<u>ABSTRACT of Dissertation</u> on "THE USE OF TRANSFORMS IN CONNECTI
WITH DIFFERENCE-DIFFERENTIAL EQUATIONS AND RELATED TOPICS."

(To be presented for the M.Sc. degree by Doris M. James.)

The theory of difference-differential equations is ex
with special reference to the problem of finding a solution by
of transforms, and the discussion is confined mainly to the li
equation, only brief references being made to the non-linear t

The question of simple exponential solutions is consid
first. Following this the published material is dealt with
chronologically, beginning with a paper by Schmidt in 1911, a
proceeding to a discussion of the main contributions from then
present day. Between 1911 and 1922 a number of German mathema
studied these equations in considerable detail, and some of th
are shown to have used methods based on transforms. They are
by Bochner and Titchmarsh, both of whom made definite use of t
Fourier transform in their work on the subject, and finally so
papers by Wright are considered, which clearly exhibit the pow
the Laplace transform method. Wright avoids certain assumptio
in the earlier papers, and his results are seen to be by far t
important and far-reaching.

The dissertation concludes with a reference to the way
these equations have been treated in practical problems of var
types.

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THE USE OF TRANSFORMS IN CONNECTION WITH DIFFERENCE-DIFFERENTIAL EQUATIONS and RELATED TOPICS.

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THE USE OF TRANSFORMS IN CONNECTION WITH DIFFERENCE-DIFFERENTIAL EQUATIONS and RELATED TOPICS.

I. INTRODUCTION.

The study of difference-differential equations has been pursued in considerable detail during the present century, and much information about these equations has been obtained by the use of transforms and similar operators. The first paper of importance was published by Schmidt (29)[*] in 1911. His method of finding a solution involves the use of a formula which is seen to be equivalent to the inversion formula of a transform. From 1911 onwards the study of the subject has developed continuously, culminating in the rigorous discussions by Wright (42-47), published in the last few years. His work is based almost entirely on the use of transforms.

By a difference-differential equation is meant here an equation of the form

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[*] References of the form (1), (2), ... are to the Bibliography, those of the form (1·1), (1·2), ... (2·1), ... are to the equations.

$$\phi\{y(x+b_0),\ y'(x+b_0),\ \cdots,\ y^{(n)}(x+b_0),\ y(x+b_1),\ \cdots,\ y^{(n)}(x+b_1),\ \cdots,\ y(x+b_m),\ \cdots,\ y^{(n)}(x+b_m)\} = 0 \tag{1.1}$$

where the b_μ are independent of x, and $y(x)$ is the unknown function. Of such equations, the type first discussed was the linear equation,

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu}(x)\, y^{(\nu)}(x+b_\mu) = v(x), \tag{1.2}$$

where each term contains only one function $y^{(\nu)}(x+b_\mu)$ and the functions $A_{\mu\nu}(x)$ and $v(x)$ are known. I shall be mainly concerned here with the linear equation, making only brief comments on the non-linear equation, since the theory of that type is still being developed.

Some of the methods used for the solution of linear differential equations may be adapted to the solution of linear difference-differential equations, although the analysis is usually more complicated. For example, when simple exponential solutions are considered, the usual auxiliary equation is found to be a transcendental equation. With the development of the Operational Calculus, however, a method of solving differential equations by Laplace transforms was evolved, and this may be applied to difference-differential equations with considerable success.

In considering the transform method a number of problems are found to arise and, in particular, it is seen that the order at infinity of a solution is of great importance to the validity of the method. One of the first steps in a rigorous approach is the proof of an existence theorem stating conditions under which the equation has solutions of a certain type. The asymptotic behaviour of solutions under certain conditions is also of interest, together with the question of obtaining an actual solution in certain simple cases.

It seems convenient in discussing linear difference-differential equations to follow a chronological scheme, beginning with Schmidt's work in 1911, but first the transcendental equation, already mentioned, is considered.

I should like to acknowledge my indebtedness to Miss B.G. Yates for the valuable help which I have received from her in frequent discussions.

II. THE TRANSCENDENTAL EQUATION.

Certain points of notation which are used throughout the dissertation will be stated here so that repetition may be avoided.

The number C is a positive constant which is not always the same at each occurrence, while k, K, c_1, c_2, ... are positive constants each of which has the same value at each occurrence. The numbers A_n, B_n, D_n $(n = 1, 2, \cdots)$ represent arbitrary constants, and δ is any small positive number.

The general linear equation is taken in the form (1.2) where $y^{(0)}(x) \equiv y(x)$ and $0 = b_0 < b_1 < \cdots < b_m$. It is also supposed that $m \geq 1$, $n \geq 1$. The linear equation which is considered in greatest detail is that with constant coefficients, namely an equation of the form

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}\, y^{(\nu)}(x + b_\mu) = v(x), \qquad (2.1)$$

where the numbers $a_{\mu\nu}$ are real or complex constants. This is referred to as the non-homogeneous equation, and the equation

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}\, y^{(\nu)}(x + b_\mu) = 0 \qquad (2.2)$$

as the homogeneous equation.

As in the case of linear differential equations it is clear that the most general solution of (2·1) is given by adding any particular solution of (2·1) to the general solution of (2·2). By analogy, the general solution of (2·2) is sometimes called the complementary function.

Occasionally it is convenient to use the operator Λ defined by

$$\Lambda\{y(x)\} \equiv \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}\, y^{(\nu)}(x+b_{\mu}), \qquad (2\cdot 3)$$

in which case equation (2·2) may be written in the form

$$\Lambda\{y(x)\} = 0.$$

The number s is a complex quantity given by $s = \sigma + it$ where σ and t are real, unless it is otherwise stated. It is seen immediately that

$$\Lambda\{e^{sx}\} \equiv \gamma(s)\, e^{sx}$$

where

$$\gamma(s) \equiv \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}\, s^{\nu} e^{b_{\mu} s}. \qquad (2\cdot 4)$$

Thus $y(x) = e^{s_n x}$ is a solution of (2·2) if s_n represents a root of the equation

$$\gamma(s) = 0. \qquad (2\cdot 5)$$

This equation (2.5) is called the associated transcendental equation of (2.2) and (2.1), and it corresponds to the auxiliary equation found in the solution of a linear differential equation.

The case when $\Upsilon(s)$ has a multiple zero is of interest, and it will now be shown that if s_λ is a zero of $\Upsilon(s)$ of order $p_\lambda + 1$, then a solution of (2.2) is given by $e^{s_\lambda x}$ multiplied by a polynomial in x. An expression of the form $x^M e^{s_\lambda x}$, where M is a positive integer, is considered first. Clearly

$$\Lambda\left\{ x^M e^{s_\lambda x} \right\} \equiv \Lambda\left\{ \frac{\partial^M}{\partial s_\lambda^M} e^{s_\lambda x} \right\}$$

$$\equiv \frac{\partial^M}{\partial s_\lambda^M} \left\{ \Lambda(e^{s_\lambda x}) \right\}$$

$$\equiv \frac{\partial^M}{\partial s_\lambda^M} \left\{ \Upsilon(s_\lambda)\, e^{s_\lambda x} \right\}$$

$$\equiv e^{s_\lambda x} \sum_{\lambda=0}^{M} \binom{M}{\lambda} \Upsilon^{(M-\lambda)}(s_\lambda)\, x^\lambda .$$

If it is supposed that $M \leq p_\lambda$, then

$$\Upsilon(s_\lambda) = \Upsilon'(s_\lambda) = \cdot \quad \cdot \quad \cdot = \Upsilon^{(M)}(s_\lambda) = 0,$$

so that

$$\Lambda\left\{ x^M e^{s_\lambda x} \right\} = 0 .$$

Thus $y(x) = x^M e^{s_\lambda x}$ is a solution of (2.2) for

$M \leq \rho_\lambda$, and it therefore follows that if $P(x)$ is a polynomial of degree less than or equal to ρ_λ then $P(x)e^{s_\lambda x}$ is also a solution of (2·2).

In general $\gamma(s)$ has an infinity of zeros, and it is clear that information about these zeros is of the utmost importance in any discussion of the exponential solutions of equation (2·2). Wright considers the question in detail in his paper (47) published in 1949. He calls mainly on results given by Langer in a paper (23) on the roots of a transcendental equation, referring also to papers (41) and (35) by Wilder and Tamarkin.

One of the most important properties is that the zeros of $\gamma(s)$ all lie to the left of some line parallel to the imaginary axis provided $a_{mn} \neq 0$. In other words, the zeros are bounded on the right, and similarly if $a_{on} \neq 0$ they are bounded on the left. Langer himself quotes this result from Wilder and Tamarkin, both of whom deal with the question in detail. The truth of the proposition can, however, be seen quite clearly from elementary considerations. If $\gamma(s)$ is written in the form

$$\gamma(s) \equiv a_{mn} e^{b_m s} s^n + a_{on} s^n + \sum_{\mu=1}^{m-1} a_{\mu n} e^{b_\mu s} s^n + \sum_{\mu=0}^{m} \sum_{\nu=0}^{n-1} a_{\mu\nu} e^{b_\mu s} s^\nu,$$

then it is clear that the term $a_{mn} e^{b_m s} s^n$ can be made to exceed the sum of all the others in absolute value provided σ is sufficiently large and positive, while the same is true of the term $a_{on} s^n$ provided σ is sufficiently large and negative. Thus positive constants C_1, C_2 exist such that

(i) If $a_{mn} \neq 0$, then $|\Upsilon(s)| > 0$ for $\sigma > C_1$,

(ii) If $a_{on} \neq 0$, then $|\Upsilon(s)| > 0$ for $\sigma < -C_2$,

and this gives the required result. If both $a_{mn} \neq 0$ and $a_{on} \neq 0$, then all the zeros of $\Upsilon(s)$ lie in the strip of finite width given by $-C_2 \leq \sigma \leq C_1$. Moreover it can be shown that, for large $|s|$, the zeros of $\Upsilon(s)$ approach asymptotically the zeros of the coefficient of s^n in $\Upsilon(s)$, namely the zeros of

$$\phi(s) \equiv \sum_{\mu=0}^{m} a_{\mu n} e^{b_\mu s} . \tag{2.6}$$

Another point which follows directly from Wilder's paper is that $\Upsilon(s)$ has only a finite number of zeros in any strip of finite width parallel to the real axis. In fact, in each of the strips

$$T < t < T + C_3, \quad -T - C_3 < t < -T,$$

the number of zeros actually lies between

$$\frac{c_3 b_m}{2\pi} \pm m$$

for large enough $|T|$.

Wright also gives a detailed account of the situation when $a_{on} = 0$. In this case zeros of $\Upsilon(s)$ exist for which $\sigma_\lambda \to -\infty$. It can be shown, however, that zeros to the left of a line $\sigma = -C_4$ lie within strips of finite width enclosing the curves

$$\sigma = -d_j \log |t| \qquad (j = 1, 2, \cdots, J)$$

where the d_j are known positive constants and J is fixed. For convenience it may be said that the zeros of $\Upsilon(s)$ lie asymptotically about curves of exponential type. From the form of these curves it is seen that for any C_5 and suitable C_6 the function $\Upsilon(s)$ has no zeros in the region

$$|\arg(-s)| < \frac{\pi}{2} - C_5, \qquad |s| > C_6 .$$

Finally, Wright points out that if

$$\beta(s) = \max_{\mu, \nu} |a_{\mu\nu} s^\nu e^{b_\mu s}|$$

then $\Upsilon(s)/\beta(s)$ is uniformly bounded from zero for all s uniformly distant from the zeros of $\Upsilon(s)$.

This completes the information needed about $\Upsilon(s)$

in the general case, but there is a further point of importance in the special case when only one term of (2.2) involves an nth derivative of y so that

$$a_{\mu n} = 0 \quad (\mu \neq \lambda), \quad a_{\lambda n} \neq 0$$

for some positive integer λ less than or equal to m. It may be supposed, without loss of generality, that $a_{\lambda n} = 1$, so that (2.2) becomes

$$y^{(n)}(x+b_\lambda) + \sum_{\mu=0}^{m} \sum_{\nu=0}^{n-1} a_{\mu\nu}\, y^{(\nu)}(x+b_\mu) = 0 \qquad (2.7)$$

and the associated transcendental equation is

$$T(s) \equiv s^n e^{b_\lambda s} + \sum_{\mu=0}^{m} \sum_{\nu=0}^{n-1} a_{\mu\nu}\, s^\nu e^{b_\mu s} = 0, \qquad (2.8)$$

In any strip $|\sigma| \leq C$ it is seen that, for $|s|$ large enough,

$$\begin{aligned} |T(s)| &\geq e^{-b_\lambda C}|s|^n - \sum_{\mu=0}^{m} \sum_{\nu=0}^{n-1} |a_{\mu\nu}|\,|s|^\nu e^{b_\mu C} \\ &\geq c\,|s|^n - C\sum_{\nu=0}^{n-1} |s|^\nu \\ &> 0. \end{aligned}$$

Further, $T(s)$ is an integral function and therefore can have only a finite number of zeros in a finite part of the plane. It follows that $T(s)$ has only a finite number of zeros in any strip $|\sigma| \leq C$.

III. SCHMIDT.

In a paper (29) published in 1911, Schmidt finds a solution for an equation of the form

$$y^{(n)}(x) + \sum_{\mu=0}^{m} \sum_{\nu=0}^{n-1} a_{\mu\nu}\, y^{(\nu)}(x - h_\mu) = v(x), \qquad (3\cdot 1)$$

where the h_μ are positive constants. For simplicity, however, he actually considers the equation

$$\Lambda_x\{y(x)\} \equiv y^{(n)}(x) + \sum_{\nu=0}^{n-1} a_\nu\, y^{(\nu)}(x - h_\nu) = v(x), \qquad (3\cdot 2)$$

in which only one difference occurs with each derivative. Both these equations have associated transcendental equations of the form (2·8) so that the results obtained by Schmidt for equation (3·2), depending as they do on this fact, may be extended to (3·1) without difficulty. As Schmidt's method of solution is clearly associated with the later applications of transforms to difference-differential equations, it is of interest to study it in some detail.

Schmidt first lays down very restrictive conditions on the nature of a function which he will recognise as a proper solution. He assumes that $v(x)$ is continuous for all real x, and of order $|x|^\alpha$ as $|x| \to \infty$, for some positive or negative α. He then considers $y(x)$

to be a proper solution only if $y^{(\nu)}(x)$ is continuous for all x and of order $|x|^{\beta}$ as $|x| \to \infty$ for some constant β, where $\nu = 0, 1, \cdots (n-1)$. In such a case it follows from the difference-differential equation itself that $y^{(n)}(x)$ must also be continuous for all x, and must satisfy a similar order condition as $|x| \to \infty$.

In considering solutions of the homogeneous equation, these restrictions clearly exclude all simple exponential solutions e^{sx}, except those for which s is purely imaginary. In fact the only exponential solutions which Schmidt considers are those of the form

$$y(x) = e^{itx}$$

where t is a real root of the associated transcendental equation, which here takes the form

$$f(t) \equiv (it)^n + \sum_{\nu=0}^{n-1} a_\nu (it)^\nu e^{-ih_\nu t} = 0. \qquad (3\cdot 3)$$

The corresponding function $\Upsilon(s)$, obtained by writing s for it in the function $f(t)$, is therefore of the form (2.8) and has only a finite number of zeros in any strip of finite width parallel to the imaginary axis. Thus, if t is thought for the moment to be complex, it is clear that in the t-plane $f(t)$ has only a finite number of zeros in any strip of finite width parallel to the real axis, and in particular it has only a finite number of real zeros. By this means, therefore, Schmidt limits himself to considering a finite number of simple exponential solutions of the homogeneous equation, instead of an infinity of such solutions. He also discusses the case when the transcendental equation has multiple roots, obtaining his results as in Section II. If $\ell(t)$ has precisely q zeros counted according to their multiplicities, he writes the general solution of the homogeneous equation in the form

$$\sum_{\lambda=1}^{q} B_\lambda \, \phi_\lambda(x) , \tag{3.4}$$

the functions $\phi_\lambda(x)$ being the q functions of the form

$$e^{itx}, \quad x e^{it'x}, \quad x^2 e^{it''x}, \quad \ldots$$

where t runs through all the real zeros of $\ell(t)$, t' through the double and higher zeros, t'' through the triple and higher zeros, and so on.

Schmidt is thus left with the problem of obtaining a particular solution of the non-homogeneous equation. In order to do this he introduces a new function, in terms of which his particular solution is eventually expressed. In the t-plane he takes first a strip of finite width given by

$$| g(t) | \leqslant K/2 .$$

Since $\ell(t)$ has only a finite number of zeros in any such strip he is able to choose a line $t = i\beta$, $(|\beta| \leq \kappa/2)$, parallel to the real axis and lying within the strip, on which there are no zeros of $\ell(t)$. The new function is then defined as an integral evaluated along this line by the formula

$$G_\beta(\xi) = \frac{1}{2\pi}\int_{-\infty+\beta i}^{\infty+\beta i} \frac{e^{i\xi t}}{\ell(t)} \cdot \frac{dt}{(t+\kappa i)^2}, \qquad (3\cdot 5)$$

where ξ is a real variable.

It is interesting to notice here the connection between this function and the inversion formula for the Laplace transform. The Laplace transform of a function $f(\xi)$ is the function $F(s)$ given by

$$F(s) = \int_0^\infty f(\xi)e^{-s\xi}\,d\xi.$$

Then the complex inversion formula, as given by Widder (40, p.66), for example, states that

$$f(\xi) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} F(s)\,e^{\xi s}\,ds, \qquad (\xi > 0)$$

for suitable k. If, therefore, s is replaced by it this gives formally

$$f(\xi) = \frac{1}{2\pi}\int_{-\infty-ki}^{\infty-ki} F(it)\,e^{i\xi t}\,dt,$$

and it can then be seen that the function $f(\xi)$ is expressed in the same form as $G_\beta(\xi)$.

Further, in considering $G_\beta(\xi)$, the presence of the factor $(t+\kappa i)^2$ in the denominator should be mentioned. The sole purpose of this factor is to ensure the convergence of the integral after differentiation under the integral sign, and, as will be shown later, its effect on the solution is eventually annulled by a device of differentiation.

Schmidt uses some straightforward inequalities to show that $G_\beta(\xi)$ represents an absolutely and uniformly convergent integral in every finite interval of ξ. Moreover, he shows that it may be differentiated at least n times under the integral sign. Thus, by operating on $G_\beta(\xi)$ by Λ_ξ, it follows that

$$\Lambda_\xi\{G_\beta(\xi)\} = \frac{1}{2\pi}\int_{-\infty+\beta i}^{\infty+\beta i} \Lambda_\xi\{e^{i\xi t}\}\frac{1}{\ell(t)}\frac{dt}{(t+\kappa i)^2}$$

$$= \frac{1}{2\pi}\int_{-\infty+\beta i}^{\infty+\beta i} e^{i\xi t}\,\frac{dt}{(t+\kappa i)^2}$$

$$= \frac{e^{\kappa\xi}}{2\pi}\int_{-\infty+(\beta+\kappa)i}^{\infty+(\beta+\kappa)i}\frac{e^{i\xi\lambda}}{\lambda^2}\,d\lambda,$$

where $\beta+\kappa>0$ since $|\beta|\leq \kappa/2$.

This integral may be evaluated by using a semi-circular contour, with centre $(\beta+\kappa)i$ and radius R, say. If $\xi \geqslant 0$, it is found that the integral round a semicircle taken above the centre will tend to zero as $R \to \infty$, while if $\xi \leqslant 0$ the semicircle must be taken below the centre for this to happen. In both cases the convergence is due to the squared factor in the denominator of the integrand. Moreover the integrand is regular apart from a double pole at the origin at which the residue is $i\xi$. Therefore, letting $R \to \infty$, it is found that

$$\Lambda_\xi \{ G_\beta(\xi) \} = \begin{cases} 0 & (\xi \geqslant 0) \\ \xi e^{\kappa \xi} & (\xi \leqslant 0) \end{cases},$$

or, putting $x - u = \xi$ where x and u are real,

$$\Lambda_x \{ G_\beta(x-u) \} = \begin{cases} 0 & (x \geqslant u) \\ (x-u) e^{\kappa(x-u)} & (x \leqslant u) \end{cases}. \qquad (3\cdot 6)$$

Schmidt next forms a solution of equation (3·2) in terms of the function G, by taking

$$\phi(x) = \int_{-\infty}^{0} v(u) G_\gamma(x-u)\, du + \int_0^\infty v(u) G_{-\gamma}(x-u)\, du \qquad (3\cdot 7)$$

and

$$y(x) = -\phi''(x) + 2\kappa \phi'(x) - \kappa^2 \phi(x),$$

where $0 < \gamma < \kappa/2$, and no zeros of $\Psi(t)$ lie on the lines $t = \pm i\gamma$. By considering $\Lambda_x \{ y(x) \}$ he shows that this function $y(x)$ satisfies the equation.

Leaving for the moment the justification of the processes and the whole question of order conditions, it is clear that formally

$$\Lambda_x\{\phi(x)\} = \int_{-\infty}^{0} v(u)\,\Lambda_x\{G_\gamma(x-u)\}\,du + \int_{0}^{\infty} v(u)\,\Lambda_x\{G_{-\gamma}(x-u)\}\,du$$

$$= \int_x^{\infty} (x-u)\,e^{K(x-u)}\,v(u)\,du$$

by (3·6), so that

$$\Lambda_x\{\phi(x)\} = v_2(x) \qquad (3\cdot 8)$$

say. Then

$$\Lambda_x\{y(x)\} = -\Lambda_x\{\phi''(x)\} + 2K\,\Lambda_x\{\phi'(x)\} - K^2\,\Lambda_x\{\phi(x)\}$$

$$= -\frac{d^2}{dx^2}\Lambda_x\{\phi(x)\} + 2K\frac{d}{dx}\Lambda_x\{\phi(x)\} - K^2\Lambda_x\{\phi(x)\}$$

$$= -v_2''(x) + 2K\,v_2'(x) - K^2\,v_2(x). \qquad (3\cdot 9)$$

But by differentiating $v_2(x)$ it is found that

$$v_2'(x) = \int_x^{\infty} e^{K(x-u)}\,v(u)\,du + K\,v_2(x)$$

and

$$v_2''(x) = -v(x) + 2K\,v_2'(x) - K^2\,v_2(x),$$

and hence it follows from (3·9) that

$$\Lambda_x\{y(x)\} = v(x).$$

Thus it is seen formally that $\psi(x)$, as defined above, satisfies equation (3·2).

Schmidt's justification of these results is quite straightforward. Using the fact that $v(x)$ is of order $|x|^{\alpha}$ as $|x| \to \infty$, he easily obtains inequalities to show that $\phi(x)$ may be differentiated at least n times under the integral sign. But $\psi(x)$ is defined in terms of $\phi'(x), \phi''(x)$ as well as $\phi(x)$, and therefore in order to show that $\psi(x)$ has n continuous derivatives, it is necessary to show that $\phi(x)$ has $(n+2)$ continuous derivatives. This is done by considering the continuity of $v_2(x)$, and then writing equation (3·8) in the form

$$\phi^{(n)}(x) = v_2(x) - a_{n-1}\phi^{(n-1)}(x-k_{n-1}) - \cdots - a_0\phi(x-k_0).$$

It is shown that the right hand side of this equation possesses at least two continuous derivatives, so that the left-hand side must have the same property and $\phi^{(n+1)}(x), \phi^{(n+2)}(x)$ do in fact exist and are continuous.

It is interesting to notice at this point how the convergence factor in the denominator of $G_{\beta}(\xi)$ is annulled by the differentiation of ϕ . If the integrals for $G_{\pm\gamma}(x-u)$ are written in full, the definition of $\phi(x)$ becomes

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{0} v(u)\,du \int_{-\infty+\gamma i}^{\infty+\gamma i} \frac{e^{i(x-u)t}}{f(t)} \cdot \frac{dt}{(t+Ki)^2}$$

$$+ \frac{1}{2\pi}\int_{0}^{\infty} v(u)\,du \int_{-\infty-\gamma i}^{\infty-\gamma i} \frac{e^{i(x-u)t}}{f(t)} \cdot \frac{dt}{(t+Ki)^2} .$$

Then, by differentiating formally under the integral sign, it is seen that the solution $y(x)$ is of the form

$$y(x) = -\phi''(x) + 2K\phi'(x) - K^2\phi(x)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} v(u)\,du \int_{-\infty+\gamma i}^{\infty+\gamma i} \frac{e^{i(x-u)t}}{f(t)}\,dt \qquad (3\cdot 10)$$

$$+ \frac{1}{2\pi}\int_{0}^{\infty} v(u)\,du \int_{-\infty-\gamma i}^{\infty-\gamma i} \frac{e^{i(x-u)t}}{f(t)}\,dt.$$

On the other hand, if G is defined without the convergence factor, so that

$$G_\beta(\xi) = \frac{1}{2\pi}\int_{-\infty+\beta i}^{\infty+\beta i} \frac{e^{i\xi t}}{f(t)}\,dt$$

then

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{0} v(u)\,du \int_{-\infty+\gamma i}^{\infty+\gamma i} \frac{e^{i(x-u)t}}{f(t)}\,dt$$

$$+ \frac{1}{2\pi}\int_{0}^{\infty} v(u)\,du \int_{-\infty-\gamma i}^{\infty-\gamma i} \frac{e^{i(x-u)t}}{f(t)}\,dt,$$

which is of precisely the same form as the solution given in (3·10). In other words, the solution is now given by $\phi(x)$ itself, instead of by the expression

$$\left\{-\phi''(x) + 2K\phi'(x) - K^2\phi(x)\right\}.$$

However, if $G_\beta(\xi)$ is defined in this way, differentiation

n times under the integral sign cannot be justified, for it is the squared factor in the denominator which ensures this.

Finally it is of interest to mention here that the solution $y(x)$ obtained in (3·10) is of the same form as that which Wright obtains later by his direct use of the Laplace transform.

It is therefore seen that the function

$$y(x) = -\phi''(x) + 2K\phi'(x) - K^2\phi(x) + \sum_{\lambda=1}^{q} B_\lambda \phi_\lambda(x) \qquad (3\cdot 11)$$

is a general solution of the non-homogeneous equation (3·2). Schmidt proceeds to prove that all possible solutions are of this form, and for this purpose the function

$$\psi(x) = \int_{-\infty}^{0} G_\gamma(x-u)\, \Lambda_u\{y(u)\}\, du + \int_0^\infty G_{-\gamma}(x-u)\, \Lambda_u\{y(u)\}\, du \qquad (3\cdot 12)$$

is considered. It is now assumed that γ is chosen so that there are no zeros of $\ell(t)$ in $|\mathcal{J}(t)| \leq \gamma$ except those on the real axis. Schmidt shows that, if the two integrals in this definition of $\psi(x)$ are integrated by parts n times, then the operators Λ are transferred from the y functions to the G functions, a sum of terms $\phi_\lambda(x)$ being added at the same time. In fact

$$\psi(x) = \int_{-\infty}^{0} \Lambda_x\{G_\gamma(x-u)\}\, y(u)\, du + \int_0^\infty \Lambda_x\{G_{-\gamma}(x-u)\}\, y(u)\, du$$

$$+ \sum_{\lambda=1}^{q} D_\lambda\, \phi_\lambda(x)$$

i.e. $$\psi(x) = \int_x^\infty (x-u)e^{K(x-u)} y(u)\,du + \sum_{\lambda=1}^{q} D_\lambda \phi_\lambda(x)$$

by (3·6), so that

$$\psi(x) = y_2(x) + \sum_{\lambda=1}^{q} D_\lambda \phi_\lambda(x) \qquad (3\cdot13)$$

say. Then, as before for $v_2(x)$, it can be proved that

$$-y_2''(x) + 2K y_2'(x) - K^2 y_2(x) = y(x).$$

Further, differentiating a sum of terms $\phi_\lambda(x)$ gives simply another sum of the same type, so that by (3·13)

$$-\psi''(x) + 2K\psi'(x) - K^2\psi(x) = y(x) - \sum_{\lambda=1}^{q} A_\lambda \phi_\lambda(x). \qquad (3\cdot14)$$

Now if $y(x)$ is any solution of the non-homogeneous equation, the definition of $\psi(x)$ in (3·12) reduces to that of $\phi(x)$ in (3·7), and so $y(x)$ as given by (3·14) is in the same form as in (3·11).

Collecting these results, Schmidt has therefore proved that, if $\ell(t)$ has q real zeros, then the general solution of the homogeneous equation is

$$y(x) = \sum_{\lambda=1}^{q} A_\lambda \phi_\lambda(x),$$

while that of the non-homogeneous equation is

$$y(x) = -\phi''(x) + 2K\phi'(x) - K^2\phi(x) + \sum_{\lambda=1}^{q} A_\lambda \phi_\lambda(x).$$

In particular, if $\ell(t)$ has no real zeros, then the homogeneous equation has one and only one solution

$$y(x) = 0,$$

while the non-homogeneous equation has one and only one solution

$$y(x) = -\phi''(x) + 2K\phi'(x) - K^2\phi(x).$$

In view of Schmidt's conditions on a proper solution, all these results have to be completed by a discussion of the order at infinity of the solutions obtained. It is sufficient to consider $y^{(\nu)}(x)$ for $\nu = 0, 1, \cdots, (n-1)$ as the result for $\nu = n$ then follows from the difference-differential ~~differentiation~~ equation itself, as was pointed out before. Thus it is sufficient to consider $\phi^{(\nu)}(x)$ for $\nu = 0, 1, \cdots, (n+1)$. Schmidt's method is to write $\phi^{(\nu)}(x)$ in the form

$$\phi^{(\nu)}(x) = \Phi^{(\nu)}(x) - \int_0^x v(u)\left\{G_\gamma^{(\nu)}(x-u) - G_{-\gamma}^{(\nu)}(x-u)\right\} du.$$

By means of simple inequalities he compares $\Phi^{(\nu)}(x), (\nu = 0, 1, \cdots, n)$ with the integral

$$\int_{-\infty}^{\infty} |v(u)|\, e^{-\eta|x-u|}\, du, \quad (\eta > 0) \qquad (3\cdot 15)$$

and he proves that, if $v(x)$ is of order $|x|^a$ as $|x| \to \infty$ then this integral is also of order $|x|^a$. The second term in the expression for $\phi^{(\nu)}(x)$, however, reduces by

Cauchy's Residue Theorem to the form

$$-\sum_{\lambda=1}^{q} E_\lambda^\nu \int_0^x v(u)\, \phi_\lambda(x-u)\, du, \qquad (\nu = 0, 1, \cdots, n)$$

where the E_λ^ν are constants.

Now if $\ell(t)$ has no real zeros this sum vanishes and then $\phi^{(\nu)}(x)$ is of order $|x|^\alpha$ for $\nu = 0, 1, \cdots, n$. By considering ~~$v_1(x)$ and~~ $v_2(x)$ it is seen that this result is also true for $\nu = n+1$. Thus $y^{(\nu)}(x)$ is of order $|x|^\alpha$ for $\nu = 0, 1, \cdots, n$.

On the other hand, if p is the greatest multiplicity of the zeros of $\ell(t)$, the q functions $\phi_\lambda(x)$ are of the form $x^{\alpha_\lambda} e^{it_\lambda x}$ where α_λ is zero or a positive integer less than or equal to $(p-1)$. It then follows, for large $|x|$, that

$$\left| \int_0^x v(u)\, \phi_\lambda(x-u)\, du \right| \leq |x|^{\alpha_\lambda} \int_0^{|x|} \{ |v(u)| + |v(-u)| \}\, du$$

$$\leq |x|^{p-1} \int_0^{|x|} \{ |v(u)| + |v(-u)| \}\, du.$$

Therefore since $v(u)$ is of order $|u|^\alpha$ as $|u| \to \infty$, it is seen that

$$y^{(\nu)}(x) = \begin{cases} O(|x|^{\alpha+p}) & (\alpha > -1) \\ O(|x|^{p-1} \log x) & (\alpha = -1), \quad (\nu = 0, 1, \cdots, \overline{n-2}) \\ O(|x|^{p-1}) & (\alpha < -1) \end{cases}$$

and by considering ~~$v_1(x)$ and~~ $v_2(x)$ these results are found to hold also for $\nu = n-1$ and $\nu = n$.

Finally, Schmidt's reference to a wider class of solutions of exponential order is interesting, as in all the later work on difference-differential equations it is solutions of this type which are most important. If the condition on $v(x)$ is relaxed, so that, instead of the original order condition, it satisfies

$$v(x) = O(e^{\eta |x|}) \quad \text{as} \quad |x| \to \infty,$$

where η is positive, then the integrals in the definition of ϕ will converge absolutely and uniformly, provided that $\eta < \gamma$. A particular solution can then be formed in terms of ϕ precisely as before, but this time it will be of order $e^{\lambda|x|}$ as $|x| \to \infty$ for some positive constant $\lambda > \eta$. This can be shown by a method similar to that which Schmidt uses in the original case.

If the general solution of the difference-differential equation is also to satisfy this last order condition, it is clear that the only simple exponential solutions which may be included in the complementary function are those which are themselves of order $e^{\lambda|x|}$ as $|x| \to \infty$. In other words, the only possible solutions of the form e^{itx} where t is a root of (3.3), are those for which

$$|\mathcal{J}(t)| \leq \lambda ,$$

but this inequality defines a strip of finite width parallel to the real t-axis, and as has been mentioned earlier $\psi(t)$ has only a finite number of zeros in any such strip. Thus the complementary function will consist of a finite number of terms only, so that again no question of the convergence of an infinite series arises.

IV. SCHÜRER.

In 1912 and 1913 Schürer published three papers, (30), (31) and (32), dealing with a difference-differential equation of the simplest possible type. The equation, of the form

$$y'(x+1) = a\,y(x), \tag{4·1}$$

is one which frequently occurs in practical problems, as will be seen in Section X.

Schürer does not use transforms to find a solution. Instead he connects equation (4·1) with an integral equation of the type studied by Herglotz in a paper (17) published in 1908, and he then uses the solution obtained for the latter. By this means a solution of (4·1) is found, which is of precisely the same form as that given by Wright's transform method. In order to compare these results later, it is convenient to outline Schürer's argument here.

Herglotz discusses the homogeneous integral equation of the form

$$0 = \phi(t) - \int_0^1 \phi(t-u)\,K(u)\,du. \tag{4·2}$$

This is a well-known integral equation, and it is interesting to notice that it may itself be solved straightforwardly by means of transforms. This is done, for example, by Titchmarsh (36).

Herglotz, however, points out that it has a solution of the form

$$\phi_\lambda(t) = e^{-\alpha_\lambda t i}$$

provided α_λ is a root of

$$\chi(\gamma) \equiv \int_0^1 e^{\gamma u i} K(u)\, du = 1.$$

He then proves by a method of contour integration, together with an application of Fourier's Integral Formula, that any arbitrary function $\phi(t)$ may be expanded in the form

$$\bar{\phi}(t) = i \sum_{\lambda=1}^{\infty} \frac{\phi_\lambda(t)}{\chi'(\alpha_\lambda)} \int_0^1 \phi(u) e^{\alpha_\lambda u i}\, du \int_{1-u}^{1} K(v) e^{\alpha_\lambda v i}\, dv, \qquad (0<t<1)$$

where

$$\bar{\phi}(t) \equiv \tfrac{1}{2}\left\{ \phi(t+0) + \phi(t-0) \right\},$$

and he finally considers the position when such a function $\phi(t)$ is a solution of (4.2).

In order to use these results Schürer connects equation (4.1) with the homogeneous integral equation in the following way. In equation (4.2) he puts

$$K(u) = -\gamma, \quad \phi(t) = \psi'(t), \quad \psi(t) = e^{\alpha t} y(t),$$

where α, γ are constants. Then it becomes

$$0 = e^{\alpha t}\{ y'(t) + \alpha y(t) + \gamma y(t) - \gamma e^{-\alpha} y(t-1)\}$$

and putting

$$\alpha = -\gamma, \qquad t = x+1,$$

this gives

$$y'(x+1) = \gamma e^{\gamma} y(x)$$

$$\text{i.e.} \quad y'(x+1) = a\, y(x)$$

where

$$a = \gamma e^{\gamma}.$$

This links the two equations and thus Schürer can express the solution $y(x)$ of (4.1) in Herglotz's expansion form. In the case of (4·1) it is easily seen that $y(x) = e^{s_\lambda x}$ is a solution provided that s_λ is a root of

$$T(s) \equiv s e^{s} - a = 0. \qquad (4\cdot 3)$$

Thus, on making the necessary changes of variable, Schürer obtains a solution of the form

$$\bar{y}(x) = \sum_{\lambda} K_\lambda e^{s_\lambda x}, \qquad (0 < x < 1)$$

where

$$K_\lambda = \frac{s_\lambda}{1+s_\lambda}\left\{ \int_0^1 y(u) e^{-s_\lambda u}\, du + \frac{1}{a} y(1)\right\}.$$

Whether this expansion represents a finite or an infinite series depends on the roots of (4.3). In the notation of Section II, it is seen that $a_{mn} \neq 0,\ a_{on} = 0$, so that the zeros of $T(s)$ are bounded on the right, while on the left they lie asymptotically about curves of exponential type. For this particular $T(s)$ it is easily shown that there is an infinity of zeros, of which at most two are real, the others occurring in conjugate complex pairs with real parts decreasing to $-\infty$. Thus if the zeros are at the points

$$s_\lambda = \sigma_\lambda + i\,t_\lambda\,, \qquad \bar{s}_\lambda = \sigma_\lambda - i\,t_\lambda\,,$$

they can be ordered so that $\sigma_{\lambda+1} < \sigma_\lambda$, $\sigma_\lambda \to -\infty$ as $\lambda \to \infty$.

There are two points to notice from these properties. First, by grouping together the terms from conjugate pairs of zeros, it is clear that the series solution may be written in terms of real quantities, since the corresponding pairs of coefficients K_λ are also conjugates. Thus Schürer's condition that a proper solution must be real, one-valued, finite and have at least one derivative in any finite interval, is not quite so restrictive as it appears.

Secondly, since there is an infinity of zeros, the question of the convergence of the series for $y(x)$ must be discussed. When the equation is solved by Laplace transforms this question is dealt with implicitly in the method, as will be seen later. Schürer's method is based on long, but elementary, arguments. He first takes the series given for $y(x)$ in $(0, 1)$, and by comparing it with $\sum 1/n^2$ he establishes its absolute and uniform convergence in that interval, provided $y(x)$ satisfies certain conditions. He next shows that the expansion for an arbitrary function $y(x)$ is unique if $y(x)$ is a solution of (4·1), and also that in any other unit interval the same convergence properties hold, so that, by repetition, they hold in every finite interval. Finally, he discusses the converse result, showing that any such infinite series, if convergent, does in fact represent a continuous solution of (4·1).

In the following section, Schürer introduces an order condition on his solutions, thus limiting himself to a finite number of simple exponential solutions. His condition is less restrictive than Schmidt's, being of the form

$$y(x) = o(e^{\sigma_{\lambda+1} x}) \quad \text{as} \quad x \to -\infty.$$

Since the zeros are ordered as above this clearly means that the exponential solutions $e^{(\sigma_{\lambda+1} \pm i t_{\lambda+1})x}$, $e^{(\sigma_{\lambda+2} \pm i t_{\lambda+2})x}$, ... must be excluded, so that any solution $y(x)$ is simply a sum of constant multiples of the finite number of simple exponential solutions $e^{(\sigma_1 \pm i t_1)x}, \cdots, e^{(\sigma_\lambda \pm i t_\lambda)x}$. An order condition for $x \to -\infty$, instead of $|x| \to \infty$, is sufficient here, because of the form of the particular $\Upsilon(s)$ given by (4·3), for since the zeros s_n are themselves bounded on the right there is no need for a condition to limit the exponential solutions in that direction. If, however, the condition

$$y(x) = o\left(e^{\sigma_{\lambda-1} x} + e^{\sigma_{\lambda+1} x}\right) \quad as \quad |x| \to \infty$$

is imposed, then the exponential solutions $e^{(\sigma_1 \pm i t_1)x}, \cdots, e^{(\sigma_{\lambda-1} \pm i t_{\lambda-1})x}$ must also be excluded, so that the only possible solution is

$$y(x) = K_\lambda e^{s_\lambda x} + \bar{K}_\lambda e^{\bar{s}_\lambda x},$$

or, writing it in real form,

$$y(x) = 2e^{\sigma_\lambda x}\left\{ \mathcal{R}(K_\lambda)\cos t_\lambda x - \mathcal{J}(K_\lambda)\sin t_\lambda x \right\}.$$

In the last part of paper (30) Schürer discusses, at great length, the zeros of a solution $y(x)$ of (4·1). These results are not important in themselves, but they are connected with the methods used in a later paper (33), in which Schürer discusses the more general equation

$$a_n y^{(n)}(x) + a_{n-1} y^{(n-1)}(x) + \cdots + a_0 y(x) = y(x-1), \qquad (4\cdot 4)$$

where $a_n \neq 0$ and $n \geqslant 1$. Basing his arguments on the number of zeros possessed by a solution $y(x)$ of (4·4), he shows that if the solution is to satisfy

$$y(x) = o(e^{\sigma_{\lambda+1} x}) \quad as \quad x \to -\infty$$

where $\sigma_{\lambda+1}$ is the real part of a root of the transcendental equation associated with (4·4), then there can be only a finite number of simple exponential solutions.

The need for this more delicate argument arises because $\Upsilon(s)$ here is more complicated than in the earlier case, and Schürer cannot order its zeros so easily. However, using the properties of $\Upsilon(s)$ given in Section II, it is interesting to see that since $a_n \neq 0$ the zeros are bounded on the right, while on the left they lie asymptotically about curves of exponential type, just as in the simple case. Moreover, since equation (4·4) is of form (2·7) there is only a finite number of zeros of $\Upsilon(s)$ in any strip of finite width, parallel to the imaginary s-axis, and, in particular, only a finite number on any line parallel to the imaginary axis. Thus the zeros may be ordered so that $\sigma_{\lambda+1} \leq \sigma_\lambda$, $\sigma_\lambda \to -\infty$ as $\lambda \to \infty$. This means that, if the order condition is

$$y(x) = o(e^{\sigma_{\lambda+1} x}) \quad as \quad x \to -\infty,$$

then the solution $y(x)$ must again be formed from a finite number of simple exponential solutions, corresponding to the finite number of zeros of $\Upsilon(s)$ with real parts greater than $\sigma_{\lambda+1}$.

Finally it is of interest to notice Schürer's comments in the case of a multiple root of the transcendental equation. For the simple equation (4·1), the only possible multiple zero of the corresponding given by (4·3), is a double zero at $s = -1$ in the case when $a = -1/e$. Schürer shows by a simple limiting process, that, if this is so, then the function xe^{-x}, as well as e^{-x}, is a solution of (4·1). Similarly if the $\Upsilon(s)$ corresponding to equation (4·4) has multiple zeros, he shows that the equation has solutions of the form obtained in Section II. His method in the latter case is to link his solutions with those of a linear differential equation.

V. HILB.

In 1918 another paper (19) was published on the subject of difference-differential equations. It is a long account by Hilb of the solution of the homogeneous linear equation with constant coefficients. He obtains a solution by considering an expansion of an arbitrary function, similar to that used by Schürer. As he does not use transforms in his work I shall not be concerned with his paper in detail, but an indication of his method is of interest as his results can be related to those obtained later by using transforms.

Hilb refers in his introduction to the results obtained by Schmidt for particular types of linear equation with constant coefficients, but he himself considers the general homogeneous equation of this type, namely the equation

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}\, y^{(\nu)}(x+b_\mu) = 0. \qquad (5.1)$$

Instead of taking the expansion given by Herglotz which Schürer used for his special equation, Hilb works with an expansion given by Picard in his "Traité d' Analyse" (25, p.167). This expansion is found by an application of Cauchy's Residue Theorem round a series of circular contours increasing indefinitely in size. Hilb shows that these circles may be replaced by any series of curves S_ρ in the s-plane on which $|s|$ is greater than any fixed number if ρ is sufficiently large. The resulting expansion of any real continuous function $y(x)$, which satisfies Dirichlet's Fourier series conditions, is of the form

$$y(x) = -\sum_{\lambda} e^{s_\lambda x} \frac{\psi(s_\lambda)}{\pi'(s_\lambda)} \int_{x_0}^{x_1} e^{-s_\lambda \mu} y(\mu)\, d\mu, \quad (x_0 < x < x_1), \qquad (5.2)$$

where $\psi(s), \pi(s)$ are two integral functions of s, the s_λ being the zeros of $\pi(s)$. Further, if $\pi(s) = \psi(s) + \chi(s)$, the following conditions for x in (x_0, x_1) are necessary for the validity of the formula:-

(A) $\lim\limits_{\sigma \to -\infty} \dfrac{\psi(s)}{\pi(s)} = 1, \quad \lim\limits_{\sigma \to +\infty} \dfrac{\psi(s)}{\pi(s)} e^{s(x-x_0)} = 0,$

(B) $\lim\limits_{\sigma \to +\infty} \dfrac{\chi(s)}{\pi(s)} = 1, \quad \lim\limits_{\sigma \to -\infty} \dfrac{\chi(s)}{\pi(s)} e^{s(x-x_1)} = 0,$

(C) $\left|\dfrac{\psi(s)}{\pi(s)}\right| < M_1, \quad \left|\dfrac{\chi(s)}{\pi(s)}\right| < M_2$, in small angles

about the imaginary axis.

It is also clear from the method of proof not only that the series for $y(x)$ is convergent, but also that it is uniformly convergent in all finite intervals of x.

Hilb first uses this expansion to find a solution of the ordinary difference equation

$$a_0 y(x) + a_1 y(x+b_1) + a_2 y(x+b_2) = 0. \tag{5·3}$$

In this case the associated transcendental equation is

$$\Upsilon(s) \equiv a_0 + a_1 e^{b_1 s} + a_2 e^{b_2 s} = 0,$$

and Hilb replaces $\mathbb{T}(s)$ in (5·2) by the function $\Upsilon(s)$ in two different ways, by taking

(i) $\psi(s) = a_0, \qquad \chi(s) = a_1 e^{b_1 s} + a_2 e^{b_2 s},$

(ii) $\psi(s) = a_0 + a_1 e^{b_1 s}, \qquad \chi(s) = a_2 e^{b_2 s}.$

Clearly these are all integral functions and it can be shown that, in the first case, the conditions (A) and (B) are satisfied for $X_0 = x_0,\ X_1 = x_0 + b_1$ while, in the second case, they are satisfied for $X_0 = x_0 + b_1,\ X_1 = x_0 + b_2$, where x_0 is any fixed number. After showing that curves S_ρ exist for which condition (C) is also satisfied, he obtains from (5·2) the expansions

$$y(x) = \sum_\lambda \left\{ -\frac{a_0}{\Upsilon'(s_\lambda)} \int_{x_0}^{x_0+b_1} e^{-s_\lambda \mu} y(\mu)\, d\mu \right\} e^{s_\lambda x}, \quad (x_0 < x < x_0 + b_1) \tag{5·4}$$

$$y(x) = \sum_\lambda \left\{ -\frac{a_0 + a_1 e^{b_1 s_\lambda}}{\Upsilon'(s_\lambda)} \int_{x_0+b_1}^{x_0+b_2} e^{-s_\lambda \mu} y(\mu)\, d\mu \right\} e^{s_\lambda x}, \quad (x_0 + b_1 < x < x_0 + b_2). \tag{5·5}$$

This, however, does not complete the problem, as the important step is to obtain one expansion valid for the whole interval (x_0, x_0+b_2). This is done by considering the new functions

$$F_1(x) = \begin{cases} y(x) & (x_0 < x < x_0+b_1) \\ 0 & (x_0+b_1 < x < x_0+b_2) \end{cases}$$

and

$$F_2(x) = \begin{cases} 0 & (x_0 < x < x_0+b_1) \\ y(x) & (x_0+b_1 < x < x_0+b_2) \end{cases}$$

Then it is seen that

$$y(x) = F_1(x) + F_2(x)$$

for $x_0 < x < x_0+b_2$, except at the point $x = x_0+b_1$. Moreover, $F_1(x)$ has an expansion of the form (5.4), while $F_2(x)$ has an expansion of the form (5.5), both of which hold for the whole interval except for the point $x = x_0+b_1$. Hence it follows that

$$y(x) = \sum_{\lambda} - \frac{K_\lambda(x_0)}{\Upsilon'(s_\lambda)} e^{s_\lambda x}$$

where

$$K_\lambda(x_0) = \left\{ a_0 \int_{x_0}^{x_0+b_2} e^{-s_\lambda \mu} y(\mu)\, d\mu + a_1 e^{b_1 s_\lambda} \int_{x_0+b_1}^{x_0+b_2} e^{-s_\lambda \mu} y(\mu)\, d\mu \right\},$$

the expansion holding for the whole interval $x_0 < x < x_0+b_2$ except at the point $x = x_0+b_1$.

So far this expansion holds for any function which satisfies the necessary conditions, but Hilb next shows by differentiation that if such a function is a solution of equation (5.3) then the coefficient $K_\lambda(x_0)$ is independent of x_0. Thus by repeatedly taking the expansion for different x_0 over intervals of lengths b_1 and $(b_2 - b_1)$ a solution of (5.3) is found in the form of a series which converges uniformly in any fixed interval of x. Further, such a series can be shown to be unique.

Hilb also obtains the result of Section II for a double zero of $\Upsilon(s)$ by using a limiting process.

His next step is to extend his results to more complicated equations. He first considers a difference equation of order n, then the particular equation of form (5.1) for which $m = 3$, $n = 4$ and finally the general equation of form (5.1). By putting $\Upsilon(s)$ in the form $\psi(s) + \chi(s)$ in a sufficient number of ways, he obtains an expansion in the case of the difference equation in precisely the same way as above. For the difference-differential equations, however, a further point must be considered. In the general case an expansion of the form

$$y(x) = \sum_\lambda - \frac{K_\lambda(x_0)}{\Upsilon'(s_\lambda)} e^{s_\lambda x}, \qquad (5.6)$$

where

$$K_\lambda(x_0) = \sum_{\mu=0}^{m-1} \sum_{\nu=0}^{n} a_{\mu\nu} s_\lambda^\nu e^{b_\mu s_\lambda} \int_{x_0+b_\mu}^{x_0+b_\lambda} e^{-s_\lambda \mu} y(\mu)\, d\mu$$

$$- \sum_{\mu=0}^{m} \sum_{\nu=1}^{n} a_{\mu\nu} s_\lambda^\nu \sum_{\nu_1=0}^{\nu-1} \frac{y^{(\nu_1)}(x_0+b_\mu)}{s_\lambda^{\nu_1+1}} e^{-s_\lambda x_0},$$

is obtained, which is valid in the interval (x_0, x_0+b_n) except at the points $x_0+b_1, \cdots, x_0+b_{n-1}$. In order to show that $K_\lambda(x_0)$ is independent of x_0 if $y(x)$ is a solution, and that the series converges uniformly in every fixed interval of x, two more conditions are found to be necessary. If neither a_{mn} nor a_{on} is zero $y(x)$ must have n continuous derivatives for all real x, while if one of a_{mn} or a_{on} is zero $y(x)$ must have continuous derivatives of every order for all real x. Then, by differentiating $K_\lambda(x_0)$ w.r.t. x_0 it is easily shown that (5·1) has a solution of the form (5·6) which converges uniformly in every fixed interval of x.

As Hilb points out, Schürer's particular equation

$$y'(x+1) = a\,y(x),$$

is of the form (5·1), but only one interval (x_0, x_0+1) is involved, and so it does not display the real difficulty of obtaining a solution which is valid over a number of intervals $(x_0, x_0+b_1), \cdots, (x_0+b_{n-1}, x_0+b_n)$. It is of interest to notice, however, that on putting

$$m = 1, \quad n = 1, \quad b_0 = 0, \quad b_1 = 1,$$

$$a_{00} = -a, \quad a_{01} = 0, \quad a_{10} = 0, \quad a_{11} = 1,$$

in (5·6) we obtain

$$K_\lambda(x_0) = -a\int_{x_0}^{x_0+1} e^{-s_\lambda \mu}\, y(\mu)\, d\mu - y(x_0+1)\, e^{-s_\lambda x_0}.$$

Then since $K_\lambda(x_0)$ is independent of x_0, we may choose x_0 to be zero, so that

$$K_\lambda(x_0) = -a\int_0^1 e^{-s_\lambda \mu} y(\mu)\, d\mu \quad - y(1) .$$

Also, in this case

$$\Upsilon(s) \equiv s e^{s} - a \qquad \text{and} \qquad s_\lambda e^{s_\lambda} = a,$$

and therefore (5·6) gives

$$y(x) = \sum_\lambda e^{s_\lambda x} \cdot \frac{s_\lambda}{s_\lambda + 1} \left\{ \int_0^1 e^{-s_\lambda \mu} y(\mu)\, d\mu \;+ \frac{1}{a}\, y(1) \right\},$$

which is of the same form as the solution obtained by Schürer.

VI. HOHEISEL.

It was not until four years later, in 1922, that Hoheisel published a paper (20) on linear difference-differential equations with polynomial coefficients, in which he uses results obtained by Schmidt for the constant coefficient equation and retains the same order condition on a solution. His method, however, is more directly related to Wright's transform method and is therefore of interest here.

Hoheisel first indicates how his equation is connected with that of Schmidt. He considers an equation of the form

$$L(y) \equiv P_0(x)y^{(n)}(x) + P_1(x)y^{(n-1)}(x-h_1) + \cdots + P_n(x)y(x-h_n) = v(x) \tag{6·1}$$

where the $P_\nu(x)$ are polynomials in x. If $P_0(x)$ is of degree q and the other coefficients $P_\nu(x)$ are of degree less than or equal to q, it is seen that

$$L(y) \equiv x^q \Lambda_q(y) + x^{q-1}\Lambda_{q-1}(y) + \cdots + \Lambda_0(y)$$

where $\Lambda_q, \cdots, \Lambda_0$ are operators with constant coefficients of the form discussed by Schmidt.

It is clear that real zeros of $P_0(x)$ will be of importance in a consideration of the solutions of (6·1), just as in the corresponding case for pure differential equations. Therefore, in confining himself to equations in which the coefficients are linear functions of x, Hoheisel still has to distinguish between the case for which $P_0(x)$ has a real zero, and the case for which it has no real zero. In the former case he considers an equation of the form

$$L(y) \equiv (\alpha x+\beta)\, y'(x) + (\gamma x+\delta)\, y(x+k) = v(x)$$

where $\alpha, \beta, \gamma, \delta$ are real, pointing out that this may be reduced by real linear transformations to the form

$$L(y) \equiv x y'(x) + (ax+b)\, y(x+1) = v(x). \qquad (6\cdot 2)$$

In the latter case he considers

$$L(y) \equiv (x+i\eta)\, y'(x) + (ax+b)\, y(x+1) = v(x) \qquad (6\cdot 3)$$

where a, b, η are real and non-zero, so that the coefficient of the highest derivative has no real zero.

The homogeneous equation associated with (6•2) is of the form

$$L(y) \equiv x\, \Lambda_1(y) + \Lambda_0(y) = 0, \qquad (6\cdot 4)$$

where

$$\Lambda_1(y) \equiv y'(x) + a\, y(x+1)$$

and

$$\Lambda_0(y) \equiv b\, y(x+1).$$

As usual it follows that

$$\Lambda_1(e^{sx}) \equiv \tau_1(s)\, e^{sx}, \qquad \Lambda_0(e^{sx}) \equiv \tau_0(s)\, e^{sx}$$

where

$$\Upsilon_1(s) \equiv s + a e^{s} \quad , \quad \Upsilon_0(s) \equiv b e^{s} ,$$

the functions $\Upsilon_1(s), \Upsilon_0(s)$ having only a finite number of zeros in any strip of finite width parallel to the imaginary axis. Moreover, it is easily seen that zeros which lie on the imaginary axis occur only for particular values of a and b, so that if these values are excluded, it is possible to choose γ_0 so that no zeros of $\Upsilon_1(s)$ or $\Upsilon_0(s)$ lie in the strip

$$|\mathcal{R}(s)| \leq \gamma_0 .$$

Hoheisel sketches formally a method of solving (6·2) and (6·4), leaving aside for the moment the question of the convergence and order of the solution obtained. He assumes that

$$y(x) = \int_{\mathcal{C}} e^{sx} w(s)\, ds , \tag{6.5}$$

where the contour $\mathcal{C}$, and the function $w(s)$ are to be found by substituting this $y(x)$ in the equation. This is equivalent to the solution of differential equations by Laplace Integrals. By substituting in (6·4) it is seen that

$$0 = \int_{\mathcal{C}} x \Lambda_1(e^{sx})\, w(s)\, ds + \int_{\mathcal{C}} \Lambda_0(e^{sx})\, w(s)\, ds$$

$$\text{i.e.} \quad 0 = \int_{\mathcal{C}} x e^{sx} \Upsilon_1(s) w(s) ds + \int_{\mathcal{C}} e^{sx} \Upsilon_0(s)\, w(s)\, ds$$

$$\text{i.e.} \quad 0 = \Big[e^{sx} \Upsilon_1(s)\, w(s) \Big]_{\mathcal{C}} - \int_{\mathcal{C}} e^{sx} A\{w(s)\}\, ds \tag{6.6}$$

where

$$A\{w(s)\} \equiv \Upsilon_1(s)w'(s) + w(s)\{\Upsilon_1'(s) - \Upsilon_0(s)\} .$$

Thus for $y(x)$ to be a solution of (6·4) it is necessary first that the contour $\mathscr{C}$ should be chosen so that the integrated term in (6·6) vanishes, and secondly that $w(s)$ should be a solution of

$$A\{w(s)\} = 0 . \qquad (6\cdot 7)$$

This is a linear differential equation, its solution being of the form

$$w(s) = C \cdot \frac{1}{\Upsilon_1(s)} e^{\int^s \frac{\Upsilon_0(z)}{\Upsilon_1(z)} dz} .$$

A solution of (6·4) is then given by substituting this value of $w(s)$ in (6·5). As Hoheisel points out, the function $w(s)$ is a Laplace transform if the contour is chosen to be a line parallel to the imaginary axis.

In order to extend this method to equation (6·2) Hoheisel uses Fourier's Integral Formula. This gives

$$v(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ix\eta} d\eta \int_{-\infty}^{\infty} e^{-i\eta u} v(u) \, du$$

and, putting $s = i\eta$, it is seen formally that

$$v(x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{xs} ds \int_{-\infty}^{\infty} e^{-su} v(u) \, du .$$

The equation corresponding to (6·6) is then considered. If γ is taken as the imaginary axis itself the integrated term vanishes, and for $y(x)$ to be a solution of (6·2) it is necessary that

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\gamma s}\,ds \int_{-\infty}^{\infty} e^{-su}\,v(u)\,du = -\int_{-i\infty}^{i\infty} A\{w(s)\}\,e^{\gamma s}\,ds.$$

Thus $w(s)$ must be a root of

$$A\{w(s)\} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-su}\,v(u)\,du. \qquad (6\cdot 8)$$

Since u is simply a parameter here, it is sufficient to find a solution $w(s,u)$ of the simple equation

$$A\{w(s)\} = e^{-su} \qquad (6\cdot 9)$$

and then take

$$w(s) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} w(s,u)\,v(u)\,du$$

as a solution of (6·8). Then the solution of (6·2) is given by substituting for $w(s)$ in (6·5), as before.

However, when he comes to justify these formal methods, Hoheisel discovers that a convergence factor must be introduced into the solution, and so, instead of (6·9), he considers the equation

$$A\{w(s)\} = e^{-su}(s+K)^{-3} \qquad (6\cdot 10)$$

where $K > \gamma, > 0$. The effect of the extra factor is annulled later in the same way as for Schmidt's equation.

Now (6·10) is another linear differential equation and its solution is of the form

$$w(s, u) = \frac{1}{\gamma_1(s)} e^{\int^s \frac{\gamma_0}{\gamma_1} dz} \int_{s_0}^{s} e^{-\zeta u} (\zeta + K)^{-3} e^{-\int^{\zeta} \frac{\gamma_0}{\gamma_1} d\sigma} d\zeta. \qquad (6\cdot 11)$$

Hoheisel chooses the two solutions, w_1 and w_2 say, given by $s = \pm i\infty$, the path of integration being taken along the imaginary axis in each case. It can be shown that w_1 and w_2 are absolutely convergent, and that, if

$$\left.\begin{aligned} U_1(x, u) &= \int_{-i\infty}^{0} e^{xs} w_1(s, u)\, ds \\ U_2(x, u) &= \int_{0}^{i\infty} e^{xs} w_2(s, u)\, ds \end{aligned}\right\}, \qquad (6\cdot 12)$$

then these integrals are also absolutely convergent, uniformly convergent with respect to x and differentiable once under the integral sign. The function $U = U_1 + U_2$ may therefore be operated on by L, and by employing the following differentiation device a solution of (6·2) can be obtained. The operation

$$\Delta_{3,u}\{v(u)\} \equiv v'''(u) + 3K v''(u) + 3K^2 v'(u) + K^3 v(u)$$

is introduced, and instead of considering a solution of the form

$$-\frac{1}{2\pi i} \int_{-\infty}^{\infty} v(u)\, U(x, u)\, du,$$

Hoheisel considers the function

$$-\frac{1}{2\pi i} W(x) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \Delta_{3,u}\{v(u)\}\, U(x, u)\, du.$$

This may be operated on by L, and it is found that

$$L\left\{-\frac{1}{2\pi i}W(x)\right\} = v(x) + C_v ,$$

where C_v is independent of x but varies with each different function $v(x)$. Thus $-\frac{1}{2\pi i}W(x)$ is a solution of

$$L(y) = v(x) + C_v , \tag{6·13}$$

or, in other words, it is a solution of (6·2) apart from an additive constant. In fact, the problem of solving (6·2) has been reduced to that of solving

$$L(y) = 1, \tag{6·14}$$

but it will be shown later that this equation is insoluble in Hoheisel's sense. First, however, since the solution of (6·13) obtained above is in the form of an infinite integral, its convergence must be discussed, and to discover whether it is a proper solution in Hoheisel's sense its order at infinity must also be considered. To deal with both of these problems Hoheisel transfers the paths of integration, in (6·11) and (6·12), from the imaginary axis to a parallel line through the point $s = \gamma$ where $|\gamma| < \gamma_0$. From the definition of γ_0 given above it is clear that the value of the integral is unchanged. In the new form a factor $e^{-\gamma u}$ is present in the integrand and it can be shown, by choosing the sign of γ properly, that $W(x)$ is absolutely and uniformly convergent, and may be differentiated once under the integral sign. In fact straightforward inequalities give the result

$$|W(x)| < C\int_{-\infty}^{\infty} |\Delta_{3,u}\{v(u)\}|\, e^{-|\gamma||x-u|}\,du + C\int_{-\infty}^{\infty}|\Delta_{3,u}\{v(u)\}|\, e^{-|\gamma u|}\,du.$$

Here the first term on the right is of the form (3·15) discussed by Schmidt. From his results it follows that if $v(u)$ is of order $|u|^{\alpha}$ as $|u| \to \infty$, then this term is of order $|x|^{\alpha}$ as $|x| \to \infty$. Moreover the second term on the right is independent of x, so that $W(x)$ itself must be of order $|x|^{\beta}$ as $|x| \to \infty$, where β is a positive constant or zero and $\beta \geqslant \alpha$. This result can be made more precise, however, by using the fact that $-\frac{1}{2\pi i} W(x)$ is a solution of (6·13). This equation may be written in the form

$$\Lambda_1(W) = -\frac{2\pi i\, v(x) - \Lambda_0(W)}{x} - \frac{2\pi i\, c_v}{x}$$

$$i.e.\quad \Lambda_1(W) = \Psi(x),$$

where $\Psi(x)$ is a function of order $|x|^{\beta-1}$ as $|x| \to \infty$. Now this is itself a difference-differential equation of Schmidt's type, and $\Upsilon_1(s)$ has no purely imaginary zeros, so by his results it follows that

$$W(x) = O(|x|^{\beta-1}), \quad W'(x) = O(|x|^{\beta-1})$$

as $|x| \to \infty$. This argument can be repeated until the stage

$$W(x) = O(|x|^{\alpha-1}), \quad W'(x) = O(|x|^{\alpha-1})$$

is reached. The function $\Psi(x)$ cannot be reduced to a lower order than this, however, because of the term involving $v(x)$, so that this is the final order result satisfied by the solution $-\frac{1}{2\pi i}W(x)$ of (6.13).

Hoheisel next discusses the same problems for a solution of (6.3) and its associated homogeneous equation. The methods used are similar, but the question of convergence is found to be much simpler because of the new form of $\Upsilon_0(s)$. In this case the factor $(s+K)^{-1}$ is sufficient to ensure convergence, and only one solution $w(s,u)$ of the equation corresponding to (6.10) need be considered. Moreover a solution of (6.3) itself is obtained without the addition of the constant c_v, and this solution satisfies the same order condition as above.

The order of a solution of the homogeneous equation (6.4) has yet to be considered, and to deal with this Hoheisel next shows that if $y(x)$ is a continuous solution of (6.4) then $y'(x)$ is also continuous, even at $x = 0$. By writing the equation in the form

$$\Lambda_1(y) = \frac{\Lambda_0(y)}{x}$$

$$\text{i.e.} \quad \Lambda_1(y) = \nu(x),$$

it is seen that $\nu(x)$ is of order $|x|^{\alpha-1}$ if $y(x)$ is of order $|x|^{\alpha}$ as $|x| \to \infty$. Thus by Schmidt's results the last equation must have a solution $y(x)$ for which

$$y(x) = O(|x|^{\alpha-1}), \quad y'(x) = O(|x|^{\alpha-1}).$$

In this case the process may be repeated any number of times and it follows that

$$y(x) = O(|x|^{-c}), \quad y'(x) = O(|x|^{-c}), \tag{6.15}$$

for any c, no matter how large.

Hoheisel proceeds to use a method similar to that employed later by Wright in his solution by Laplace transforms, the only difference being that he uses a two-sided instead of a one-sided transform. He supposes that $y(x)$ is a solution of (6.4) of the correct order at infinity and he defines $\phi(\alpha)$ by the equation

$$\phi(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x} y(x)\, dx.$$

From (6.15) it follows that this integral, and all its derivatives, will converge if α is purely imaginary. It is therefore justifiable to multiply (6.4) by $e^{-\alpha x}$ and integrate w.r.t. x over $(-\infty, \infty)$ giving

$$\int_{-\infty}^{\infty} e^{-\alpha x} L(y)\, dx = 0.$$

This integral may be written in terms of Λ_1 and Λ_2 and simplified, and it is found to reduce to

$$A\{\phi(\alpha)\} = 0.$$

Thus

$$\phi(\alpha) = \frac{1}{\gamma_1(\alpha)}\, e^{\int^{\alpha} \frac{\gamma_0}{\gamma_1} dz},$$

and this is bounded on the imaginary α-axis. It therefore follows that

$$|\alpha^2 \phi(\alpha)| \to \infty$$

as $\alpha \to \infty$ along the imaginary axis. On the other hand

$$\alpha^2 \phi(\alpha) = \int_{-\infty}^{\infty} \alpha^2 e^{-\alpha x} y(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-\alpha x} y''(x)\, dx,$$

and it can be shown by elementary arguments that $y''(x)$ is continuous and of order $|x|^{-c}$, so that $\int_{-\infty}^{\infty} |y''(x)|\, dx$ exists. This means that on the imaginary axis

$$|\alpha^2 \phi(\alpha)| < \int_{-\infty}^{\infty} |y''(x)|\, dx$$

$$< C,$$

which contradicts the result found above. Thus the only possibility is for $\phi(\alpha)$ to be zero if α is purely imaginary. It then follows from Fourier's Integral Formula that

$$y(x) \equiv 0.$$

In other words the only solution of (6·4) which satisfies the necessary order condition is the zero solution.

Finally, Hoheisel considers the order of solutions of (6·2); the problem of solving this equation has been reduced to that of solving (6·14). By a method similar to that just used for the homogeneous equation, it is shown that if $y(x)$ is a solution satisfying the proper order condition, then a contradiction can be avoided only if

$$y''(x) \equiv 0.$$

Moreover, from the equation itself, it is easily seen that the given condition for $y(x)$ can be reduced here to the condition

$$y(x) = O(\tfrac{1}{x}).$$

It therefore follows that

$$y(x) \equiv 0,$$

which does not satisfy (6.14). In other words, (6.14) has no solution of the correct order at infinity.

Hoheisel has therefore shown that equation (6,2) has no solution of the correct order at infinity, and that the only such solution of (6.4) is the zero solution. A solution of (6.3) of the correct order does exist, however, although the homogeneous equation associated with (6.3) again has only the zero solution, as may be shown by a method similar to that used for (6.4).

The results of these simple equations which Hoheisel has considered in detail may be extended, without difficulty, to the cases for which

$$L(y) \equiv x y^{(n)}(x) + \sum_{\mu=0}^{m} \sum_{\nu=0}^{n-1} P_{\mu\nu}(x)\, y^{(\nu)}(x - h_\mu)$$

and

$$L(y) \equiv (x+i\eta)y^{(n)}(x) + \sum_{\mu=0}^{m}\sum_{\nu=0}^{n-1} P_{\mu\nu}(x)\, y^{(\nu)}(x-h_\mu),$$

where the $P_{\mu\nu}(x)$ are linear functions of x. The only possible difficulty arises if, in the first case, there is a term involving $y^{(n-1)}(x)$ whose coefficient has a real zero. In such a case, however, it can be shown, by choosing a different convergence factor, that the same results are true.

Hoheisel does not discuss the case of quadratic coefficients in any detail, but he makes a few comments on equations for which

$$L(y) \equiv (x^2 + a_0)y'(x) + (b_2x^2 + b_1x + b_0)y(x+1).$$

If $a_0 < 0$, the coefficient of $y'(x)$ has real zeros, and a solution of

$$L(y) = v(x) + c_1x + c_2$$

can be found. The equation

$$L(y) = c_1x + c_2$$

can be shown to be insoluble, however, so that in general the equation

$$L(y) = v(x)$$

is also insoluble. On the other hand, if $a_0 > 0$, the coefficient of $y'(x)$ has no real zeros, and a solution of

$$L(y) = v(x)$$

can be shown to exist. If $v(x)$ is of order $|x|^q$ as

$|x| \to \infty$, then this solution is of order $|x|^{n-2}$ as one would expect. Further, the homogeneous equation has no proper solution other than the zero solution.

These results may, in turn, be generalized for the case when

$$L(y) \equiv (x^2 + a_0)\, y^{(n)}(x) + \sum_{\mu=0}^{m} \sum_{\nu=0}^{n-1} P_{\mu\nu}(x)\, y^{(\nu)}(x - h_\mu)$$

where the $P_{\mu\nu}(x)$ are quadratics in x.

VII. BOCHNER.

The next important contribution to the subject is due to Bochner, who makes explicit use of transforms in finding a solution of a difference-differential equation. He confines himself entirely to linear equations with constant coefficients of the form (2·1), and in his first three papers on the subject, (5), (6) and (7), published in 1930 and 1931, he assumes further that the function $v(x)$ is almost periodic. In his book (9) on Fourier integrals, published in 1932, however, he establishes similar results in the case where $v(x)$ is any function whose Fourier transform is readily calculable. The latter case will be considered in some detail here, with only a brief reference to the earlier papers.

Bochner's methods are based on the systematic theory of Fourier transforms as developed in his book (9). In order to use these methods it is necessary to give a brief account of his notation. The Fourier transform $E(t)$ of a function $f(x)$ is given by the formula

$$E(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-itx} \, dx,$$

the integral being taken in the Lebesgue sense, and then the corresponding inversion formula is

$$f(x) = \int_{-\infty}^{\infty} E(t)\, e^{ixt}\, dt$$

under certain specified conditions. These are the definitions taken by Doetsch (13) and Titchmarsh (36), apart from a change of sign and a change in the constant $1/2\pi$ respectively. Bochner also defines the resultant of $f_1(x), f_2(x)$ to be the function

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} f_1(\xi)\, f_2(x-\xi)\, d\xi$$

in the usual way, so that the properties of a resultant may be assumed.

For convenience he then makes the following definitions. The class $\mathfrak{F}_0$ is the class of all functions $f(x)$ which are absolutely integrable in $(-\infty, \infty)$ and $\mathfrak{J}_0$ is the class of all functions which are Fourier transforms of functions of $\mathfrak{F}_0$. Further, the notation

$$f(x) \sim \int_{-\infty}^{\infty} E(t)\, e^{ixt}\, dt$$

is taken to mean that $f(x)$ is the function from $\mathfrak{F}_0$ whose Fourier transform is $E(t)$, whether the integral converges or not.

Finally, Bochner introduces three special definitions in order to state his methods and results as neatly as possible. First, a function $f(x)$ is said to be differentiable λ times in $\mathfrak{F}_0$ if, together with its first

λ derivatives, it is absolutely integrable in $(-\infty, \infty)$. Secondly, he supposes that to a function $f(x)$ there corresponds an λth integral $F_\lambda(x)$ such that

$$F_\lambda^{(\lambda)}(x) = f(x),$$

where $F_\lambda(x)$ is unique apart from a polynomial of degree $(\lambda - 1)$ in x. Then $f(x)$ is said to be integrable λ times in $\mathfrak{F}_0$ if $f(x)$ belongs to $\mathfrak{F}_0$ and if $F_\lambda(x)$ is differentiable λ times in $\mathfrak{F}_0$. Thirdly, a function $\Gamma(t)$ is said to be a multiplicator w.r.t. $E(t)$ if $\Gamma(t)E(t)$ is the transform of some function $f_\Gamma(x)$, say, and if this is true for every transform $E(t)$ then $\Gamma(t)$ is said to be a general multiplicator.

The question of the existence of a solution of (2·1) can now be discussed. Bochner makes his own restriction on the type of function $y(x)$ which he will consider as a solution, requiring in this case that it shall be differentiable n times in $\mathfrak{F}_0$. He supposes that

$$y(x) \sim \int_{-\infty}^{\infty} \phi(t)\, e^{ixt}\, dt. \qquad (7.1)$$

Then it is known that

$$y^{(\nu)}(x) \sim \int_{-\infty}^{\infty} (it)^\nu \phi(t)\, e^{ixt}\, dt, \qquad (\nu = 0, 1, \cdots, n)$$

and therefore

$$y^{(\nu)}(x + b_\mu) \sim \int_{-\infty}^{\infty} (it)^\nu e^{ib_\mu t} \phi(t)\, e^{ixt}\, dt, \qquad (\nu = 0, 1, \cdots, n)$$

It follows that

$$\Lambda(y) \sim \int_{-\infty}^{\infty} \ell(t)\, \phi(t)\, e^{ixt}\, dt, \tag{7.2}$$

where

$$\ell(t) \equiv \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} (it)^{\nu} e^{ib_{\mu} t}.$$

The function $\ell(t)$ thus corresponds to the transcendental function $\Upsilon(s)$, in which s is replaced by it , as in Schmidt's paper. The reason for using this form is again due to the fact that only the real zeros of $\ell(t)$ are of interest, since in (7·1) t is a real, not a complex, variable.

If now

$$v(x) \sim \int_{-\infty}^{\infty} V(t)\, e^{ixt}\, dt$$

then it is seen from (7·2) that $y(x)$ is a solution of (2·1) if and only if $\phi(t)$ satisfies

$$\ell(t)\, \phi(t) = V(t) \tag{7.3}$$

This means that $V(t) / \ell(t)$ must be a function of $\mathcal{J}_0$ and further, that $y(x)$, given by

$$y(x) \sim \int_{-\infty}^{\infty} \frac{V(t)}{\ell(t)}\, e^{ixt}\, dt, \tag{7.4}$$

must be differentiable n times in $\mathcal{J}_0$.

Before discussing these conditions in greater detail the homogeneous equation (2·2) is considered. In this case (7·3) becomes

$$f(t)\ \phi(t) = 0.$$

Thus if $f(t)$ has no real zeros, it follows that $\phi(t)$ must be zero for all real t, and consequently that the only solution of (2·2) is $y(x) \equiv 0$. Further, it is clear that $f(t)$ is a regular function whose zeros are isolated, so that $\phi(t)$ must in any case be zero almost everywhere. It therefore follows again that the only possible solution $y(x)$ is the zero solution. The usual simple exponential solutions given by real zeros of $f(t)$ are omitted here because they do not belong to $\mathfrak{F}_0$ and so they are not solutions in Bochner's restricted sense.

Returning to the non-homogeneous equation (2·1), it is next supposed that $f(t)$ has zeros at

$$t_1,\ t_2,\ \cdots,\ t_\lambda,\ \cdots$$

of multiplicities

$$p_1,\ p_2,\ \cdots,\ p_\lambda,\ \cdots$$

respectively. Then in a small interval $(\alpha_\lambda, \beta_\lambda)$, including the zero t_λ, but no others, it is assumed that

$$f(t) \equiv \{ i(t - t_\lambda) \}^{p_\lambda}\ \ell_*(t)$$

where $\ell_*(t)$ is non-zero and differentiable any number of times in this interval. Thus, in $(\alpha_\lambda, \beta_\lambda)$

$$\frac{V(t)}{\ell(t)} = \frac{1}{\ell_\varkappa(t)} \{i(t-t_\lambda)\}^{-p_\lambda} V(t) \tag{7.5}$$

Then if (2·1) has a solution $y(x)$ of form (7·4) it can be seen formally that the function

$$\{i(t-t_\lambda)\}^{-p_\lambda} V(t)$$

must belong to $\mathcal{J}_0$ for each λ. Bochner proves this result rigorously, using the technique of multiplicators. He also shows that an equivalent condition is that the function

$$e^{-it_\lambda x} v(x)$$

should be integrable p_λ times in $\mathcal{F}_0$.

In order to consider sufficient conditions for (2.1) to have a solution $y(x)$ it must be remembered that the solution $y(x)$ itself is to be differentiable n times in $\mathcal{F}_0$. From (7·4) it therefore follows that the functions

$$\frac{(it)^\nu V(t)}{\ell(t)}, \quad (\nu = 0, 1, \cdots, n) \tag{7.6}$$

must belong to $\mathcal{J}_0$, and this is so if it can be proved that the functions

$$H_\nu(t) \equiv \frac{(it)^\nu}{\ell(t)}, \quad (\nu = 0, 1, \cdots, n) \tag{7.7}$$

are general multiplicators.

Bochner points out that there are considerable simplifications if, instead of the general equation (2·1), the equation

$$y^{(n)}(x) + \sum_{\mu=0}^{m} \sum_{\nu=0}^{n-1} a_{\mu\nu}\, y^{(\nu)}(x+b_\mu) = v(x) \qquad (7\cdot 8)$$

is considered. The corresponding function $\ell(t)$ is then of the same form as that discussed by Schmidt, and has only a finite number of real zeros. Bochner supposes first that $\ell(t)$ has no real zeros, and it is then easily proved that $H_\nu(t)$ is a general multiplicator. His method is quite straightforward, involving a lemma in which he proves that the functions $H_\nu(t)$ are absolutely integrable for all t, together with the general properties of multiplicators. It therefore follows that (7·8) has a solution if and only if

$$\{i(t-t_\lambda)\}^{-\rho_\lambda}\ V(t)$$

belongs to $\mathcal{J}_0$. By the theory of resultants, the solution is of the form

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(\xi)\, v(x-\xi)\, d\xi\,, \qquad (7\cdot 9)$$

where

$$K(\xi) = \int_{-\infty}^{\infty} \frac{1}{\ell(t)}\, e^{i\xi t}\, dt$$

Secondly, Bochner supposes that $f(t)$ has a finite number of real zeros $t_\lambda, (\lambda = 1, 2, \cdots, q)$. In order to obtain the same results here, he introduces a series of general multiplicators $\Gamma_\lambda(t)$, each of which has the value unity in a small interval about the corresponding zero, and vanishes outside a slightly larger interval. Then by considering a multiplicator of $1/f(t)$ given by

$$\Gamma_0(t) = 1 - \sum_{\lambda=1}^{q} \Gamma_\lambda(t)$$

it is possible to change $f(t)$ in the neighbourhood of each zero into a non-vanishing function $f_0(t)$ without affecting the result. The proof then follows in the same way as before.

A particular case of an equation of this type is a pure differential equation.

Returning to the general equation (2·1), Bochner is again concerned with the real zeros of $f(t)$, which may now be infinite in number. In order to limit himself to a finite number, he considers the function which corresponds to $\phi(s)$, as given by (2·6). In the present notation this is

$$\gamma(t) = \sum_{\mu=0}^{m} a_{\mu n} e^{i b_\mu t},$$

and it is called by Bochner the "principal part" of $f(t)$. He supposes that

$$|\gamma(t)| \geqslant c > 0$$

for all t , so that $\gamma(t)$ is uniformly bounded from zero. Then

$$f(t) = \gamma(t)\left\{(it)^n + \sum_{\mu=0}^{m}\sum_{\nu=0}^{n-1} a_{\mu\nu}(it)^\nu \frac{e^{it_\mu t}}{\gamma(t)}\right\}$$

$$= \gamma(t)\, f_1(t)$$

say, the function $f_1(t)$ being of the same form as $\gamma(s)$ in (2.8). Thus $f_1(t)$, and consequently $f(t)$, have only a finite number of real zeros. Bochner then shows in the same way as before that the functions

$$\frac{(it)^\nu}{f(t)} = \frac{(it)^\nu}{f_1(t)\,\gamma(t)} \qquad (\nu = 0, 1, \cdots, n)$$

are general multiplicators. Therefore, provided that the principal part of $f(t)$ is uniformly bounded from zero, it follows that (2.1) has a solution if and only if

$$\left\{ i(t - t_\lambda) \right\}^{-p_\lambda} V(t)$$

belongs to $\mathfrak{J}_0$.

A particular case of an equation of this type is a pure difference equation.

In the last part of his book, Bochner considers certain generalizations of the classes $\mathfrak{F}_0$ and $\mathfrak{J}_0$, and a corresponding extension in the results for difference-differential equations. The class $\mathfrak{F}_k$ is defined as the class of all functions $\phi(x)$ for which $\phi(x)/x^k$ is absolutely integrable in $(-\infty, \infty)$. Also, the functions $\phi(t)$, $\psi(t)$ are said to be k-equivalent if the difference between them is a polynomial in t of degree $(k-1)$ at most. Bochner denotes this relation by

$$\phi(t) \overset{k}{\asymp} \psi(t).$$

Then the k-transform of $f(x)$ is the function $E(t,k)$ given by

$$E(t,k) \overset{k}{\asymp} \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\,\frac{e^{-itx} - L_k}{(-ix)^k}\,dx,$$

where L_k is a polynomial of degree $(k-1)$ in x for $|x| \leq 1$ and zero elsewhere. Further, $\mathfrak{F}_k$ is the class of all transforms $E(t,k)$. It follows from these definitions that a function of the form

$$x^{\lambda}\, e^{it_\lambda x}, \qquad (0 \leq \lambda \leq k-2)$$

where t_λ is real, belongs to the class $\mathfrak{F}_k$.

Returning to the equation (2·1), Bochner now restricts his solutions to be functions of $\mathfrak{F}_k$, and so it is clear that if $f(t)$ has q real zeros t_λ of multiplicities p_λ then the complementary function is of the form

$$\sum_{\lambda=1}^{q} e^{it_\lambda x}\left(\sum_{\lambda=0}^{m_\lambda} c_{\lambda\lambda}\, x^{\lambda}\right)$$

where

$$m_\lambda = \min\,(p_\lambda - 1,\ k-2), \quad (\lambda = 1, 2, \cdots, q).$$

The particular solution of (2·1) is then discussed by methods similar to those used before, and, in particular, if $f(t)$ has no real zeros, it is found to be of form (7·9) again. From this it is seen that certain properties of $v(x)$ will also be possessed by the corresponding solution $y(x)$, for example, if $v(x)$ is almost periodic, then $y(x)$ will also be almost periodic. This is the case discussed by Bochner in his earlier papers (5), (6) and (7). In these papers Bochner obtains results on the existence of solutions which are the same as those found when $v(x)$ is a general function, but he goes further in discussing the question of convergence of solutions.

In (5) he considers an equation of the form (2·1) where $v(x)$ is almost periodic, meaning by this that it has a Fourier series given by

$$v(x) \sim \sum_{\lambda} a_{\lambda} e^{i \lambda_{\lambda} x},$$

the λ_{λ} being real. He does not restrict his solutions in the same way as above, supposing instead that the solution, together with its first n derivatives, is almost periodic. Thus, provided that the complementary function of (2·1) satisfies this condition, it may be included in the solution, so that terms of the form $e^{it_{\lambda}x}$ where t_{λ} is real, can arise. Bochner's method of obtaining his results here is based on Fourier series instead of transforms, but apart from that the proofs follow precisely the same lines as before. He shows that if the principal part of $\ell(t)$ is uniformly bounded from zero, for all t , then the necessary and sufficient condition for the existence of a solution is that the function

$$e^{-it_\lambda x} v(x)$$

should be integrable p_λ times.

In the case when $\ell(t)$ has an infinity of zeros, the complementary function becomes an infinite series, and it is necessary to discover whether it is a Fourier series. This is shown to be the case if the exponents in the complementary function are bounded.

In paper (6) it is shown further that every solution which, together with its first n derivatives, is convergent and uniformly continuous, is also almost periodic. In this paper Bochner uses transforms again. Finally in (7) these convergence results are extended to the solution of a finite set of difference-differential equations of the form (2.1).

VIII. TITCHMARSH.

Titchmarsh makes a brief reference to difference-differential equations in his book on Fourier Integrals (36, p.298), published in 1937, and the results established there are extended in a paper (37) published two years later.

Titchmarsh considers particular linear equations with constant coefficients of the form (2·1), and he finds solutions by means of generalized Fourier transforms. His method follows the same lines as Bochner's, but is less restrictive, with the result that the complementary function does not have to be omitted as in the first case discussed in Section VII.

The most general equation discussed in (36) is of the form

$$y^{(n)}(x) + \sum_{\nu=0}^{n-1} a_\nu y^{(\nu)}(x+b_\nu) = v(x). \tag{8·1}$$

In the notation of Section II it is seen that a_{mn} is zero, and thus the zeros of $\gamma(s)$ are unbounded on the right. This means that the complementary function will be an infinite series of form $\sum_\lambda c_\lambda e^{s_\lambda x}$ which will certainly not converge for positive x. In order to overcome this difficulty Titchmarsh limits himself to solutions which are of order $e^{c|x|}$ as $|x| \to \infty$, so that exponential terms in the complementary function are confined to those for which $|\sigma_\lambda| \leq c$, and these are finite in number.

This order condition on solutions is less restrictive than that of Schmidt and Hoheisel, and it is of importance, since the exponential solutions of difference-differential equations are of considerable interest in applications, as will be seen later. Further, this condition is essential for justifying the use of transforms, and in order to see this, Titchmarsh's definitions of generalized Fourier transforms must now be given. These definitions are an extension of Bochner's because they are given in terms of the complex variable $t = u + iv$, say, instead of in terms of real t.

The generalized Fourier transforms of $f(x)$ are given by

$$F_+(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(x)\, e^{itx}\, dx,$$

when v is sufficiently large and positive, and

$$F_-(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 f(x)\, e^{itx}\, dx$$

when v is sufficiently large and negative. Then the corresponding inversion formula given by Titchmarsh is of the form

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{ia-\infty}^{ia+\infty} F_+(t)\, e^{-ixt}\, dt + \frac{1}{\sqrt{2\pi}} \int_{ib-\infty}^{ib+\infty} F_-(t)\, e^{-ixt}\, dt,$$

where a is sufficiently large and positive, and b is sufficiently large and negative. Thus if $y(x)$ is of order $e^{\alpha|x|}$ it is clear that $F_+(t)$ will converge for $v > \alpha$, and $F_-(t)$ will converge for $v < -\alpha$.

These transforms will now be used to find a solution of (8·1). The method depends entirely on the application of a general theorem on transforms, proved by Titchmarsh in his book (36, p.255). This states that if

$$\int_{ia-\infty}^{ia+\infty} F_1(t)\, e^{-ixt}\, dt + \int_{ib-\infty}^{ib+\infty} F_2(t)\, e^{-ixt}\, dt = 0,$$

then it must follow, under certain specified conditions, that

$$F_1(t) = -F_2(t)$$

and also that both $F_1(t)$ and $F_2(t)$ tend to zero as $u \to \pm\infty$ in a certain strip of the t-plane parallel to the real axis.

For convenience, equation (8·1) is put in terms of the new function $\phi(x)$ given by

$$\phi(x) = y(x) - y(0) - x\,y'(0) - \cdots - x^{n-1} y^{(n-1)}(0).$$

Then it is seen that $\phi(x)$ and its first $(n-1)$ derivatives all vanish at $x = 0$, and the equation itself becomes

$$\phi^{(n)}(x) + \sum_{\nu=0}^{n-1} q_\nu\, \phi^{(\nu)}(x + b_\nu) = \psi(x) \qquad (8\cdot 2)$$

where $\psi(x)$ differs from $v(x)$ by a polynomial of degree $(n-1)$ in x .

Now if $\Phi_{\pm}(t)$ represent the generalized Fourier transforms of $\phi(x)$, the inversion formula gives

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty}\Phi_{+}(t)\,e^{-ixt}\,dt + \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty}\Phi_{-}(t)\,e^{-ixt}\,dt \qquad (8.3)$$

where $a > c,\ b < -c$. Titchmarsh shows further that, if the integrals are taken in the L^2 sense, then the formulae for $\phi^{(\nu)}(x),\ (\nu = 1, 2, \cdots, n)$, may be obtained by differentiating (8.3) under the integral sign. Further by putting $(x+b_\nu)$ for x , it is seen that

$$\phi^{(\nu)}(x+b_\nu) = \frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty}(-it)^{\nu}e^{-ib_\nu t}\,\Phi_{+}(t)\,e^{-ixt}\,dt$$

$$+ \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty}(-it)^{\nu}e^{-ib_\nu t}\,\Phi_{-}(t)\,e^{-ixt}\,dt,$$

from which it follows that

$$\phi^{(n)}(x) + \sum_{\nu=0}^{n-1} a_\nu\,\phi^{(\nu)}(x+b_\nu) \qquad (8.4)$$

$$= \frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty}\ell(t)\,\Phi_{+}(t)\,e^{-ixt}\,dt + \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty}\ell(t)\,\Phi_{-}(t)\,e^{-ixt}\,dt$$

where

$$\ell(t) \equiv (-it)^{n} + \sum_{\nu=0}^{n-1} a_\nu\,(-it)^{\nu}e^{-ib_\nu t}.$$

Thus $\ell(t)$ corresponds to the usual transcendental function. It is now supposed that $v(x)$ is of order $e^{c|x|}$ as $|x| \to \infty$. Then $\psi(x)$ is also of order $e^{c|x|}$, and thus it has generalized Fourier transforms $\Psi_\pm(t)$, which satisfy the usual inversion formula

$$\psi(x) = \frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty} \Psi_+(t)\, e^{-ixt}\,dt + \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty} \Psi_-(t)\, e^{-ixt}\,dt \qquad (8\cdot5)$$

where $a > c$, $b < -c$.

Thus, by substituting from (8·4) and (8·5) into the equation (8·2), it is seen that

$$\int_{ia-\infty}^{ia+\infty} \{\Phi_+(t)\,\ell(t) - \Psi_+(t)\}\, e^{-ixt}\,dt + \int_{ib-\infty}^{ib+\infty} \{\Phi_-(t)\,\ell(t) - \Psi_-(t)\}\, e^{-ixt}\,dt = 0.$$

From this it follows by the theorem referred to above that

$$\Phi_\pm(t)\,\ell(t) - \Psi_\pm(t) = \pm\,\chi(t),$$

where $\chi(t)$ is regular for $b \le v \le a$, and $\chi(t)$ tends to zero as $u \to \pm\infty$ in this strip.

Since $\psi(x)$ differs from $v(x)$ by a polynomial of degree $(n-1)$ it follows that, if $V_\pm(t)$ are transforms of $v(x)$, then

$$\Psi_\pm(t) = V_\pm(t) \pm \pi\left(\frac{1}{t}\right)$$

where π is a polynomial of degree n. Thus

$$\Phi_\pm(t) = \frac{1}{\ell(t)}\left\{\pm\,\chi(t) + V_\pm(t) \pm \pi\left(\frac{1}{t}\right)\right\},$$

and the solution $\phi(x)$ is found by substituting this in (8·3), giving

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty}\frac{V_+(t)}{\ell(t)}e^{-ixt}dt + \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty}\frac{V_-(t)}{\ell(t)}e^{-ixt}dt$$

$$+\frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty}\frac{\chi(t)+\pi(\frac{1}{t})}{\ell(t)}e^{-ixt}dt - \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty}\frac{\chi(t)+\pi(\frac{1}{t})}{\ell(t)}e^{-ixt}dt.$$

Here $a > c$, $b < -c$, but a, b are now chosen so close to $+c, -c$ respectively that no zeros of $\ell(t)$ lie in $b \leq v < -c$, $c < v \leq a$. Cauchy's Residue Theorem is next applied to the last two terms in the expression for $\phi(x)$, it being noticed that the poles of the integrands are at the zeros, t_λ say, of $\ell(t)$. It is therefore seen that these two terms have the value

$$\frac{1}{\sqrt{2\pi}}\cdot 2\pi i\left\{p(x) + \sum c_\lambda e^{-it_\lambda x}\right\}$$

where c_λ is a constant if t_λ is a simple zero of $\ell(t)$, a linear function of x if t_λ is a double zero, and so on; $p(x)$ is a polynomial of degree $\leq (n-1)$ in x; and the sum is taken over all zeros t_λ for which $|v_\lambda| \leq c$.

Thus it follows that a solution $y(x)$ of the original equation (8·1) is of the form

$$y(x) = \frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty}\frac{V_+(t)}{\ell(t)}e^{-ixt}dt + \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty}\frac{V_-(t)}{\ell(t)}e^{-ixt}dt$$

$$+\sum_\lambda c_\lambda e^{-it_\lambda x} + q(x)$$

where $q(x)$ is a polynomial of degree $\leq (n-1)$ in x, and the sum is taken over the same zeros. Clearly the first two terms represent a particular solution of (8·1), while the last two terms represent the complementary function and are thus solutions of the corresponding homogeneous equation. Further, Titchmarsh shows quite simply that $q(x)$ can be included in the function $\sum c_\lambda e^{-it_\lambda x}$, so that altogether the solution is of the form

$$y(x) = \frac{1}{\sqrt{2\pi}}\int_{ia-\infty}^{ia+\infty} \frac{V_+(t)}{\ell(t)} e^{-ixt}\,dt + \frac{1}{\sqrt{2\pi}}\int_{ib-\infty}^{ib+\infty} \frac{V_-(t)}{\ell(t)} e^{-ixt}\,dt$$
$$+ \sum c_\lambda e^{-it_\lambda x},$$

where c_λ is a polynomial of degree $(p_\lambda - 1)$ if t_λ is a zero of $\ell(t)$ of order p_λ. It is interesting to compare this with the form of solution found by Bochner in (7·4) for there is a clear analogy between the two. There is an even closer link with the solution found by Schmidt, however, and the result here is almost identical in form with that given by (3·11), (3·4) and (3·10).

In his later paper (37), Titchmarsh avoids making any assumption on the order of a solution, assuming instead the necessary differentiability of a solution. In this case he considers the simple equation

$$y'(x) - \frac{1}{2h}\{y(x+h) - y(x-h)\} = 0 \qquad (8\cdot6)$$

and instead of using transforms as defined above he works with the function

$$Y_{\alpha,\beta}(t) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} y(x) e^{itx}\, dx.$$

His method is to multiply equation (8.6) by e^{itx} and integrate w.r.t. x over (α, β), finally expressing the result in terms of $Y_{\alpha,\beta}(t)$. This gives

$$-\sqrt{2\pi}\, \ell(t)\, Y_{\alpha,\beta}(t) = \Phi_\alpha(t) - \Phi_\beta(t)$$

where

$$\Phi_\alpha(t) = y(\alpha) e^{it\alpha} - \frac{\sqrt{2\pi}}{2h} \left\{ e^{-ith} Y_{\alpha,\alpha+h}(t) - e^{ith} Y_{\alpha-h,\alpha}(t) \right\}$$

and $\Phi_\beta(t)$ is of similar form. In this case the inversion formula is given by

$$y(x) = \frac{1}{\sqrt{2\pi}} \int_{ia-\infty}^{ia+\infty} Y_{\alpha,\beta}(t) e^{-ixt}\, dt, \qquad (\alpha < x < \beta)$$

where a is any real number. Titchmarsh takes a to be small and positive, so that $\ell(t)$ has no zeros on the line of integration and $y(x)$ is given by

$$y(x) = -\frac{1}{2\pi} \int_{ia-\infty}^{ia+\infty} \frac{\Phi_\alpha(t)}{\ell(t)} e^{-ixt}\, dt + \frac{1}{2\pi} \int_{ia-\infty}^{ia+\infty} \frac{\Phi_\beta(t)}{\ell(t)} e^{-ixt}\, dt.$$

These integrals are evaluated by Cauchy's Residue Theorem, the contours in both cases being rectangles with one side along the line of integration. For the second integral, a rectangle lying in the upper half-plane is chosen, and by letting its dimensions tend to infinity the value of the integral is found to be equal to $2\pi i$ times the sum of the residues of the integrand at poles lying above the line $v=a$. In other words, if the poles are all simple ones, the second integral is equal to

$$\sum \frac{\Phi_\beta(t_\lambda)\, e^{-it_\lambda x}}{\cos t_\lambda k - 1}$$

and this is convergent for $x<\beta-k$, from the order of Φ_β. Similarly, the first integral is found to be equal to

$$A + Bx + Dx^2 + \sum \frac{\Phi_\alpha(t_\lambda)\, e^{-it_\lambda x}}{\cos t_\lambda k - 1}$$

where the sum is convergent for $x > \alpha+k$, the extra terms here being due to a triple pole at the origin, which lies below the line $v=a$ since a is positive. It is next shown, by differentiation, that $\Phi_\alpha(t_\lambda)$, $\Phi_\beta(t_\lambda)$ are independent of α, β respectively, and so the solution is of the form

$$y(x) = A + Bx + Dx^2 + \sum c_\lambda e^{-it_\lambda x}$$

where A, B, D and c_λ are constants, and t_λ runs through all the zeros of $\sin tk - tk$ except $t=0$. The series clearly converges uniformly in any finite interval. It is interesting to notice that Hilb found the same result in the general case by a different method.

Titchmarsh proceeds to solve a well-known integral equation by a similar use of Fourier transforms, and this method is extended in a paper (10) by Miss Busbridge, published in 1939. The equation considered in the latter paper is of the form

$$f(x) = \int_{-\infty}^{\infty} k(t)\, f(x-t)\, dt, \tag{8·7}$$

and it is assumed that

$$k(t) = O(e^{-at^2}), \quad f(x) = O(e^{bx^2})$$

where $0 < b < a$. It is pointed out that, if

$$k(t) = \begin{cases} \frac{1}{2k} & (|t| \leq k) \\ 0 & (|t| > k) \end{cases}$$

and

$$\psi(x) = \int^{x} f(\xi)\, d\xi,$$

then the integral equation (8·7) reduces to a difference-differential equation of the form (8·6). In this case a is infinitely large and so b may also be as large as we please. It is seen that the solution found for (8·7) in this paper reduces in this particular case to

$$f(x) = Ax + B + \sum c_\lambda e^{-it_\lambda x}$$

and thus it follows that

$$\psi(x) = A'x^2 + Bx + D + \sum c'_\lambda e^{-it_\lambda x},$$

which is the same result as that found above.

Further connections with integral equations are to be found in Pitt's work. He actually deals with integro-differential equations, considering the homogeneous case in a paper (26) published in 1944, and the non-homogeneous case in (27) published in 1947. In the former, the homogeneous linear difference-differential equation occurs as a particular case, and in finding a solution there Pitt makes the hypothesis $|y^{(\nu)}(x)| < Ce^{c|x|}$, which corresponds to that in Titchmarsh's first method. The solution found is of the same form as that obtained by Titchmarsh, but as will be seen later, Pitt's work actually has closer connections with Wright's.

IX. WRIGHT.

The most recent, and by far the most important developments in the study of difference-differential equations, are due to Wright. His first paper (43) on the non-linear equation was published in 1946 and this will be considered briefly at the end of this section, but of chief interest here are the papers (44), (45) and (47) on the linear equation, published in 1948 and 1949.

In order to indicate the way in which Wright uses Laplace transforms to obtain a solution, it seems convenient to outline the main steps first, leaving justification of the formal processes until later. As an example the homogeneous linear equation with constant coefficients will be considered. This is of the form

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}\, y^{(\nu)}(x+b_\mu) = 0 \tag{9.1}$$

and it will be assumed that $a_{mn} \neq 0$. The equation is multiplied by e^{-sx} $(\sigma > 0)$, and integrated w.r.t. x from 0 to ∞. This gives

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} \int_0^\infty e^{-sx}\, y^{(\nu)}(x+b_\mu)\, dx = 0, \tag{9.2}$$

and by putting x in place of $x+b_\mu$, it is seen that the integral becomes

$$e^{b_\mu s}\left\{\int_0^\infty e^{-sx}y^{(\nu)}(x)\,dx-\int_0^{b_\mu} e^{-sx}y^{(\nu)}(x)\,dx\right\}.$$

Further,

$$\int_0^\infty e^{-sx}y^{(\nu)}(x)dx=\left[e^{-sx}y^{(\nu-1)}(x)\right]_0^\infty+s\int_0^\infty e^{-sx}y^{(\nu-1)}(x)\,dx.$$

Thus, by assuming that the order at infinity of $y(x)$ is such that

$$e^{-sx}y^{(\nu)}(x)\to 0 \qquad \text{as} \qquad x\to\infty \qquad (9.3)$$

for $\nu=0,1,\cdots,n$, it follows that

$$\int_0^\infty e^{-sx}y^{(\nu)}(x)dx=-y^{(\nu-1)}(0)+s\int_0^\infty e^{-sx}y^{(\nu-1)}(x)dx.$$

By repetition of this argument it is seen that

$$\begin{aligned}\int_0^\infty e^{-sx}y^{(\nu)}(x)dx &= -y^{(\nu-1)}(0)-sy^{(\nu-2)}(0)\\ &\quad -\cdots-s^{\nu-1}y(0)\\ &\quad +s^\nu\int_0^\infty e^{-sx}y(x)\,dx\\ &= -\sum_{\lambda=0}^{\nu-1}s^{\nu-1-\lambda}y^{(\lambda)}(0)+s^\nu Y(s),\end{aligned}$$

where $Y(s)$ is the Laplace transform of $y(x)$.

Thus (9·2) becomes

$$\sum_{\mu=0}^{m}\sum_{\nu=0}^{n} a_{\mu\nu} e^{b_\mu s}\left\{-\sum_{\lambda=0}^{\nu-1} s^{\nu-1-\lambda} y^{(\lambda)}(0) + s^{\nu} Y(s) - \int_0^{b_\mu} e^{-sx} y^{(\nu)}(x)\,dx\right\}$$
$$= 0$$

i.e.
$$\Upsilon(s)\; Y(s) = H(s) \tag{9·4}$$

where

$$H(s) \equiv \sum_{\mu=0}^{m}\sum_{\nu=0}^{n} a_{\mu\nu} e^{b_\mu s}\left\{\int_0^{b_\mu} e^{-sx} y^{(\nu)}(x)\,dx + \sum_{\lambda=0}^{\nu-1} s^{\nu-1-\lambda} y^{(\lambda)}(0)\right\} \tag{9.5}$$

and $\Upsilon(s)$ is the transcendental function associated with equation (9·1).

In order to obtain formulae for the derivatives of $y(x)$ at the same time, the function $Y_\nu(s)$ is now defined by the equation

$$Y_\nu(s) = \int_0^\infty \{y^{(\nu)}(x) - y^{(\nu)}(0)\} e^{-sx}\,dx. \tag{9.6}$$

Then

$$Y_0(s) = Y(s) - \frac{1}{s} y(0)$$

and so equation (9·4) may be written in the form

$$\Upsilon(s)\; Y_0(s) = H(s) - \frac{1}{s}\Upsilon(s)\, y(0) \tag{9·7}$$

To deduce the corresponding relation for $Y_\nu(s)$, (9·6) is integrated by parts ν times. Since by (9·3) the integrated terms vanish at the upper limit, it is found that

$$Y_\nu(s) = s^\nu Y_0(s) - \sum_{\lambda=1}^{\nu} s^{\nu-\lambda-1} y^{(\lambda)}(0),$$

and so (9·7) may be written in the form

$$Y_\nu(s) = \frac{s^\nu H(s)}{\Upsilon(s)} - \sum_{\lambda=0}^{\nu} s^{\nu-\lambda-1} y^{(\lambda)}(0). \qquad (9.8)$$

It is therefore seen that the solution of (9·1) depends on the solution of (9·8), which is an equation in the function $Y_\nu(s)$ instead of the original function $y(x)$. In order to find $y^{(\nu)}(x)$ from (9·8), the complex inversion formula for $Y_\nu(s)$ is used, as given by Widder (40, p.66), for example. This states, under certain conditions on $y^{(\nu)}(x)$, that

$$y^{(\nu)}(x) - y^{(\nu)}(0) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} Y_\nu(s) e^{xs}\, ds \quad (x > 0) \qquad (9.9)$$

for suitable k. Thus, by (9·8)

$$y^{(\nu)}(x) - y^{(\nu)}(0) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \left\{ \frac{s^\nu H(s)}{\Upsilon(s)} - \sum_{\lambda=0}^{\nu} s^{\nu-\lambda-1} y^{(\lambda)}(0) \right\} e^{xs}\, ds \qquad (9.10)$$

The importance of the assumption $a_{mn} \neq 0$ now becomes evident, for it ensures that the zeros of $\Upsilon(s)$ are bounded on the right, and it is therefore possible to choose k so large that all the zeros lie to the left of the line of integration $\sigma = k$.

The integral in (9·10) is evaluated by means of Cauchy's Residue Theorem, the contour being a rectangle with sides parallel to the real and imaginary s-axes, the right hand side lying along $\sigma = k$. This side is kept fixed, but the other three sides are allowed to move off to infinity, so that finally, since all the zeros of $\Upsilon(s)$ are to the left of $\sigma = k$, they will all lie within the rectangle. It is necessary, of course, to prove the existence of such an infinite sequence of rectangles, having no zeros of $\Upsilon(s)$ actually on their sides.

It is next shown that as the three sides of the rectangles tend to infinity, the values of the corresponding integrals tend to zero for certain values of x. Thus, in the limit, the integral along the fourth side, which is now the whole line $\sigma = k$ is equal to the sum of the residues at the poles of the function

$$\left\{ \frac{s^{\nu} H(s)}{\Upsilon(s)} - \sum_{\lambda=0}^{\nu} s^{\nu-\lambda-1} y^{(\lambda)}(0) \right\} e^{xs}.$$

The residue at a simple zero s_λ of $T(s)$ is easily found to be

$$\frac{H(s_\lambda)\, s_\lambda^{\nu}}{T'(s_\lambda)}\, e^{s_\lambda x}$$

and the residue at the origin is $y^{(\nu)}(0)$, so that if $T(s)$ has no multiple zeros it follows from (9·10) that

$$y^{(\nu)}(x) = \sum_\lambda \frac{H(s_\lambda)\, s_\lambda^{\nu}}{T'(s_\lambda)}\, e^{s_\lambda x}$$

summed over all the zeros s_λ of $T(s)$. On the other hand, if s_λ is a double zero of $T(s)$, the corresponding residue is found to contain a factor which is a linear function of x, while, in general, if s_λ is a zero of multiplicity $(p_\lambda + 1)$, the solution of (9·1) is given by

$$y^{(\nu)}(x) = \sum_\lambda P(\lambda, \nu, x)\, e^{s_\lambda x}, \qquad (9.11)$$

where $P(\lambda, \nu, x)$ is a polynomial of degree p_λ in x. This agrees, of course, with the results for multiple zeros mentioned in Section II.

Thus a solution $y(x)$ has been found in the form of an infinite series, convergent for certain values of x.

It is interesting to notice here how the method applies to an equation of the form

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} (a_{\mu\nu}\, x + a_{\mu\nu}')\, y^{(\nu)}(x + b_\mu) = 0,$$

where the coefficients are linear functions of x. In this case the equation corresponding to (9·4) becomes

$$\Upsilon(s)\, Y'(s) + \phi(s)\, Y(s) = H(s)$$

where $\phi(s)$ is a function of the same form as $\Upsilon(s)$. This is a linear differential equation which may be solved for $Y(s)$ by using an integrating factor. However, the subsequent problem of finding the solution $y(x)$ from the inversion formula becomes very much more complicated since the zeros of $\Upsilon(s)$ are now branch points instead of poles of $Y(s)$. Similarly, the equation whose coefficients are polynomials of degree q, say, in x, depends for its solution on a linear differential equation of order q in $Y(s)$.

Before proceeding to discuss the method more precisely, Wright's assumptions on the nature of a solution must be considered. He assumes that his solution $y(x)$ satisfies certain boundary conditions, namely that the values of $y^{(\nu)}(0)$ be given for $\nu = 0, 1, \cdots, (n-1)$, and also that $y^{(n)}(x)$ be given and integrable in Lebesgue's sense in the initial interval $(0, b_m)$. The functions $y^{(\nu)}(x), (\nu = 0, 1, \cdots, \overline{n-1})$ are then defined by the equation

$$y^{(\nu)}(x) = y^{(\nu)}(0) + \int_0^x y^{(\nu+1)}(\xi)\, d\xi,$$

so that (9·1) may in fact be considered as a difference-integral equation in $y^{(n)}(x)$. It is evident that these properties assumed by Wright on the nature of a solution are much less restrictive than those assumed by the earlier writers, Schmidt and Hoheisel, for example. In particular he imposes no restriction on the order at infinity of a solution, but shows in (44) that such a property follows from certain conditions laid down in the initial interval.

In this paper (44), he considers instead of (9·1), the more general linear equation

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} A_{\mu\nu}(x)\, y^{(\nu)}(x+b_\mu) = v(x) \qquad (9{\cdot}12)$$

where the coefficients $A_{\mu\nu}(x)$ are known functions of x. By methods which are somewhat long, but of an elementary nature, he proves that under certain simple conditions on the coefficients $A_{\mu\nu}(x)$, on the function $v(x)$, and on $y(x)$ in the initial interval $(0, b_m)$, the solution $y(x)$ itself must be of exponential order as $x \to \infty$.

First it is supposed that $A_{mn}(x) = 1$, that $A_{\mu\nu}(x)$ and $v(x)$ are integrable, and that $A_{\mu n}(x)$, $A_{m\nu}(x)$ are bounded, all in the closed interval $(0, X_0)$. Wright then shows that (9·12) may be written in the form

$$\phi(\xi) = u(\xi) + \sum_{\ell=0}^{L} B_\ell(\xi) \int_0^\xi t^\ell \phi(t)\,dt$$

where $u(\xi)$ is integrable, and $B_\ell(\xi)$, $(\ell = 0, 1, \cdots, L)$ is bounded and integrable in $0 \leq \xi \leq a$, for some finite positive number a. It is proved that such an integral equation always has a unique integrable solution $\phi(\xi)$ for $0 \leq \xi \leq a$, and hence it is established that (9.12) always has a solution in the interval $(0, b_m + a)$ where $a \leq b_m - b_{m-1}$. Repetition of this argument extends the result to the interval $(0, X_0 + b_m)$.

Wright also considers the case when the functions $A_{\mu\nu}(x)$, $v(x)$ are bounded, continuous, of bounded variation or of integrable square in the interval $(0, X_0)$. If $y^{(n)}(x)$ also possesses any one of these properties in the initial interval $(0, b_m)$, it is shown that this must hold in the whole interval $(0, X_0 + b_m)$.

Finally it is shown that information on the order at infinity of a solution $y(x)$ can be obtained by making the conditions on $v(x)$ a little more exacting. The following three cases are considered:-

(i) $\int_0^X |v(x)|\,dx < Ce^{cX}$ for all $X > 0$,

(ii) $\int_0^X |v(x)|^2\,dx < Ce^{cX}$ for all $X > 0$,

(iii) $|v(x)| < Ce^{cx}$ for all $x > 0$.

Provided $y^{(n)}(x)$ behaves suitably in the initial interval Wright proves, under any one of these conditions, that the solution, which has been shown to exist by the earlier theorems, must be of order e^{cx} as $x \to \infty$. This result holds also for the first $(n-1)$ derivatives of $y(x)$, while $y^{(n)}(x)$ satisfies the same condition as $v(x)$.

These order results are true in the case of the equation with constant coefficients, for, provided that $a_{mn} \neq 0$, it is easily seen that the necessary hypotheses are satisfied. With this information it is now possible to justify the method of transforms used in (47), and outlined at the beginning of this section. C_γ is chosen so that the solution $y(x)$ is of order $e^{C_\gamma x}$ as $x \to \infty$. Then it is seen that both $Y(s)$ and $Y_\nu(s)$ $(\nu = 0, 1, \cdots, n)$ exist for $\sigma > C_\gamma$, and, further, that condition (9·3) is satisfied. Also the inversion formula (9·9) holds for $k > C_\gamma$ provided $y^{(\nu)}(x)$, $(\nu = 0, 1, \cdots, n)$ is continuous and of bounded variation in the neighbourhood of x. As was seen earlier this is true for $\nu = 0, 1, \cdots, (n-1)$ since $y^{(\nu)}(x)$ is an integral for these values of ν, and it is also true for $\nu = n$ provided $y^{(n)}(x)$ has these properties in the initial interval $(0, b_m)$.

The next step is to discuss the sequence of rectangular contours required for the application of

Cauchy's Residue Theorem. A typical rectangle of the sequence is $\Gamma(M)$ which has sides $\Gamma_1(M)$, $\Gamma_2(M)$, $\Gamma_3(M)$, $\Gamma_4(M)$ along the lines $\sigma = k$, $t = T_M$, $\sigma = -T_M$, $t = -T_M$ respectively, M being a positive integer, and T_M increasing with M. It is required to show that no zeros of $T(s)$ lie on the sides of $\Gamma(M)$ provided M is sufficiently large. The simplest situation occurs if T_M can be chosen as a constant multiple of M, for example $T_M = M\ell$, but this is only possible if no zeros lie on the parts of the three lines $t = \pm M\ell$, $\sigma = -M\ell$, which form three sides of the rectangle. It has been pointed out in Section II however, that in strips of finite width about the lines $t = \pm M\ell$, there can be only a finite number of zeros of $T(s)$, and so in such strips lines $\Gamma_2(M)$, $\Gamma_4(M)$ can be chosen, parallel to the real axis, which do not pass through any of these zeros. It was also seen that, for any C_5 and suitable C_6, no zeros lie in an angular region of the form

$$|\arg(-s)| < \frac{\pi}{2} - C_5, \qquad |s| > C_6,$$

and further that the zeros actually lie asymptotically about curves of exponential type. Thus if M is taken sufficiently large it is clear that no zeros lie on $\Gamma_3(M)$.

In particular cases the choice of these rectangles may be quite simple. For example, for the equation (4.1) discussed by Schürer it is easily shown that $T_M = \pi M$ is a suitable choice.

Finally it is proved that the integrals of $Y_\nu(s)e^{xs}$ along $\Gamma_2(M)$, $\Gamma_3(M)$ and $\Gamma_4(M)$ tend to zero as $M \to \infty$ for x greater than a fixed number. This ensures the convergence of the infinite series in the expression (9·11) found for $y^{(\nu)}(x)$. It may be seen that this is not necessarily true for all x, by considering the particular case when $a_{on} = 0$. Then the zeros of $T(s)$ are unbounded on the left and it is clear that the series (9·11) cannot possibly converge for negative x. In fact, if μ_1 is the least μ for which $a_{\mu n} \neq 0$, it is now shown that the series converges only for $x > b_{\mu_1} = b$, say.

For convenience the function $H(s)$, given in (9·5), is written in the form

$$H(s) \equiv H_1(s) + \sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} e^{b_\mu s} \sum_{\lambda=0}^{\nu-1} s^{\nu-1-\lambda} y^{(\lambda)}(0)$$

where

$$H_1(s) \equiv \sum_{\mu=1}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} e^{b_\mu s} \int_0^{b_\mu} y^{(\nu)}(x) e^{-sx} \, dx.$$

It therefore follows from (9·8) that, for $\nu = 0, 1, \cdots, n$,

$$T(s) Y_\nu(s) - s^\nu H_1(s) = s^\nu \{H(s) - H_1(s)\} - T(s) \sum_{\lambda=0}^{\nu} s^{\nu-1-\lambda} y^{(\lambda)}(0)$$

$$= \sum_{\mu=0}^{m} \left\{ \sum_{\rho=1}^{n} \sum_{\lambda=0}^{\rho-1} - \sum_{\rho=0}^{n} \sum_{\lambda=0}^{\nu} \right\} a_{\mu\rho} e^{b_\mu s} s^{\nu+\rho-\lambda-1} y^{(\lambda)}(0).$$

From this it is found that

$$\Upsilon(s)\, y_\nu(s) - s^\nu H_1(s) = \begin{cases} O(s^{n-1}) & (\nu = n) \\ O(s^{n-2}) & (\nu = 0, 1, \cdots, \overline{n-1}) \end{cases}$$

as $s \to \infty$, so that

$$|y_n(s)| \leq \frac{C|s|^{n-1}}{|\Upsilon(s)|} + \left|\frac{s^n H_1(s)}{\Upsilon(s)}\right| \tag{9.13}$$

and

$$|y_\nu(s)| \leq \frac{C|s|^{n-2}}{|\Upsilon(s)|} + \left|\frac{s^\nu H_1(s)}{\Upsilon(s)}\right|, (\nu = 0, 1, \cdots, \overline{n-1}). \tag{9.14}$$

The next step is to find an inequality for the function $H_1(s)$ and this is done by putting $x = b_\mu - v$ in the definition of $H_1(s)$, giving

$$H_1(s) = \sum_{\mu=1}^{m} \sum_{\nu=0}^{n} a_{\mu\nu} \int_0^{b_\mu} f^{(\nu)}(b_\mu - v)\, e^{sv}\, dv.$$

Thus, if $\sigma < C$, it follows that

$$|H_1(s)| < C \tag{9.15}$$

and, further, by the Riemann-Lebesgue Theorem, that

$$H_1(s) \to 0 \tag{9.16}$$

uniformly in σ as $|t| \to \infty$.

It is now supposed that $b+\delta \leq x \leq C$, and the integral along $\Gamma_3(M)$ is considered first. Since no zeros of $\Upsilon(s)$ lie on this line it follows from Section II that $|\tau(s)| > C\beta(s)$. Further, it is clear in this case that $\beta(s)$, the term of maximum modulus, is uniformly bounded from zero for $|s|$ sufficiently large, and so $|\tau(s)| > C$ for $|s| > C$. Therefore, by (9.13), (9.14) and (9.15), it follows that

$$| y_\nu(s) | \leq C|s|^n$$

$$\text{i.e. } | y_\nu(s) | \leq C M^n$$

on $\Gamma_3(M)$. Thus

$$\left| \int_{\Gamma_3(M)} y_\nu(s) e^{xs} ds \right| \leq C M^n e^{-CM\delta} \cdot C M$$

$$\to 0$$

uniformly for x in $b+\delta \leq x \leq C$ as $M \to \infty$.

On the other hand, on $\Gamma_2(M)$ it is seen that

$$|\tau(s)| > C\beta(s) > C|s^n e^{bs}|,$$

and so, by (9.13) and (9.14),

$$|y_\nu(s) e^{bs}| \leq \left| \frac{s^\nu e^{bs} H_1(s)}{\tau(s)} \right| + C\left| \frac{s^{n-1} e^{bs}}{\tau(s)} \right|$$

$$\leq C\left\{ |H_1(s)| + \frac{1}{|s|} \right\}$$

$$\to 0$$

uniformly in σ as $M \to \infty$ by (9.16). Thus

$$\left| \int_{\Gamma_2(M)} y_\nu(s) e^{xs} ds \right| \leq o(1) \int_{-T_M}^{C} e^{\sigma(x-b)} d\sigma$$

$$\leq Ce^c o(1)$$

$$\to 0$$

uniformly for x in $b+\delta \leq x \leq c$ as $M \to \infty$. Similarly, the integral along $\Gamma_4(M)$ tends uniformly to zero.

It remains to collect the results which have been found. Wright has finally proved that if $a_{mn} \neq 0$ and if b is the least b_μ for which $a_{\mu n} \neq 0$, then the solution of (9·1) and its first $(n-1)$ derivatives are given by (9·11) for $\nu = 0, 1, \cdots, (n-1)$, provided $x > b$. If, further, $\psi^{(n)}(x)$ is continuous and of bounded variation in the initial interval, this result holds also for $\nu = n$.

The last point which is discussed in this paper is the question of the uniform convergence of the series in (9·11). For $\nu = 0, 1, \cdots, (n-1)$ this series converges uniformly in any finite interval $b+\delta \leq x \leq c$, as is seen quite simply by the following considerations. From its definition $H_1(k+it)$ is seen to be the Fourier transform of a function, $h_1(x)$ say, which vanishes for $x<0$ and $x > b_m$. Hence

$$\int_{-\infty}^{\infty} |h_1(x)|^2 \, dx$$

converges, and therefore by Parseval's Theorem it follows that

$$\int_{-\infty}^{\infty} |H_1(k+it)|^2 \, dt$$

must converge. Thus, by (9·14),

$$|Y_\nu(k+it)| < \frac{c}{c^2+t^2} + c\left|\frac{H_1(k+it)}{k+it}\right|$$

for $\nu = 0, 1, \ldots, (n-1)$ and

$$\left\{\int_{-T}^{T}\left|\frac{H_1(k+it)}{k+it}\right|dt\right\}^2 \le \int_{-T}^{T}|H_1(k+it)|^2\,dt\int_{-T}^{T}\frac{dt}{k^2+t^2}$$

$$< C.$$

Hence $\int_{-\infty}^{\infty}|Y_\nu(k+it)|\,dt$ converges, and so the integral for $y^{(\nu)}(x)$ given by (9·9), is seen to be uniformly convergent in the required interval. Since the integrals along $\Gamma_2(M)$, $\Gamma_3(M)$ and $\Gamma_4(M)$ have been shown to tend uniformly to zero, the result follows.

Similar results clearly hold for the case when a_{0n} is assumed to be non-zero. In fact, if b' is the greatest b_μ such that $a_{\mu n} \neq 0$, then the solution of (9·1) and its first $(n-1)$ derivatives are again given by (9·11), the series converging uniformly in any finite interval $-C \le x \le b'-\delta$. Further, if both $a_{mn} \neq 0$ and $a_{0n} \neq 0$ the two results may be put together and it is found that the solution of (9·1) is of the form (9·11) for all x, the convergence being uniform in any finite interval. This completes the information given by Wright on the solution of the homogeneous equation with constant coefficients.

It is interesting to notice the distinction between Wright's results and those obtained previously for such an equation. The simple exponential solutions $e^{s_\lambda x}$ discussed by the earlier writers may be thought of as a fundamental set of solutions, which gives a general solution of the form $\sum_\lambda c_\lambda e^{s_\lambda x}$, the c_λ being arbitrary constants. On the other hand, Wright obtains a general solution in which the coefficients are evaluated in terms of the functions $P(\lambda, \nu, x)$, and these are seen to be determined by the boundary conditions set down initially.

Turning next to the equation

$$\sum_{\mu=0}^{m} \sum_{\nu=0}^{n} a_{\mu\nu}\, y^{(\nu)}(x+b_\mu) = v(x) \qquad (9\cdot 17)$$

it is clear, in view of the above discussion on the homogeneous equation, that the general solution of this non-homogeneous equation will be known provided that a particular solution can be found. For a simple equation such a solution can sometimes be found by inspection, but, in general, the problem is best dealt with by transforms. Formally it is seen, by substituting in (9·17), that the function

$$y(x) = \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty} \frac{V(s)}{\gamma(s)}\, e^{xs}\, ds$$

represents a solution for positive x, $V(s)$ being the Laplace transform of $v(x)$.

Wright considers the problem in detail in his paper (45) on the equation with asymptotically constant coefficients, and it is this section of that paper which is of greatest interest in the present context. In contrast to his other papers, this one is based entirely on the L^2 theory of Fourier transforms, by means of which he obtains a particular solution of (9·17) valid for almost all x. There are, of course, considerable changes in his assumptions on the nature of a solution. He now takes $y^{(n)}(x)$ to be of integrable square in the initial interval, so that it is of integrable square over every finite interval, but assumes only that the equation is satisfied for almost all x. Clearly for such a solution to exist the function $v(x)$ must also be of integrable square.

The theory of Fourier transforms as developed in Titchmarsh's book (36) is now used, together with the notation l.i.m. to denote limit in mean square. It is supposed that $V(t)$ is the Fourier transform of $v(x)$ so that

$$V(t) = \underset{X \to \infty}{\mathrm{l.i.m.}} \int_{-X}^{X} v(x) e^{-itx}\, dx. \tag{9.18}$$

Then this function is known to belong to L^2. From the information in Section II, it is clear that there exists a certain strip $\sigma_1 < \sigma < \sigma_2$, parallel to the imaginary axis, within which $\Upsilon(s)$ has no zeros. Wright proves that there is no loss of generality if the imaginary axis is actually taken to lie within this strip, and in this case it follows that $\Upsilon(it)$ is never zero for real t. Then the function $t^n V(t)/\Upsilon(it)$ belongs to L^2 and so it is possible to define $y^{(n)}(x)$ by the equation

$$y^{(n)}(x) = \underset{T\to\infty}{l.\,i.\,m.} \frac{1}{2\pi}\int_{-T}^{T} \frac{(it)^n V(t)}{\Upsilon(it)} e^{ixt}\, dt.$$

It then follows by Plancherel's Theorem that, for almost all x,

$$y^{(n)}(x) = \frac{1}{2\pi} \frac{d}{dx}\int_{-\infty}^{\infty} \frac{(it)^n V(t)}{\Upsilon(it)} \frac{e^{ixt}-1}{it}\, dt$$

$$= \frac{1}{2\pi} \frac{d}{dx}\int_{-\infty}^{\infty} \frac{(it)^{n-1} V(t)}{\Upsilon(it)} e^{ixt}\, dt.$$

Further, from the fact that

$$y^{(\nu)}(x) = y^{(\nu)}(x_0) - \int_{x}^{x_0} y^{(\nu+1)}(\xi)\, d\xi, \qquad (\nu = 0, 1, \cdots (n-1))$$

it is seen that, if $y^{(\nu)}(x_0)$ is defined by

$$y^{(\nu)}(x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(it)^\nu V(t)}{\Upsilon(it)} e^{ix_0 t}\, dt,$$

then

$$y^{(\nu)}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(it)^{\nu}\, V(t)}{\tau(it)}\, e^{ixt}\, dt$$

for $\nu = 0, 1, \cdots, (n-1)$, both integrals being absolutely and uniformly convergent. Thus

$$y^{(\nu)}(x) = \frac{1}{2\pi} \operatorname*{l.i.m.}_{T\to\infty} \int_{-T}^{T} \frac{(it)^{\nu}\, V(t)}{\tau(it)}\, e^{ixt}\, dt$$

for $\nu = 0, 1, \cdot\cdot\cdot, n$ so that differentiation of $y(x)$ n times under the integral sign is justified. It follows that

$$\Lambda(y) = \frac{1}{2\pi} \operatorname*{l.i.m.}_{T\to\infty} \int_{-T}^{T} V(t)\, e^{ixt}\, dt$$

and therefore

$$\Lambda(y) = v(x)$$

for almost all x , using the usual inversion formula for $v(x)$. Thus the function

$$y(x) = \frac{1}{2\pi} \operatorname*{l.i.m.}_{T\to\infty} \int_{-T}^{T} \frac{V(t)}{\tau(it)}\, e^{ixt}\, dt \qquad (9.19)$$

represents a solution of (9•17) for almost all x , and the information on the solution of this equation is complete.

There is a clear analogy between this particular solution and that obtained by Schmidt, for (9•19) can be written formally as

$$y(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{V(t)}{\Upsilon(it)} e^{ixt}\,dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{ixt}}{\Upsilon(it)}\,dt \int_{-\infty}^{\infty} v(u)\, e^{-itu}\,du$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} v(u)\,du \int_{-\infty}^{\infty} \frac{e^{i(x-u)t}}{\Upsilon(it)}\,dt ,$$

and this is closely comparable with Schmidt's solution (3.10). Also the solution (7·4) obtained by Bochner is of the same form.

A brief comment may be made here on the connection between Wright's constant coefficient equation and the integral equations discussed in papers (26) and (27) by Pitt. In the first of these papers Pitt is concerned with the equation

$$\sum_{\nu=0}^{n} \int_{-\infty}^{\infty} y^{(\nu)}(x-y)\, d k_\nu(y) = 0 , \qquad (9.20)$$

and in the second with the equation

$$\sum_{\nu=0}^{n} \int_{-\infty}^{\infty} y^{(\nu)}(x-y)\, d k_\nu(y) = v(x).$$

As he points out, if each $k_\nu(y)$ is a step function with a finite number of steps, equation (9·20) reduces to an equation of the form (9·1). Thus the linear difference-differential equation with constant coefficients is a particular case of Pitt's integral equation. Under certain stated conditions Pitt proves that a solution of (9·20) must be of exponential order at infinity, and he then uses a two-sided Laplace transform in order to find an expression for such a solution. In particular he obtains the same result as Wright for the solution of (9·1) provided that both $a_{mn} \neq 0$ and $a_{on} \neq 0$, but his method does not cover the case when one of these coefficients is zero.

The main problem with which Wright is concerned in his paper (45) on the equation with asymptotically constant coefficients will now be mentioned briefly. The equation discussed is of the form (9·12) with the additional condition that

$$A_{\mu\nu}(x) \rightarrow a_{\mu\nu}$$

as $x \rightarrow \infty$, the behaviour at infinity of its solution being considered, and also the relation of these solutions to those of (9·17). The theory follows the same lines as that for similar problems in the case of pure difference and pure differential equations as discussed by Bochner (8) and Poincaré (28), for example, but the methods used are necessarily rather more elaborate. The behaviour at infinity of the solution is measured by the function $\omega(y)$

which is such that $y e^{-\sigma x}$ is $L^2(x_0, \infty)$ for all $\sigma > \omega(y)$ but for no $\sigma < \omega(y)$. This function $\omega(y)$ corresponds exactly to the "characteristic number" used by Poincaré in the parallel theory for differential equations. By a method of successive approximations, Wright shows that the behaviour at infinity of the solution $y(x)$ of the difference-differential equation corresponds to that of $v(x)$.

This information is of some importance in the discussion of certain non-linear equations, to which a brief reference will now be made. As an example, the equation

$$y'(x+1) = -\alpha y(x)\{1 + y(x+1)\} \tag{9.21}$$

is considered. This is mentioned by Wright in (42), and it originally arose in connection with Lord Cherwell's investigation (12) into the distribution of prime numbers. If the condition $y(x) \to 0$ as $x \to \infty$ is imposed on the solutions, it is seen that the equation may be written in the form

$$y'(x+1) = A(x)\,y(x)$$

where $A(x) \to -\alpha$ as $x \to \infty$. Thus, by the properties just mentioned, it follows that the solution of (9·21) may be related asymptotically to the solution of the equation

$$y'(x+1) = -\alpha y(x)$$

first discussed by Schürer.

As Wright points out in his paper (43), there are three stages in the discussion of this problem of small solutions, namely, the existence of such solutions, the proof that any such solution must be exponentially small, and the determination of an asymptotic expansion for such a solution. In this paper, Wright considers the third problem, assuming the other results for the moment. He takes an equation of the form

$$\Lambda_1(y) + \Lambda_2(y) = v(x)$$

where

$$\Lambda_1(y) \equiv y^{(n)}(x) + \sum_{\mu=1}^{m} \sum_{\nu=0}^{n-1} a_{\mu\nu}\, y^{(\nu)}(x+b_\mu)$$

and

$$\Lambda_2(y) \equiv \sum_{\lambda} A_\lambda\, y^{(\beta_{\lambda,1})}(x+b_{\lambda,1})\, y^{(\beta_{\lambda,2})}(x+b_{\lambda,2}) \cdots$$

In Λ_2 there is a finite number of terms each containing at least two y functions, the coefficients A_λ are constants, and the numbers $\beta_{\lambda,\lambda}$ are less than or equal to n. Assuming the solution to be of order $e^{-(c-\epsilon)x}$, Wright proves that it takes the form of a finite sum of exponential terms related to the zeros of $T(s)$ in the strip $-C \leq \sigma \leq -c$, together with an error term of order $e^{-(C-\epsilon)x}$.

X. APPLICATIONS.

Linear difference-differential equations have occurred in a variety of practical problems, the most important of which will be discussed in this section. The main question of interest in such cases is the stability of the solutions, as Wright points out in an article (46) published in "Nature" in 1948. Since an exponential function e^{sx} is small, periodic or large as $x \to \infty$, according as σ is negative, zero or positive, it follows that Wright's solution

$$y(x) = \sum_{n} \frac{H(s_n)}{T'(s_n)} e^{s_n x}$$

of equation (9·1) depends for its behaviour as on the signs of the σ_n.

Many of the equations found in practice can be reduced by simple transformations to the equation

$$y'(x+1) + \alpha\, y(x) = 0, \qquad (\alpha > 0) \qquad (10\cdot 1)$$

which is of the form originally discussed by Schürer. In this case it is easily found that if $\alpha < \pi/2$ all the zeros s_n of $T(s)$ have $\sigma_n < 0$, so that $y(x) \to 0$ as $x \to \infty$. On the other hand, if $\alpha > \pi/2$ at least two zeros have $\sigma_n > 0$, and so $y(x)$ oscillates with increasing amplitude except under certain very special boundary conditions. Finally, if $\alpha = \pi/2$, there are two imaginary roots giving a periodic solution

$$y(x) = A \sin\left(\frac{\pi}{2} x + B\right)$$

whilst for all the other roots $\sigma_n < 0$. In this case the general solution approaches the periodic solution as $x \to \infty$.

Since the problem of stability depends on the zeros of $\Upsilon(s)$ it is a discussion of these zeros which constitutes the chief topic in the applications which will now be considered.

In 1933, in a lecture to the Econometric Society of Leyden, it was shown by Kalecki that certain problems in economic dynamics depend for their solution on a difference-differential equation. He produces an equation of the form

$$y'(t) = a\, y(t) - c\, y(t-\theta) \qquad (10.2)$$

where t represents the time, and a, c, θ are positive constants, and this equation is discussed mathematically by Frisch and Holme in a paper (15) published in 1935. Their treatment is subsequently extended to the case by James and Belz, (21), who point out that in some economic problems this case may arise. In both cases the discussion is confined entirely to the simple exponential solutions and the roots of the transcendental equation.

It is immediately seen that e^{st} is a solution of (10·2) provided s is a root of

$$\frac{a}{c} - \frac{s}{c} = e^{-s\theta} . \tag{10·3}$$

Thus real exponential solutions are given by real roots of (10·3), which may be found from the intersections of the straight line

$$y = \frac{a}{c} - \frac{s}{c}$$

with the exponential curve

$$y = e^{-s\theta}$$

Clearly there will be $0, 1$ or 2 such roots depending on the relative sizes of a, c and θ. If a greater degree of accuracy is required, the roots obtained may be improved by some method of successive approximations.

In discussing complex roots of (10·3) it is convenient to make the substitution

$$s\theta = v + iu$$

where u and v are real. Then equating real and imaginary parts of (10·3) gives

$$v = a\theta - c\,\theta \cos u \cdot e^{-v} \tag{10·4}$$

and

$$1 = c\theta \, \frac{\sin u}{u} \, e^{-v} . \tag{10·5}$$

It is clear that if, for a given v, a certain u satisfies these equations, then $-u$ also satisfies them, so that the roots of (10·3) occur in conjugate pairs. Also, from (10·5), it is seen by taking logarithms that

$$0 = \log|c\theta| - v + \log\left|\frac{\sin u}{u}\right| ,$$

so that, on substituting for v, (10·4) reduces to

$$f(u) = K$$

where

$$f(u) \equiv u\cot u + \log\left|\frac{\sin u}{u}\right|$$

and

$$K = a\theta - \log|c\theta| .$$

This equation, in the real variable u, may now be solved graphically. Since the roots of (10·3) occur in conjugate pairs, it is sufficient to draw the graph $y = f(u)$ for $u \geq 0$, and since $f(u)$ is independent of the constants a, c, θ, this graph may be used for all equations of the form (10·2).

Values of u which are roots of (10·4) and (10·5) are now given by certain intersections of this graph with the line $y = K$, but care must be taken in choosing the correct intersections. Correct values of u clearly depend on the sign of $\sin u/u$, and this is determined by the sign of $c\theta$, as is seen from (10·5). In fact if

$\alpha\theta > 0$, then $\sin u/u$ must be positive and so u must lie in $[2n\pi, (2n+1)\pi]$, while if $\alpha\theta < 0$, u must lie in $[(2n+1)\pi, (2n+2)\pi]$, n being a positive integer. Having found the correct values of u , the corresponding values of v are then obtained from (10·5). If these values are not sufficiently accurate, a method of approximation may again be used to improve them. Thus it is seen that solutions of the form

$$y(x) = A e^{v/\theta\, x} \cos\left(\frac{u}{\theta} x + B\right)$$

where A, B are constants, may be calculated to the degree of accuracy required.

Another interesting application of difference-differential equations is found in the field of X-ray irradiation. In 1941, in a paper (34), Sievert developed a mathematical theory to explain the results obtained experimentally by Forssberg (14) in his work on a certain mould called Phycomyces Blakesleeanus. This mould was subjected to X-ray irradiation and the experimental graph of its subsequent rate of growth was found by Forssberg to have a wave-like formation. Sievert attempted to find a mathematical explanation of this result by making certain assumptions on the nature of the mould and the actions taking place in it.

It is known that, in a living cell, the quantity of each substance X is in a constant ratio to the total quantity of substances constituting the cell. If X is partly destroyed by irradiation, the natural reaction of the cell is to restore the substance X in its original proportion. After such a destructive action, however, a certain time τ elapses before the restoration process begins. This time is known as the initial period. In Forssberg's experiment the mould is subjected to the action of X-rays for a certain period of time T. It is assumed that the rate of destruction of the substance is proportional to the quantity of rays absorbed, and thus to the quantity of X itself, and also that the rate of restoration of X is proportional to $(X_{t-\tau} - X_0)$ where X_t represents the quantity of X in existence at time t. These facts may be expressed by the following difference-differential equations:-

$$\frac{dX_t}{dt} = -I X_t + R(X_0 - X_{t-\tau}), \quad (0 \leq t \leq T) \qquad (10\cdot6)$$

and

$$\frac{dX_t}{dt} = R(X_0 - X_{t-\tau}), \quad (t > T) \qquad (10\cdot7)$$

where I, R are constants, known as the destruction and restoration coefficients respectively, and $X_{t-\tau}$ is assumed to be equal to X_0 for $t < \tau$.

In order to compare the solutions of these equations with the experimental results, Sievert uses a method of "graphical integration", finding the solutions in graphical form for a very large number of particular cases. For the initial period the equation reduces to the ordinary differential equation

$$\frac{dX_t}{dt} = - I X_t$$

which has the exponential solution $X_t = e^{-It}$ and thus the graph of X_t may be drawn for the interval $(0, \tau)$. At a time $\tau + s$ a line-element of curve may then be drawn by using equation (10·6), for the quantity $(X_0 - X_s)$ can be calculated from the graph already drawn. By proceeding in this way step by step, a rough sketch of the solution may be obtained, equation (10·7) being used instead of (10·6) after time τ. In many of the particular cases calculated by Sievert a wave-like form of graph is found to occur, and for these cases the theory fits with the experiments.

It is interesting to notice that, by means of simple transformations, equation (10·7) may be reduced to the form (10.1) with $\alpha = R/\tau$. Thus the solutions of (10·7) are stable, periodic or unstable according as R/τ is less than, equal to or greater than $\pi/2$.

A more general attack was made on the problem in 1944 by van der Werff (38). Instead of (10·6) and

(10·7) he considers the equations

$$\frac{dx_t}{dt} = -F(I, x)_t + G(R, x)_{t-\tau}, \quad (0 < t \leq T)$$

and

$$\frac{dx_t}{dt} = G(R, x)_{t-\tau}, \quad (t > T)$$

where the functions F, G represent general biological reactions. In the particular cases then discussed curves of a wave-like form are again obtained in a similar way.

The last application to be considered in this section concerns the theory of control mechanisms. This was first discussed mathematically in a paper (11) by Callender, Hartree and Porter, published in 1936. In many physical operations some form of controlling gear is required in order to keep a given physical quantity as nearly constant as possible, and it is shown in this paper that the effect of such a mechanism is represented mathematically by a difference-differential equation.

The function $\theta(t)$ represents the departure from its standard value of the physical quantity at time t, $C(t)$ represents the effect of the controlling gear, and $D(t)$ the effect of random disturbances. It is supposed that the rate of increase of $\theta(t)$ depends not only on

$C(t)$ and $D(t)$ but also on $\theta(t)$ itself, so that the equation

$$\frac{d\theta(t)}{dt} = D(t) + C(t) - m\,\theta(t), \qquad (10\cdot 8)$$

where m is a constant, is obtained. It is also known that there is a time-lag, T say, between the controlling gear being called into play and its consequent effect on the physical quantity, and so a law of control is taken of the form

$$\frac{dC(t+T)}{dt} = -n_1\,\theta(t) - n_2\,\frac{d\theta(t)}{dt} - n_3\,\frac{d^2\theta(t)}{dt^2} \qquad (10\cdot 9)$$

where n_1, n_2, n_3 are constants. Further, it may be assumed, without loss of generality, that $T = 1$ for this equation may easily be transformed into a similar equation in which T is replaced by 1. In this case, eliminating C between equations (10·8) and (10·9), it is found that $\theta(t)$ satisfies the linear difference-differential equation

$$n_1\,\theta(t) + n_2\,\frac{d\theta(t)}{dt} + n_3\,\frac{d^2\theta(t)}{dt^2} + \frac{d^2\theta(t+1)}{dt^2} + m\,\frac{d\theta(t+1)}{dt} = \frac{d}{dt}D(t+1)\,. \qquad (10\cdot 10)$$

Clearly, for the mechanism to be of any practical value, it is necessary for this equation to have stable solutions. Thus the next step is to discuss the possibility of simple exponential solutions of the form e^{st} for which $\sigma < 0$.

If there is no outside disturbance $\mathcal{D}(t)$ is zero, and the transcendental equation associated with (10·10) becomes

$$n_1 s^2 + n_2 s + n_3 + (m+s)se^s = 0. \qquad (10 \cdot 11)$$

It is seen that the roots of this equation occur in conjugate pairs, and, in a number of particular examples, their approximate values are found by graphical methods of the usual type. The corresponding exponential solutions of (10·10) are called the normal modes, and the problem is to discover the values of the constants n_1, n_2, n_3, m which produce stable modes. These indicate suitable values of the constants in the more general case when $\mathcal{D}(t)$ is not zero.

In 1937 a further paper (16) on the subject was published by Hartree, Porter, Callender and Stevenson. In this case they were concerned with a more general law of control which gave more effective results. Here the transcendental equation corresponding to (10·11) is found to be of the form

$$(n_1 s^2 + n_2 s + n_3) \frac{b_1 + b_2 s}{b_1 + s} + (m+s)se^s = 0,$$

where b_1, b_2 are also constants. Again, particular values of the constants are considered with a view to finding those which will give stable solutions. The paper ends with an account of the way in which a differential analyser may be used to facilitate the calculations.

XI. CONCLUSION.

The main developments in the study of difference-differential equations having been considered, a few brief references may now be made to some other papers on the subject.

There is a short paper (24) by Neufeld, published in 1934, in which a linear equation of form (9·17) is solved by means of the Laplace transform. The results, however, are less general than those of Wright, since instead of proving the order conditions which are necessary for justification of the method, Neufeld merely assumes them.

In a paper (18), published in 1935, Herzog obtains results for a finite set of difference-differential equations, using an intricate notation in order to state these results neatly. He is chiefly interested in sets of non-linear equations, however, discussing the possible exponential solutions.

Another writer to consider non-linear equations is Bellman, who discusses the existence of bounded solutions of pure difference and pure differential equations in (2), extending his results to difference-differential equations in (3), published last year. In the latter he is ultimately concerned with the equation

$$u'(t+1) = a_1 u(t) + a_2 u(t+1) + f[u(t), u(t+1)], \qquad (11\cdot1)$$

but he deals with this equation by considering first the linear equation

$$u'(t+1) = a_1 u(t) + a_2 u(t+1). \qquad (11\cdot2)$$

As boundary conditions for the latter equation he supposes that the function $u(t)$ is known in the initial interval $(0,1)$ and that $u'(1)$ is given, and then by a formal application of the Laplace transform method he obtains a solution. The method is not justified directly by consideration of the order at infinity of a solution. Instead, Bellman proceeds to show that the function obtained does in fact satisfy the equation and therefore provides a true solution. Further, to ensure that this solution is stable, he points out that it must be assumed that all the roots of the associated transcendental equation lie to the left of some line $\sigma = -c$. A solution of the non-homogeneous equation is found in a similar way, and finally the results are extended to equation (11·1).

Another recent paper (4) on the subject, by de Bruijn, concerns a linear equation whose coefficients are functions of x. In particular the writer considers the equation

$$x^{-\alpha} f'(x) + f(x) - f(x-1) = 0,$$

examining the behaviour at infinity of its solution for certain values of α. His methods, however, are not of general application and will not therefore be considered here.

Bateman published a paper (1) in 1943 on equations of the form

$$M\frac{d^2u_n}{dt^2} + 2K\frac{du_n}{dt} + Su_n = (na+a+b)(u_{n+1} - u_n) - (na+c)(u_n - u_{n-1})$$

Clearly such equations conform to the definition of difference-differential equations given in (1·1) only in the particular case when $n \equiv t$. It is interesting to notice that Bateman mentions the definite integral and the Laplace transform as means of solving his equations; he is chiefly concerned, however, with particular equations which have occurred in practical problems.

Finally, it has been pointed out to me by Professor McCrea that Whittaker's solution of differential equations by definite integrals may also be applied to difference-differential equations. The basic theorem of this method is stated in Whittaker's paper (39) and proved by Kermack and McCrea in (22).

Using their notation, the particular case of the Laplace transform is obtained by taking

$$W = -q\, Q$$

as generating function. Then the corresponding contact transformation is given by the equations

$$P = \frac{\partial W}{\partial Q}, \quad p = -\frac{\partial W}{\partial q}$$

which reduce here to

$$P = -q, \quad Q = p.$$

The auxiliary function χ is given by the partial differential equations

$$\frac{\partial \chi}{\partial q} = t\,\chi, \quad \frac{\partial \chi}{\partial t} = q\,\chi,$$

so that

$$\chi = e^{qt}.$$

The theorem then states that the solution of

$$F\left(q, \frac{\partial}{\partial q}\right)\psi = 0$$

is of the form

$$\psi(q) = \int \chi(q, t)\, \phi(t)\, dt$$

$$= \int e^{qt}\, \phi(t)\, dt \qquad (11 \cdot 3)$$

where $\phi(t)$ is a solution of

$$F\left(-\frac{\partial}{\partial t}, t\right)\phi = 0.$$

As an example the theorem may be applied formally to the equation

$$\psi'(q+1) = a\,\psi(q). \qquad (11\cdot 4)$$

Writing the equation in terms of operators it is seen to be of the form

$$F\left(q, \frac{\partial}{\partial q}\right)\psi = 0$$

where

$$F\left(q, \frac{\partial}{\partial q}\right) \equiv \frac{d}{dq}\, e^{\frac{d}{dq}} - a.$$

Hence

$$F\left(-\frac{\partial}{\partial t}, t\right) \equiv t e^{t} - a,$$

so that the solution of (11·4) is of the form (11·3) where $\phi(t)$ is a solution of

$$(te^{t} - a)\,\phi = 0.$$

It follows that $\phi(t)$ may be arbitrary, provided t_λ is a root of the transcendental equation associated with (11·4), so that (11·3) becomes

$$\psi(q) = \sum_{\lambda} e^{qt_\lambda}\,\phi(t_\lambda).$$

This is the same form of solution as was found earlier for this equation.

If the method is applied to equation (6·4) it is found that the function $\phi(t)$ is given by the same equation

$$A\{\phi(t)\} = 0$$

as was obtained by Hoheisel in Section VI.

The dissertation will now be concluded with a brief survey of the chief points of interest in the study of the linear difference-differential equation. The gradual development of the use of transforms has been seen to be closely connected with the order at infinity of a solution and it has also been remarked that the situation is considerably simpler for an equation with only one term containing an nth derivative of $y(x)$. Schmidt, Hoheisel and Titchmarsh confined themselves to equations of this type, while Bochner indicated the advantage of doing so, although he considered the general equation as well.

The order conditions laid down for a solution by the early writers were of a very restrictive nature. Schmidt and Hoheisel, for example, confined themselves to solutions of order $|x|^a$ as $|x| \to \infty$, so that only the purely imaginary zeros of $\gamma(s)$ could be used for the simple exponential solutions. The same restriction was made in effect by Bochner. Schmidt, however, briefly mentioned the possibility of solutions of order $e^{\alpha|x|}$ as $|x| \to \infty$, a condition which was also taken at first by Titchmarsh, although he omitted it in his later work. As was seen in Section III, the purpose of such order conditions was to limit the complementary function to being a finite instead of an infinite series. Hilb and Schürer, however, did not restrict themselves to solutions of any particular order and they obtained, in consequence, solutions in the form of infinite series whose convergence had to be discussed. Finally, in Wright's work, an order condition was no longer assumed but instead it was actually proved that a solution must be of exponential order provided certain conditions were laid down in the initial interval.

With regard to the use of transforms, it was pointed out earlier that Schmidt used a function closely connected with a transform in order to obtain a particular solution of his equation. As was seen later, the solution he found was comparable with those obtained by Bochner, Titchmarsh and Wright, all of whom made explicit use of transforms. Hilb, on the other hand, did not use transforms at all, but based his results on the expansion of an arbitrary function as an infinite series. He considered the homogeneous equation, recognising that the solution depended on the behaviour of $y(x)$ in the initial interval, and obtaining a solution in the same form as that found by Wright later on. He also mentioned the importance of the non-vanishing of one of the coefficients a_{mn} or a_{on}. Schürer's approach to his own simple equation was of a similar nature.

The real use of transforms began with Hoheisel, who solved his equation by assuming a solution in the form of a Laplace Integral. He was followed by Bochner and Titchmarsh, both of whom used Fourier transforms. Bochner, however, was restricted by his definition of a Fourier transform which confined him to a consideration of the purely imaginary zeros of $\Upsilon(s)$ only. His conditions on the integrability of a solution also led to the exclusion of the complementary function, although he later remedied this by extending his class of integrable functions. Titchmarsh, by using generalized Fourier transforms, was able to consider complex zeros of $\Upsilon(s)$ and thus he obtained more general results.

Finally, Wright was able, by using his result on the order of a solution, to justify the use of the Laplace transform in obtaining a solution. Further, provided that one of the coefficients a_{mn}, a_{on} was non-zero, he proved results on the convergence of this series solution which he obtained. Thus it was with the publication of his papers on the subject that the real power of transforms in this connection was finally appreciated.

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