis we have

i) $\Omega(\vec{x}_1\vec{y}_2)$ is a formula in $S_2[S_1[W]]$

ii) $\gamma \models \Omega(\vec{a}\ \vec{b})$ (this makes sense as $\Omega(\vec{a}\ \vec{b})$ is a sentence and so a T_R for some R)
and $\vec{a}\ \vec{b}$ are γ consistent for those variables in $\vec{x}_1$ of form $v_{1ik\rho}$ for $1 \leqslant i \leqslant m$ which do not occur in a sub-formula of Ω of type 3 .

iii) $T_1^\gamma \vdash OP_1\ \Omega(\overline{\phi_{i1}})(\vec{a}\ \vec{a}_{x_3})$

iv) $T_2^\gamma \vdash OP_2\ \Omega(\overline{\phi_{i2}})(\vec{b}\ \vec{b}_{y_4})$

where we assume the free variables of $OP_1\Omega$ are $\vec{x}_1\ \vec{x}_3$ and the free variables of $OP_2\Omega$ are $\vec{y}_2\ \vec{y}_4$ and $\vec{a}\ \vec{a}_{x_3}\ \vec{b}\ \vec{b}_{y_4}$ are γ consistant for those variables in $\vec{x}_1\ \vec{x}_3$ of form $v_{1ik\rho}$ for $1 \leqslant i \leqslant m$ which do not occur free in a sub-formula of Ω of type 3

Clearly i) ii) iii) iv) hold for Φ in place of Ω.
We now show how to reduce Ω by induction .

Case 1

Suppose Ω is of form

$\forall \vec{x}_3 \vec{x}_5 \in U_1 \forall \vec{y}_4 \vec{y}_6 \in U_2 (\theta_3 \to \Theta)$ where Θ is in $S_2[S_1[W]]$

$\vec{x}_3$ $\vec{y}_4$ occur in the formula of type 3 θ_3

and $\vec{x}_5$ $\vec{y}_6$ do not .

Then by iii) and iv)

$T_1^{\gamma} \vdash \exists \vec{x}_5 (OP_1(\Theta)[\overline{\phi_{i1}}][\vec{a}\ \vec{a}_{x_3}]$

$T_2^{\gamma} \vdash \exists \vec{y}_6 (OP_2(\Theta)[\overline{\phi_{i2}}][\vec{b}\ \vec{b}_{y_4}]$

Hence we can find $\vec{a}_{x_5} \in Const(T_1^{\gamma})$ and $\vec{b}_{y_6} \in Const(T_2^{\gamma})$

s.t.

$T_1^{\gamma} \vdash OP_1(\Theta)[\overline{\phi_{i1}}][\vec{a}\vec{a}_{x_3} \vec{a}_{x_5}]$

$T_2^{\gamma} \vdash OP_2(\Theta)[\overline{\phi_{i2}}][\vec{b}\vec{b}_{y_4} b_{y_6}]$.

Since Ω is a subformula of Φ a Π-sentence , the variables $\vec{x}_5 \vec{y}_6$ of form v_{jikp} for $1 \leqslant i \leqslant m$ occur free in a subformula of Θ of type 3 . So iii) iv) hold for Θ in place of Ω . It is easy to see that i) ii) hold for Θ in place of Ω , since $\gamma \models \Theta[\vec{a}\vec{a}_{x_3} \vec{a}_{x_5} \vec{b}\vec{b}_{y_4} \vec{b}_{y_6}]$

Case 2

Suppose Ω is of the form

$\bigvee_{k \in t} \exists \vec{x}_{k_1} \in U_1 \exists \vec{y}_{k_2} \in U_2 (\theta_{k_1}^{U_1} \cap \theta_{k_2}^{U_2} \cap \theta_{k_3} \cap \Theta_k)$

where $\Theta_k \in S_1[W]$ for $k \in t$

Then by assumption

a) $T_1^{\gamma} \vdash \bigwedge_{k \in t} \forall \vec{x}_{k_1} (\theta_{k_1} \to OP_1(\Theta))[\overline{\phi_{i1}}][\vec{a}]$

b) $T_2^{\gamma} \vdash \bigwedge_{k \in t} \forall \vec{y}_{k_2} (\theta_{k_2} \to OP_2(\Theta))[\overline{\phi_{i2}}][\vec{b}]$

and since ii) holds , for some $m \in t$

$\gamma \models \exists \vec{x}_{m_1} \in U_1 \exists \vec{y}_{m_2} \in U_2 (\theta_{m_1}^{U_1} \cap \theta_{m_2}^{U_2} \cap \theta_{m_3} \cap \Theta_m)[\vec{a}\vec{b}]$

So we may chose $\vec{a}_{x_{m_1}}$ ($\vec{b}_{y_{m_2}}$) $\in Const(T_1^{\gamma})$ (T_2^{γ})

s.t.

$\gamma \models (\theta_{m_1}^{U_1} \cap \theta_{m_2}^{U_2} \cap \theta_{m_3} \cap \Theta_m)[\vec{a}\vec{a}_{x_{m_1}} \vec{b}\vec{b}_{y_{m_2}}]$

As in Case 1 those variables in $\vec{x}_{m_1}$ $\vec{y}_{m_2}$ which do not occur in θ_{m_3}, occur free in some subformula of Θ of type 3 . So by induction hypothesis ii) holds for Θ , as does i) .

It follows from a) b) that iii) iv) hold , since $T_1^\gamma \vdash OP_1(\Theta)[\overline{\phi_{i1}}][\mathfrak{a}\mathfrak{a}_{x_m}]$ and

$T_2^\gamma \vdash OP_2(\Theta)[\overline{\phi_{i2}}][\mathfrak{b}\mathfrak{b}_{y_m}]$.

Case 3

Suppose Ω is of the form

$$\forall \vec{x}_1 \in U_1 \forall \vec{y}_2 \in U_2 \ (\theta_1^{U_1} \cap \ \theta_2^{U_2} \longrightarrow \Theta) \quad \text{where } \Theta \in S_1[W] ,$$

so

a) $T_1^\gamma \vdash \exists \vec{x}_1(\theta_1 \cap OP_1(\Theta))[\overline{\phi_{i1}}][\mathfrak{a}]$

b) $T_2^\gamma \vdash \exists \vec{y}_2(\theta_2 \cap OP_2(\Theta))[\overline{\phi_{i2}}][\mathfrak{b}]$.

So we may choose $\mathfrak{a}_{x_1}$ ($\mathfrak{b}_{y_2}$) $\in$ Const(T_1^γ) (T_2^γ)

s.t.

$T_1^\gamma \vdash (\theta_1 \cap OP_1(\Theta))[\overline{\phi_{i1}}][\mathfrak{a}\mathfrak{a}_{x_1}]$

$T_2^\gamma \vdash (\theta_2 \cap OP_2(\Theta))[\overline{\phi_{i2}}][\mathfrak{b}\mathfrak{b}_{y_2}]$

so

$\gamma \models \Theta[\mathfrak{a}\mathfrak{a}_{x_1} \mathfrak{b}\mathfrak{b}_{y_2}]$

and since the variables $\vec{x}_1$ $\vec{y}_2$ occur in a subformula of Θ of type 3 , i) ii) iii) iv) clearly hold .

Case 4

Suppose Ω is of the form

$$\bigvee_{k \in t} \exists x_{k_1} \in U_1 \exists \vec{y}_{k_2} \in U_2 (\theta_{k_1}^{U_1} \cap \ \theta_{k_2}^{U_2} \cap \theta_{k_3})$$

then

a) $T_1^\gamma \vdash \bigwedge_{k \in t} \forall \vec{x}_{k_1} (\theta_{k_1} \longrightarrow \phi_{k1})[\mathfrak{a}]$

b) $T_2^\gamma \vdash \bigwedge_{k \in t} \forall \vec{y}_{k_2} (\theta_{k_2} \longrightarrow \phi_{k2})[\mathfrak{b}]$

where $\phi_{k1} \in \overline{\phi_{i1}}$

Again since ii) holds , for some $m \in t$

$\gamma \models \exists \vec{x}_{m_1} \in U_1 \exists \vec{y}_{m_2} \in U_2 (\theta_{m_1}^{U_1} \cap \theta_{m_2}^{U_2} \cap \theta_{m_3})$

So we may choose $\vec{a}_{x_{m_1}}$ $(\vec{b}_{y_{m_2}})$ $\in$ Const(T_1^γ) (T_2^γ)
s.t.
$\gamma \models (\theta_{m_1}^{U_1} \cap \theta_{m_2}^{U_2} \cap \theta_{m_3})[\vec{a}\vec{a}_{x_{m_1}} \vec{b}\vec{b}_{y_{m_2}}]$
so $T_1^\gamma \vdash \phi_{m1}[\vec{a}\vec{a}_{x_{m_1}}]$ and $T_2^\gamma \vdash \phi_{m2}[\vec{b}\vec{b}_{y_{m_2}}]$.
Now the variables corresponding to $\vec{a}\vec{a}_{x_{m_1}}$ $\vec{b}\vec{b}_{y_{m_2}}$ are x-variables y-variables resp. of the form $v_{jik\rho}$ where $1 \leq i \leq m$. They form a complete corresponding sequence , as can be seen from the fact that Φ is a Π -sentence and the reduction in the proof.
It can also be seen from the proof that $\vec{a}\vec{a}_{x_{m_1}}$ and $\vec{b}\vec{b}_{y_{m_2}}$ are γ consistent for the variables corresponding to $\vec{a}\vec{a}_{x_{m_1}}$. It thus follows by induction that our result is proved.

□

CHAPTER 3

3.1

In this chapter we give a model theoretic description of those n-simple relations which we can show have a S.C. (See 2.58).
First we prove some theorems.

Def

In section 2.53 we defined S_1 as a set of (1)- operators of form a) and b). Let S_1b be the set of form b).

A Π_2 -sentence is a Π-sentence in $S_2[S_1b[W]$.
For example:-

$\forall x_{111}\epsilon U_1 \forall y_{111}\epsilon U_2(x_{111}R_1y_{111} \longrightarrow \forall x_{112}\epsilon U_1(\theta(x_{111}x_{112})^{U_1} \longrightarrow$
$\longrightarrow \exists y_{112}\epsilon U_2(x_{112}R_1y_{112} \cap \theta(y_{111}y_{112})^{U_2})))$
is a Π_2-sentence. Externally it is an $\forall\exists$ sentence, as the 2 in" Π_2" is to suggest, though it can be very complex when one considers θ.

3.11 Theorem

Let Δ be an n-set and Φ be a Π_2-sentence. If γ is an n-sequence which is $OP(\Phi)[[\Delta]]$ good, then $\exists$ δ s.t.

i) $\gamma\subset\delta$

ii) $\delta\models\Phi$

iii) δ is $OP(\Phi)[[\Delta]]$ good.

Proof

I) and ii) say γ is an approximation to Φ, which by Remark 2.55 is meaningful.
Since Φ is a Π_2-sentence we may suppose w.l.o.g. that Φ is of the form :-

$\forall \vec{x}_1 \in U_1 \forall \vec{y}_1 \in U_2 (\theta_3 \vec{x}_1 \vec{y}_1 \to \forall \vec{x}_3 \in U_1 \forall \vec{y}_4 \in U_2 (\theta_1^{U_1} \cap \theta_2^{U_2} \to$

$\to \bigvee_{\kappa \in t} \exists \vec{x}_{\kappa_1} \in U_1 \exists \vec{y}_{\kappa_2} \in U_2 (\theta_{\kappa_1}^{U_1} \cap \theta_{\kappa_2}^{U_2} \cap \theta_{\kappa_3}))) .$

This has the effect of tidying up our proof without significantly altering $OP(\Phi)[[\Delta]]$.

<u>Claim</u>

Suppose β is an $\underline{n}$-sequence which is $OP(\Phi)[[\Delta]]$ good, s.t. for some $\vec{a}_{x_1} \vec{a}_{x_3} \in Const(T_1^{\beta})$ and $\vec{b}_{y_1} \vec{b}_{y_4} \in Const(T_2^{\beta})$,
$\vec{a}_{x_1}$ and $\vec{b}_{y_1}$ are β consistent for $\vec{x}_1$
and
$T_1^{\beta} \vdash \theta_1(\vec{a}_{x_1} \vec{a}_{x_3})$ and $T_2^{\beta} \vdash \theta_2(\vec{b}_{y_1} \vec{b}_{y_4})$.

Then choosing , for $\kappa \in t$, new distinct constants $\vec{c}_{x_{\kappa_1}}, \vec{d}_{y_{\kappa_2}}$, we have for some $\kappa \in t$ β^{κ} is $OP(\Phi)[[\Delta]]$ good ; where

β^{κ} is $T_1^{\beta} \cup \{ \theta_{\kappa_1}(\vec{a}_{x_1} \vec{a}_{x_3} \vec{c}_{x_{\kappa_1}}) \}, (R_1^{\beta})', \ldots, (R_m^{\beta})', T_2^{\beta} \cup$
$\cup \{\theta_{\kappa_2}(\vec{b}_{y_1} \vec{b}_{y_4} \vec{d}_{y_{\kappa_2}})\}$,

where for $1 \leq i \leq m$ $(R_i^{\beta})'$ is formed from R_i^{β} by adding the subset of $(\vec{a}_{x_3} \cup \vec{c}_{x_{\kappa_1}})^{n_i} \times (\vec{b}_{y_4} \cup \vec{d}_{y_{\kappa_2}})^{n_i}$ consisting of those sequences of constants for which the corresponding variables are of the form

$x_{i1\rho}, \ldots, x_{in_i\rho}, y_{i1\rho}, \ldots, y_{in_i\rho}$ for some $\rho \in \omega$.

Suppose not , there will be , for $\kappa \in t$

$<\chi_1^{\kappa} \vec{x}_{\kappa_3}, \chi_2^{\kappa} \vec{y}_{\kappa_4}> \in OP(\Phi)[[\Delta]]$ s.t.

$T_1^{\beta} \vdash \theta_1(\vec{a}_{x_1} \vec{a}_{x_3}) \cap (\theta_{\kappa_1}(\vec{a}_{x_1} \vec{a}_{x_3} \vec{c}_{x_{\kappa_1}}) \to \chi_1^{\kappa}(\vec{f}_{x_{\kappa_3}}))$

$T_2^{\beta} \vdash \theta_2(\vec{b}_{y_1} \vec{b}_{y_4}) \cap (\theta_{\kappa_2}(\vec{b}_{y_1} \vec{b}_{y_4} \vec{d}_{y_{\kappa_2}}) \to \chi_2^{\kappa}(\vec{g}_{y_{\kappa_4}}))$

where $\vec{f}_{x_{\kappa_3}}$ and $\vec{g}_{y_{\kappa_4}}$ are β^{κ} consistent for $\vec{x}_{\kappa_3}$,
we thus have

$T_1^{\beta} \vdash \exists \vec{x}_3 (\theta_1(\vec{a}_{x_1}) \cap \bigwedge_{\kappa \in t} \forall \vec{x}_{\kappa_1} (\theta_{\kappa_1}(\vec{a}_{x_1}) \to \chi_1^{\kappa}(\vec{f}_{x_{\kappa_3}} - (x_{\kappa_1} \cup x_3))))$

$T_2^{\beta} \vdash \exists \vec{y}_4 (\theta_2(\vec{b}_{y_1}) \cap \bigwedge_{\kappa \in t} \forall \vec{y}_{\kappa_2} (\theta_{\kappa_2}(\vec{b}_{y_1}) \to \chi_2^{\kappa}(\vec{g}_{y_{\kappa_4}} - (y_{\kappa_2} \cup y_4))))$

which shows that β is not $OP(\Phi)[[\Delta]]$ good .

Contradiction, so claim holds . Claim □

We now proceed as follows.

Suppose β is OP(Φ)[[Δ]] good , we well-order those sequences of constants $\mathfrak{a}_{x_1}\ \mathfrak{b}_{y_1}\ \mathfrak{a}_{x_3}\ \mathfrak{b}_{y_4}$ which satisfy the conditions of the above claim , as $t_\alpha : \alpha<\mu$ for some ordinal μ .

We define a sequence $\beta_\alpha : \alpha\leqslant\mu$ of n-sequences

s.t. a) $\beta_\alpha \subset \beta_\gamma \quad \alpha<\gamma \leqslant \mu$

b) β_α is OP(Φ)[[Δ]] good for $\alpha\leqslant\mu$

by

i) $\beta_0 = \beta$

ii) If β_α is defined, then $\beta_{\alpha+1}$ is obtained from β_α using t_α ,as β^{k} was obtained from β in the claim.

iii) If α is a limit ordinal and β_γ is defined for $\gamma < \alpha$ then $\beta_\alpha = \bigcup_{\gamma<\alpha} \beta_\gamma$ (See 2.42 (c) for def.)

It is easy to see that a) and b) hold .

(For iii) use 2.42 (c))

By Theorem 2.42 (b) we can extend β_μ to β^* which is OP(Φ)[[Δ]] good, where $T_1^{\beta^*}$ and $T_2^{\beta^*}$ are H.C.C. theories . Thus we have defined an operation from β to β^* .

We now define a denumerable sequence $\gamma_n : n \in \omega$

by $\gamma_0 = \gamma$

$\gamma_{n+1} = (\gamma_n)^*$

Let $\delta = \bigcup_{n\in\omega} \gamma_n$

1) $\gamma\subset\delta$: The operation β to β^* has the property that $\beta \subset \beta^*$

2) $\delta \models \Phi$: The whole point of our claim and construction was to guarantee that this held. The details are left to the reader.

3) δ is $OP(\Phi)[[\Delta]]$ good : Each γ_n for $n \in \omega$ is $OP(\Phi)[[\Delta]]$ good, so by Theorem 2.42 (c) the result follows.

□

Def

Suppose $\{\gamma_\alpha\}_{\alpha<\mu}$ is a sequence of n-sequences s.t. for some T_R of an n-simple relation R,

a) $\gamma_\alpha \subset \gamma_\beta \qquad \alpha \leqslant \beta < \mu$

b) $\gamma_\alpha \models T_R \qquad \alpha < \mu$

Then we say $\{\gamma_\alpha\}_{\alpha<\mu}$ is a T_R - sequence.

Notice that there is a natural elementary embedding of $[T_i{}^{\gamma_\alpha}]$ into $[\bigcup_{\beta<\mu} T_i{}^{\gamma_\beta}]$ for $\alpha < \mu \quad i = 1,2$

Def

We say R is preserved in T_R sequences if the union of every T_R sequence is a model of T_R.

3.12 Theorem

Let T be a set of Π_2-sentences (in L^n); let R_T be the relation defined by T. (which is defined, see Remark 2.55)

Then i) R_T is n - simple

ii) R_T is preserved in T sequences ($T = T_{R_T}$)

iii) R_T is S.C. with a notion of goodness $[[OP(T)]]$.

Proof

i) This is a restatement of Remark 2.55 .

ii) This is left to the reader .

(See the definition of Π_2-sentences .)

iii)

Let γ be a n-sequence which is [[OP(T)]] good.
We show that γ is an approximation to T .
Well-order T as $\{\Phi_\mu\}_{\mu<\kappa}$
For $\mu < \kappa$

$$OP(\Phi_\mu)[\ [[OP(T)]]\ ] = [[OP(T)]]$$

Suppose β is $OP(\Phi_\mu)[\ [[OP(T)]]\ ]$ good .
By Theorem 3.11 we can extend β to β^μ s.t.

$\beta^\mu \models \Phi_\mu$

β^μ is [[OP(T)]] good .

We define γ_μ for $\mu < \kappa$ s.t.

a) $\gamma_\mu \subset \gamma_\nu$ for $\mu \leqslant \nu < \kappa$

b) $\gamma_{\mu+1} \models \Phi_\mu$ $\mu < \kappa$

c) γ_μ is [[OP(T)]] good for $\mu < \kappa$.

by

$$\gamma_0 = \gamma$$

$$\gamma_{\mu+1} = (\gamma_\mu)^\mu$$

for limit μ

$$\gamma_\mu = \bigcup_{\alpha<\mu} \gamma_\alpha$$

Let $\gamma^* = \bigcup_{\mu<\kappa} \gamma_\mu$

So γ^* is [[OP(T)]] good by Theorem 2.42 (c) .

We now define $\{{}_n\gamma\}_{n\in\omega}$ by

$${}_0\gamma = \gamma$$

$${}_{n+1}\gamma = ({}_n\gamma)^*$$

Then set

$$\delta = \bigcup_{n\in\omega} {}_n\gamma$$

Clearly $\delta \leq \gamma$ and is $[[OP(T)]]$ good .

We claim that $\delta \models T$.

It can easily be seen that T_1^δ and T_2^δ are H.C.C. theories .

For each $\mu < \kappa$

$\delta = \bigcup_{n\in\omega} {}_n(\gamma_{\mu+1})$ where ${}_n(\gamma_{\mu+1})$ is the $\mu+1^{th}$ element in the chain used to construct ${}_n\gamma$.

This is a Φ_μ- sequence , and since Φ_μ is a Π_2-sentence we have by ii) of this theorem that $\delta \models \Phi_\mu$. Hence $\delta \models T$ so γ is an approximation to T.

Suppose now that γ is an approximation to T.

We show that γ is $[[OP(T)]]$ good .

It suffices to show that if $\gamma \models T$ then γ is $[[OP(T)]]$ good .

By $\{ OP(\Phi_\mu) \}_{\mu<\eta}[[\Delta]]$ we mean the union of $OP(\Phi_{\alpha_1})[[OP(\Phi_{\alpha_2})[[\ldots[[OP(\Phi_{\alpha_s})[[\Delta]]\ ]]\ \ldots]]$ for all finite subsets $\{\alpha_1,\ldots,\alpha_s\}$ of η where

$$\alpha_1 > \alpha_2 > \cdots > \alpha_s \ .$$

$\gamma \models \Phi_0$ and is $\{ \langle t,f\rangle , \langle f,t\rangle \}$ good .

By Theorem 2.6 γ is $OP(\Phi_0)[[\{ \langle t,f\rangle,\langle f,t\rangle \}]]$good.

If $\eta < \kappa$ and we assume γ is $\{ OP(\Phi_\mu) \}_{\mu<\eta}[[\{ \langle t,f\rangle,\langle f,t\rangle \}]]$good , then since $\gamma \models \Phi_\mu$,again by Theorem 2.6 , γ is $\{ OP(\Phi_\mu) \}_{\mu\leq\eta}[[\{ \langle t,f\rangle,\langle f,t\rangle \}]]$ good .

It follows by transfinite induction that γ is

$$\{ OP(\Phi_\mu) \}_{\mu<\kappa}[[\{ \langle f,t\rangle,\langle t,f\rangle \}]] \text{ good.}$$

Iterating with $\{ OP(\Phi_\mu) \}_{\mu<\kappa}[[\{ \langle f,t\rangle,\langle t,f\rangle \}]]$ in place of $\{ \langle t,f\rangle,\langle f,t\rangle \}$, we find that γ is

$$\{ OP(\Phi_\mu) \}_{\mu<\kappa}[[\{OP(\Phi_\mu) \}_{\mu<\kappa}[[\{ \langle f,t\rangle,\langle t,f\rangle \}]]]] \text{ good.}$$

Repeating this denumerably many times gives our result .

□

3.2

Theorem 3.12 is syntactic in nature . It allows us to find a great many n-simple relations which are S.C. . We now prove that if R is n-simple and preserved in T_R - sequences for some T_R then R is S.C. .

By Theorem 3.12 it suffices to show that if R is n-simple and preserved in T_R - sequences for some T_R then we can find a T_R^*, say , which is a set of Π_2-sentences .

As might be expected we rely heavily on our previous results . We also adapt a type of proof developed by Keisler in [K_1] Theorem 6 .

3.21

We need to consider two binary relations, N_1 and N_2 between L^n - structures, (rather than L - structures) . They will be (1,1) - simple relations.

To avoid confusion we suppose that the unary predicates added to L^n to obtain $(L^n)^{(1,1)}$ are V_1 and V_2, and the added relation predicates are F_1^1 and F_2^1.

N_1 is the (1,1) - simple relation asserting the existence of two relations F_1 and F_2 s.t.

i) F_1 is functional from U_1^C to U_1^D

ii) F_2 is functional from U_2^C to U_2^D

iii) F_1 "preserves" L formulae from U_1^C to U_1^D

IV) F_2 "preserves" L'($\mathcal{B}$) formulae from U_2^C to U_2^D

v) for $1 \leqslant i \leqslant m$ if

F_1 relates $\vec{a}$ to $\vec{c}$ (pointwise)

F_2 relates $\vec{b}$ to $\vec{d}$ (pointwise)

and $C \models \vec{a} R^n_{i\iota} \vec{b}$ then $D \models \vec{c} R^n_{i\iota} \vec{d}$

3.22

If for an L^n - structure $C \models \Sigma$ (See 2.21) we define an $\underline{n}$ - sequence

$\gamma_c = Th(C_1{}^+), (R^n_{11})^C, \ldots, (R^n_{mm})^C, Th(C_2{}^+)$

then for $C, D \models \Sigma$

$C N_1 D$ iff γ_c is included in some "copy" of γ_d.

(By "copy" we mean, obtained from γ_d by changing individual constants .)

As is to be expected N_1 is (1,1) - simple. In fact a suitable T_{N_1} is the following set of sentences in $(L^{\underline{n}})^{(1,1)}$.

$$\forall x_{k11} \in V_1 (U_k(x_{k11})^{V_1} \to \exists y_{k11} \in V_2 (U_k(y_{k11})^{V_2} \cap x_{k11} F_k^1 y_{k11})) ,$$

$$\forall x_{k1\bar{\rho}} \in V_1 \forall y_{k1\bar{\rho}} \in V_2 (\bigwedge\bigwedge x_{k1\bar{\rho}} F_k^1 y_{k1\bar{\rho}} \to ((\phi^{U_k}(x_{k1\bar{\rho}}))^{V_1} \to$$
$$\to (\phi^{U_k}(y_{k1\bar{\rho}}))^{V_2})) ,$$

where $k = 1,2$, and if $k = 1$ $\phi \in L$

$k = 2$ $\phi \in L'(\hat{B})$

Our conventions stated in 2.53 hold for ϕ in $(L^{\underline{n}})^{(1,1)}$.

** By $x_{k1\bar{\rho}}$ we mean a sequence of variables of form $x_{k1\rho_1}, \ldots, x_{k1\rho_t}$.

For each $1 \leqslant i \leqslant m$

$$\forall x_{11\bar{\rho}}\ x_{21\bar{\rho}} \in V_1 \forall y_{11\bar{\rho}}\ y_{21\bar{\rho}} \in V_2 (\bigwedge\bigwedge x_{11\bar{\rho}} F_1^1 y_{11\bar{\rho}} \cap \bigwedge\bigwedge x_{21\bar{\rho}} F_2^1 y_{21\bar{\rho}} \to$$
$$\to ((x_{11\bar{\rho}} R_i^{n_i} x_{21\bar{\rho}})^{V_1} \to (y_{11\bar{\rho}} R_i^{n_i} y_{21\bar{\rho}})^{V_2}))$$

It is easy to see that each of the above sentences is a Π_2- sentence and so N_1 is S.C. .

We have that $[[OP(T_{N_1})]]$ is :-

$\{< \exists x_{111}(U_1(x_{111}) \cap X_1) , \forall y_{111}(U_1(y_{111}) \to X_1) > ,$

$< \exists x_{211}(U_2(x_{211}) \cap X_1) , \forall y_{211}(U_2(y_{211}) \to X_1) > ,$

$< \phi^{U_1}(x_{11\bar{\rho}}) \cap X_1 , \phi^{U_1}(y_{11\bar{\rho}}) \to X_1 > ,$

$< \phi^{U_2}(x_{21\bar{\rho}}) \cap X_1 , \phi^{U_2}(y_{21\bar{\rho}}) \to X_1 > ,$

$< x_{11\bar{\rho}} R_i^{n_i} x_{21\bar{\rho}} \cap X_1, y_{11\bar{\rho}} R_i^{n_i} y_{21\bar{\rho}} \to X_1 >$

for $1 \leqslant i \leqslant m$ with the relevant conditions on $\phi \}$ $[[\{<t,f>,<f,t>\}]]$.

<u>3.23</u> It is not difficult to see that $[[OP(T_{N_1})]]$ is the set of those pairs of formulae of the form $< \theta_1 , \text{n.n.f.} (\neg \theta_1)>$ where

$\theta_1 \in \{<\exists x_{111} \in U_1(X_1)>, <\exists x_{211} \in U_2(X_1)>\}[[Z]]$

where Z is the set of all formulae of the form

$\phi^{U_1}(x_{11\bar{\rho}})$ for $\phi \in L$, $\phi^{U_2}(x_{21\bar{\rho}})$ for $\phi \in L'(\hat{B})$

and for $1 \leqslant i \leqslant m$ $\quad x_{11\bar{\rho}} R_i^{n_i} x_{21\bar{\rho}}$.

Where n.n.f.(ψ) is the negation normal form of ψ, and in this case, we make suitable changes of the constants and the variables . (e.g. $x_{11\rho} \rightarrow y_{11\rho}$ etc.)

We now describe N_2 , the reason for considering this relation will be seen shortly . N_2 is the (1, 1)- simple binary relation between $L^{\underline{n}}$ - structures C and D asserting the existence of two relations F_1 and F_2 s.t.

i) F_1 is functional from U_1^C to U_1^D

ii) F_2 is functional from U_2^C to U_2^D.

If F_1 relates $\vec{a}$ and $\vec{c}$ (pointwise) and F_2 relates $\vec{b}$ and $\vec{d}$ (pointwise) then ;

iii) For $1 \leqslant i \leqslant m$

If $C \models \vec{a} R_i^{n_i} \vec{b}$ then $D \models \vec{c} R_i^{n_i} \vec{d}$

and if $D \models \vec{c} R_i^{n_i} \vec{d}$ then $C \models \vec{a} R_i^{n_i} \vec{b}$.

iv) If $<\theta_1(x_{11\bar{\rho}} x_{21\bar{s}}) , \theta_2(y_{11\bar{\rho}} y_{21\bar{s}})> \in [[OP(T_{N_1})]]$

and $C \models \theta_2(\vec{a}\vec{b})$ then

$D \models \theta_2(\vec{c}\vec{d})$ (equivalently $D \models \neg\, \theta_1(\vec{c}\vec{d})$)

Again N_2 is (1,1)-simple , it is easily shown that there is a set of Π_2- sentences (in $(L^{\underline{n}})^{(1,1)}$ T_{N_2} describing N_2 .

We find that $[[OP(T_{N_2})]]$ can be described as the set of all pairs of formulae of the form

$<\psi_1 , \psi_2>$ where $\psi_2 =$ n.n.f.$(\neg\, \psi_1)$ (again with suitable changes of the variables and constants) and ψ_1 is any formula in

$\{< \exists x_{111} \in U_1(X_1)> \quad <\exists x_{211} \in U_2(X_1)> \}[[\{<\neg X_1>\}[V]]]$

where V consists of all formulae θ where

for some $<\theta_1 \ \theta_2> \in [[OP(T_{N_1})]]$

$\theta = n.n.f.(\neg \theta_1)$ (i.e. θ is θ_2 with suitable changes of the variables and individual constants .)

We leave the details to the reader .

3.24 Remark

In both N_1 and N_2 the relations F_1 and F_2 are 1 to 1 functional from U_1^C to U_1^D and U_2^C to U_2^D resp. (This is because in both cases $(x_{111} \neq x_{112})^{U_1}$ is " preserved " etc.)

If the (1,1)-sequence (in L^n) γ is s.t. $\gamma \models T_{N_2}$ then

$T_2^\gamma, (F_1^\gamma)^{-1}, (F_2^\gamma)^{-1}, T_1^\gamma$ is an approximation to N_1 (Since it is $[[OP(T_{N_1})]]$ good by iv) in the definition of N_2) .

Where by $(F_k^\gamma)^{-1}$ we mean $\{<ba> : <ab> \in (F_k^\gamma)\}$ for $k = 1,2$ (i.e. the inverse relation). This property is the main point of the definition of N_2 . Notice that $(F_k^\gamma)^{-1}$ will also be 1 to 1 functional for $k = 1,2$.

Def

We let $\Delta_0 = \{ \phi :$ for some sentences ψ_1, ψ_2 s,t.

$<\psi_1, \psi_2> \in [[OP(T_{N_2}]]$

$\phi = n.n.f.(\neg \psi_1) \quad \}$

3.3 Lemma

Let Γ be any L^n theory . Let A be any L^n - structure s.t. $A \models \Gamma \cap \Delta_0$ then the (1,1) -sequence

$Th(A^+), \phi, \phi, \Gamma$ is $[[OP(T_{N_2})]]$ good .

Proof

Suppose not , there will be sentences θ_1 and θ_2 where $\langle\theta_1, \theta_2\rangle \in [[OP(T_{N_2})]]$ and

$A \models \theta_1$ and $\Gamma \vdash \theta_2$

but θ_2 is equivalent to a member of Δ_0 (infact modulo change of bound variables $\theta_2 \in \Delta_0$)

So $A \models \theta_2$, but this is not possible since $A \models \theta_1$ and $\vdash \theta_1 \longleftrightarrow \neg\theta_2$

□

Def

If $\{A^i\}_{i\in\omega}$ is a sequence of (L^n) structures where for $i \in \omega$ $A^i \models \Sigma$ (For def. of Σ see 2.21)

$A^i \; N_1 \; A^{i+1}$ and the relations asserted to exist F_1 and F_2 are the inclusion functions $F_1 : U_1^{A^i} \rightarrow U_1^{A^{i+1}}$ and $F_2 : U_2^{A^i} \longrightarrow U_2^{A^{i+1}}$ then $\{A^i\}_{i\in\omega}$ is called an N_1 - chain .

(Notice the similarity to the Def. given prior to Theorem 3.12 and the Remark in 3.22 .)

Its union is defined to be any L^n - structure C s.t.

$$C_1 = \bigcup_{n\in\omega} (A^n)_1$$

$$C_2 = \bigcup_{n\in\omega} (A^n)_2$$

$$(R_i^{n_i})^C = \bigcup_{n\in\omega} (R_i^{n_i})^{A^n} \quad \text{for } 1 \leqslant i \leqslant m \; .$$

Notice that this is a reasoable definition since $\{ (A^n)_k : n\in\omega \}$ is an elementary chain for $k = 1,2$

Def

We say a L^n theory Γ is preserved in $\aleph_1$ - chains if whenever $\{A^i\}_{i\in\omega}$ is an $\aleph_1$ - chain s.t. $A^i \models \Sigma \cup \Gamma$ for $i\in\omega$ then (all of) its union(s) is (are) also a model of Γ .

3.31 Theorem

If Γ is a theory in L^n which is preserved in $\aleph_1$ - chains , then there is a set of sentences $\Gamma' \subset \Delta_0$ s.t.

$$\Sigma \cup \Gamma \vdash \Gamma' \quad \text{and} \quad \Sigma \cup \Gamma' \vdash \Gamma \ .$$

Proof

Suppose there is no such set of sentences Γ' so that the above conditions hold .

Let $\Gamma'' = \{ \phi : \Sigma \cup \Gamma \vdash \phi \text{ and } \phi \in \Delta_0 \}$

Clearly

$$\Sigma \cup \Gamma \vdash \Gamma''$$

so $\Sigma \cup \Gamma'' \nvdash \Gamma$

hence there is $\phi \in \Gamma$ s.t.

$$\Sigma \cup \Gamma'' \nvdash \phi$$

Now $(\Sigma \cup \Gamma) \cap \Delta_0 \subset \Gamma''$ so

$$\Sigma \cup ((\Sigma \cup \Gamma) \cap \Delta_0) \nvdash \phi$$

Let A be an L^n - structure s.t.

$$A \models \Sigma \cup ((\Sigma \cup \Gamma) \cap \Delta_0) \cup \{ \neg \phi \}$$

By Lemma 3.3

3.32 $Th(A^+) , \phi , \phi , \Sigma \cup \Gamma$ is $[[OP(T_{N_2})]]$ good .

So we can find ${}_1A$, F_1 , F_2 and ${}_1B$ s.t.

<u>3.33</u> $Th({}_1A^+)$, F_1 , F_2 , $Th({}_1B^+) \models T_{N_2}$ by 3.12 iii) , which extends 3.32 .

Since F_1 and F_2 are 1 to 1 functional over $U_1{}^{{}_1A}$ to $U_1{}^{{}_1B}$ and $U_2{}^{{}_1A}$ to $U_2{}^{{}_1B}$ resp. we may assume w.o.l.g. that F_1 and F_2 are infact functions from

$F_1 : U_1{}^{{}_1A} \rightarrow U_1{}^{{}_1B}$,

$F_2 : U_2{}^{{}_1A} \rightarrow U_2{}^{{}_1B}$,

$A \leqslant {}_1A$ and ${}_1B$ is chosen so that F_1 and F_2 are infact inclusion maps .

Since 3.33 holds , it follows from Remark 3.24 that :

$Th({}_1B^+)$,$(F_1)^{-1}$,$(F_2)^{-1}$,$Th({}_1A^+)$ is $[[OP(T_{N_1}]]$ good which , therefore , is an approximation to T_{N_1} , so there is an extension of the form :-

$Th({}_2B^+)$, G_1 , G_2 , $Th({}_2A^+) \models T_{N_1}$

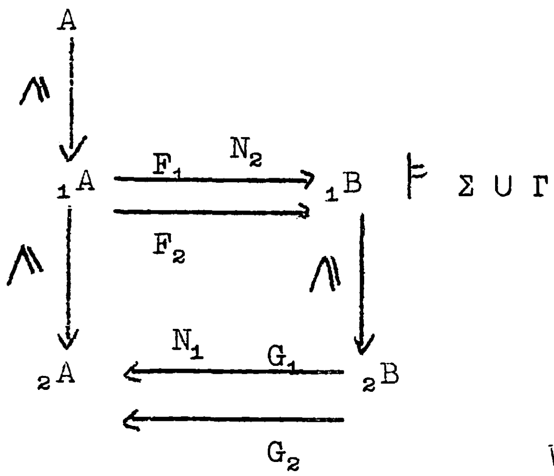
Where, again w.o.l.g. we may assume

${}_1A \leqslant {}_2A$ and ${}_1B \leqslant {}_2B$,

and since G_k extends $(F_k)^{-1}$ (still considered as relations) $k = 1,2$, .. we may suppose that

$G_k : U_k{}^{{}_2B} \rightarrow U_k{}^{{}_2A}$ $k = 1,2$ and are inclusion maps .

We thus have the following situation :-



Where all the maps are inclusion maps on their domain .

From the definition of N_1 it easily follows that $_1B\ N_1\ _2A$ where the relations asserted to exist by N_1 are simply

$G_1 \cap (\ U_1\,{}_1{}^B \times U_1\,{}_2{}^A\ )$ and $G_2 \cap (U_2\,{}_1{}^B \times U_2\,{}_2{}^A\ )$

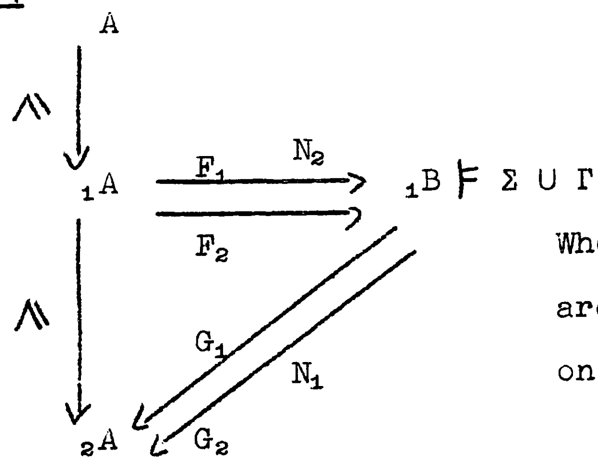
which we continue to call G_1 and G_2 resp.

Hence from the fact that

$Th(A^{+})\ ,\phi\ ,\phi\ ,\ \Sigma \cup \Gamma$ is $[[OP(T_{N_2})]]$ good

and the preceding argument we have

3 34



Where all the maps are inclusion maps on their domains .

Clearly

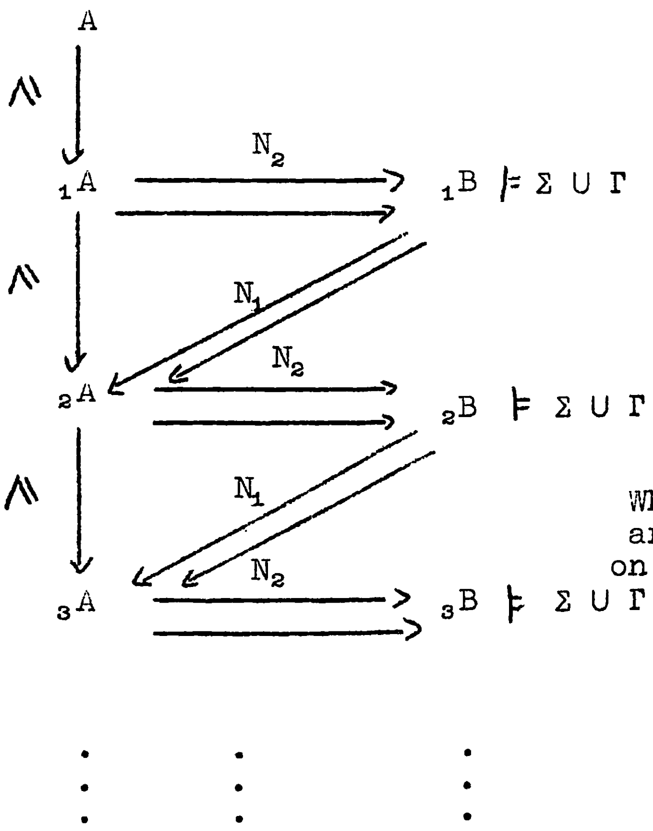
$Th({}_2A^{+}), \phi, \phi, \Sigma \cup \Gamma$ is $[[OP(T_{N_2})]]$ good,

since

$Th(A^{+}), \phi, \phi, \Sigma \cup \Gamma$ was.

Repeating the above argument with ${}_2A$ in place of A, we again obtain a situation similar to 3.34 .

Iterating we find :-



Where all the maps are inclusion maps on their domains .

It is easy to check that $\{ {}_iB \}_{i \in \omega}$ is an N_1 - chain .

Let $_{\omega}A = \bigcup_{n \in \omega} {}_nA$

ωA is a union of $\{ {}_{i}B \}_{i \in \omega}$

This fact is the whole point of the construction . The details are left to the reader .

Since $A \leqslant \omega A$ and $A \models \neg \phi$

$$\omega A \not\models \Gamma$$

but ${}_{i}B \models \Gamma$ for $i \in \omega$.

It follows that Γ is not preserved in N_1 - chains .

□

3.35 MAIN THEOREM

If R is a n - simple binary relation such that there is a T_R which is preserved in N_1 - chains , then R is Syntactically Characterizable .

Proof

By Theorem 3.31 we may suppose T_R is a set of Δ_0 sentences (See Def 2.21)

In view of Theorem 3.12 it suffices to show that for each $\phi \in \Delta_0$ there are a finite number of sentences $\phi_1 , \ldots , \phi_s$ in Π_2 s.t.

$$\vdash \phi \longleftrightarrow \bigwedge_{1 \leqslant i \leqslant s} \phi_i$$

Now Δ_0 is the set of sentences in

$\{ <\forall x_{111} \in U_1(X)>, <\forall x_{211} \in U_2(X)>, <X_1 \cap X_2>, <X_1 \cup X_2> \} \ldots$
$[\{<\neg X>\} [\{ <\exists x_{111} \in U_1(X)>, <\exists x_{211} \in U_2(X)>, <X_1 \cap X_2>,$
$< X_1 \cup X_2 >\}[W]]]$

where W is the set of all formulae of the form
$\theta_1^{U_1}(x_{11\bar{\rho}})$, $\theta_2^{U_2}(x_{21\bar{\rho}})$, $x_{11\bar{\rho}} R_i^{n_i} x_{21\bar{\rho}}$
for $1 \leqslant i \leqslant m$. (see Def after 3.24)

By the usual normal form theorems , see Keisler[K_1], this set is the same as the sentences in

$\{ <X_1 \cap X_2 >\} [\{ <\forall x_{111} \in U_1(X)> , <\forall x_{211} \in U_2(X) >\} \ldots$
$[\{ <X_1 \cup X_2 >\} [\{ <\neg X> \} [\{< X_1 \cup X_2 >\} \ldots$
$[\{ <\exists x_{111} \in U_1(X)> , <\exists x_{211} \in U_2(X)> \}[\{< X_1 \cap X_2 >\}[W]]]]]]]$.

Which in turn can be seen to be the set of sentences in

$\{< X_1 \cap X_2 >\} [\{ <\forall x_{111} \in U_1(X)> , <\forall x_{211} \in U_2(X) >\}[T]]$

where T is the set of formulae of the form

$\psi \to \chi$ where

$\psi \in \{< X_1 \cap X_2 >\} [W]$

$\chi \in \{< X_1 \cup X_2 >\} [\{< \exists x_{111} \in U_1(X)> , < \exists x_{211} \in U_2(X) >\} \ldots$
$[\{< X_1 \cap X_2 >\} [W]]$.

Which is equivalent to the finite conjunction of the sentences in

<u>3.36</u> $\{ <\forall x_{111} \in U_1(X)> , <\forall x_{211} \in U_2(X) >\} [T]$.

<u>Claim</u>

Each of the sentences in 3.36 is equivalent to a Π_2- sentence .

The proof of this fact is left to the reader . We only have to change the variables $x_{21\rho}$ to $y_{11\rho}$ and check that sub-formulae of the form $x_{11\bar{\rho}} R_i^{n_i} y_{11\bar{\rho}}$ have their variables changed suitably .

Example:

$$\forall x_{114} \in U_1 \forall x_{214} \in U_2 (x_{114} x_{114} R_1^2 x_{214} x_{214} \cap \theta^{U_1} x_{114} \rightarrow \phi^{U_2} x_{214})$$

becomes

$$\forall x_{114} x_{124} \in U_1 \forall y_{114} y_{124} \in U_2 (x_{114} x_{124} R_1^2 y_{114} y_{124} \rightarrow$$
$$(((\theta x_{114} \cap x_{114} = x_{124})^{U_1} \cap \ y_{114} \overset{U_2}{=} y_{124}) \rightarrow$$
$$\rightarrow \phi^{U_2} y_{114}))$$

With the claim we have proved our result .

□

The natural question to ask now is whether the converse holds . That is :

If R is n̲ - simple and S.C. then is there a T_R which is preserved in N_1 - chains ?

Alternatively , is there any subset Ω of Π s.t. if R is n̲ - simple and S.C. then T_R can be chosen to be a set of sentences in Ω ?

3.4

After Def 2.22 we suggested that we could give a syntactic condition on those T for which R_T is defined : this is left as an exercise for the reader .

CHAPTER 4

4.1 Introduction

Suppose Δ is a notion of goodness for some binary $\underline{n}$ - simple relation R in L , s.t. whenever $<\theta_1,\theta_2> \in \Delta$ $\theta_2 = n.n.f.(\neg\theta_1)$ (with the usual suitable conditions on the variables and individual constants . (See eg. 3.23))

Let $\pi_1\Delta = \{\theta_1 : \theta_1$ is a sentence and $\exists\, \theta_2(<\theta_1,\theta_2> \in \Delta)\}$

Provided $\pi_1\Delta$ can be described in a syntactically simple way , we have an interpolation theorem for R . (See 2.1 b))

For let ψ , χ be any sentences in L .
If the L.H.S. of 2.11 holds , then

$$\psi ,\phi ,\ldots,\phi ,\neg\chi \quad \text{is } \Delta \text{ bad .}$$

So there are sentences θ_1, θ_2 s.t. $<\theta_1,\theta_2> \in \Delta$
and

$$\psi \vdash \theta_1 \quad \text{and} \quad \neg\chi \vdash \theta_2$$

but $\theta_2 = n.n.f.(\neg\theta_1)$ (We have suppressed mention of the individual constants , as we shall continue to do)
Therefore

$$\psi \vdash \theta_1 \quad \text{and} \quad \neg\chi \vdash \neg\theta_1$$

ie. $\psi \vdash \theta_1$ and $\theta_1 \vdash \chi$ where $\theta_1 \in \pi_1\Delta$
So the R.H.S. holds .
If the R.H.S. holds then clearly

$$\psi ,\phi ,\ldots,\phi ,\neg\chi \quad \text{is } \Delta \text{ bad .}$$

So the L.H.S. holds .

It follows that we have an interpolation theorem .

In this chapter we use our previous results , and the above comments , to obtain interpolation theorems.

In particular we consider :-

Direct Roots of Direct Powers .

Direct Factors (See $[K_3]$)

A new interpolation theorem concerning "cofinal" embeddings .

An extended version of Craig's Interpolation Theorem .

Towards the end of the chapter we consider certain ternary relations .

4.2

The following sentences have nice properties , as we shall see .

Symmetric Sentences

Let V be the set of all formulae in L of the form :-

$$\forall \vec{x} \in U_1 (\phi^{U_1} \longrightarrow \exists \vec{y} \in U_2 (\phi^{U_2} \cap \theta)) \quad \text{or}$$

$$\forall \vec{y} \in U_2 (\phi^{U_2} \longrightarrow \exists \vec{x} \in U_1 (\phi^{U_1} \cap \theta))$$

where ϕ is a formula in L (of type 1 when relativized to U_1 and type 2 when relativized to U_2 (See 2.53))

θ is a formula of type 3 in L and the variables in θ are precisely the variables $\vec{x}\ \vec{y}$.

Let T_1 be the set of (1) - operators (in L) of the form

$$\forall \vec{x} \in U_1 (\phi^{U_1} \longrightarrow \exists \vec{y} \in U_2 (\phi^{U_2} \cap \theta \cap X_1)) \qquad \text{or}$$

$$\forall \vec{y} \in U_2 (\phi^{U_2} \longrightarrow \exists \vec{x} \in U_1 (\phi^{U_1} \cap \theta \cap X_1))$$

where the above conditions on ϕ and θ hold , and the variables in θ are precisely the variables in $\vec{x}\,\vec{y}$ of the form $v_{jik\rho}$ where $1 \leqslant i \leqslant m$.
(There may be variables of the form $v_{j00\rho}$ in $\vec{x}\,\vec{y}$.)

Let T_2 be the set of (1) - operators of the form

$$\forall \vec{x} \in U_1 . \forall \vec{y} \in U_2 (\theta \longrightarrow X_1)$$

where θ is a formula of type 3 in L and the variables in θ are precisely the variables $\vec{x}\,\vec{y}$
(Our usual conventions still hold so , for example , in all cases $\vec{x}$ corresponds to $\vec{y}$.)

<u>4.21</u> <u>Def</u>

Symmetric Sentences are all the sentences in $T_2[T_1[V]]$

<u>4.22</u> <u>Remark</u>

Comparing the above Def. with Def 2.54 of Π - sentences , we see that every Symmetric Sentence is also a Π - sentence . In consequence Theorem 2.6 holds for Symmetric Sentences .

It is not difficult to see that for a Symmetric Sentence Φ we have $OP_2(\Phi) = \text{n.n.f.}(\neg OP_1(\Phi))$ where again we have to change the variables and individual constants but also $\neg X_k$ is replaced by X_k .

Example

Consider the Symmetric Sentence Φ :-

$$\forall y_{001}\epsilon U_2 \exists x_{001}\epsilon U_1 \forall x_{111}\epsilon U_1 ((x_{001}\leq x_{111}) \xrightarrow{U_1} \exists y_{111}\epsilon U_2 ((y_{001}\leq y_{111})^{U_2} \cap x_{111}R_1^1 y_{111}))$$

$$OP_1(\Phi) = \forall x_{001}\exists x_{111}(x_{001}\leq x_{111} \cap X_1)$$

$$OP_2(\Phi) = \exists y_{001}\forall y_{111}(y_{001}\leq y_{111} \rightarrow X_1)$$

So $n.n.f.(\neg OP_1(\Phi)) = \exists x_{001}\forall x_{111}(x_{001}\leq x_{111} \rightarrow \neg X_1)$

and with our conventions this becomes :-

$$\exists y_{001}\forall y_{111}(y_{001}\leq y_{111} \rightarrow X_1)$$

which is $OP_2(\Phi)$.

4.23 Remark

From now on we make the further convention , for ease of reading , that provided there is no ambiguity we omit the symbols U_1 and U_2. There is no real problem since all x-variables are relativized to U_1 and all y-variables are relativized to U_2 . Thus , for example , the above Φ becomes :-

$$\forall y_{001}\exists x_{001}\forall x_{111}(x_{001}\leq x_{111} \rightarrow \exists y_{111}(y_{001}\leq y_{111} \cap x_{111}R_1^1 y_{111}))$$

We shall also be fairly loose with our subscripts. The reader will be able to substitute more suitable subscripts easily . For example , we might have written the above as :-

$$\forall y_0\exists x_0 \forall x_1(x_0\leq x_1 \rightarrow \exists y_1(y_0\leq y_1 \cap x_1R_1^1 y_1)$$

4.3 Def

Let R be any <u>n</u> - simple binary relation in L ; we say there is an Interpolation Theorem for R if for some set Γ of symmetric sentences $[[OP(\Gamma)]]$ is a notion of goodness for R .

If Γ is as above , then $\pi_1[[OP(\Gamma)]]$ is described syntactically and simply . It thus serves as the set required in the usual definition , (See 2.1 and 4.1) to show that R has an interpolation theorem .

4.31

It is easy to check that every sentence in $T_2[V]$ is also a Π_2- sentence . (See 3.1) It follows from Theorem 3.12 that for any set of sentences Γ in $T_2[V]$, R_Γ is defined and is S.C. with a notion of goodness $[[OP(\Gamma)]]$. It follows from Def 4.3 that R_Γ has an Interpolation Theorem . (Both in our sense and the usual sense)

Example

Consider the relation H of "onto homomorphism" between L - structures . It can be thought of as the (1)-simple relation with a T_H :-

$$\forall x_1 \exists y_1 (x_1 R_1^1 y_1)$$

$$\forall y_1 \exists x_1 (x_1 R_1^1 y_1)$$

$$\forall \vec{x}_1 \vec{y}_1 (\bigwedge\!\!\bigwedge \vec{x}_1 R_1^1 \vec{y}_1 \longrightarrow (\theta(\vec{x}_1) \longrightarrow \theta(\vec{y}_1)))$$

for $\theta(\vec{v})$ any atomic formula in L .

These sentences are all in $T_2[V]$, so H has an Interpolation Theorem .

$$[[OP(T_H)]] = \{ <\exists x_1 (X_1), \forall y_1 (X_1)> , <\forall x_1 (X_1), \exists y_1 (X_1)> ,$$
$$<\theta(x_1) \cap X_1, \theta(y_1) \rightarrow X_1 > \}[[\{<t,f>,<f,t>\}]]$$

It is easy to check that $\pi_1[[OP(T_H)]]$ is simply the set of positive sentences in $L_\}$ together with f .

Def 4.3 is not suitable if we replace "interpolation" by "preservation" . For in view of Lemma 2.44 it is not difficult to see that all n-simple relations would have a preservation theorem under this definition , in quite a trivial way .
In the case of Interpolation Theorems , there appears to be no cause for treating Def 4.3 as trivial .

In [Mo], Proof Theory is used to obtain many interpolation theorems in a wide class of languages . The work which refers to First Order Languages is roughly equivalent to 4.31 ; though the proof is , of course , much different .

<u>4.32</u> <u>Def</u>

Let Δ and Δ' be two n-sets , we say $\Delta \Rrightarrow \Delta'$ if whenever $<\theta_1, \theta_2> \in \Delta$ there is $<\phi_1, \phi_2> \in \Delta'$ s.t.

$$\vdash \theta_1 \longrightarrow \phi_1$$
$$\vdash \theta_2 \longrightarrow \phi_2$$

We say that

$$\Delta \equiv \Delta' \text{ if } \Delta \Rrightarrow \Delta' \text{ and } \Delta' \Rrightarrow \Delta .$$

<u>4.33</u> <u>Theorem</u>

If Δ is a notion of goodness for the $\underline{n}$-simple binary relation R by T_R in L and Δ' is an $\underline{n}$ - set, then $\Delta \equiv \Delta'$ iff Δ' is a notion of goodness for R by T_R.

<u>Proof</u>

Suppose $\Delta \equiv \Delta'$

If γ is a T_R approximation then γ is Δ good, but then γ is Δ' good (Since $\Delta' \Rrightarrow \Delta$)

If γ is Δ' good then γ is Δ good, so γ is a T_R approximation .

Suppose now Δ' is a notion of goodness for R by T_R .

Let $<\theta_1\vec{x}, \theta_2\vec{y}> \in \Delta$

Choose new individual constants $\vec{a}_x$ $\vec{b}_y$, for $1 \leq i \leq m$ let R_i be the set of those $2n_i$ sequences in $\vec{a}_x\vec{b}_y$ whose corresponding variables are of the form :-

$$x_{i1\rho}, \ldots, x_{in_i\rho}, y_{i1\rho}, \ldots, y_{in_i\rho}$$

Then $\delta = \theta_1\vec{a}_x, R_1, \ldots, R_m, \theta_2\vec{b}_y$ is not Δ good, so is not an approximation to T_R and hence is not Δ' good .

So there is $<\phi_1\vec{x}_1, \phi_2\vec{y}_1> \in \Delta'$ and some

$\vec{a}_{x_1} \subset \vec{a}_x$ $\quad \vec{b}_{y_1} \subset \vec{b}_y$ s.t.

$\vec{a}_{x_1}$ and $\vec{b}_{y_1}$ are δ consistent for $\vec{x}_1$ s.t.

$$\theta_1\vec{a}_x \vdash \phi_1\vec{a}_{x_1}$$

$$\theta_2 b_y \vdash \phi_2 b_{y_1}$$

W.l.o.g. we may suppose that the variables corresponding to $\vec{a}_{x_1}$ in $\vec{a}_x$ are in fact $\vec{x}_1$

and those corresponding to $\mho_{y_1}$ in $\mho_y$ are $\vec{y}_1$.
We thus have

$$\vdash \theta_1 \vec{x} \rightarrow \phi_1 \vec{x}_1 \quad \text{and} \quad \vdash \theta_2 \vec{y} \longrightarrow \phi_2 \vec{y}_1$$

So $\Delta \not\Rightarrow \Delta'$

Symmetry gives the result .

□

Suppose R is an $\underline{n}$ - simple binary relation between L - structures which has an Interpolation Theorem . So there is a set of Symmetric sentences Γ s.t. $[[OP(\Gamma)]]$ is a notion of goodness for R . It follows easily that R has an interpolation theorem , in the usual sense , between models of T (a theory in L) .
That is to say

4.34

For all ψ , χ sentences in L

$$\forall A \forall B \,(A , B \models T \text{ and } A \, R \, B \text{ and } A \models \psi \text{ imply } A \models \chi) \text{ iff there is } \theta_1 \in \pi_1[[OP(\Gamma)]] \text{ s.t. } T \cup \{\psi\} \vdash \theta_1 \text{ and } T \cup \{\theta_1\} \vdash \chi$$

Now the L.H.S. of the above describes an $\underline{n}$ - simple relation R(T)
i.e.

$$A \, R(T) \, B \text{ iff } A \, R \, B \text{ and } A , B \models T .$$

We cannot in general expect R(T) to have an Interpolation Theorem (in our sense , 4.3).
We extend our earlier definitions .

By T^{U_k} we mean $\{ \phi^{U_k} : \phi \in T \}$ for $k = 1,2$.

4.35 Def

We say a binary $\underline{n}$ - simple relation R between L - structures has an Interpolation Theorem between models of T if for some Symmetric Theory Γ , $[[OP(\Gamma \cup T^{U_1} \cup T^{U_2})]]$ is a notion of goodness for R(T) .

In order to show that we can deduce 4.34 from this definition we need to know that we can simplify (suitably) a notion of goodness in the correct way .

4.36 Theorem

Let Δ be an $\underline{n}$-set and Γ be a set of Π - sentences . Suppose $\phi^{U_1} \in \Gamma$ where ϕ is a sentence in L . Then

$$OP(\Gamma)[[\Delta]] \equiv \{<\phi \rightarrow X, X>\}[OP(\Gamma - \{\phi^{U_1}\}[[\Delta]]] .$$

Proof

R.H.S. $\subset$ L.H.S.

It suffices to show that L.H.S.$\Rightarrow$ R.H.S. .

This can easily be shown by induction on the complexity of the formulae involved , by using the following obvious facts :

$$[\ \exists \vec{x}(\psi \cap_{k \in t} \Delta_t \forall \vec{x}_k (\psi_k \rightarrow \Theta_k))\ ] \rightarrow [\phi \rightarrow \exists \vec{x}(\psi \cap_{k \in t} \Delta_t \forall \vec{x}_k (\psi_k \rightarrow \Theta_k'))\ ]$$

Where we suppose for (say) $m \in t$ Θ_m is of the form $\phi \rightarrow \Theta_m'$ and for $k \neq m$ Θ_k' is Θ_k ;-

and if $\vdash \Theta_1 \rightarrow \Theta_2$

$$[\ \exists \vec{x}_1 (\psi \cap_{k \in t} \Delta_t \forall \vec{x}_k (\psi_k \rightarrow \Theta_1))\ ] \rightarrow [\exists \vec{x}(\psi \cap_{k \in t} \Delta_t \forall \vec{x}_k (\psi_k \rightarrow \Theta_2))] .$$

□

The above Theorem , with its obvious corollary for sentences of form ϕ^{U_2} , where ϕ is a sentence in L , allows us to "pull" a theory out of the notion of goodness .

<u>4.37</u> <u>Theorem</u>

Suppose R has an Interpolation Theorem between models of T , and $[[OP(\Gamma' \cup T^{U_1} \cup T^{U_2})]]$ is a notion of goodness for R(T) , where Γ' is a Symmetric theory . Then 4.34 is satisfied for R and T by $\pi_1[[OP(\Gamma')]]$.

<u>Proof</u>

The proof is straightforward and relies heavily on Theorem 4.36 .

□

We can also simplify $\underline{n}$ - sets , and so notions of goodness in another direction .

<u>4.38</u> <u>Theorem</u>

Let Δ be an $\underline{n}$ - set in L and Γ be a Π - theory . Let Φ be of form

$\forall x_1 y_1 (\theta \longrightarrow (\phi^{U_1} \longrightarrow \phi^{U_2} .)$ where θ is a formula of type 3 containing precisely the variables $x_1\ y_1$.

Then if $\Phi \in \Gamma$

$OP(\Gamma)[[\Delta]] \equiv OP(\Gamma - \{\Phi\})[[\Delta \cup \{<\phi , \neg\phi>\}]]$.

<u>Proof</u>

Trivial □

4.4 Interpolation Theorems

4.41 Consider the binary relation DR between L - structures A , B s.t.

A DR B iff $A \times A \simeq B \times B$.

In $[K_3]$ Keisler calls this relation Direct Roots of Direct Powers . He uses infinitely long formulae to obtain his results and expresses the difficulty experienced in finding the necessary sentences to obtain an interpolation theorem .

DR is the (2) - simple relation defined by the following sentences Γ .

$$\forall x_1 x_2 \exists y_1 y_2 (x_1 x_2 R^2_1 y_1 y_2)$$
$$\forall y_1 y_2 \exists x_1 x_2 (x_1 x_2 R^2_1 y_1 y_2)$$
$$\forall \vec{x}_1 \vec{x}_2 \vec{y}_1 \vec{y}_2 (\Lambda\Lambda\ \vec{x}_1 \vec{x}_2 R^2_1 \vec{y}_1 \vec{y}_2 \longrightarrow (\theta \vec{x}_1 \cap \theta \vec{x}_2 \longrightarrow (\theta \vec{y}_1 \cap \theta \vec{y}_2)))$$
$$\forall \vec{x}_1 \vec{x}_2 \vec{y}_1 \vec{y}_2 (\Lambda\Lambda\ \vec{x}_1 \vec{x}_2 R^2_1 \vec{y}_1 \vec{y}_2 \longrightarrow (\theta \vec{y}_1 \cap \theta \vec{y}_2 \longrightarrow (\theta \vec{x}_1 \cap \theta \vec{x}_2)))$$

where θ is an atomic formula in L and all the variables in $\vec{x}_1 \vec{x}_2 \vec{y}_1 \vec{y}_2$ are supposed to be of the form $v_{j1k\rho}$ for $j , k = 1,2$ $p \in \omega$

Clearly $\Gamma \subset T_1[V]$ so R has an Interpolation Theorem. Indeed a notion of goodness for R is the set of pairs of formulae of the form :

$\langle \phi_1 , \phi_2 \rangle$ where ϕ_2 = n.n.f.$(\neg \phi_1)$

(with the usual conditions on the variables and constants) and $\phi_1 \in \{<\exists x_1 x_2 (X_1)>, <\forall x_1 x_2 (X_1)>\}$ ...

$[[\{<\theta\bar{x}_1 \cap \theta\bar{x}_2>, <\neg\theta\bar{x}_1 \cup \neg\theta\bar{x}_2>, <t>, <f>\}]]$

(We have used Theorem 4.38)

Translating this into the usual form , we obtain Keisler's Theorem $[K_3]$ Cor. 4.2 .

We use the following Theorem to simplify the proof of a new Interpolation Theorem .

4.42 Theorem

Let R(T) be an n - simple binary relation defined by a set of Π_2- sentences $T_{R(T)}$. Let Γ be a Symmetric Theory s.t.

a) $[[OP(T_{R(T)})]] \Rrightarrow [[OP(\Gamma \cup T^{U_1} \cup T^{U_2})]]$

b) $T_{R(T)} \vdash \Gamma$

Then R(T) has an Interpolation Theorem between models of T (See 4.34 for def. of R(T))

Proof

It suffices to show that if γ is any n - sequence which is $[[OP(T_{R(T)})]]$ good , then γ is $[[OP(\Gamma \cup T^{U_1} \cup T^{U_2})]]$ good . That this is so follows from b) and Theorem 2.6

□

Let L contain in particular a binary relation $\leqslant$. Let $T(\leqslant)$ state that $\leqslant$ is a partial ordering .

4.43 Let COF be the binary relation between models A , B of $T(\leqslant)$ s.t.

A COF B iff $\exists f : A \longrightarrow B$ which is an embedding and for $b \in B$ $\exists a \in A$ $\exists c \in B$ where

$f(a) = c$ and $b \leqslant c$ in B

i.e. f is a " co-final " embedding .

A T_{COF} is :-

$$\forall \bar{x}_1 \bar{y}_1 (\wedge\wedge\ \bar{x}_1 R_1^1 \bar{y}_1 \longrightarrow (\ \theta\bar{x}_1 \longrightarrow \theta\bar{y}_1\ ))$$

$$\forall x_1 \exists y_1 (\ x_1 R_1^1 y_1\ )$$

$$\forall y_0 \exists x_1 y_1 (\ x_1 R_1^1 y_1 \cap y_0 \leqslant y_1\ )$$

$$T(\leqslant)^{U_1}$$

$$T(\leqslant)^{U_2}$$

for θ an atomic or negated atomic formula in L .

So $[[OP(T_{COF})]]$ is :-

$$\{<\phi \longrightarrow X, \phi \longrightarrow X> : \phi \in T(\leqslant)\ \}\ [\{<\exists x_1 (X), \forall y_1 (X)>, \ldots$$
$$< \forall x_1 (X),\ \exists y_0 \forall y_1 (\ y_0 \leqslant y_1 \longrightarrow X)>\}[[\ <\theta\bar{x}_1, \neg\theta\ \bar{y}_1>\ ]]\ ]$$

Where $\theta\bar{x}_1$ is atomic or negated atomic in L .

It is easy to see that this $\Rrightarrow$

$$\{<\phi \rightarrow X, \phi \longrightarrow X> : \phi \in T(\leqslant)\}[\{<\ \exists x_1 (X)\ , \forall y_1 (X)>\ , \ldots$$
$$< \forall x_0 \exists x_1 (\ x_0 \leqslant x_1 \cap X), \exists y_0 \forall y_1 (y_0 \leqslant y_1 \longrightarrow X)>\ \}\ \ldots$$
$$[[<\ \theta\bar{x}_1 \neg \theta\bar{y}_1\ >]]\ ]$$

Working backwards we see that the above <u>n</u>-set is $[[OP(\Gamma \cup T(\leqslant)^{U_1} \cup T(\leqslant)^{U_2}\ )]]$ where Γ is

$$\forall x_1 \exists y_1 (x_1 R_1^1 y_1)$$

$$\forall \bar{x}_1 \bar{y}_1 (\wedge\wedge\ \bar{x}_1 R_1^1 \bar{y}_1 \longrightarrow (\ \theta\bar{x}_1 \longrightarrow \theta\bar{y}_1\ )$$

$$\forall y_0 \exists x_0 \forall x_1 (\ x_0 \leqslant x_1 \longrightarrow \exists y_1 (\ y_0 \leqslant y_1 \cap x_1 R_1^1 y_1\ ))$$

Clearly $T_{COF} \models \Gamma$

<u>4.44</u> It follows from Theorem 4.42 that the relation of co-final embedding 4.43 has an Interpolation Theorem between models with a Partial Ordering .

We have the new interpolation theorem using the above notation

4.45 Theorem

$\forall \phi$, ψ sentences in L

$\forall A \forall B$ (A , B $\models T(\leqslant)$ A COF B and $A \models \phi$

imply $B \models \psi$)

iff there is a $\theta \in \Delta_{COF}$ s.t. θ is a sentence and $T(\leqslant) \vdash \phi \rightarrow \theta \cap \theta \rightarrow \psi$

Where Δ_{COF} is the least set of formulae s.t.

a) If θ is atomic or negated atomic in L then $\theta \in \Delta_{COF}$

b) if $\theta_1, \theta_2 \in \Delta_{COF}$ so does $\theta_1 \cap \theta_2$, $\theta_1 \cup \theta_2$

c) if $\theta \in \Delta_{COF}$ then $\exists x\theta \in \Delta_{COF}$ and $\forall x_0 \exists x_1 (x_0 \leqslant x_1 \cap \theta) \in \Delta_{COF}$ providing x_1 does not occur in θ.

Proof

This is a simplification of 4.44

□

It should be fairly clear that a large portion of the known interpolation theorems will be amenable to our methods , Indeed all the interpolation theorems expressable in a First Order Language in [K_1] , [K_3] and [Ma.] are easily proved by our methods , except the next result .

The following variant of Keisler's Theorem on Direct Factors (See [K_3]) is given here . It is the only Interpolation Theorem I have attempted and found difficulty with . I have been unable to prove the original result .

Let A DF B iff $\exists C \; A \times C \simeq B$ and the cardinality of the dom(A) (Card A) is equal to Card B , where A , B and C are L-structures .

4.46 Theorem

DF is a (2) - simple binary relation with a T_{DF} :-

1* $\forall x_1 x_2 \exists y_1 y_2 (x_1 x_2 R_1^2 y_1 y_2)$

2* $\forall y_1 y_2 \exists x_1 x_2 (x_1 x_2 R_1^2 y_1 y_2)$

3* $\forall \vec{x}_1 \vec{x}_2 \vec{y}_1 \vec{y}_2 (\bigwedge\bigwedge \vec{x}_1 \vec{x}_2 R_1^2 \vec{y}_1 \vec{y}_2 \rightarrow (\theta \vec{y}_1 \rightarrow \theta \vec{x}_1))$

4* $\forall \vec{x}_1 \vec{x}_2 \vec{x}_3 \vec{x}_4 \vec{y}_1 \vec{y}_2 \vec{y}_3 \vec{y}_4 (\bigwedge\bigwedge \vec{x}_1 \vec{x}_2 R_1^2 \vec{y}_1 \vec{y}_2 \cap \bigwedge\bigwedge \vec{x}_3 \vec{x}_4 R_1^2 \vec{y}_3 \vec{y}_4 \rightarrow$
$(\vec{x}_2 = \vec{x}_4 \rightarrow \vec{y}_2 = \vec{y}_4 \cap (\theta \vec{y}_1 \cap \neg \theta \vec{y}_3 \rightarrow \theta \vec{x}_1 \cap \neg \theta \vec{x}_3)))$

where $\theta \vec{x}$ is an atomic formula in L .

Hence DF has an UInterpolation Theorem .

Proof

Suppose $f : A \times C \rightarrow B$ is an isomorphism and Card A = Card B

Case 1 : Card B is finite .

Then Card C is 1 so

$g : A \rightarrow B$ defined by

$g(a) = b$ iff $\exists c$ s.t. $f(ac) = b$ is a bijection .

Let R be defined by

abRcd iff $g(a) = c$ and $g(b) = d$

It is a simple matter to check that

$A \; R \; B \models T_{DF}$.

Case 2 : Card B is infinite .

Define $h : B \to C$ by

$h(b) = c$ iff $\exists\, a \in A$ s.t. $f(ac) = b$

We define an equivalence relation $\sim$ over B by

$b \sim b'$ iff $h(b) = h(b')$

let $\hat{b} = \{ b' : b \sim b' \}$ for $b \in B$

$B^h = \{ \hat{b} : b \in B \}$ and

$\hat{h} : B^h \to C$ be the induced map $\hat{h}(\hat{b}) = h(b)$.

Let $g : A \to B^h$ be any onto function s.t.

for $\hat{b} \in B^h$

$\text{Card}\, \{ a : g(a) = \hat{b} \} = \text{Card}\, A$

Such a function exists because by assumption

$\text{Card}\, B^h \leqslant \text{Card}\, B = \text{Card}\, A \geqslant \aleph_0$

For $\hat{b} \in B$ let

$j_{\hat{b}} : \{ a : g(a) = \hat{b} \} \to B$ be any bijection

(which clearly exist)

We define R by

abRcd iff $f(a,\hat{h}(g(b))) = c$ and $j_{\hat{c}}(b) = d$

<u>Claim</u> $A,R,B \models T_{DF}$.

Consider sentence 1*

Suppose $a , b \in A$ then

$abR(f(a,\hat{h}(g(b))))(j_{\widehat{(f(a,\hat{h}(g(b))))}}(b)$

Consider sentence 2*.

Suppose $c , d \in B$

$\exists\ b$ s.t. $j_{\hat{c}}(b) = d$

$\exists\ a$ s.t. $f(a,\hat{h}(g(b))) = c$ (since $g(b) = \hat{c}$)

So $abRcd$.

Consider a sentence of form 3*.

If $f(\vec{x}_1 ,\hat{h}(g(\vec{x}_2))) = \vec{y}_1$

then $B \models \theta[\vec{y}_1] \implies A \models \theta[\vec{x}_1]$ for θ atomic .

Consider a sentence of form 4*.

If $f(\vec{x}_1 ,\hat{h}(g(\vec{x}_2))) = \vec{y}_1$ and

$f(\vec{x}_3 ,\hat{h}(g(\vec{x}_4))) = \vec{y}_3$

and $\vec{x}_2 = \vec{x}_4$ then

$\Lambda\Lambda$ ($\hat{h}(g(\vec{x}_2)) = \hat{h}(g\vec{x}_4))$.

So $\bar{\vec{y}}_1 = \bar{\vec{y}}_3$

Now since

$\Lambda\Lambda$ ($j_{\bar{\vec{y}}_1}(\vec{x}_2) = j_{\bar{\vec{y}}}(\vec{x}_4)$) and

$\Lambda\Lambda$ ($j_{\bar{\vec{y}}_1}(\vec{x}_2) = \vec{y}_2$) and $\Lambda\Lambda$($j\bar{\vec{y}}_3(\vec{x}_4) = \vec{y}_4$)

we have $\vec{y}_2 = \vec{y}_4$

Suppose further $B \models \theta[\vec{y}_1] \cap \neg\theta[\vec{y}_3]$ it follows fairly easily from the definitions that

$A \models \theta[\vec{x}_1] \cap \neg\theta[\vec{x}_3]$

(Hint $B \models \theta[\vec{y}_1] \implies C \models \theta[\hat{h}(g(\vec{x}_2))]$

$\implies C \models \theta[\hat{h}(g(\vec{x}_4))]$ **

Now $B \models \neg\theta[\vec{y}_3] \implies A \models \neg\theta[\vec{x}_3]$

<u>or</u> $C \models \neg\theta[\hat{h}(g(\vec{x}_4))]$ **)

Claim □

Suppose now $A \; R \; B \models T_{DF}$.

We show A DF B and Card A = Card B.

For each $a \in A$ define $B_a = \{ b \in B : \exists cd \, (caRbd) \}$

By 1* for $a \in A$ $B_a \neq \phi$

We now define an equivalence relation τ over A by $a \, \tau \, b$ iff $B_a = B_b$.

For $a \in A$ let $\hat{a} = \{ b : a \, \tau \, b \}$

We now define the L - structure C as follows .

$\text{dom } C = \{ \hat{a} : a \in A \}$

For $\theta\vec{v}$ atomic in L we let :

$\theta\bar{\hat{a}}$ holds in C iff $\exists \, \vec{b} \vec{c} \vec{d}$ s.t.

$\Lambda\Lambda \; \vec{b}\vec{a}R\vec{c}\vec{d}$ and $B \models \theta\vec{c}$.

In order to check that this is a well - defined definition , it suffices to show that if $\vec{a}\tau \vec{e}$ then $C \models \theta\bar{\hat{a}}$ iff $C \models \theta\bar{\hat{e}}$

But suppose $\Lambda\Lambda \; \vec{a} \, \tau \, \vec{e}$

$C \models \theta\bar{\hat{a}} \implies \exists \, \vec{b} \, \vec{c} \, \vec{d}$ s.t.

$\Lambda\Lambda \; \vec{b}\vec{a}R\vec{c}\vec{d}$ and $B \models \theta\vec{c}$

$\implies \Lambda\Lambda \; \vec{c} \in B_{\vec{a}}$ and $B \models \theta\vec{c}$

$\implies \Lambda\Lambda \; \vec{c} \in B_{\vec{e}}$ (since $\Lambda\Lambda \; \vec{a} \, \tau \, \vec{e}$) and $B \models \theta\vec{c}$

$\implies \exists \, \vec{m} \, \vec{n}$ s.t.

$\Lambda\Lambda \; \vec{m}\vec{e}R\vec{c}\vec{n}$ and $B \models \theta\vec{c}$

$\implies C \models \theta\bar{\hat{e}}$

Symmetry gives the result .

We define a function $f : A \times C \rightarrow B$ by

$f(a\hat{b}) = c$ iff $\exists\, d$ (abRcd)

This is a valid definition for if $b \;\tau\; e$ and $\exists\, d$ (abRcd) then $\exists\, a'\, d$ (a'eRcd) , and so by 3* we have $a' = a$ (taking equality for θy_1)

<u>Claim</u> $f \,;\, A \times C \rightarrow B$ is an isomorphism .

i) f is a function by 4*

ii) f is a function from $A \times C$ to B

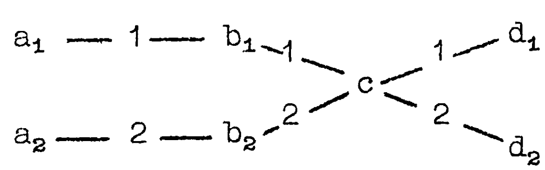
For let $a\hat{b} \in A \times C$, by 1*. $\exists\, c\, d$ s.t. abRcd so by definition $f(a\hat{b}) = c$

iii) f is onto , for let $c \in B$ and $d \in B$ by 2* $\exists\, a\, b$ s.t. abRcd and $f(a\hat{b}) = c$ by definition .

iv) f is 1 to 1 for suppose

$f(a_1\hat{b}_1) = c = f(a_2\hat{b}_2)$

In pictures

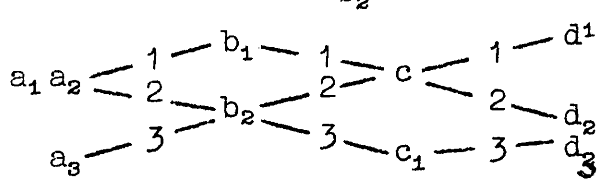


Where a sequence a — n — b — n — c — n — d "means" abRcd

By 3* $a_1 = a_2$

It suffices to show $b_1 \;\tau\; b_2$

Suppose $c_1 \in B_{b_2}$ so we have



for some $a_3\, d_3$.

To show that $c_1 \in B_{b_1}$ it suffices to show that $\exists\, d_4$ st. $a_3 b_1 R c_1 d_4$ By 1* $\exists ef$ s.t. $a_3 b_1 Ref$

Suppose if possible $e \neq c_1$.

We have

$a_1 b_1 Rcd$ and $a_1 b_2 Rcd_2$

$a_3 b_1 Ref$ and $a_3 b_2 Rc_1 d_3$

and $c = c$ and $e \neq c_1$ and $b_1 = b_1$ and $b_2 = b_2$.

Hence by 4*, $a_1 = a_1$ and $a_3 \neq a_3$. Contradiction.

Therefore $e = c_1$

By symmetry it follows that

$$b_1 \ \tau \ b_2$$

v) f is an isomorphism.

Suppose $\Lambda\Lambda\, f(\bar{a}\bar{b}) = \bar{c}$ and $B \models \theta\bar{c}$ then by 3*

$A \models \theta\bar{a}$ and $C \models \theta\bar{b}$ by definition.

Suppose $\Lambda\Lambda\, f(\bar{a}\bar{b}) = \bar{c}$ and $A \models \theta\bar{a}$ and $C \models \theta\bar{b}$,

and assume $B \models \neg\,\theta\bar{c}$

Since $C \models \theta\bar{b}$, by definition, there are

$\bar{a}_1$, $\bar{c}_1$ s.t. $\Lambda\Lambda\, f(\bar{a}_1\bar{b}) = \bar{c}_1$ and $B \models \theta\bar{c}_1$.

Then by 4* $A \models \neg\,\theta\bar{a}$ which gives a contradiction.

Claim □

We have only to show that Card A = Card B.

Since Card $\mathcal{C}$ $\leqslant$ Card B, it follows that

Card A $\leqslant$ Card B so it is sufficient to show that there exists a 1 to 1 function

$g : B \longrightarrow A$, which follows easily using 2* and 4*.

□

Let $\{ x_{1\rho} : p \in \omega \}$ and $\{ x_{2\rho} : p \in \omega \}$ be sets of variables s.t.

$$\{ x_{1\rho} : p \in \omega \} \cap \{ x_{2\rho} : p \in \omega \} = \phi .$$

Let F_{DF} be the least set of formulae containing :

$\neg \theta(\vec{x}_{1\rho})$ where θ is atomic in L

$\vec{x}_{2\rho} = \vec{x}_{2r} \cap (\neg \theta(\vec{x}_{1\rho}) \cup \theta(\vec{x}_{1r}))$ where θ is atomic in L s.t. if $\theta \in F_{DF}$ then

$\exists x_{1\rho} x_{2\rho} \theta \in F_{DF}$ and $\forall x_{1\rho} x_{2\rho} \theta \in F_{DF}$ and F_{DF} is closed under conjunction and disjunction .

<u>4.47</u> <u>Theorem</u>

The binary relation DF defined above has an interpolation theorem in the usual sense (See 2.11) , Where the set of interpolants consist of the sentences in F_{DF} .

<u>Proof</u>

This is a simplification of Theorem 4.46 .

□

<u>4.5</u>

In [F] Feferman using Proof Theoretic techniques in a many-sorted infinitary language proves an extended variant of Craig's Interpolation Theorem . We shall look at the problem in the case of First Order languages .

Let L be a First Order Language not containing function symbols (for simplicity) . Let $M_1, \ldots, M_s$ be new unary predicate symbols . $\phi^{\underline{M}}$ will denote a sentence in $L \cup \{M_1, \ldots, M_s\}$ where $\phi^{\underline{M}}$ is obtained from the sentence ϕ in L, by relativizing each occurrence of a quantifier in ϕ to one of $M_1,\ldots,M_s$.

For simplicity we assume w.l.o.g. that ϕ is taken in negation normal form . That is each negation symbol occurring in ϕ negates an atomic formula of ϕ and the implication sign does not occur .

Clearly , in general , for each $\phi \in L$ there will be several $\phi^{\underline{M}}$ obtainable from ϕ.

Let $J_1(\phi^{\underline{M}})$ be the set of those $i \in \{1,\ldots,n\}$ s.t. some universal quantifier in ϕ is relativized to M_i in $\phi^{\underline{M}}$.

Let $J_2(\phi^{\underline{M}})$ be the set of those $i \in \{1,\ldots,n\}$ s.t some existential quantifier in ϕ is relativized to M_i in $\phi^{\underline{M}}$.

Thus , for instance , if $\theta^{\underline{M}} = \text{n.n.f.}(\neg(\phi^{\underline{M}}))$ then $J_1(\theta^{\underline{M}}) = J_2(\phi^{\underline{M}})$.

Suppose $\vdash \phi^{\underline{M}} \to \psi^{\underline{M}}$ then Craig's Interpolation Theorem states that there will be a $\theta \in L \cup \{M_1,..,M_s\}$ s.t. $\vdash (\phi^{\underline{M}} \to \theta) \cap (\theta \to \psi^{\underline{M}})$

** where the relation symbols and constants in θ occur in both $\phi^{\underline{M}}$ and $\psi^{\underline{M}}$.

i.e. $L(\theta) \subset L(\phi^{\underline{M}}) \cap L(\psi^{\underline{M}})$.

We cannot deduce directly that we can find such a θ of the form $\chi^{\underline{M}}$ for some $\chi \in L$.

That this is indeed the case was proved by Feferman using proof theory in a many-sorted language . He expresses some doubt that this theorem is amenable to single sorted First Order methods . Feferman's Theorem 4.2 in [F] in the First Order case amounts to the following :-

4.51 Theorem

Suppose $\phi^{\underline{M}} \longrightarrow \psi^{\underline{M}}$ then there is an interpolant of the form $\theta^{\underline{M}}$ satisfying the conditions ** above and further

$J_1(\theta^{\underline{M}}) \subset J_1(\phi^{\underline{M}})$

$J_2(\theta^{\underline{M}}) \subset J_2(\psi^{\underline{M}})$.

We prove this theorem using our methods . Let A , B be $L \cup \{ M_1, \ldots, M_s \}$ structures and $J \subset \{1,\ldots,n\}$.

We say $A \xrightarrow[J]{L} B$ if $\exists f : A \longrightarrow B$ s.t.

1) f is an embedding of $A|L \longrightarrow B|L$
2) $f[M_i^{\,A}] \subset M_i^{\,B}$ for $1 \leqslant i \leqslant n$.
3) For $i \in J$ $f[M_i^{\,A}] = M_i^{\,B}$

4.52 Theorem

Suppose $\chi^{\underline{M}}(\vec{v})$ is given , where $\chi \in L$, $f : A \xrightarrow[J]{L} B$ and $J_1(\chi^{\underline{M}}) \subset J$ then for $\vec{a} \in A$ $A \models \chi^{\underline{M}}[\vec{a}]$ implies $B \models \chi^{\underline{M}}[f\vec{a}]$.

Proof

By induction on the complexity of $\chi^{\underline{M}}$.

□

Consider the following relation F defined by T_F:-

$$\forall x(M_i x \rightarrow \exists y(M_i y \cap xF_1^1 y)) \quad \text{for } i \in J_2(\psi^{\underline{M}})$$
$$\forall y(M_i y \rightarrow \exists x(M_i x \cap xF_1^1 y)) \quad \text{for } i \in J(\phi^{\underline{M}})$$
$$\forall \vec{x}\vec{y}(\Lambda\Lambda\ \vec{x}F_1^1\vec{y} \rightarrow (\theta(\vec{x}) \rightarrow \theta(\vec{y}))) \quad \text{where } \theta \text{ is}$$

an atomic or negated atomic formula in $L(\phi) \cap L(\psi)$.

Clearly F is (1)-simple with a S.C. by $[[OP(T_F)]]$

<u>Claim</u>

$\phi^{\underline{M}}$, ϕ , $\neg \psi^{\underline{M}}$ is $[[OP(T_F)]]$ bad .

For otherwise we can find A , F_1 , B s.t.
$A\ F\ B$ and $A \models \phi^{\underline{M}}$ and $B \models \neg(\psi^{\underline{M}})$ where

$L(A) = L(\phi) \cup \{M_1,\ldots,M_s\}$

$L(B) = L(\psi) \cup \{M_1,\ldots,M_s\}$

and since F_1 is a 1 to 1 function we may assume it is the inclusion map on its domain . We may also assume

$$\text{dom}(A) \cap \text{dom}(B) = \text{dom } F_1$$

We define an $L(\phi\cap\psi) \cup \{M_1,\ldots,M_s\}$ structure as follows .

$$\text{dom } (C) = \text{dom}(A) \cup \text{dom}(B)$$

and for atomic $\theta\vec{v} \in L(C)$ amd $\vec{a} \in \text{dom}(C)$

$C \models \theta\vec{a}$ iff $\vec{a} \in \text{dom}(A)$ $\theta\vec{v} \in L(A)$ and $A \models \theta\vec{a}$
or $\vec{a} \in \text{dom}(B)$ and $\theta\vec{v} \in L(B)$ and $B \models \theta\vec{a}$

It is easy to check that this is a valid definition of C .

It also follows easily that

$$A \xrightarrow[J_1(\phi^{\underline{M}})]{L(\phi)} C \mid L(\phi) \cup \{M_1,\dots,M_s\}$$

$$B \xrightarrow[J_2(\psi^{\underline{M}})]{L(\Psi)} C \mid L(\psi) \cup \{M_1,\dots,M_s\}$$

For if $i \in J_1(\phi^{\underline{M}})$ then from the definition of F and C we have :

$$M_i{}^B \subset M_i{}^A \qquad \text{and}$$

$$M_i{}^C = M_i{}^A \cup M_i{}^B = M_i{}^A .$$

If $i \in J_2(\psi^{\underline{M}})$ then a similar argument holda .

Now since $A \models \phi^{\underline{M}}$ and $B \models \neg\psi^{\underline{M}}$ it follows from Theorem 4.52 that $C \models \phi^{\underline{M}} \cap \neg\psi^{\underline{M}}$, which gives us our contradiction .

It follows that for some pairs of sentences

$$<\theta_1 \theta_2> \in [[OP(T_F)]]$$

$$\vdash \phi^{\underline{M}} \longrightarrow \theta_1$$

$$\vdash \neg\psi^{\underline{M}} \longrightarrow \theta_2$$

A closer inspection of T_F will give us our theorem.

□

<u>4.6</u> If R is a binary relation which is S.C. then we shall denote a notion of goodness of R by Δ_R . In particular if R is the relation asserting the existence of an embedding, we denote it by $\subset$ and $\Delta_\subset$ is the " natural " notion of goodness .

If R and S are binary relations which are S.C. and (for simplicity) are (1) - simple , then by

$R(S)$ we mean the binary relation defined by

$$T_R \cup \{ \forall x_1 y_1 (\bigwedge\bigwedge x_1 R_1^1 y_1 \longrightarrow (\theta_1 x_1 \longrightarrow \neg \theta_2 y_1)) \}$$

for $\langle \theta_1 x_1 , \theta_2 y_1 \rangle \in \Delta_S$.

(To be $\Delta_{R(S)}$ good is to be an approximation to R which is Δ_S good .)

It follows that if T_1 , ϕ , T_2 is $\Delta_{R(S)}$ good then there are A,B,C and D s.t.

$$\begin{array}{ccc} C & S & D \\ \curlyvee\!\!/ & & \curlyvee\!\!/ \\ A & R & B \\ \models T_1 & & \models T_2 \end{array}$$

where R_1, the relation asserted to exist by R between A and B is included in S_1, the relation asserted to exist by S between C and D .

In $[K_1]$ Keisler proved that a sentence ψ is equivalent to a $\forall\exists$ sentence iff

whenever

$$\begin{array}{ccccc} A & & \leqslant & & B \\ & \diagdown & & \diagup & \\ & & C \models \psi & & \end{array} \qquad \text{then } A \models \psi$$

We can easily obtain this result , for ψ is not equivalent to a $\forall\exists$ sentence iff $\psi , \phi , \neg\psi$ is $\Delta_{\supset(\subset)}$ good , iff there are A , B , A' and B' s.t.

$$\begin{array}{ccc} A' & \subset & B' \\ \curlyvee\!\!/ & & \curlyvee\!\!/ \\ A & \supset & B \\ \models \psi & & \models \neg\psi \end{array}$$

(We can assume $\subset$ are really inclusions since one of the embeddings "extends" the other .)

iff

there are A , B and B' s.t.

$$\begin{array}{ccc} & & B' \\ & \subseteq & \vee\!/ \\ A_{\models \psi} & \supset & B_{\models \neg\psi} \end{array}$$

Which is Keisler's result .

It will be remembered that these ideas were employed in Chapter 3.

It is a well-known fact (See [R] page 232) that if ψ is a sentence containing at least one constant which is equivalent to both an existential sentence and a universal sentence , then ψ is equivalent to an open sentence (one not containing any quantifiers).

Indeed if R is defined by

$\forall x \, (x = \sigma \rightarrow \exists y \, (xR_1^1 y \cap y = \sigma) \quad$ for each closed term σ in $L(\psi)$

$\forall \vec{x}\vec{y} \, (\Lambda\Lambda \, \vec{x}R_1^1\vec{y} \rightarrow (\theta\vec{x} \rightarrow \theta\vec{y})) \quad$ for θ atomic or negated atomic in $L(\psi)$.

Then it is easy to see that if

$\psi , \phi , \neg\psi$ is Δ_R bad (not Δ_R good)

then ψ is equivalent to an open sentence .

(Simply eliminate quantifiers)

Thus if ψ is not equivalent to an open sentence we can find A and B s.t.

A R B where $A \models \psi$ and $B \models \neg\psi$.

Consider the minimal substructure A' and B' of A and B respectively. (These exist , i.e. are not empty since Const($L(\psi)$) $\neq \phi$)

Clearly

$$A' \; R \; B'$$

and the natural relation to take R_1 is an isomorphism of A' onto B' .

Now since ψ is equivalent to both an existential and a universal sentence we have

$$A \models \psi \implies A' \models \psi \implies B' \models \psi \implies B \models \neg\psi \cap \psi$$

A contradiction .

We can easily prove many simple results of this kind . For example :

<u>4.61</u> <u>Theorem</u>

If ψ is equivalent to a $\forall\exists$ sentence and a positive sentence , then ψ is equivalent to a sentence which is at the same time a $\forall\exists$ sentence and a positive sentence .

<u>Proof</u>

We sketch the proof .

Assume ψ is positive but ψ is not equivalent to a sentence which is both positive and $\forall\exists$.

Consider the relation R defined by

$$\forall y \exists x \, (x R_1 y)$$

$$\forall \vec{x}\vec{y} \, (\vec{x} R_1 \vec{y} \longrightarrow (\theta\vec{x} \longrightarrow \theta\vec{y}))$$

where $\theta\vec{x}$ is both existential and positive .

Clearly $\psi , \phi , \neg\psi$ is Δ_R good .

So there are A , B and C s.t.

$$A \models \psi \qquad\qquad \longrightarrow C \models \neg\psi$$

$$\cup \qquad f_1 \text{ onto homomorphism}$$

$$B$$

where $Th(A^+)$, f_1, $Th(C^+)$ is $\Delta_{(homomorphism)}$ good .

So we have D , E and f_2 s.t.

$$f_2 \subset f_2$$

$$f_2 : A \xrightarrow[\text{onto}]{\text{homomorphism}} D$$

where

$$\begin{array}{ccc} & & E \\ & \subset & \\ D & & \not\!\!\!/ \\ & \supset & \\ & & C \end{array}$$

That $C \subset D$ follows since $f_1 \subset f_2$.

We thus have

$$\begin{array}{ccc} & & D \models \psi \\ & \subset & \\ C_{\models \neg \psi} & & \cap \\ & \leqslant & \\ & & E_{\models \neg \psi} \end{array}$$

$D \models \psi$ since ψ is positive and $A \models \psi$

So ψ is not equivalent to an $\forall\exists$ sentence .

(A picture helps to follow the proof !)

The theorem follows ;.

□

Suppose R is a ternary relation s.t. for some binary relations R_1 and R_2 we have

$$\langle A,B,C\rangle \in R \text{ iff } A\ R_1\ B \text{ and } B\ R_2\ C \ .$$

If R_1 and R_2 are preserved in T_{R_1} - sequences and T_{R_2} - sequences resp. and are <u>n</u> - simple, then R has a notion of goodness . In fact by generalising the results of chapter 2 and 3 we could prove that R has a "S.C.". However , this is unnecessary as the following shows .

4.62 Theorem

If R_1 and R_2 are $\underline{n}$ - simple binary relations preserved in T_{R_1} - sequences and T_{R_2} - sequences respectively , then a notion of goodness for the ternary relation R defined by :

$<A,B,C> \in R$ iff $A\ R_1\ B$ and $B\ R_2\ C$ is :

$$\Delta_R = \{ <\theta_1, \theta_2 \cup \phi_1, \phi_2> : <\theta_1,\theta_2> \in \Delta_{R_1} \text{ and } <\phi_1,\phi_2> \in \Delta_{R_2} \}$$

Proof

Strictly Δ_R is not an $\underline{n}$ - set , we ignore this complication .

Suppose $\gamma = T_1, \bar{R}_1, T_2, \bar{R}_2, T_3$ (We have extended the notion of an $\underline{n}$ - sequence in the natural way)

If γ is an R approximation then clearly γ is Δ_R good .

Suppose now γ is Δ_R good .

We may assume w.l.o.g. that T_2 is complete . (c.f. the proof of Theorem 2.42)

Now $T_1, \bar{R}_1, T_2$ is Δ_{R_1} good and

$T_2, \bar{R}_2, T_3$ is Δ_{R_2} good .

So we may find A_1 , B_1 s.t.

$A_1\ R_1\ B_1$ where the relations asserted to exist are $\bar{R}_1^1$ (say) where

$A_1 \models T_1$, $B_1 \models T_2$ and $\bar{R}_1^1 \supset \bar{R}_1$ (pointwise)

Now $Th(B^+)$, $\bar{R}_2$, T_3 is Δ_{R_2} good . (Since T is complete.)

So we can find B_2 , C_2 s.t.

$B_2 R_2 C_2$ where the relations asserted to exist are $\bar{R}_2^2$ (say) where

$B_1 \leq B_2$, $C_2 \models T_3$ and $\bar{R}_2^2 \supset \bar{R}_2$.

We thus have

$$\begin{array}{ccccc} & & B_2 & R_2 & C_2 \models T_3 \\ & & \vee\!\!/ & & \\ A_1 \models T_1 & R_1 & B_1 \models T_2 & & \end{array}$$

But $Th(A_1{}^+)$, $\bar{R}_1^1$, $Th(B_2{}^+)$ is Δ_{R_1} good .

Thus we may iterate this process denumerably often . Since R_1 and R_2 are each preserved in T_{R_1} - sequences and T_{R_2} - sequences resp, it easily follows that γ is an R approximation .

□

CHAPTER 5

5.1 We consider in this chapter the characterization of those theories, which have models satisfying various complicated relations between several L - structures . We show that the results of the previous chapters can be successfully applied to a range of such problems .

5.11 The Amalgamation Properties

We say that a theory T has the Amalgamation Property (A.P.) if whenever

$$B \models T$$
$$\subseteq$$
$$A \models T$$
$$\subseteq$$
$$C \models T$$

there is $D \models T$ and embeddings

$f : B \to D \qquad g : C \to D$ s.t.

the " diagram commutes " . (That is ; if $a \in A$ then $fa = ga$.)

In view of the fact that there are many conditions on a theory T of a similar " shape " as above , we generalize the above as follows .

For simplicity we restrict our attention to

(1) - simple binary relations between L structures.

5.12 Def

A (1)-simple binary relation R is s.t.b. Diagrammatic if there is a T_R consisting of sentences of the form :

$$\forall x_1 y_1 (\Lambda\Lambda\ x_1 R_1^1 y_1 \rightarrow \forall x_2 (\theta_1 x_1 x_2 \longrightarrow \exists y_2 (\Lambda\Lambda\ x_2 R_1^1 y_2 \cap \theta_2 y_1 y_2)))$$

With our conventions these are all Π_2 sentences .

If the above sentence is in T_R we simply say $(\theta_1 x_1 x_2 , \theta_2 y_1 y_2)$ is in T_R .

So , for example , to say $(x_2 = x_2 , y_2 = y_2)$ is in T_R, means

$$\forall x_2 (x_2 = x_2 \rightarrow \exists y_2 (x_2 R_1^1 y_2 \cap y_2 = y_2))$$ or

equivalently $\forall x \exists y (x R_1^1 y)$, is in T_R.

We have the following basic facts about Diagrammatic relations .

5.13 1) If A R B and $C \leqslant A$ then C R B , the natural relation to take being $R_1 \cap (C X C)$. As usual R_1 is the binary relation included in AXB asserted to exist by R .

2) If R is Diagrammatic and A is an L-structure we let Diag[R,2](A) (the notation is supposed to be suggestive) be ;

$$\{ \phi_2 \vec{a} : (\phi_1 , \phi_2) \text{ is in } T_R \text{ and } A \models \phi_1 [\vec{a}] \} .$$

If there is a B , an L-structure , s.t. for some $f : A \longrightarrow B$ $(B\ fa)_{a \in A} \models \text{Diag}[R,2](A)$ then A R B .

The converse , in practice , will often occur .

(We need $(x_2 = x_2 , y_2 = y_2)$ and $(x_1 = x_1 , y_1 = y_1)$ in T_R)

These results follow easily from the definitions. Keisler's Generalized Subsystems and Homomorphisms (see [K_1]) are Diagrammatic.
In particular if R is $\subset$ or a homomorphism then R is Diagrammatic.
The relation $\subset_{poss}$ defined by

$$\forall \vec{x}_1 \vec{y}_1 (\bigwedge\!\!\bigwedge \vec{x}_1 R_1^1 \vec{y}_1 \longrightarrow (\theta \vec{x}_1 \longrightarrow \theta \vec{y}_1)) \text{ for } \theta \text{ atomic}$$

or negated atomic is Diagrammatic.
Note that for any L - structures A and B

$$A \subset_{poss} B \quad \text{since} \quad Th(A^+), \phi , Th(B^+) \models T_{\subset_{poss}}$$

<u>5.14</u> <u>Def</u>

We say (T_1, T_2, T_3, T_4) has the (R_1, R_2, R_3, R_4) - A.P. iff whenever

$B \models T_2$

R_1

$A \models T_1$

R_2

$C \models T_3$

$\exists\ D \models T_4$ s.t.

$B\ R_3 D$ and $C\ R_4\ D$ and the diagram commutes.
i.e. if $a\ R_1\ b$ and $a\ R_2\ c$ then there is a $d \in D$ s.t. $b\ R_3\ d$ and $c\ R_4\ d$. We do not distinquish between the binary relation R_1 between L - structures and the relation R_1 asserted to exist by R_1.

If R is a binary relation between L- structures then R^{-1} is the relation defined by

$$A\ R^{-1}\ B \quad \text{iff} \quad B\ R\ A.$$

If R_1 and R_2 are binary relations, then (R_1, R_2) is the ternary relation defined by ;

$< A,B,C> \in (R_1,R_2)$ iff $A \, R_1 \, B$ and $B \, R_2 \, C$.

5.15 Theorem

If R_3 and R_4 are Diagrammatic and R_1 and R_2 are S.C. then the following are equivalent ;

1) (T_1,T_2,T_3,T_4) has the (R_1,R_2,R_3,R_4) - A.P.

2) Whenever

$$\begin{array}{ccc} & & B \models T_2 \\ & R_1 & \\ A \models T_1 & & \\ & R_2 & \\ & & C \models T_3 \end{array} \qquad (A)$$

then $T_h(B^+)$, S_1 , $T_4 \cup \{e=e : e \in E\}$, S_2 , $Th(C^+)$ is a (R_3,R_4^{-1}) approximation . Where S_1 , S_2 and E are s.t. whenever $a \, R_1 b$ and $aR_2 \, c$ then a new constant e is chosen and

$e \in E$ $\quad <b \; e> \in S_1$ $\quad <e \; c> \in S_2$

Thus E = Range S_1 = Domain S_2 .

3) Whenever $T_4 \vdash \forall \vec{x}(\phi_2\vec{x} \cup \theta_2\vec{x})$ where $<\phi_1\vec{x} , \phi_2\vec{y}> \in \Delta_{R_3}$ and $<\theta_1\vec{x} , \theta_2\vec{y}> \in \Delta_{R_4}$ then there are $<\gamma_1\vec{x} , \gamma_2\vec{y}> \in \Delta_{R_1}$ and $<\delta_1\vec{x} , \delta_2\vec{y}> \in \Delta_{R_2}$ s.t.

$$\begin{array}{ll} T_1 & \vdash \forall \vec{x}(\gamma_1\vec{x} \cup \delta_1\vec{x}) \\ T_2 & \vdash \forall \vec{x}(\neg \phi_1\vec{x} \cup \gamma_2\vec{x}) \\ T_3 & \vdash \forall \vec{x}(\neg \theta_1\vec{x} \cup \delta_2\vec{x}) \end{array} \qquad (B)$$

Proof

That $1) \Leftrightarrow 2)$ is trivial , 2) was written by way of explanation .
Note that we use strongly the fact that R_3 and R_4 are Diagrammatic.

3) $\Longrightarrow$ 2) :

Suppose (A) holds but

** $Th(B^+)$, S_1 , $T_4 \cup \{e=e : e \in E\}$, S_2 , $Th(C^+)$

is not an (R_3, R_4^{-1}) approximation .

So for some $\langle \phi_1 \vec{x}, \phi_2 \vec{y} \rangle \in \Delta_{R_3}$ and some

$\langle \theta_1 \vec{x}, \theta_2 \vec{y} \rangle \in \Delta_{R_4}$ (this holds iff $\langle \theta_2 \vec{x}, \theta_1 \vec{y} \rangle \in \Delta_{R_4^{-1}}$)

we have $T_4 \vdash \phi_2 \vec{e} \cup \theta_2 \vec{e}$ and

$$\left.\begin{array}{ll} B \models \phi_1 \vec{b} & C \models \theta_1 \vec{c} \\ \text{where } \Lambda\Lambda \vec{b} S_1 \vec{e} & \Lambda\Lambda \vec{e} S_2 \vec{c} \end{array}\right\} \quad (C)$$

since ** is $\Delta_{(R_3 R_4^{-1})}$ bad . (See Theorem 4.62)

We thus have $T_4 \vdash \forall \vec{x}(\phi_2 \vec{x} \cup \theta_2 \vec{x})$, because $Const(L(T_4)) \cap E = \phi$.

In view of 3) it is clear that (B) holds ;

so as $A \models T_1$, we have $A \models \forall \vec{x}(\gamma_1 \vec{x} \cup \delta_1 \vec{x})$.

Now by the definition of S_1 and S_2 there are $\vec{a} \in A$ s.t. $\Lambda\Lambda \vec{a} R_1 \vec{b}$ and $\Lambda\Lambda \vec{a} R_2 \vec{c}$ (by (C)).

So $A \models \gamma_1 \vec{a} \cup \delta_1 \vec{a}$.

W.l.o.g. suppose $A \models \gamma_1 \vec{a}$

$A R_1 B$ $\quad B \models \neg \gamma_2 \vec{b}$ and since $B \models T_2$

$B \models \neg \phi_1 \vec{b}$ which contradicts (C) .

It follows that 3) $\Longrightarrow$ 2) .

2) $\Longrightarrow$ 3) :

Suppose for some $\langle \phi \vec{x}, \phi_2 \vec{y} \rangle \in \Delta_{R_3}$ and $\langle \theta \vec{x}, \theta_2 \vec{y} \rangle \in \Delta_{R_4}$ $\quad T_4 \vdash \forall \vec{x}(\phi_2 \vec{x} \cup \theta_2 \vec{x})$ but (B) does not hold .

It is not difficult to see that for new constants $\vec{a}$

$T_2 \cup \{\phi_1 \vec{a}\}$, $\{ \langle aa \rangle : a \in \vec{a}\}$, $T_1 \cup \{\Lambda\Lambda\ \vec{a}=\vec{a}\}$, ...

$\{ \langle aa \rangle : a \in \vec{a}\}$, $T_3 \cup \{\theta_1 \vec{a}\}$ is

$\Delta(R_1^{-1}, R_2)$ good . So we can find :-

$$B \models T_2 \cup \{\phi_1[\vec{a}]\}$$

$$R_1$$

$$A \models T_1$$

$$R_2$$

$$C \models T_3 \cup \{\theta_1[\vec{a}]\} \ ,$$

but then by construction 2) cannot hold .

□

Clearly T has the A.P. iff

(T,T,T,T) has the $(\subset, \subset, \overrightarrow{\text{emb}}, \overrightarrow{\text{emb}})$ - A.P.

<u>5.16</u> <u>Corollary</u>

T has the A.P. iff

whenever

$$T \vdash \forall \vec{x}(\phi \vec{x} \cup \theta \vec{x})$$

where $\phi \vec{x}$ and $\theta \vec{x}$ are universal formulae ,

then there are existential formulae $\gamma \vec{x}$ and $\delta \vec{x}$

s. t.

$$T \vdash \forall \vec{x}(\gamma \vec{x} \cup \delta \vec{x})$$

$$T \vdash \forall \vec{x}(\gamma \vec{x} \rightarrow \phi \vec{x})$$

$$T \vdash \forall \vec{x}(\delta \vec{x} \rightarrow \theta \vec{x}) \ .$$

□

We say injections are transferable in T if (T,T,T,T) has the $(\subset, \xrightarrow{\text{hom}}, \xrightarrow{\text{hom}}, \subset)$ - A.P. , where $A \xrightarrow{\text{hom}} B$ iff there is a homomorphism of A into B . Clearly we can characterize such T , as was pointed out to me by Paul Bacsich .He also suggested consideration of the following problem .

5.17 Def

We say T has the Congruence Extension Property (C.E.P.) iff

(T,T,T,T) has the $(\subset, \xrightarrow[\text{onto}]{\text{hom}}, \xrightarrow[\text{onto}]{\text{hom}}, \subset)$ - A.P.

where $A \xrightarrow[\text{onto}]{\text{hom}} B$ iff there is a homomorphism of A onto B .

The problem here , of course , is that $\xrightarrow[\text{onto}]{\text{hom}}$ is not Diagrammatic.

We ask ourselves , under what conditions on T do we have , whenever $f : A \xrightarrow{\text{hom}} B$ where $A,B \models T$ that the image of A under f is a model of T ? That is , what conditions on T prevents :

for some $\psi \in T$

$T , \phi , \neg\psi, \phi , T$ is $(\xrightarrow[\text{onto}]{\text{hom}}, \subset)$ good ?

The answer easily pops out that T is the union of a positive theory and a universal theory . Which was , perhaps , the expected answer .

5.18 Theorem

Let T be the union of a universal theory and a positive theory .

T has the C.E.P. iff

* (T,T,T,T) has the $(\subset, \xrightarrow[\text{onto}]{\text{hom}}, \xrightarrow{\text{hom}}, \subset)$ - A.P. .

Proof

Obvious from the definitions and our restriction on T .

□

Thus we can easily find a characterization of such T .

What is not so obvious is :

5.19 Theorem

Let T be the union of a universal theory and a positive theory.

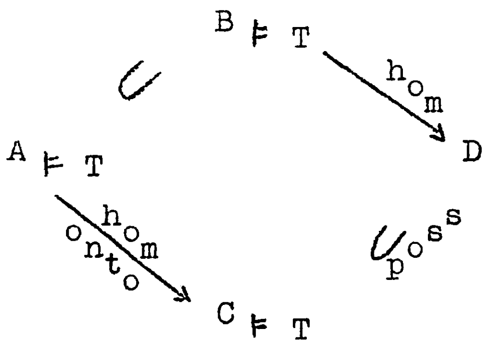
T has the C.E.P. iff

** (T,T,T,T) has the $(\subset, \xrightarrow{hom}, \xrightarrow{hom}, \subset_{poss})$ - A.P.

Proof

$** \longrightarrow *$

Suppose we have

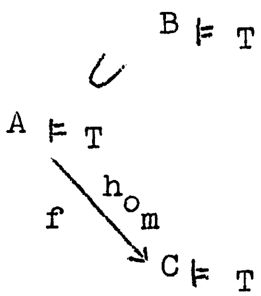


such that the diagram commutes.

It follows that each member of C is mapped to some member of D, so we may assume $C \subset D$.

C.E.P. $\longrightarrow$ **

Suppose



It suffices to show

$T \cup \{ \phi\vec{b} : \phi\vec{v}$ is positive and $B \models \phi[\vec{b}]$ for $\vec{b} \in B \}$

$\cup \{ \phi f\vec{a} : \phi\vec{v}$ is atomic or negated atomic

$\vec{a} \in A$ and $C \models \phi[f\vec{a}] \}$ is consistent.

In view of the condition on T

$C' = C | \{ fa : a \in A \} \models T$

So it suffices to show that

$T \cup \{ \phi\vec{b} : \phi\vec{v}$ is positive and $B \models \phi[\vec{b}]$ for $\vec{b} \in B \}$
$\cup$ Diag C' is consistent , but this follows
from C,E.P.

□

<u>5.110</u> Assuming T is the union of a positive theory and a universal theory ,

T has the C.E.P. iff whenever $T \vdash \forall \vec{x}(\theta\vec{x} \cup \phi\vec{x})$ where $\theta\vec{x}$ is the negation of an existential positive formula and $\phi\vec{x}$ is quantifier free , there are $\gamma\vec{x}$ and $\delta\vec{x}$ s.t. $\gamma\vec{x}$ is existential and $\delta\vec{x}$ is existential positive s.t.

$T \vdash \forall \vec{x}(\gamma\vec{x} \cup \delta\vec{x})$

$T \vdash \forall \vec{x}(\gamma\vec{x} \rightarrow \phi\vec{x})$

$T \vdash \forall \vec{x}(\delta\vec{x} \rightarrow \theta\vec{x})$

<u>5.2 The Strong Amalgamation Property</u>

<u>5.21 Def</u>

We say that a theory T has the Strong Amalgamation Property (S.A.P.) if whenever

$B \models T$

$\cup\!\!/$

$A \models T$

$\subseteq$

$C \models T$

there is $D \models T$ s.t.

$B \subseteq D$ and $C \subseteq D$.

Here, by $\subseteq$, we really mean inclusion , so again the diagram commutes .

If T has the S.A.P. then T has the A.P. . The S.A.P. is a stronger condition than the A.P. In fact the theory of Fields has the A.P. but not the S.A.P. .

5.22

For convenience we define

$\mathrm{Meet}(\theta_1\vec{x}\vec{y}\ ,\ \theta_2\vec{x}\vec{z}\ )$ to be

$$\forall\vec{x}\vec{y}\vec{z}((\theta_1\vec{x}\vec{y} \cap \theta_2\vec{x}\vec{z}\ ) \longrightarrow \bigvee_{y\in\vec{y}}\bigvee_{z\in\vec{z}} y=z\ )$$

5.23 Theorem

A theory T has the S.A.P. iff whenever $T \vdash \mathrm{Meet}(\theta_1\vec{x}\vec{y}\ ,\ \theta_2\vec{x}\vec{z}\ )$ where $\theta_1\vec{x}\vec{y}$ and $\theta_2\vec{x}\vec{z}$ are the conjunctions of atomic and negated atomic formulae in L(T) , there are quantifier free formulae $\phi_1\vec{x}\vec{t}$ and $\phi_2\vec{x}\vec{t}'$ in L(T) s.t.

$$T \vdash \forall\vec{x}(\ \exists\vec{t}\phi_1\vec{x}\vec{t} \cup \nexists\vec{t}'\phi_2\vec{x}\vec{t}'\ )$$

$$T \vdash \mathrm{Meet}(\theta_1\vec{x}\vec{y}\ ,\phi_1\vec{x}\vec{t}\ ) \qquad *$$

$$T \vdash \mathrm{Meet}(\theta_2\vec{x}\vec{z}\ ,\phi_2\vec{x}\vec{t}'\ )$$

Proof $\Longleftarrow$

Suppose T has not the S.A.P. , so there are A , B and C $\models$ T s.t. A = B $\cap$ C and $\mathrm{Diag}(B) \cup \mathrm{Diag}(C) \cup T \cup \{b\neq c : b\in \mathrm{dom}B-\mathrm{dom}A\ ,\ c\in\mathrm{dom}C-\mathrm{dom}A\ \}$ is inconsistent . That is to say , there are conjunctions of atomic and negated atomic formulae , $\theta_1\vec{x}\vec{y}$ and $\theta_2\vec{x}\vec{z}$ in L(T) and constants $\vec{a}\in A$, $\vec{b}\in\mathrm{dom}B-\mathrm{dom}A$ and $\vec{c}\in\mathrm{dom}C-\mathrm{dom}A$ s.t.

$T \vdash \mathrm{Meet}(\theta_1\vec{x}\vec{y}\ ,\theta_2\vec{x}\vec{z}\ )$ where $B \models \theta_1[\vec{a}\vec{b}]$ and $C \models \theta_2[\vec{a}\vec{c}]$.

I claim there are no quantifier free formulae $\phi_1\vec{x}\vec{t}$ and $\phi_2\vec{x}\vec{t}'$ in L(T) s.t. * holds .

For otherwise, since $A \models \exists \vec{t}\phi_1[\vec{a}] \cup \exists \vec{t}'\phi_2[\vec{a}]$
there are $\vec{d}_t$ and $\vec{d}_{t'}$ in A s.t.

$A \models \phi_1[\vec{a}\vec{d}_t]$ or $A \models \phi_2[\vec{a}\vec{d}_{t'}]$.

W.l.o.g. we assume $A \models \phi_1[\vec{a}\vec{d}_t]$.
It follows that $B \models \phi_1[\vec{a}\vec{d}_t]$, and since $B \models T$ and $T \vdash \mathrm{Meet}(\theta_1\vec{x}\vec{y}, \theta_2\vec{x}\vec{z})$ and $B \models \theta_1\vec{a}\vec{b}$ there is $d \in \vec{d}_t$ $b \in \vec{b}$ s.t. $B \models d=b$. This gives us a contradiction since $b \in \mathrm{dom}B-\mathrm{dom}A$ and $d_t \in A$.

$\Longrightarrow$

Assume that there are conjunctions of atomic and negated atomic formulae $\theta_1\vec{x}\vec{y}$ and $\theta_2\vec{x}\vec{z}$ in L(T) s.t. $T \vdash \mathrm{Meet}(\theta_1\vec{x}\vec{y}, \theta_2\vec{x}\vec{z})$, but no open formulae $\phi_1\vec{x}\vec{t}$ and $\phi_2\vec{x}\vec{t}'$ in L(T) s.t. * holds.
i.e. s.t.

$$*** \begin{cases} T \vdash \forall \vec{x}(\exists \vec{t}\phi_1\vec{x}\vec{t} \cup \exists \vec{t}'\phi_2\vec{x}\vec{t}') \\ T \vdash \forall \vec{x}\vec{y}\vec{t}((\theta_1\vec{x}\vec{y} \cap \bigwedge_{y \in \vec{y}} \bigwedge_{t \in \vec{t}} y \neq t) \longrightarrow \neg\phi_1\vec{x}\vec{t}) \\ T \vdash \forall \vec{x}\vec{z}\vec{t}'((\theta_2\vec{x}\vec{z} \cap \bigwedge_{z \in \vec{z}} \bigwedge_{t \in \vec{t}'} z \neq t') \longrightarrow \neg\phi_2\vec{x}\vec{t}') \end{cases}$$

Choose sequences of new distinct constants $\vec{a}_x$, $\vec{b}_y$ and $\vec{c}_z$. Consider the binary relation $R_{\vec{b}_y}$ between $L(T)(\vec{a}_x\vec{b}_y\vec{c}_z)$ - structures , claiming the existence of an embedding f

$f : A|L(T)(\vec{a}_x) \longrightarrow B|L(T)(\vec{a}_x)$ s.t. for all $a \in A$ $fa \notin \vec{b}_y$.

Clearly the above relation has a S.C. ; it is not difficult to see that a notion of goodness for $R_{\vec{b}_y}$ consists of those pairs of formulae of the form $<\exists \vec{x}_1 \theta\vec{x}_1\vec{x}_2 , \forall \vec{y}_1 (\bigwedge_{y \in \vec{y}_1 \cup \vec{y}_2} \bigwedge_{b \in \vec{b}_y} y \neq b \longrightarrow \neg\theta\vec{y}_1\vec{y}_2 >$ where $\theta\vec{x}$ is an open formula in $L(T)(\vec{a}_x)$.

(In fact letting $\Delta_{\vec{b}_y}$ be the above notion of goodness , $\Delta_{R_{\vec{b}_y}} \supset \Delta_{\vec{b}_y}$, so it suffices to show that $\Delta_{R_{\vec{b}_y}} \subset \Delta_{\vec{b}_y}$.)

Similarly we define $R_{\vec{c}_z}$.

We now consider the ternary relation $(R^{-1}_{\vec{b}_y}, R_{\vec{c}_z})$. By Theorem 4.62 we see that a notion of goodness for R is Δ_R consisting of triples of the form :

$< \psi_2 , \psi_1 \cup \chi_1 , \chi_2 >$ where

$< \psi_1 \psi_2 > \in \Delta_{\vec{b}_y}$ and $< \chi_1 \chi_2 > \in \Delta_{\vec{c}_z}$.

I claim that

$T \cup \{ \theta_1 \vec{a}_x \vec{b}_y \} , \phi , T \cup \{ a=a : a \in \vec{a}_x \} , \phi , \ldots$
$T \cup \{ \theta_2 \vec{a}_x \vec{c}_z \}$ is Δ_R good .

For otherwise , there are open formulae $\phi_1 \vec{x}\vec{t}$ and $\phi_2 \vec{x}\vec{t}'$ in L(T) s.t.

$$T \vdash \exists \vec{t} \phi_1 \vec{a}_x \vec{t} \cup \exists \vec{t}' \phi_2 \vec{a}_x \vec{t}'$$

$$T \cup \{ \theta_1 \vec{a}_x \vec{b}_y \} \vdash \forall \vec{t} (\bigwedge_{t \in \vec{t}} \bigwedge_{b \in \vec{b}_y} t \neq b \longrightarrow \neg \phi_1 \vec{a}_x \vec{t})$$

$$T \cup \{ \theta_2 \vec{a}_x \vec{c}_z \} \vdash \forall \vec{t}' (\bigwedge_{t \in \vec{t}'} \bigwedge_{c \in \vec{c}_z} t \neq c \longrightarrow \neg \phi_2 \vec{a}_x \vec{t}')$$

Which is easily seen to contradict *** .

It follows that we can find L(T) structures A ,B and C s.t. w.l.o.g. A , B and C $\models$ T $A = B \cap C$, $B \models \theta_1[\vec{a}\vec{b}]$ and $C \models \theta_2[\vec{a}\vec{c}]$ where $\vec{a} \in A$, $\vec{b} \in$ domB-domA and $\vec{c} \in$ domC-domA .

So , by construction , T does not have the S.A.P.

□

5.3 We turn now to an open problem of G.Grätzer, (see [G] page 299, 74) stated as follows :

5.31 " Which sentences Θ have the property that the substructures satisfying Θ of a structure A form a sublattice of the lattice of all substructures of A ? "

In [R] A.Robinson defines a sentence Θ to be convex if whenever

$$\begin{array}{lll} A \models \Theta & & \\ & \subseteq \quad B \models \Theta & \quad * \\ C \models \Theta & \subseteq & \end{array}$$

then if $A \cap C$ is a structure i.e. non-empty, we have $A \cap C \models \Theta$.

He proves the important result that if Θ is convex then Θ is $\forall\exists$.

Let us say that a sentence Θ has the join property if whenever * holds, $[A \cup B]_C \models \Theta$ where $[A \cup B]_C$ is the substructure of C generated by domA $\cup$ domB .

It is easy to see that Θ has the property in the open problem iff Θ is convex and has the join property .

In [Ra_1] M.O.Rabin gives a syntactic characterization of a sentence to be convex . In a further paper [Ra_2], he proves an alternative characterization . We give here a further characterization.

Let Θ be a given sentence , which we assume is of the form $\forall x_1 \exists x_2 A(x_1 x_2)$ where $A(x_1 x_2)$ is open . Consider the (2)-simple relation R_Θ defined by :

3) $\forall x_1 x_2 y_1 y_2 (\Lambda\Lambda\ x_1 x_2 R y_1 y_2 \longrightarrow ((\Lambda\Lambda\ y_1 = y_2 \cap \phi y_1) \longrightarrow (\Lambda\Lambda\ x_1 = x_2 \cap \phi x_1)))$ for ϕ atomic or negated atomic in $L(\Theta)$.

4) $\forall x_1 x_2 y_1 y_2 (\Lambda\Lambda\ x_1 x_2 R y_1 y_2 \longrightarrow \exists x_1' x_2' y_1' y_2' \ \ldots (\Lambda\Lambda\ x_1' x_2' R y_1' y_2' \cap A(y_1 y_1') \cap A(y_2 y_2')))$

5) $\forall x_1 x_2 y_1 y_2 (\Lambda\Lambda\ x_1 x_2 R y_1 y_2 \longrightarrow \exists x_1' x_2' y_1' y_2' (x_1' x_2' R y_1' y_2' \cap \cap\ t y_1 = y_1' \cap t y_2 = y_2'))$ for $t y_1$ a term in $L(\Theta)$.

Clearly R_Θ has a S.C. with a notion of goodness Δ_{R_Θ} .

We consider now a further (2)-simple relation R_{con} , defined by :

1) $\forall x_1 x_2 (x_1 = x_2 \longrightarrow \exists y_1 y_2 (x_1 x_2 R y_1 y_2 \cap y_1 = y_2))$

2) $\forall x_1 x_2 y_1 y_2 (\Lambda\Lambda\ x_1 x_2 R y_1 y_2 \longrightarrow (\psi_1 x_1 x_2 \longrightarrow \neg \psi_2 y_1 y_2))$

where $\langle \psi_1 x_1 x_2 , \psi_2 y_1 y_2 \rangle \in \Delta_{R_\Theta}$.

To justify these relations we show ;

<u>5.32</u> <u>Theorem</u>

Θ is convex iff $\neg\Theta, \phi , \Theta$ is R_{con} bad.

<u>Proof</u> $\Longrightarrow$

Suppose $\neg\Theta, \phi , \Theta$ is R_{con} good , then there are A , B and $R \subset A^2 \times B^2$ s.t.

$A \models \neg\Theta \quad B \models \Theta \quad A\ R_{con}\ B$ by R.

W.l.o.g. we may assume that if abRcd then a=b and c=d . (Simply cut R down to size !)

It follows that $Th(A^+)$, R $Th(B^+)$ is Δ_{R_Θ} good , and so we can find

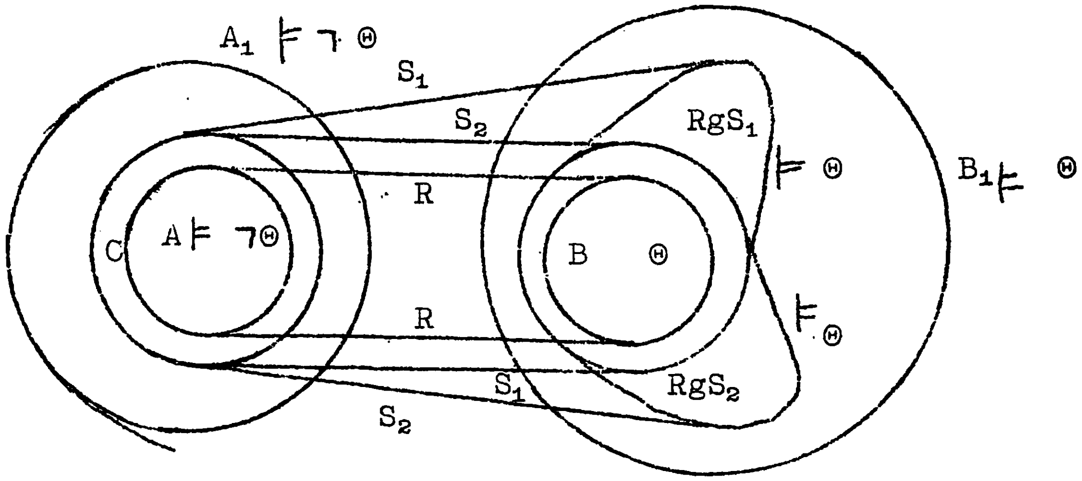
$A_1 \leqslant A_1 \quad B \leqslant B_1 \quad R_1 \supset R$ s.t. $A_1\, R_{\Theta}\, B_1$ by R_1 .

We define S_1 and S_2 as follows .

a S_1b iff $\exists$ c d s.t. acR_1bd ,

c S_2d iff $\exists$ a b s.t. acR_1bd .

A picture might help .



Which we now explain .

By 5) RgS_1 is closed under functions in B_1 , so $B_1|RgS_1 \subset B_1$ with domain RgS_1 ; similarly $B_1|RgS_2 \subset B_1$ with domain RgS_2 .

By 4) $B_1|RgS_i \models \Theta$ for $i \in \{1,2\}$.

It was here , of course , that we required Θ to be $\forall\exists$.

By 3) the domain of $B_1|RgS_1 \cap B_1|RgS_2$ corresponds to $Rg(S_1 \cap S_2)$, and since $S_1 \cap S_2 \supset \{ \langle a,b\rangle : aaRbb \}$ * $Rg(S_1 \cap S_2) \neq \phi$.

Again by 3) and 5) $S_1 \cap S_2$ represents an isomorphism from some $C \subset A_1$ (see picture) onto $B_1|RgS_1 \cap B_1|RgS_2$.

That $C \supset A$ follows from * .

Since $A \leqslant A_1 \quad A$, $A_1 \models \neg\Theta$ and $A \subset C \subset A_1$ and Θ is $\forall\exists$, it follows that $C \models \neg\Theta$ (see 4.6) and hence that $B_1|RgS_1 \cap B_1|RgS_2 \models \neg\Theta$, but then Θ is not convex !

$\Longleftarrow$

Suppose Θ is not convex , so there are A , B , C , D s.t. $A \not\models \neg\Theta$, $A = B \cap C$ and B ,C ,$D \models \Theta$ where

$$\begin{array}{ccccc} & & B & & \\ & \subseteq & & \supseteq & \\ A & & & & D \\ & \supseteq & & \subseteq & \\ & & C & & \end{array}$$

Let $S_1 = \{ \langle aa \rangle : a \in A \} \cup \{ \langle ab \rangle : A \in A \quad b \in (\mathrm{dom}B - \mathrm{dom}A) \}$

$S_2 = \{ \langle aa \rangle : a \in A \} \cup \{ \langle ac \rangle : a \in A \quad c \in (\mathrm{dom}C - \mathrm{dom}A) \}$

we define

$$abRcd \text{ iff } aS_1c \text{ and } bS_2d .$$

I claim that

$$\neg\Theta,\ \phi\ ,\ \Theta \text{ is } R_{con} \text{ good .}$$

It suffices to show that $A\ R\ D \models \Sigma$ where Σ consists of all the sentences 1) 3) 4) 5) . Which is not at all difficult to prove .

□

It follows that if Θ is convex there will be a universal existential sentence $\neg\theta_1$ and a universal sentence θ_2 s.t. $\langle \theta_1, \theta_2 \rangle \in \Delta_{R_{con}}$ ** and $\neg\theta_1 \vdash \Theta \vdash \theta_2$ ..

The reader may wonder how we came upon such a result . In fact the method was quite simple . We drew the picture and described it using 1) 2) 3) 4) 5) quite naturally . We then separated, again naturally, into two relations in order to ensure that in the above ** θ_1 could be chosen existential universal . For the detailed proof we simply had faith !

5.33 We turn now to the condition that Θ should have the join property . Once again we assume Θ is $\forall\exists$ and so of the form $\forall\vec{x}_1\exists\vec{x}_2 A(\vec{x}_1\vec{x}_2)$ where $A(\vec{x}_1\vec{x}_2)$ is open .

Consider the (1,1,1)-simple relation R_{join} defined by :

1) $\forall x_1\exists y_1(x_1R_1y_1)$

2) $\forall x_2y_2x_1(x_2R_2y_2 \cap x_1=x_2 \longrightarrow \exists y_1(x_1R_1y_1 \cap y_1=y_2))$

3) $\forall x_3y_3x_1(x_3R_3y_3 \cap x_1=x_3 \longrightarrow \exists y_1(x_1R_1y_1 \cap y_1=y_3))$

4) $\forall\vec{x}_1\vec{y}_1(\Lambda\Lambda\ \vec{x}_1R_1\vec{y}_1 \cap (\phi\vec{x}_1 \longrightarrow \phi\vec{y}_1))$ where ϕ is atomic or negated atomic in $L(\Theta)$.

5) $\exists x_2y_2(x_2R_2y_2)$

6) $\exists x_3y_3(x_3R_3y_3)$

7) $\forall\vec{x}_2\vec{y}_2(\Lambda\Lambda\ \vec{x}_2R_2\vec{y}_2 \longrightarrow \exists\vec{x}_2'\vec{y}_2'\ \Lambda\Lambda\ \vec{x}_2'R\vec{y}_2' \cap A(\vec{y}_2\vec{y}_2')))$

8) $\forall\vec{x}_3\vec{y}_3(\Lambda\Lambda\ \vec{x}_3R_3\vec{y}_3 \longrightarrow \exists\vec{x}_3'\vec{y}_3'(\Lambda\Lambda\ \vec{x}_3'R_3\vec{y}_3' \cap A(\vec{y}_3\vec{y}_3')))$

9) $\forall\vec{x}_2\vec{y}_2(\Lambda\Lambda\ \vec{x}_2R_2\vec{y}_2 \longrightarrow \exists x_2'y_2'(x_2'R_2y_2' \cap t\vec{y}_2=y_2'))$ for $t\vec{y}_2$ a term in $L(\Theta)$.

10) $\forall\vec{x}_3\vec{y}_3(\Lambda\Lambda\ \vec{x}_3R_3\vec{y}_3 \longrightarrow \exists x_3'y_3'(x_3'R_3y_3' \cap t\vec{y}_3=y_3'))$ for $t\vec{y}_3$ a term in $L(\Theta)$.

5.34 Theorem

Θ has the join property iff for all sequences of terms of length $lg(\vec{x}_1) = n$ (say) , $t_1(\vec{z}_1\vec{y}_1),\dots,t_n(\vec{z}_n\vec{y}_n)$ and all sequences of new distinct constants $\vec{a}_{z_1},\dots,\vec{a}_{z_n},\vec{b}_{y_1},\dots,\vec{b}_{y_n}$ $\forall\vec{x}_2A(t_1(\vec{a}_{z_1}\vec{b}_{y_1}),\dots t_n(\vec{a}_{z_n}\vec{b}_{y_n}),\vec{x}_2)$, ϕ , R_2 , R_3 , Θ is R_{join} bad .Where $R_2 = \bigcup_{1\leq i\leq n}\{ \langle aa\rangle : a\in\vec{a}_{z_i} \}$ and $R_3 = \bigcup_{1\leq i\leq n}\{ \langle bb\rangle : b\in\vec{b}_{y_i} \}$.

Proof $\Longrightarrow$

Suppose the R.H.S. does not hold , so there $L(\Theta)$-structures A and B s.t.

$$A \models \forall \vec{x}_2 A[t_1(\vec{a}_{z_1} \vec{b}_{y_1}), \ldots, t_n(\vec{a}_{z_n} \vec{b}_{y_n}), \vec{x}_2] \qquad *$$

$B \models \Theta$ where w.l.o.g. we may assume $A \subset B$.

Since , for $i \in \{2,3\}$, $Rg(R_i) \subset B$ is closed under functions $B|Rg(R_i) \subset B$ with domain $Rg(R_i)$.

In fact it is easy to see that

$$B|Rg(R_i) \models \Theta \text{ and } B|Rg(R_i) \subset A \text{ for } i \in \{2,3\} .$$

We thus have the following situation ,

$$\begin{array}{lcl} B|Rg(R_2) & \subseteq & \\ & & A \subseteq B \\ B|Rg(R_3) & \subseteq & \end{array}$$

It follows that

$$[\, B|Rg(R_2) \cup B|Rg(R_3) \,]_B \models \neg\Theta \quad \text{by } * \quad \text{and}$$

hence that Θ has not the join property .

$\Longleftarrow$

Suppose now that Θ has not the join property . So for some A , B and C we have

$$\begin{array}{lcl} A \models \Theta & \subseteq & \\ & & [A \cup B]_C \models \neg\Theta \subseteq C \models \Theta \\ B \models \Theta & \subseteq & \end{array}$$

Let R_1 be the identity over $[A \cup B]_C$

R_2 be the identity over A and

R_3 be the identity over B .

It is now easy (though tedious) to show that the R.H.S. of the theorem fails . □

It follows that we can now characterize the sentences satisfying 5.31 . It is not a very enlightening result , but does show the existence of a solution . We think that the methods employed are more important than the results themselves , and hope that they will be further developed .

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