

ROYAL HOLLOWAY COLLEGE

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COLOURINGS OF HYPERGRAPHS

A THESIS

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ABSTRACT

In Chapter 2, we describe some generalized chromatic numbers of graphs. In Chapter 3, we describe how these may be regarded as chromatic numbers of associated hypergraphs.

In Chapter 4, we consider some upper bounds for the chromatic number of a hypergraph, and attempt to characterize those hypergraphs for which these bounds are attained.

Chapter 5 is devoted to a study of the chromatic polynomials of hypergraphs; and an algorithm for their evaluation is described.

In Chapter 6, we are concerned with planar hypergraphs and some of their colouring properties. We introduce the face-chromatic number of a hypermap.

Chapter 7 consists of notes on the previous chapters.



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I Mam

"A tree is a connected forest"

CHAPTER 1

DEFINITIONS

Most of our terms will be defined, as they arise, in the text. This chapter contains some basic terminology. Here, as throughout the thesis, the first appearance of a word being defined will be indicated by *italics*.

A *hypergraph* H (sometimes written (V, E)) consists of a finite non-empty set V (sometimes written $V(H)$) of *vertices*; together with a finite family E (or $E(H)$) of *edges*. Each edge is a subset of V and contains at least two vertices. Let us emphasize: Hypergraphs may contain multiple edges, but may not contain loops.

An edge containing r vertices may be called an r -*edge*: we reserve the name *hyperedge* for edges containing more than two vertices.

A hypergraph, all of whose edges contain the same number of vertices, is called *uniform* (k -*uniform* if this number is k). An r -*graph* is an r -uniform hypergraph. A *graph* is a 2-graph. A graph is *simple* if there are no multiple edges (the edge family E is a subset of the power set $P(V)$).

We shall sometimes represent hypergraphs by drawings. Vertices are represented by points, and hyperedges each by a closed curve enclosing just

those vertices contained in that edge; 2-edges are represented by lines joining their vertices. (In Chapter 5, a different mode of representation will be used for hypergraphs derived from the Fano plane. More will be said about plane representations of some hypergraphs in Chapter 6.)

The hypergraph $H' = (V', E')$ is a *subhypergraph* of $H = (V, E)$ if $V' \subset V$ and $E' \subset E$. A subhypergraph which is a graph may be called a *subgraph*. If $S \subset V$ the subhypergraph of H induced by S , denoted $\langle S \rangle$, is that hypergraph whose vertex set is S , and whose edges are all those edges of H contained in S (symbolically $\{ e : e \in E, e \subset S \}$).

The hypergraph $H' = (V', E')$ is a *part* of H if $V' \subset V$ and there is a one-to-one function $f: E' \rightarrow E$ with the property that $e' \subset f(e')$ for each $e' \in E'$.

The subhypergraph, or part, is *proper* if $H' \neq H$.

A *chain* (of length l) in a hypergraph H is a sequence $v_1, e_1, v_2, e_2, \dots, e_l, v_{l+1}$ such that

- i. v_1, v_2, \dots, v_l are distinct vertices of H ,
- ii. e_1, e_2, \dots, e_l are distinct (but not necessarily unequal) edges of H ,
- iii. $v_k, v_{k+1} \in e_k$ for $k = 1, 2, \dots, l$.

If $l > 1$ and $v_{l+1} = v_1$ the chain is called a *cycle*.

In a simple graph, a chain is completely determined

by specifying either all its vertices or all its edges. But in a hypergraph it is essential, in general, to specify all the vertices and all the edges.

A hypergraph is *connected* if there is a chain joining every pair of its vertices. Those subhypergraphs of H which are connected and maximal with respect to this property (i.e. are not proper subhypergraphs of any connected subhypergraph of H) are the *connected components* of H .

Definitions not given here, or in the text, may be found in Berge (1973), the translated and revised edition of Berge (1970).

CHAPTER 2

PARTITION NUMBERS OF GRAPHS

2.1: Introduction.

This chapter is based on some of the ideas contained in my M.Sc. thesis, RPJ(1973).

For certain families P of graphs, we shall define P -chromatic numbers and P -chromatic indexes. We shall mention some of the results known about these partition numbers; and show how some of the well known parameters of graphs (e.g. arboricity, thickness, and point-thickness) are included in this theory.

2.2: Hereditary Families of Graphs.

A family P of graphs is *hereditary* if:

- H1) P contains at least one non-null graph,
- H2) there is a graph which is not a member of P ,
- H3) whenever a graph G is a member of P , and H is a subgraph of G , then H is also a member of P .

Some examples of hereditary families are: the family of planar graphs; the family of graphs with no edges (totally disconnected graphs); the family of graphs with at most N vertices, for some positive integer N .

Proposition 2.2.1: Let P be a hereditary family.

There is a non-negative integer $k = k(P)$ such that the complete graph $K_{k+1} \in P$, but $K_{k+2} \notin P$.

Proof: H1 and H3 ensure that $K_1 \in P$.

Suppose the graph, not a member of P , whose existence is guaranteed by H2, has n vertices.

It follows from H3 that $K_n \notin P$. (Clearly $n > 1$.)

An integer $k(P)$ with the required property must lie between 0 and $n-1$. //

This parameter $k(P)$ is called the *completeness* of the hereditary family P . The completeness of the family of planar graphs is 3, since K_4 is planar whereas K_5 is not. The family of outerplanar graphs has completeness 2; the family of graphs with no cycles has completeness 1; and the completeness of the family of totally disconnected graphs is 0. For any positive integer N , the family of graphs with at most N vertices has completeness $N-1$.

2.3: P -chromatic Numbers.

Let P be a hereditary family of graphs, and let $G = (V, E)$ be a graph. An m -(P -colouring) of G is a partition of V into m parts such that the

subgraph induced by each of the parts is a member of P ; i.e.

$$V = S_1 \cup S_2 \cup \dots \cup S_m ; \langle S_i \rangle \in P, i = 1, 2, \dots, m.$$

We note that any graph on n vertices has an n -(P -colouring) in which each part of the partition consists of just a single vertex.

The smallest m for which G has an m -(P -colouring) is called the P -chromatic number of G , denoted $\chi_P(G)$. If we denote by D_0 the family of totally disconnected graphs, it is apparent that $\chi_{D_0}(G)$ is simply the familiar chromatic number of G .

The next theorem establishes upper and lower bounds for $\chi_P(G)$. Firstly, we need to define the *point-independence number* $M_P(G)$ of G with respect to P ; this is the largest number of vertices of G which can induce a subgraph which is a member of P .

Theorem 2.3.1 (RPJ 1973):

Let P be any hereditary family of completeness k . For any graph G with n vertices and $M_P(G) = M$:

$$\left\{ \frac{n}{M} \right\} \leq \chi_P(G) \leq 1 + \left\{ \frac{n-M}{k+1} \right\}. \quad //$$

2.4: Some Examples.

If P is a hereditary family, let us denote by P^m that family of graphs whose P -chromatic number does not exceed m . P^m is a hereditary family of completeness $m(k(P)+1)-1$.

With this notation, we may state the five-colour theorem for planar graphs in the form:

$$Q_3 \subset D_0^5$$

where Q_3 is the family of planar graphs, and D_0 is the family of totally disconnected graphs.

The four-colour conjecture asserts that:

$$Q_3 \subset D_0^4 .$$

Lick and White (1970) have studied the families of k -degenerate graphs. The *strength*, $\sigma(G)$, of a graph G is defined to be the maximum, over all subgraphs, of the minimum valency of the subgraphs:

$$\sigma(G) := \text{Max}\{\delta(H) : H \leq G\} .$$

A graph whose strength does not exceed k , for some non-negative integer k , is said to be *k-degenerate*. The family, D_k , of all k -degenerate graphs is hereditary of completeness k .

D_0 is the family of totally disconnected graphs, D_1 is the family of graphs without cycles; D_2 contains every outerplanar graph, and D_5 contains every planar graph.

$\chi_{D_0}(G)$ is the chromatic number of G ; $\chi_{D_1}(G)$ is the *point arboricity*. Every planar graph is a member of D_5 ; thus $\chi_{D_5}(G)$ does not exceed the *point thickness* of G .

Proposition 2.4.1 (Lick and White (1970)):

$$\chi_{D_k}(G) \leq 1 + \left\lceil \frac{\sigma(G)}{k+1} \right\rceil . \quad //$$

Particular instances of this proposition include the fact that the point arboricity of a planar graph does not exceed 3, or that the chromatic number of an outerplanar graph does not exceed 3.

Simões-Pereira (1976) has compiled a survey of results concerning k -degenerate graphs. Other examples of hereditary families and their associated P -chromatic numbers may be found in Chartrand, Geller and Hedetniemi (1968) and (1971).

2.5: P -chromatic Indexes.

If $S \subseteq E(G)$ the (edge) *induced subgraph* $\langle S \rangle$ has vertex set $\bigcup_{e \in S} e$ and edges S . Let P be a hereditary family of completeness at least 1. An m -(P -edge-colouring) of a graph $G = (V, E)$ is a partition of E into m parts, each of which induces a subgraph which is a member of P :

$$E = S_1 \cup S_2 \cup \dots \cup S_m ; \quad \langle S_i \rangle \in P, \quad i=1, 2, \dots, m.$$

The smallest integer m for which the graph G has an m -(P -edge-colouring) is called the P -chromatic index of G , denoted $\gamma_P(G)$. We denote by \underline{P}^m the family of graphs whose P -chromatic indexes do not exceed m . If P is hereditary of completeness at least 1, then so is \underline{P}^m for any positive integer m .

2.6: Some Examples.

For any positive integer k , let C_k be the family of graphs whose maximum valency does not exceed k . C_k is a hereditary family of completeness k . $\gamma_{C_1}(G)$ is simply the chromatic index of the graph G . Relationships between the families \underline{C}_k^m for various values of k and m have recently been studied by Hilton and Jones (1976). We determined those values of m and k for which $\underline{C}_k^m = \underline{C}_m^k$, and those for which $\underline{C}_{km} = \underline{C}_k^m$.

Recall that D_1 is the family of graphs without cycles, and Q_3 is the family of planar graphs. Let us denote by Q_2 the family of outerplanar graphs. $\gamma_{D_1}(G)$, $\gamma_{Q_3}(G)$, and $\gamma_{Q_2}(G)$ are, respectively, the *arboricity*, the *thickness*, and the *outerthickness* of the graph G .

CHAPTER 3

COLOURINGS OF HYPERGRAPHS

3.1: Introduction.

In Sections 3.2, 3.3, and 3.4 we shall consider three different chromatic numbers for hypergraphs. One of these, the *weak chromatic number*, will be discussed further in Section 3.5, where its relevance to the generalized chromatic numbers of Chapter 2 will be described. In Section 3.6 we shall explain why we do not generalize the weak chromatic number of a hypergraph as we did the chromatic number of a graph in Chapter 2.

3.2: The Strong Chromatic Number.

A *strong m -colouring* of the hypergraph $H = (V, E)$ is a partition of V into m parts:

$$V = S_1 \cup S_2 \cup \dots \cup S_m$$

such that:

$$|e \cap S_i| \leq 1 \quad \text{for each } e \in E \text{ and for each } 1 \leq i \leq m.$$

The *strong chromatic number* of H is the smallest integer m for which there is a strong m -colouring of H .

If H is a graph, the strong chromatic number is simply the (graph theoretic) chromatic number of H .

It happens that the strong chromatic number of a hypergraph H is the same as the (graph theoretic) chromatic number of an associated graph. Let G be the graph whose vertex set is $V(H)$ and whose edges are those pairs of vertices which are subsets of at least one edge of H . It is readily seen that the strong chromatic number of H is precisely $\chi(G)$, the chromatic number of the graph G .

3.3: The Equitable Chromatic Number.

Berge (1973) describes another type of colouring for hypergraphs: An *equitable m -colouring* of a hypergraph $H = (V, E)$ is a partition of V into m parts:

$$V = S_1 \cup S_2 \cup \dots \cup S_m$$

such that, for each $e \in E$ and for any positive integers $i, j \leq m$, we have:

$$-1 \leq |e \cap S_i| - |e \cap S_j| \leq 1.$$

The *equitable chromatic number* of a hypergraph H is the smallest integer $m \geq 2$ such that H has an equitable m -colouring.

We note that any strong m -colouring of a hypergraph is automatically an equitable m -colouring. We deduce that the equitable chromatic number of a hypergraph never exceeds the strong chromatic number.

3.4: The Weak Chromatic Number.

A *weak m -colouring* of the hypergraph $H = (V, E)$ is a partition of V into m parts:

$$V = S_1 \cup S_2 \cup \dots \cup S_m$$

such that for each positive integer $i \leq m$ induced subhypergraph $\langle S_i \rangle$ has no edges. (If we regard each of the parts of the partition as a set of vertices of the same colour, with a different colour corresponding to each part, we note that a weak m -colouring has the property that none of the edges of H has all its vertices the same colour.)

The *weak chromatic number* of H , denoted $\chi(H)$ is the smallest integer m for which H has a weak m -colouring.

We note that the weak chromatic number of a graph is the same as its (graph theoretic) chromatic number. We note also that since an equitable m -colouring ($m \geq 2$) of a hypergraph is already a weak colouring, the weak chromatic number of a hypergraph does not exceed its equitable chromatic number.

There is no straightforward construction which will generally associate with a hypergraph a graph whose chromatic number is the same as the weak chromatic number of the hypergraph.

3.5: P -Chromatic Numbers.

In Chapter 2, we discussed the P -chromatic number, $\chi_P(G)$, of a graph G . For a hereditary family P , this was the smallest number of parts into which we could partition the vertex set of G so that the subgraph induced by each of the parts was a member of P .

Given a graph G and a hereditary family P , let us construct a hypergraph H in the following manner: The vertex set $V(H)$ shall be the same as the vertex set $V(G)$; the edges of H shall be those subsets of $V(G)$ which induce subgraphs of G which are *not* members of P .

Proposition 3.5.1: $\chi_P(G) = \chi(H)$.

Proof: Any weak m -colouring of H is also a partition of $V(G)$ with the property that each of the m parts induces a subgraph of G which is a member of the family P . So a weak m -colouring of H is an m -(P -colouring) of G . And *vice versa*.//

Notice that had we, in defining H , insisted that the edges of H be only those subsets of $V(G)$ minimal with respect to the property of inducing subgraphs of G not in P , the assertion of 3.5.1

would still be true, with virtually the same proof.

Let us now look at the P -chromatic index of a graph, also defined in Chapter 2. This was the smallest number of parts into which we could partition the edge set of the graph so that each of the parts induced a subgraph with property P .

Given a graph G and a hereditary family P of completeness at least 1, let us associate with G and P a hypergraph H' defined as follows: There shall be a one-to-one correspondence between the edges $E(G)$ and the vertex set $V(H')$; the edges of H' shall be those subsets of $V(H')$ which correspond to edge families of G which induce subgraphs of G not in P and which are minimal with respect to this property. (I.e. Any proper subset of an edge of H' corresponds to a family of edges of G which induces a subgraph in P .)

Proposition 3.5.2: $\gamma_P(G) = \chi(H')$. //

In Chapter 2 we noted that many of the partition numbers encountered in Graph Theory may be regarded as P -chromatic numbers or indexes. Propositions 3.5.1 and 3.5.2 now indicate that all these may be regarded as weak chromatic numbers

of associated hypergraphs.

From now on, we propose to drop the adjective *weak*. An m -colouring of a hypergraph is understood to be a weak m -colouring; and the chromatic number of a hypergraph is its weak chromatic number.

3.6: A Generalized Chromatic Number?

If we define a hereditary family of hypergraphs in the obvious way, and proceed, as we did in Chapter 2 with graphs, to define m -(P -colourings) of hypergraphs, we find there is no difficulty in defining the P -chromatic number of a hypergraph. Similarly, there is a very natural way to define the P -chromatic index of a hypergraph.

There is no significant increase in generality to be obtained by so doing: For by using ideas similar to those used in Section 3.5, we find that the P -chromatic number (or index) of a hypergraph is identical with the ordinary chromatic number of an associated hypergraph.

CHAPTER 4

SOME BOUNDS ON $\chi(H)$ - BROOKS' THEOREM

4.1: Introduction.

R.L. Brooks (1941) has proved:

Theorem 4.1.1: Let G be a connected graph with maximum valency $\Delta(G)$. Then:

- a) $\chi(G) \leq 1 + \Delta(G)$;
- b) $\chi(G) \leq \Delta(G)$, unless G is a complete graph or a cycle with an odd number of vertices. //

Our aim in this chapter will be to attempt to generalize this result to hypergraphs. Our approach will be rather different from that of Lovász (1968) or Gardner (1975). Their versions of Brooks' theorem will be mentioned in the notes on this chapter in Chapter 7.

4.2: The Valency of a Hypergraph.

Our first problem is to decide how to extend to hypergraphs the concept of *valency*. We shall do this in three different ways:

Definition 4.2.1: The *degree* $d_H(v)$ of the vertex v in the hypergraph H is the number of edges of H which contain v . $d(H)$ will denote the smallest of the degrees of the vertices of H , and $D(H)$ the largest.

Suppose that two hypergraphs H and H' differ only in that H' contains additional edges all of which contain edges already present in H . It is apparent from the definitions that any m -colouring of H is also an m -colouring of H' , and *vice versa*. In other words, the colouring properties of a hypergraph are not affected by the addition of non-minimal edges. This discussion motivates our:

Definition 4.2.2: The edge e of a hypergraph H is *minimal* if it contains properly no other edge of H .

The *minimal-degree* $d_H^*(v)$ of the vertex v in the hypergraph H is defined to be the number of distinct minimal edges containing v . $d^*(H)$ will denote the smallest, and $D^*(H)$ the largest of the minimal-degrees of the vertices of H .

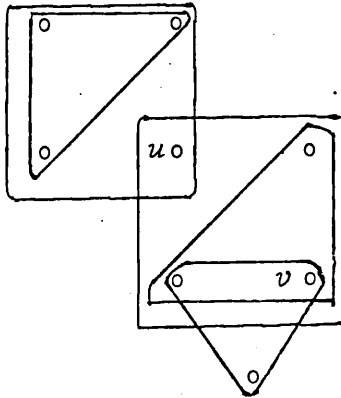
Note that if H^* is the hypergraph obtained from H by removing all those edges which properly contain

another edge, and then replacing all those sets of identical edges by a single representative from each set, we have for any vertex v of H (or H^*):

$$d_{H^*}(v) = d_H^*(v).$$

Definition 4.2.3: Our third generalization of graph-theoretic valency is due to Lovász (1968). A set F of edges of the hypergraph H will be called a v -star if the intersection of any two edges $e, f \in F$ is precisely $\{v\}$, the single vertex v . We define the *valency* $\delta_H(v)$ of the vertex v in the hypergraph H to be the largest number of edges in a v -star. $\delta(H)$ will denote the smallest and $\Delta(H)$ the largest of the valencies of the vertices of H .

Example:



$$d_H(u) = \delta_H(u) = 2; \quad d_H^*(u) = 0$$

$$d_H(v) = 3; \quad d_H^*(v) = 2; \quad \delta_H(v) = 1.$$

We would remark that in a simple graph, the degree, the minimal-degree, and the valency of any vertex are all equal to its graph-theoretic valency.

For any vertex v of any hypergraph H it is always true that:

$$\begin{aligned} d_H^*(v) &\leq d_H(v) && \text{and} \\ \delta_H(v) &\leq d_H(v) ; \end{aligned}$$

but, as our example on the previous page has shown, $d_H^*(v)$ and $\delta_H(v)$ are not, in general, comparable.

4.3: Upper Bounds for $\chi(H)$.

If v is a vertex of the hypergraph H , we shall denote by $H-v$ that hypergraph obtained from H by removing v and all the edges of H which contain v . $H-v$ is, in fact, the induced subhypergraph $\langle V(H) - \{v\} \rangle$.

The hypergraph H will be called *n-critical* if $\chi(H) = n$, but for any vertex v of H , $\chi(H-v) < n$.

Lemma 4.3.1: Let H be a hypergraph with chromatic number n . There is a subset $S_c \subset V(H)$ such that the induced subhypergraph $\langle S_c \rangle$ is *n-critical*.

Proof: Either H is already *n-critical* (in which case we may take $S_c = V(H)$), or there is a vertex

v_1 such that $\chi(H-v_1) = n$.

Unless $H-v_1$ is n -critical (in which case we take $S_c = V(H) - \{v_1\}$), there is a vertex v_2 such that $\chi((H-v_1)-v_2) = n$.

And so on. Since $V(H)$ is finite, the process must terminate with an n -critical hypergraph $H-v_1-\dots-v_r$, say. This is a subhypergraph induced by the set $S_c = V(H) - \{v_1, \dots, v_r\}$. //

Lemma 4.3.2: If K is an n -critical hypergraph,
 $\delta(K) \geq n-1$.

Proof: Let v be any vertex of K .

Since K is n -critical, there is an $(n-1)$ -colouring of $K-v$, say:

$$V(K-v) = S_1 \cup S_2 \cup \dots \cup S_{n-1}.$$

Since K itself cannot have an $(n-1)$ -colouring, there must be, for each $i=1,2,\dots,n-1$, an edge e_i of K , and

$$v \in e_i \subset S_i \cup \{v\}.$$

We know that the sets S_i are disjoint; it follows that, whenever $i \neq j$, $e_i \cap e_j = \{v\}$.

We have found a set of $n-1$ edges, the intersection of any pair of which is precisely $\{v\}$. This proves that $\delta_K(v) \geq n-1$.

Since v was an arbitrary vertex of K , it follows that $\delta(K) \geq n-1$. //

Theorem 4.3.3: Let H^* be the hypergraph obtained from a hypergraph H by deleting all those edges which are not minimal. For any subset $S \subset V(H)$ ($=V(H^*)$), let $\langle S \rangle^*$ denote the subhypergraph of H^* induced by S . Then:

$$\chi(H) \leq 1 + \text{Max } \delta(\langle S \rangle^*),$$

where the Maximum is taken over all subsets $S \subset V(H)$.

Proof: Let $\chi(H) = n$. We note that also $\chi(H^*) = n$.

By Lemma 4.3.1, there is a subset $S_e \subset V(H)$ such that $K = \langle S_e \rangle^*$ is n -critical.

By Lemma 4.3.2, $\delta(K) \geq n-1 = \chi(H) - 1$.

Thus $\chi(H) \leq 1 + \delta(K) = 1 + \delta(\langle S_e \rangle^*)$
 $\leq 1 + \text{Max } \delta(\langle S \rangle^*)$. //

The next three Corollaries provide us with weaker upper bounds. Each is a generalization to hypergraphs of Part a) of Theorem 4.1.1. (Brooks).

Corollaries: Let H be a hypergraph:

4.3.4: $\chi(H) \leq 1 + \Delta(H)$;

4.3.5: $\chi(H) \leq 1 + D^*(H)$;

4.3.6: $\chi(H) \leq 1 + D(H)$. //

We end this section by mentioning another upper bound due to Tomescu (1968):

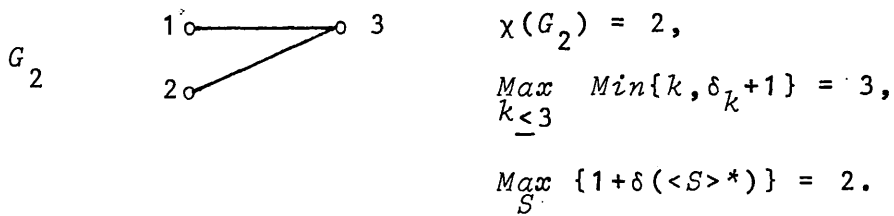
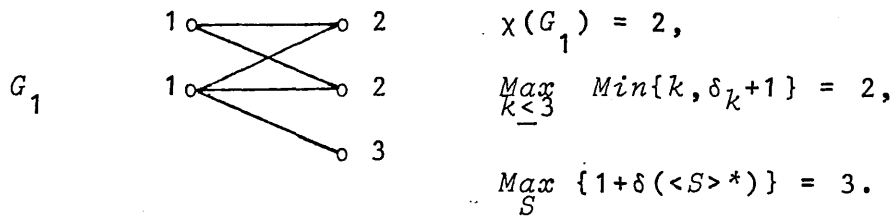
Theorem 4.3.7 (Tomescu 1968):

Let $V(H) = S_1 \cup \dots \cup S_m$ be an m -colouring of the hypergraph H , and let

$$\delta_i = \max_{v \in S_i} \delta_H(v). \text{ Then:}$$

$$\chi(H) \leq \max_{i < m} \min\{i, \delta_i + 1\}. //$$

The two hypergraphs (in fact, they are graphs) shown below indicate that neither of the bounds of Theorems 4.3.3 and 4.3.7 is, in general, better than the other.



In both cases we have a graph and a 3-colouring (indicated by the numbers next to the vertices; the vertices labelled i are the vertices of S_i , $i=1,2,3$). For G_1 , the bound of 4.3.7 is "better" than that of 4.3.3. The opposite is true for G_2 .

4.4: Brooks' Theorem for Hypergraphs.

In the last section we proved the corollary:

4.3.6 $\chi(H) \leq 1 + D(H)$

where H is any hypergraph, and $D(H)$ is the maximum of the degrees of its vertices. Our next theorem characterizes those hypergraphs whose chromatic numbers attain this upper bound.

Theorem 4.4.1: If H is a connected hypergraph,

$$\chi(H) \leq D(H) , \text{ unless:}$$

- i. H has at most one edge, or
- ii. H is a complete graph, or
- iii. H is a cycle (graph) with an odd number of vertices.

Proof: We use the word *suitable* to describe a connected hypergraph K which satisfies $\chi(K) = 1 + D(K)$.

Let H be a suitable hypergraph, and let e be an edge of H containing at least three vertices. (If no such edge exists, then H is a graph, and our result follows immediately from Theorem 4.1.1.)

We construct a new hypergraph H' from H as follows:

Consider all the two-element subsets of e . If each of these pairs is already a 2-edge of H , we form

our new hypergraph H' simply by removing the edge e from H . Otherwise, we select a pair u, v of vertices of e which do not form a 2-edge in H . We replace the edge e by a new 2-edge $\{u, v\}$. If the new hypergraph is not connected, we make it so by removing all but one of the connected components, that component having the largest possible chromatic number. The resulting hypergraph is H' .

The chromatic number of H' is at least as large as that of H ; the degree of any vertex of H' cannot exceed the degree of the same vertex in H .

Combining these two facts with the result of Corollary 4.3.6 applied to H' , we obtain:

$$\chi(H) \leq \chi(H') \leq 1 + D(H') \leq 1 + D(H).$$

But $\chi(H) = 1 + D(H)$, since H is suitable. It follows that H' is also suitable.

A hyperedge e has been either removed altogether or replaced by a 2-edge. We repeat this procedure, at each stage producing a suitable hypergraph. Since the number of hyperedges in H is finite, we shall eventually obtain a suitable graph, say G . By Theorem 4.1.1, we know that G is either a complete graph or a cycle with an odd number of vertices.

The only graph of this type which may be obtained by our construction from a *suitable* hypergraph is the complete graph on two vertices;

and the only suitable hypergraph from which the complete graph on two vertices may be formed by our construction is a hypergraph with only one edge. There is no way in which a hypergraph consisting of a single hyperedge and its vertices could be constructed from a suitable hypergraph, using the above procedure.

Since H is a suitable hypergraph with at least one hyperedge, we have proved that H consists of a single hyperedge and its vertices. //

We may deduce that the bound in Corollary 4.3.6 is attained only by hypergraphs H , some of whose connected components are suitable of chromatic number $\chi(H)$, and whose other components have maximum degree $\chi(H)+1$ or less. By adding non-minimal edges (i.e. edges which contain an already-present edge) to such a hypergraph, we obtain a hypergraph whose chromatic number attains the bound of Corollary 4.3.5. Our next theorem assures us that all hypergraphs whose chromatic numbers attain this stronger bound are of this form. Recall that if H is a hypergraph, then H^* is that hypergraph obtained from H by deleting all the non-minimal edges.

Theorem 4.4.2: If H is a hypergraph, then

$$\chi(H) \leq D^*(H)$$

unless, when the connected components C_1, \dots, C_k of H^* are labelled so that

$$\chi(C_1) = \dots = \chi(C_r) > \chi(C_{r+1}) \geq \dots \geq \chi(C_k),$$

at least one of the following conditions holds:

- i. Each of C_1, \dots, C_r has exactly one edge.
 - ii. Each of C_1, \dots, C_r is a cycle (graph) on an odd number of vertices.
 - iii. Each of C_1, \dots, C_r is a complete graph.
- together with the further condition:
- iv. None of the vertices of any of the connected components C_{r+1}, \dots, C_k has degree greater than $\chi(H)+1$.

Proof: Let H be a hypergraph satisfying

$$\chi(H) = 1 + D^*(H).$$

For each $i=1, \dots, r$ we have:

$$\begin{aligned} \chi(C_i) &= \chi(H^*) = \chi(H) = 1 + D^*(H) = 1 + D(H^*) \\ &\geq 1 + D(C_i). \end{aligned}$$

We know from Corollary 4.3.6 that $\chi(C_i) \leq 1 + D(C_i)$.

We have proved that C_i is a *suitable* (as in the proof of Theorem 4.4.1) hypergraph; our result now follows from our characterization of *suitable* hypergraphs. //

CHAPTER 5

THE CHROMATIC POLYNOMIAL OF A HYPERGRAPH

5.1: Introduction.

G. Birkhoff (1912) obtained an expression involving determinants for the number of colourings of a map with λ colours. A theory of chromatic polynomials of graphs gradually developed, significant contributions being made by Whitney (1932), and Birkhoff and Lewis (1946). An introduction to the theory may be found in the survey article by Read (1968).

In Section 5.2, we shall define a chromatic function for hypergraphs; we shall show later that this function is a polynomial, a generalization to hypergraphs of the chromatic polynomial of a graph. We shall also define a rank polynomial for hypergraphs, and note why some of the techniques developed by Whitney for the study of chromatic polynomials of graphs cannot be used for hypergraphs. We shall end the chapter by describing an algorithm for determining the chromatic polynomial of a hypergraph.

5.2: A Chromatic Function.

Let H be a hypergraph and let λ be a non-negative integer. We recall that a λ -colouring of H (if it exists) is a partition of the vertex set $V(H)$ into λ parts, some of which may be empty, but none of them contains an edge of H .

Our definition of λ -colouring is, in a sense, "colour-indifferent". For let us regard our λ -colouring as an assignment of one colour, from a set $\{c_1, \dots, c_\lambda\}$ of available colours, to each of the vertices of H ; this assignment of colours is *proper* in that no edge of H has all its vertices assigned the same colour. A permutation of the colours will not lead to a distinct λ -colouring, since a partition is essentially an *unordered* dissection.

To each λ -colouring of H with exactly μ non-empty parts, there correspond:

$$\lambda_{(\mu)} = \lambda(\lambda-1) \dots (\lambda-\mu+1)$$

distinct *proper* assignments of μ *distinguishable* colours chosen from a set of λ available colours.

There are $\lambda_{(\mu)}$ distinct functions:

$$f : V(H) \rightarrow \{1, \dots, \lambda\}$$

whose images have cardinality μ , and which are *proper* in that for any edge $e \in E(H)$, there are two vertices

$u, v \in e$ with $f(u) \neq f(v)$.

The total number of distinct *proper* functions:

$$f: V(H) \rightarrow \{1, \dots, \lambda\}$$

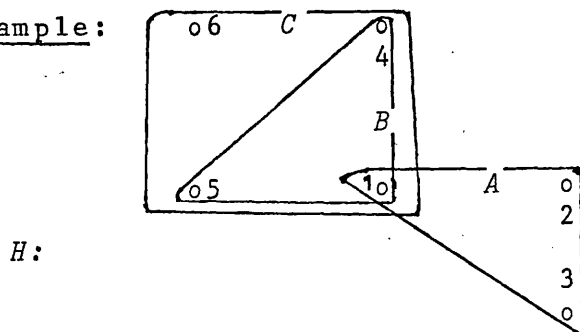
is, therefore:

$$\psi(H; \lambda) = \sum_{\mu=1}^{\lambda} \lambda_{(\mu)} T(H; \lambda, \mu)$$

where $T(H; \lambda, \mu)$ is the number of distinct λ -colourings of H with exactly μ non-empty parts (colour indifferent).

$\psi(H; \lambda)$ is the number of *proper* assignments of λ distinguishable colours to $V(H)$. ψ may be regarded as a function of the non-negative integer variable λ ; and we shall, temporarily, refer to $\psi(H; \lambda)$ as the *chromatic function* of H . (It is well known that, when G is a graph, $\psi(H; \lambda)$ is a polynomial in λ - the *chromatic polynomial* of G .)

Example:



Colour vertex 1:

(λ choices).

Colour vertices

2 and 3, not both

the same as 1:

($\lambda^2 - 1$ choices)

(Similarly $\lambda^2 - 1$ choices for 4 and 5). Edge C is already properly coloured since B is. (λ choices for 6).

Thus: $\psi(H; \lambda) = \lambda^2 (\lambda^2 - 1)^2$.

5.3: The Rank Polynomial.

For any hypergraph H , we define the *rank* $r(H)$ of H to be the difference between the number of vertices of H and the number $c(H)$ of connected components:

$$r(H) = |V(H)| - c(H).$$

(This rank function, acting on the edge-induced subhypergraphs of H , is the *hypperrank function* of the *chromatic hypermatroid* of H , defined by Helgason (1974).)

We also define the *co-rank* $s(H)$ by:

$$s(H) = |E(H)| - r(H) = |E(H)| - |V(H)| + c(H).$$

Let $F \subset E(H)$. The subhypergraph *induced* by F , denoted $\langle F \rangle$, has vertex set $\bigcup_{e \in F} e$ and edge family F . We write v^F for the number of its vertices, r^F for its rank, s^F for its co-rank, and c^F for the number of its connected components.

Let $H = (V, E)$ be a hypergraph. Following Biggs (1974), we define:

$$R(H; z, w) = \sum_{F \subset E} z^{r^F} w^{s^F}$$

When G is a graph, $R(G; z, w)$ is a polynomial in z and w called the *rank polynomial* of G . However, when H is a hypergraph, there is the possibility that s^F is a negative integer for some subsets F . r^F is always a non-negative integer; so $R(H; z, w_0)$ is a polynomial

expression in z for any fixed value w_0 . We shall call $R(H; z, w)$ the *rank polynomial* of the hypergraph H , although it is to be understood that $R(H; z, w)$ is not necessarily a polynomial expression in the variable w .

5.4: The Chromatic Polynomial.

Our aim in this section is to prove that the chromatic function $\psi(H; \lambda)$ defined in Section 5.2 is a polynomial in λ . This we do by demonstrating that ψ is a partial evaluation of the rank polynomial defined in the last section. We proceed to deduce results concerning the coefficients of ψ . We shall follow roughly the exposition given by Biggs (1974) for graphs.

Let λ be a positive integer, and let X be a set. We write λ^X to denote the set of all functions

$$\xi : X \rightarrow \{1, \dots, \lambda\}.$$

If $H = (V, E)$ is a hypergraph, then with each function $\xi \in \lambda^V$ we associate a function

$$\hat{\xi} : E \rightarrow \{0, 1\}, \text{ given by:}$$

$$\hat{\xi}(e) = \begin{cases} 1 & \text{if there are vertices } u, v \in e, \xi(u) \neq \xi(v). \\ 0 & \text{otherwise.} \end{cases}$$

We introduce a function $W(H; z, \lambda)$ of the complex variable z and the non-negative integer λ , whose value is 0 when $\lambda=0$, and is otherwise given by:

$$W(H; z, \lambda) = \lambda^{-|V|} \cdot \sum_{\xi \in \lambda^V} \prod_{e \in E} (\hat{\xi}(e) - z) .$$

Lemma 5.4.1: For any hypergraph $H = (V, E)$ and any non-negative integer λ :

$$\psi(H; \lambda) = \lambda^{|V|} \cdot W(H; 0, \lambda) .$$

Proof: $\lambda^{|V|} \cdot W(H; 0, \lambda) = \sum_{\xi \in \lambda^V} \prod_{e \in E} (\hat{\xi}(e)) \dots (1)$

Let $V = S_1 \cup \dots \cup S_\lambda$ be a λ -colouring of H .

The function $\xi \in \lambda^V$ given by:

$\xi(v) = i$, where i is the index for which $v \in S_i$, contributes 1 to the right-hand side of (1).

So does the function ξ given by:

$\xi(v) = \pi(i)$, where $v \in S_i$, and π is a permutation of the set $\{1, \dots, \lambda\}$.

Thus every λ -colouring of H defines $\lambda!$ functions in λ^V , each of which contribute 1 to the right-hand side of (1). These $\psi(H; \lambda)$ functions are all distinct. Furthermore, any function in λ^V which does not arise in this way from a λ -colouring of H contributes 0 to the right-hand side of (1). //

Lemma 5.4.2: $\psi(H; \lambda) = \lambda^{|V|} \cdot z^{|E|} \sum_{F \subseteq E} W(\langle F \rangle; z, \lambda) z^{-|F|}.$

Proof:

$$\begin{aligned} \psi(H; \lambda) &= \lambda^{|V|} \cdot W(H; 0, \lambda) \\ &= \sum_{\xi \in \lambda^V} \prod_{e \in E} \hat{\xi}(e) \\ &= \sum_{\xi \in \lambda^V} \prod_{e \in E} ((\hat{\xi}(e) - z) + z) \\ &= \sum_{\xi \in \lambda^V} \sum_{F \subseteq E} \prod_{e \in F} ((\hat{\xi}(e) - z) z^{|E| - |F|}). \end{aligned}$$

Let us write $VF = V(\langle F \rangle)$ and $vF = |VF|$. Since any function in λ^{VF} is the restriction of precisely $\lambda^{|V| - vF}$ functions in λ^V , we may reverse the order of the summations to obtain:

$$\begin{aligned} \psi(H; \lambda) &= \sum_{F \subseteq E} \lambda^{|V| - vF} \sum_{\xi \in \lambda^{VF}} \prod_{e \in F} ((\hat{\xi}(e) - z) z^{|E| - |F|}) \\ &= \lambda^{|V|} \cdot z^{|E|} \sum_{F \subseteq E} (\lambda^{-vF} \sum_{\xi \in \lambda^{VF}} \prod_{e \in F} (\hat{\xi}(e) - z) z^{-|F|}) \\ &= \lambda^{|V|} \cdot z^{|E|} \sum_{F \subseteq E} W(\langle F \rangle; z, \lambda) z^{-|F|} . \quad // \end{aligned}$$

We may now proceed to prove the main theorem of this section; that the chromatic function ψ is a partial evaluation of the rank polynomial.

Theorem 5.4.3: Let $H = (V, E)$ be a hypergraph. Then:

$$\psi(H; \lambda) = \lambda^{|V|} \cdot R(H; -\frac{1}{\lambda}, -1)$$

where $R(H; z, w)$ is the rank polynomial defined in Section 5.3.

Proof: Setting $z = 1$ in the result of Lemma 5.4.2,

we have:

$$\begin{aligned} \psi(H; \lambda) &= \lambda^{|V|} \cdot \sum_{F \subset E} W(\langle F \rangle; 1, \lambda) \\ &= \lambda^{|V|} \cdot \sum_{F \subset E} \lambda^{-v_F} \sum_{\xi \in \lambda^{V_F}} \prod_{e \in F} (\hat{\xi}(e) - 1), \end{aligned}$$

where $V_F = V(\langle F \rangle)$ and $v_F = |V_F|$.

The product $\prod_{e \in F} (\hat{\xi}(e) - 1)$ is non-zero only if

ξ is constant on every connected component of $\langle F \rangle$, and then the value of the product is $(-1)^{|F|}$.

There are precisely λ^{c_F} such functions in λ^{V_F} .

Thus:

$$\begin{aligned} \psi(H; \lambda) &= \lambda^{|V|} \cdot \sum_{F \subset E} \lambda^{-v_F} (-1)^{|F|} \lambda^{c_F} \\ &= \lambda^{|V|} \cdot \sum_{F \subset E} (-1)^{|F| - v_F + c_F} (-\lambda^{-1})^{v_F - c_F} \\ &= \lambda^{|V|} \cdot \sum_{F \subset E} (-1)^{s_F} (-\lambda^{-1})^{r_F} \\ &= \lambda^{|V|} \cdot R(H; (-\lambda)^{-1}, -1). \quad // \end{aligned}$$

Corollary 5.4.4: Let $H = (V, E)$ be a hypergraph.

$\psi(H; \lambda)$ is a monic polynomial in λ of degree $|V|$ whose constant term is 0.

Proof: From the theorem, and the definition of the rank polynomial, we have:

$$\psi(H; \lambda) = \lambda^{|V|} \sum_{F \subset E} (-\lambda)^{-r_F} (-1)^{s_F}.$$

For any subset $F \subset E$, r_F is an integer and

$$0 \leq r_F \leq |V| - 1.$$

Furthermore $r_F = 0$ if, and only if, $F = \phi$ the empty set. //

Having thus established that $\psi(H; \lambda)$ is, for any hypergraph H , a polynomial expression in λ , we shall henceforth refer to $\psi(H; \lambda)$ as the *chromatic polynomial* of H . We shall also consider $\psi(H; \lambda)$ to have been defined for all complex numbers λ .

Let us express ψ in its polynomial form:

$$\psi(H; \lambda) = b_0 \lambda^n + b_1 \lambda^{n-1} + \dots + b_{n-1} \lambda + b_n$$

(where $n = |V|$, and by the Corollary $b_0 = 1$ and $b_n = 0$).

Let us also write:

$$R(H; z, w) = \sum_{r, s} \rho_{rs} z^r w^s$$

(where ρ_{rs} is the number of edge-induced subhypergraphs of rank r and co-rank s . The summation extends over

all admissible values of r and s , including some negative values of s).

Proposition 5.4.5: For any integer i , $0 \leq i \leq n$,

$$(-1)^i b_i = \sum_s (-1)^s \rho_{rs}.$$

Proof: By Theorem 5.4.3:

$$\begin{aligned} \sum_{i=0}^n b_i \lambda^{n-i} &= \lambda^n \sum_{r,s} \rho_{rs} (-\lambda)^{-r} (-1)^s \\ &= \sum_{r=0}^n \sum_s (-1)^{s-r} \rho_{rs} \lambda^{n-r}. \end{aligned}$$

Equating the coefficients of λ^{n-i} , we have:

$$b_i = \sum_s (-1)^{s-r} \rho_{rs}, \text{ and our result follows. } //$$

Corollary 5.4.6: $\psi(H; \lambda) = \sum_{F \subset E} (-1)^{|F|} \lambda^{|V|-rF}.$

Proof:

$$\begin{aligned} \psi(H; \lambda) &= \sum_r b_r \lambda^{n-r} \\ &= \sum_r \sum_s (-1)^{s-r} \rho_{rs} \lambda^{n-r} \\ &= \sum_{r,s} (-1)^{r+s} \rho_{rs} \lambda^{n-r} \\ &= \sum_{F \subset E} (-1)^{|F|} \lambda^{n-rF}. \quad // \end{aligned}$$

Whitney (1932) proved that, when G is a graph, it is possible to divide the edge-induced subgraphs of G which have rank r into three disjoint classes:

- (i) Those which contain no *broken cycles* (which occur when a particular edge is removed from a cycle).
- (ii) Those which contain broken cycles, but whose co-rank is even.
- (iii) Those which contain broken cycles but whose co-rank is odd.

Whitney established a correspondence between the members of (ii) and (iii) which demonstrated that the contributions of their members to the sum $\sum_s (-1)^s \rho_{rs}$ cancel each other out.

Any edge-induced subgraph in (i) is a forest. Its rank is the number of its edges, and its co-rank is 0. By Proposition 5.4.5, we know that

$$(-1)^r b_r = \sum_s (-1)^s \rho_{rs}.$$

Thus Whitney established the following result concerning the coefficients of the chromatic polynomial of a graph G :

$(-1)^r b_r$ is the number of edge-induced subgraphs of G having r edges and containing no broken cycles.

It follows that $(-1)^r b_r$ is a non-negative integer: The coefficients of the chromatic polynomial

of a graph alternate in sign. This is not true in general for hypergraphs. For consider the hypergraph consisting of a single 3-edge and its vertices. The chromatic polynomial of this hypergraph is $\lambda^3 + 0\lambda^2 - \lambda$, whose coefficients certainly do not alternate in sign.

It also follows from Whitney's result that we can put upper and lower bounds on $(-1)^r b_r$ when the b_r are the coefficients of the chromatic polynomial of a connected graph:

$$\binom{n-1}{r} \leq (-1)^r b_r \leq \binom{m}{r}$$

where $0 \leq r \leq n = |V(G)|$, and $m = |E(G)|$.

Neither of these bounds apply in general to the coefficients of the chromatic polynomial of a hypergraph. The coefficient b_1 of the chromatic polynomial of the hypergraph consisting of a single 3-edge and its vertices is 0; so $(-1)b_1$ is certainly less than the proposed lower bound of 2. The chromatic polynomial of the hypergraph consisting of a single 4-edge and its vertices is $\lambda^4 - \lambda$. $(-1)^3 b_3$ is 1, which exceeds the proposed upper bound which is $\binom{1}{3} = 0$.

It is not possible to apply to hypergraphs analysis similar to that applied by Whitney to graphs. One reason for this we have already noted: It is

possible for a hypergraph to have a negative co-rank; this eventuality can never occur for a graph.

Another reason is that, whereas the addition to, or removal from, a graph of a 2-edge can change the values of the rank and co-rank by at most 1, there is no limit to the amount by which the rank and co-rank of a hypergraph may be varied by the addition of a hyperedge.

We cannot, therefore, extract from Proposition 5.4.5 some of the powerful results which can be proved for graphs. We can, however, deduce the following theorem concerning the coefficients of the chromatic polynomial of a hypergraph:

Theorem 5.4.7: Let H be a hypergraph with n vertices none of whose edges contain less than k (>2) vertices. Write:

$$\psi(H; \lambda) = \lambda^n + b_1 \lambda^{n-1} + \dots + b_{n-1} \lambda. \quad \text{Then:}$$

$b_i = 0$, for $1 \leq i < k-1$, and

$-b_{k-1}$ is the number of k -edges in H .

Proof: Since each edge contains at least k vertices, there are no edge-induced subhypergraphs of rank less than $k-1$. The only edge-induced subhypergraphs of rank $k-1$ are those induced by single

k -edges; these subhypergraphs have co-rank $-k$. Our theorem now follows from Proposition 5.4.5. //

5.5: Another Expansion.

Helgason (1974) has defined the *Poincaré polynomial* of a hypermatroid. If r is the hyperrank function of a hypermatroid on a set E , the Poincaré polynomial is:

$$\tau(E, r; \lambda, \mu) = \sum_{F \subseteq E} (\mu - 1)^{|F|} \lambda^{rE - rF}.$$

Helgason proves that if E is the edge family of the hypergraph H , and r is the rank function we defined in Section 5.3, then:

$$\psi(H; \lambda) = \lambda \tau(E, r; \lambda, 0).$$

From this follows the expansion:

$$\psi(H; \lambda) = \sum_{F \subseteq E} (-1)^{|F|} \lambda^{|V| - rF}$$

which we proved as Corollary 5.4.6.

5.6: An Algorithm for Calculating $\psi(H;\lambda)$.

Read (1968) describes an algorithm for calculating $\psi(G;\lambda)$ when G is a graph. With the graph G he associates two graphs G' and G'' with the property that:

$$\psi(G;\lambda) = \psi(G';\lambda) + \psi(G'';\lambda).$$

Repeated application of this process eventually enables us to express $\psi(G;\lambda)$ as a sum of chromatic polynomials of complete graphs. It is, of course, well known that the chromatic polynomial of the complete graph on n vertices is:

$$\lambda(\lambda-1) \dots (\lambda-n+1).$$

In this section, we shall describe a process which enables us to express the chromatic polynomial of a hypergraph as a sum of chromatic polynomials of graphs. Combined with Read's process, this enables us to calculate the chromatic polynomial of any hypergraph.

Proposition 5.6.1: The chromatic polynomial of a disconnected hypergraph is the product of the chromatic polynomials of its connected components.

Proof: Immediate from the definitions. //

Proposition 5.6.2: Let H^* be the hypergraph obtained from H by deleting a non-minimal edge (i.e. an edge which contains another edge):

$$\psi(H^*; \lambda) = \psi(H; \lambda).$$

Proof: The removal of non-minimal edges does not in any way affect the colouring properties of a hypergraph. //

By the *identification* of two vertices u and v in a hypergraph we mean the replacement of u and v by a single vertex w ; edges which previously contained u or v will instead, in the new hypergraph, contain the vertex w .

Let u and v be vertices, both contained in the hyperedge e of the hypergraph H ; and let us suppose that $\{u, v\}$ is not a 2-edge of H . Denote by H' that hypergraph obtained from H by replacing e with the new 2-edge $\{u, v\}$. Denote by H'' that hypergraph obtained from H after the identification of u and v .

Proposition 5.6.3:

$$\psi(H; \lambda) = \psi(H'; \lambda) + \psi(H''; \lambda).$$

Proof: There is a natural one-to-one correspondence between the set of λ -colourings of H which assign different colours to u and v , and the set of

all λ -colourings of H' .

There is another natural one-to-one correspondence between the set of λ -colourings of H in which u and v have the same colour, and the set of all λ -colourings of H'' . //

H' has fewer hyperedges than has H . Of the edges of H'' , at least one contains fewer vertices than does the corresponding hyperedge of H . In this sense, both H' and H'' are "more like graphs" than was H .

The Algorithm:

We begin with a hypergraph H . Proposition 5.6.2 allows us, without affecting the chromatic polynomial, to remove any edges that properly contain another edge, and also to remove all but one of any collection of multiple edges.

We then choose two vertices of a hyperedge, and form two hypergraphs (as we did for Proposition 5.6.3) the sum of whose chromatic polynomials is the chromatic polynomial of H .

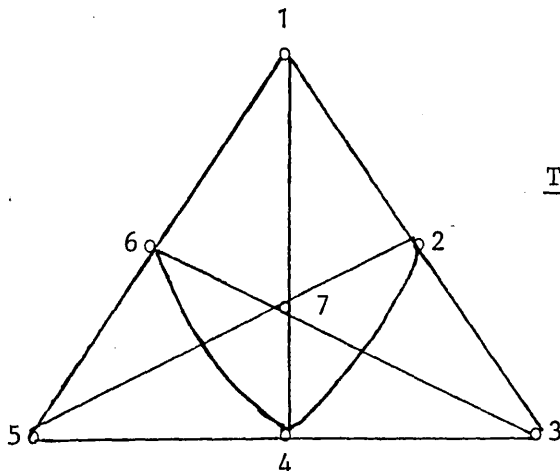
Each of these hypergraphs may now be dealt with in a similar manner:

- (i) Remove "superfluous" edges;
- (ii) Associate with the hypergraph two new hypergraphs,

each of which is "more like a graph" than was the original. The sum of the chromatic polynomials of the new hypergraphs is the chromatic polynomial of the original.

We continue with this procedure. If, at any stage, we obtain a disconnected hypergraph, we may invoke Proposition 5.6.1 and treat the connected components separately. Since $V(H)$ is finite, our procedure will eventually enable us to express $\psi(H; \lambda)$ as a sum of products of chromatic polynomials of graphs. The process, incidentally, provides an alternative proof of Corollary 5.4.4.

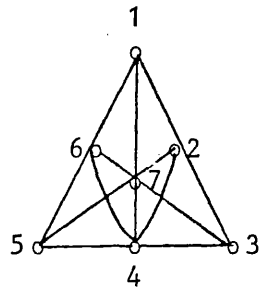
Let us illustrate our procedure by determining the chromatic polynomial of the Fano plane. All the edges of this hypergraph contain just three vertices; in the picture below, edges are represented by lines joining the points representing their vertices.



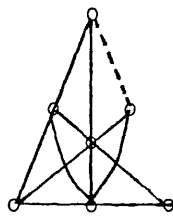
The Fano Hypergraph F .

Edges:

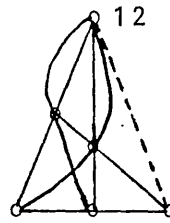
123, 345, 561, 174,
376, 572, 246.



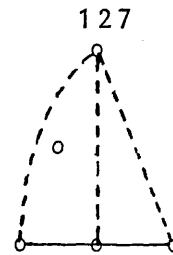
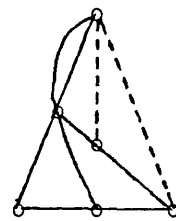
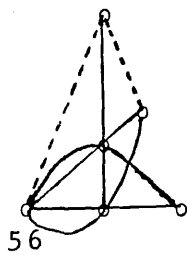
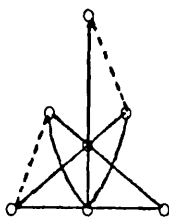
(1&2)



(5&6)



(12&7)



* * * * *

ETC.

This diagram shows how we might start applying our algorithm to F . Continuous lines represent 3-edges, while broken lines represent 2-edges.

Although we do not reproduce the calculations here, we would assure the reader that it is a relatively simple task to continue the process. There is not, of course, a unique way to apply the algorithm - there is considerable latitude in the choice of vertex-pairs for the application of Proposition 5.6.3. Intuitively, it seems to be useful to select vertex-pairs which appear in a large number of hyperedges. The bottom right hypergraph on the previous page has many fewer hyperedges than its predecessor: In that hypergraph, the vertex-pair $\{12,7\}$ appears in two hyperedges.

When we did our calculations, we expressed $\psi(F;\lambda)$ as the sum of the chromatic polynomials of twenty seven graphs; all of these polynomials were easy to find by inspection. The main result of our labours is:

$$\psi(F;\lambda) = \lambda^7 - 7\lambda^5 + 21\lambda^3 - 21\lambda^2 + 6\lambda.$$

The symmetry of F enables us to check this result quickly using Corollary 5.4.6.

$$\psi(F;\lambda) \text{ factorizes as: } \lambda(\lambda-1)(\lambda-2)(\lambda^4+3\lambda^3-6\lambda+3).$$

We see that $\psi(F;\lambda)$ vanishes when $\lambda = 0, 1$, or 2 ; and is a positive integer whenever λ is an integer ≥ 3 .

We deduce that the Fano plane has chromatic number

$$\chi(F) = 3.$$

CHAPTER 6
PLANAR HYPERGRAPHS

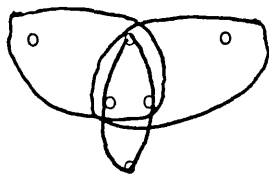
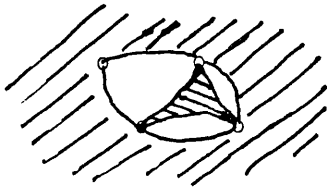
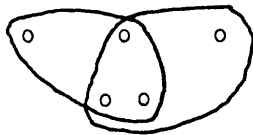
6.1: Definition.

Let us represent the vertices of a hypergraph, each by its own distinct point in the plane. Let us then represent each edge by a subset of the plane homeomorphic to a closed disc and containing all those points representing vertices contained in that edge.

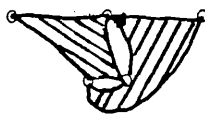
If, in such a representation, the subsets representing any two edges intersect only in points representing vertices common to both edges, we call this representation a *plane imbedding* of the hypergraph. A hypergraph which has a plane imbedding is called *planar*.

Examples:

Hypergraph



Plane Imbedding



The shaded regions represent edges.

Not Planar

(Proof later)

From now on, we shall use the shorter expression "closed disc" for any subset of the plane homeomorphic to a closed disc.

The left-hand representation of the second example on the previous page would be regarded by some authors (including Zykov) as a plane imbedding; they would regard the subset extending to infinity as a closed disc. We prefer not to allow this. (In any case, a subset extending to infinity can always be "folded over" into a finite closed disc.)

Our definition of planarity is rather unwieldy: The reader is invited to prove directly that the hypergraph of our third example on the last page is not planar. We proceed to remedy this situation.

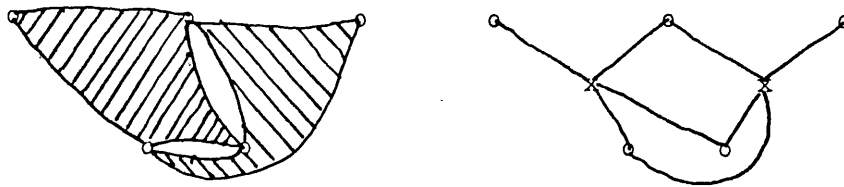
Let H be a hypergraph. Its *König graph* $K(H)$ is that graph with vertex set $V \cup E$ whose edges are those vertex-edge pairs $\{v, e\}$ for which $v \in e$. Every König graph is, of course, bipartite and simple.

Theorem 6.1.1: A hypergraph H is planar if, and only if, $K(H)$ is planar.

Proof: Given a plane imbedding of H , place a new vertex inside each of the closed discs representing edges. Join each new vertex to all the

"old" vertices contained in the closed disc containing that new vertex by non-intersecting lines lying in that disc. Totally ignoring the original closed discs, we find that we have a plane imbedding of $K(H)$.

Illustration:



To prove the converse, we may simply use the reverse procedure to obtain from a plane imbedding of $K(H)$ a plane imbedding of H . //

6.2: The Four-Colour Theorem.

Bulitko has established the equivalence of the four-colour conjecture for planar graphs and the conjecture that any planar hypergraph admits a 4-colouring (See Zykov (1974)). It has recently been announced that K.Appel and W.Haken have verified the four-colour conjecture for graphs.

In this section, we describe an iterative procedure which associates with any planar hypergraph a planar graph whose chromatic number is not less than that of the original hypergraph. This will establish Bulitko's result. If there is a flaw in the proof of Appel and Haken, we will at least know that any planar hypergraph admits a 5-colouring.

The Procedure:

We first of all remove from the hypergraph any edges which properly contain another edge; and remove all but one of any set of multiple edges. This cannot destroy the planarity, neither can it decrease the chromatic number, of our hypergraph.

Now take any hyperedge, and select any two of its vertices, say u and v . Replace the hyperedge by the new 2-edge $\{u, v\}$. Again this cannot destroy the planarity or decrease the chromatic number of our hypergraph.

By repeating this process, we will eventually obtain a planar graph whose chromatic number is not less than that of the original hypergraph.

Any upper bound on the chromatic number of the class of planar graphs will therefore apply also to the class of planar hypergraphs.

Let us say that a hypergraph *can be imbedded in* a surface if its König graph can. Let us define the *(hyper)graph-chromatic number of a surface* to be the largest of the chromatic numbers of (hyper)graphs which can be imbedded in that surface. An argument almost identical to the above may be used to prove:

Theorem: The hypergraph-chromatic number of a surface is the same as its graph-chromatic number. //

6.3: The Two-Colour Theorem.

M.I.Burstein (1975) has proved that any planar hypergraph with at most two 2-edges may be 2-coloured. The proof presented here will make use of the four-colour theorem for planar graphs; although it must be stressed that Burstein's proof is independent of that result.

Theorem 6.3.1: Let H be a planar hypergraph with at most two 2-edges. Then

$$\chi(H) \leq 2.$$

Proof: As with the procedure described in the last section, the first step is to remove all non-minimal edges. The planarity is not destroyed, and the chromatic number cannot decrease.

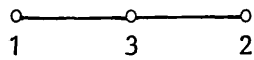
The second step shows that we may restrict our attention to hypergraphs all of whose edges contain exactly three vertices, except possibly two 2-edges. Let u, v, w be three vertices all contained in an edge e , with $|e| > 3$. The hypergraph obtained by replacing e by the new 3-edge $\{u, v, w\}$ is planar and its chromatic number is not less than that of H . All edges with more than three vertices may be similarly replaced by 3-edges.

We may now suppose that H is a planar hypergraph

all of whose edges, except possibly two 2-edges, contain exactly three vertices.

Let G be the graph obtained from H by replacing each 3-edge $\{u, v, w\}$ by the three new 2-edges $\{u, v\}$, $\{v, w\}$ and $\{w, u\}$. G is planar, and the 3-edges of H correspond to triangles in G .

By the four-colour theorem for planar graphs, the vertices of G may be properly coloured with the four colours 1, 2, 3 and 4. We may replace two of these colours by the single colour α , and the other two by the single colour β , in such a way that the end-vertices of each of the (at most two) 2-edges of H receive different colours (α and β). For example,



if the two 2-edges are adjacent and coloured as

shown, we might take α to replace 1 and 2, and β to replace 3 and 4.

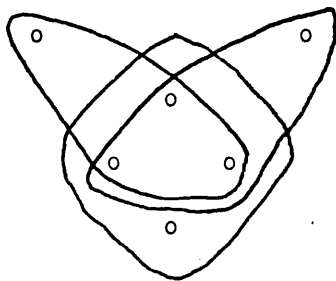
We have ensured that each 2-edge of H receives two colours (α and β); and since each 3-edge of H corresponds to a triangle in G , we may be sure that each 3-edge of H also receives two colours. //

6.4: The Blocks of a Hypergraph.

Blocks of hypergraphs have been defined by Zykov (1974). Unfortunately, because of our differing definition of hypergraph, some of the objects which Zykov would call "blocks" are not even hypergraphs to us. (They contain edges with fewer than two vertices.)

Rather than adapt Zykov's definition, we prefer to proceed from the definition of *block* of a graph (as may be found, for example, in Harary (1969)). A block of a graph is *non-trivial* if it contains at least three vertices.

If B is a part of the hypergraph H with the property that the König graph $K(B)$ is a non-trivial block of $K(H)$, we say that B is a *non-trivial block* of H .



The hypergraph shown here contains one non-trivial block: (contract each edge to 'enclose' only the three central vertices; then delete the resulting isolated vertices). Incidentally, we promised earlier to prove that this hypergraph is not planar. To see this, note that its König graph contains the (non-planar) complete bipartite graph $K_{3,3}$ as a subgraph.

Theorem 6.4.1: A hypergraph H is planar if, and only if, all its non-trivial blocks are planar.

Proof: $K(H)$ is planar if, and only if, all its blocks are planar.

Any trivial (i.e. with less than three vertices) block in a graph must be planar.

So $K(H)$ (and hence H) is planar if, and only if, all the non-trivial blocks of $K(H)$ (and hence of H) are planar. //

It is desirable to obtain a characterization of the non-trivial blocks of a hypergraph without reference to its König graph. Let us first describe some notation and introduce a definition.

If e is an edge of the hypergraph $H = (V, E)$ we denote by $H-e$ the hypergraph $(V, E-\{e\})$.

Let v be a vertex of the hypergraph $H = (V, E)$. For any edge $e \in E$ let:

$$e|v = \begin{cases} e-\{v\} & \text{if } v \in e \text{ and } |e| > 2, \\ \phi & \text{if } v \in e \text{ and } |e| = 2, \\ e & \text{if } v \notin e. \end{cases}$$

Write $E|v = \{ e|v : e \in E; e|v \neq \phi \}$ and denote by $H|v$ the hypergraph $(V-\{v\}, E|v)$.

A hypergraph H is 2-connected if:

- (i) H is connected and has at least three vertices,
- (ii) $H|v$ is connected for each vertex v , and
- (iii) $H-e$ is connected for each edge e .

Notice that a graph G is 2-connected if, and only if, $G|v$ (which most graph theorists would write $G-v$) is connected and has at least two vertices, for each vertex v . So our definition of 2-connectedness, when applied to graphs, accords with that of Berge (1973).

Theorem 6.4.2: A part, with at least three vertices, of a hypergraph is a non-trivial block if, and only if, it is a maximal 2-connected part (i.e. is not a proper part of any other 2-connected part). Non-trivial blocks with less than three vertices consist of two vertices, and at least two edges.

Proof: It is immediate from the definitions that a non-trivial block with two vertices must have more than one edge.

Let B be a non-trivial block, with at least three vertices, of the hypergraph H . B is 2-connected. (Otherwise $K(B)$ cannot be a block of $K(H)$.) B is also maximal. (Otherwise, there would be a part B' ; and $K(B)$ would be a proper subgraph of the 2-connected subgraph $K(B')$ of $K(H)$. But $K(B)$ is a block.)

Conversely, let M be a maximal 2-connected part.

$K(M)$ is certainly a 2-connected subgraph of $K(H)$ and has at least three vertices. That $K(M)$ is maximal (and hence a block) follows from the maximality of M .

6.5: The Faces of a Hypergraph.

Suppose we have a plane imbedding of a hypergraph; let us shade those subsets of the plane which represent edges. The unshaded portion of the plane will consist of several connected open subsets (*regions*). By analogy with Graph Theory, we should like to be able to regard these regions as the "faces" of our hypergraph.

Given a plane imbedding of a 2-connected graph, it is well known that the boundaries of the faces, including the infinite face, form what we shall call a *MacLane system*: That is a system of cycles with the property that each edge of the graph appears in exactly two of the cycles. MacLane (1937) has proved that a graph is planar if, and only if, each of its non-trivial blocks has a MacLane system.

Let H be a 2-connected hypergraph with at least two vertices. A system of cycles of H is called a *face system* if each vertex-edge pair $\{v, e\}$ of the hypergraph with $v \in e$ appears in exactly two of the cycles. It follows that each r -edge ($r \geq 2$) appears in exactly r cycles; and hence that a face system in a graph is a MacLane system.

As a justification for our terminology, notice that the cycles which form the boundaries of the unshaded regions of a plane imbedding of a 2-connected hypergraph form a face system.

Theorem 6.5.1: A 2-connected hypergraph has a face system if, and only if, its König graph has a MacLane system.

Furthermore, given a plane imbedding of the 2-connected hypergraph H , there is a natural one-to-one correspondence between the cycles of the face system formed by the boundaries of its unshaded regions and the cycles of the MacLane system formed by the boundaries of the faces in the associated (as in the proof of Theorem 6.1.1) imbedding of $K(H)$.

Proof: Let S be a face system of H .

Let $C = v_0, e_1, v_1, e_2, \dots, v_{l-1}, e_l, v_0$ be a cycle in S . Then $v_0e_1, e_1v_1, v_1e_2, \dots, v_{l-1}e_l, e_lv_0$ are the edges of a cycle C' of $K(H)$.

(If C forms the boundary of an unshaded region in a plane imbedding of H , then C' forms the boundary of the "naturally corresponding" face of the associated imbedding of $K(H)$.)

With each cycle C in S , let us similarly associate a cycle C' of $K(H)$. Let S' denote the system of cycles

of $K(H)$ thus obtained.

Since each vertex-edge pair $\{v, e\}$ of H with $v \in e$ appears in exactly two cycles of S , it follows that each edge of $K(H)$ appears in exactly two cycles of S' . So S' is a MacLane system.

Conversely, by reversing our argument, we can associate with each cycle of a MacLane system of $K(H)$ a cycle of a face system of H . //

Corollary 6.5.2: A hypergraph is planar if, and only if, each of its non-trivial blocks admits a face system.

Proof: Follows from the theorem above, Theorem 6.4.1, and MacLane's theorem. //

Corollary 6.5.3: Every face system of a 2-connected planar graph contains the same number of cycles.

Proof: Given a face system of H , associate with it (as in the proof of the above theorem) a MacLane system of $K(H)$. A MacLane system consists of a cycle basis together with the (modulo 2) sum of the cycles of that basis. Every cycle basis of a graph contains the same number of cycles. //

Suppose we have a plane imbedding, with the edges as usual shaded, of a (not necessarily 2-connected) planar hypergraph. The regions of the unshaded portion of the plane are the *faces* of the hypergraph.

Proposition 6.5.4: Let B be a non-trivial block of the planar hypergraph H . If B has three or more vertices, let $f(B)$ denote the number of cycles in a face system of B ; if B has but two vertices, $f(B)$ will denote the number of edges of B . The number of faces in any plane imbedding of H is: $1 + \sum (f(B)-1)$

where the summation extends over all the non-trivial blocks of H .

Proof: By induction on the number of non-trivial blocks. //

Corollary 6.5.5: The number of faces in any plane imbedding of a planar hypergraph H is equal to the number of faces of $K(H)$. //

Corollary 6.6.4: The bounds of Corollaries 6.6.2 and 6.6.3 are incompatible if

$\delta r \geq 2(r+\delta)$. In particular, we may deduce that any planar hypergraph, all of whose edges contain at least 3 (respectively 4,6) vertices, has a vertex of degree at most 5 (respectively 3,2).

Proof: The bounds are incompatible if

$$(r-2)((\delta-2)n + 2k + 2) > 4(n-k-1).$$

This is certainly the case if $\delta r \geq 2(r+\delta)$.

If r is at least 3 (respectively 4,6) then any value of δ exceeding 5 (respectively 3,2) leads to an incompatibility in our bounds. //

An Interpretation: Let us define an r -tile to be any plane figure bounded by a closed Jordan curve, r distinct points of which are chosen and called the *corners* of the tile. (Notice that we do not insist that the *sides*, those portions of the curve lying between alternate distinguished points, be straight lines.)

We have proved that it is impossible to arrange any finite collection of 3-tiles (respectively 4-tiles, 6-tiles) in the plane so that:

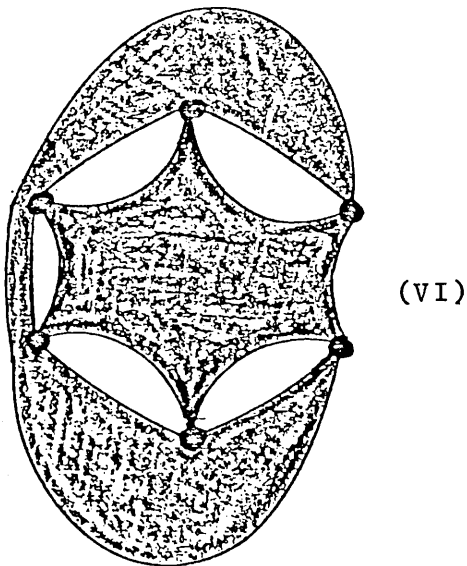
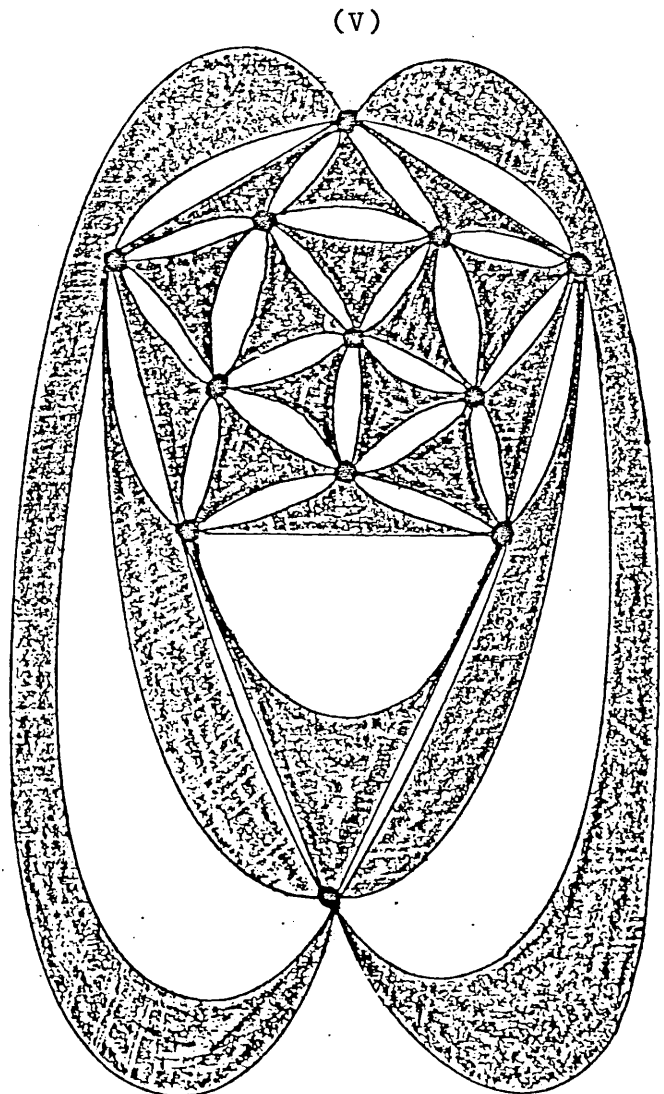
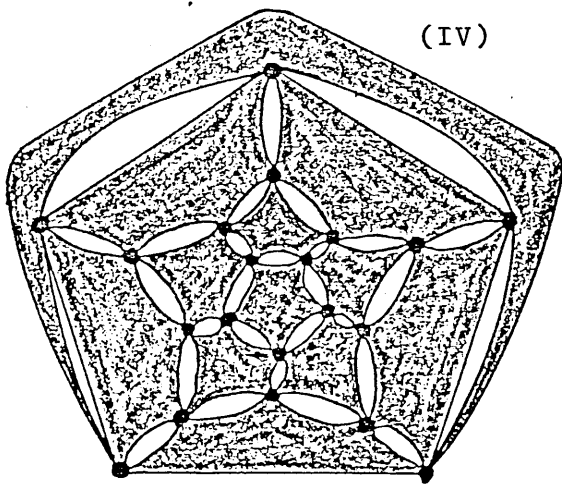
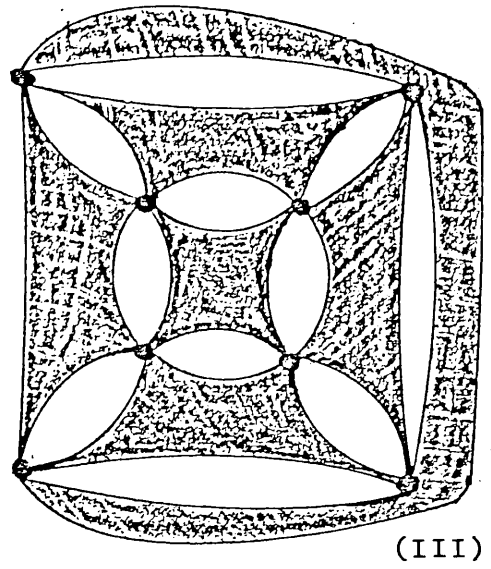
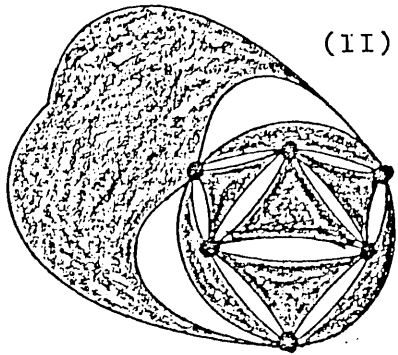
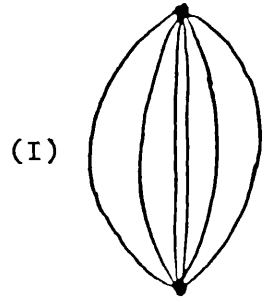
- (i) they do not overlap, except at corners, and
- (ii) each corner is coincident with at least 5 (respectively 3,2) others.

These results are not true for arrangements of infinitely many tiles: Counterexamples may easily be constructed from well-known regular tessellations of the plane.

Over the page are shown some representations of planar, regular, uniform hypergraphs.

- (I) A 2-graph of degree 6;
- (II) A 3-graph of degree 4;
- (III) A 4-graph of degree 3;
- (IV) A 5-graph of degree 3;
- (V) A 3-graph of degree 5;
- (VI) A 6-graph of degree 2.

Notice that Corollary 6.6.2 does not give us an upper bound for the number of edges in a planar graph. This is because our definition of graph allows for multiple edges. It is, of course, possible to obtain an upper bound for the number of edges in a planar *simple* graph.



Definition: A hypergraph is *simple* if, for each pair
 e, e' of its edges, $|e \cap e'| \leq 1$.

Proposition 6.6.5: Each face of a planar simple
hypergraph is bounded by at least
three distinct edges.

Proof: If a face is bounded by just two edges, the
intersection of those two edges contains more
than one vertex. //

Corollary 6.6.6: Let H be a planar simple hypergraph
with n vertices, k connected
components, and m edges each containing at least r
vertices.

$$m \leq \frac{3(n - (k+1))}{2r-3}.$$

Proof: From the Proposition, $3f \leq \sum_{e \in E} r(e)$.

Substituting into the result of Theorem 6.6.1:

$$\sum_{e \in E} r(e) + 3n \geq 3 \sum_{e \in E} r(e) - 3m + 3(k+1).$$

As before, we have $\sum_{e \in E} r(e) \geq rm$, and the result

follows from this substitution. //

Corollary 6.6.7: Let H be a planar simple hypergraph with m edges, k connected components, and n vertices, each of degree at least δ .

$$m \geq \frac{(2\delta-3)n}{3} + k + 1.$$

Proof: Similar to Corollary 6.6.3. //

Corollary 6.6.8: The bounds of Corollaries 6.6.6 and 6.6.7 are incompatible if

$2\delta r \geq 3(r+\delta)$. In particular, we may deduce that any planar simple hypergraph, all of whose edges contain at least 2 (respectively 3,6) vertices, has a vertex of degree at most 5 (respectively 2,1). //

An Interpretation: If we trace the Jordan curve boundary of an r -tile in one direction, we effectively order the corners of the tile. Let us understand by a *side* of an r -tile any section of the bounding curve lying between two successive corners.

We have proved that it is impossible to arrange any finite collection of 3-tiles (respectively 6-tiles) in the plane so that:

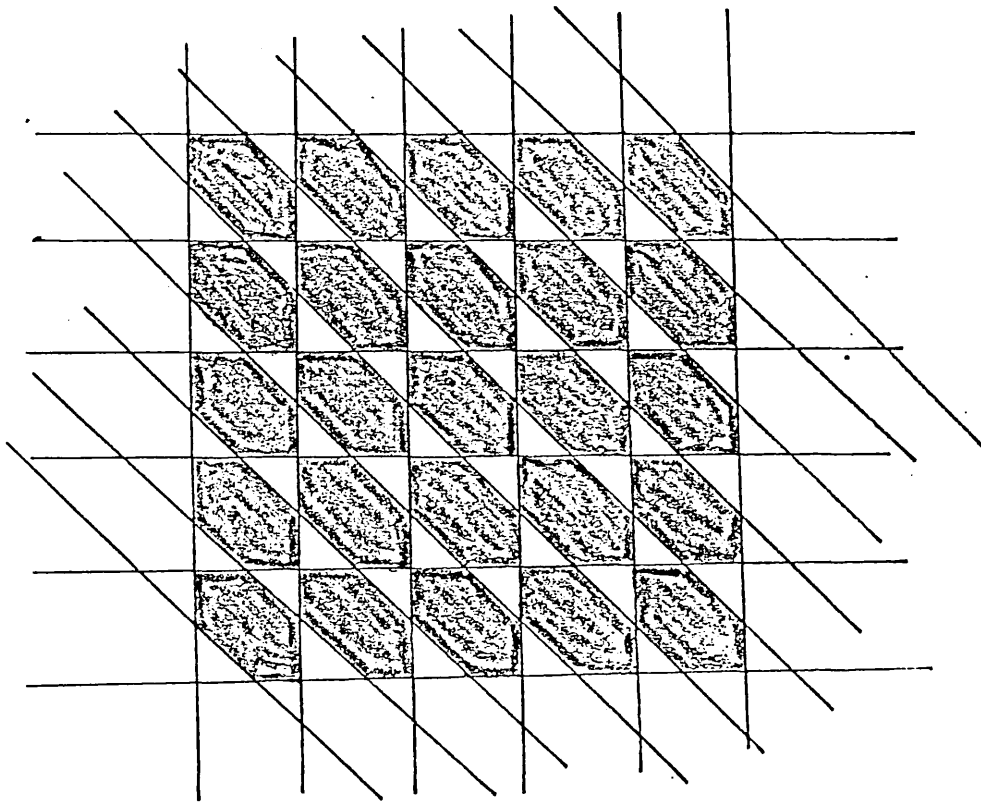
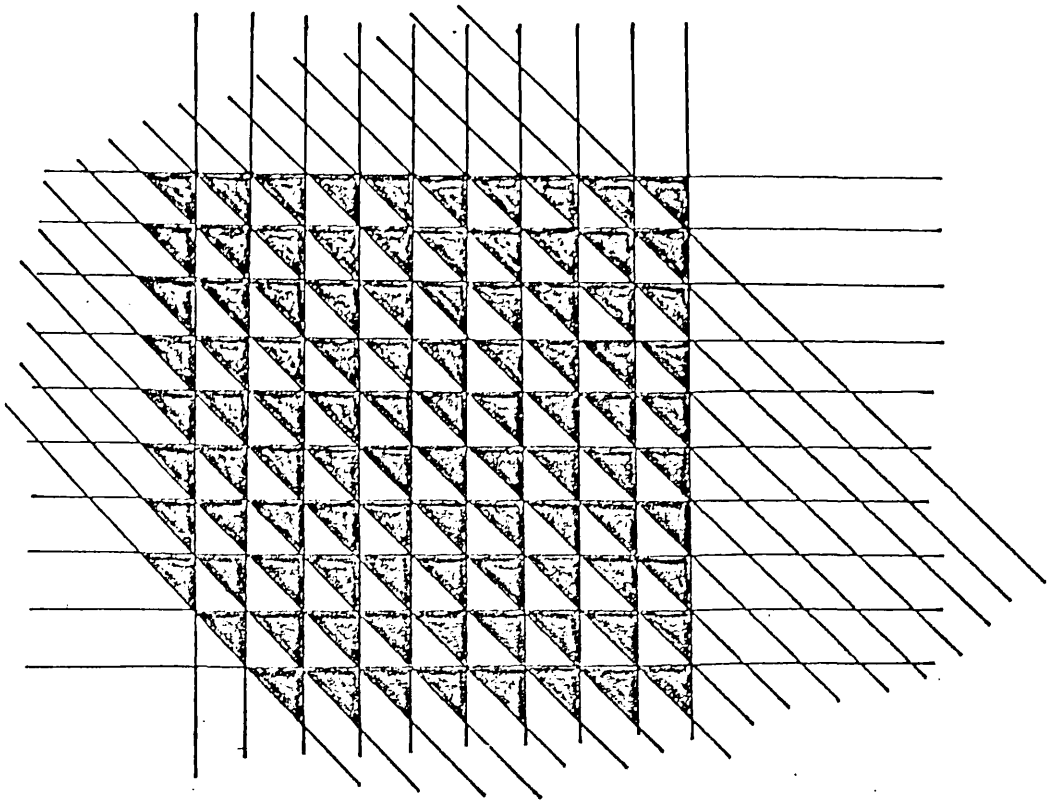
- (i) they do not overlap, except at corners, and
- (ii) each untiled area of the plane has a boundary consisting of at least three sides of tiles, and

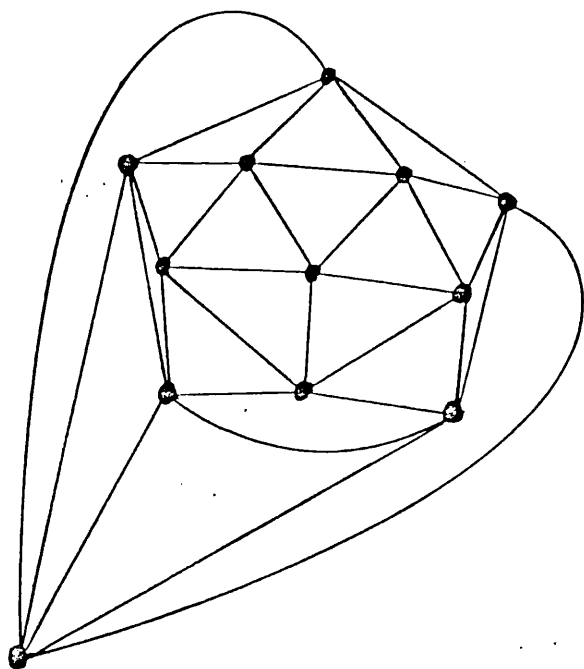
- (iii) each corner coincides with at least 2 others
(respectively 1 other) from different tiles.

This result is not true for arrangements of infinite collections of tiles; (see the illustrations over the page).

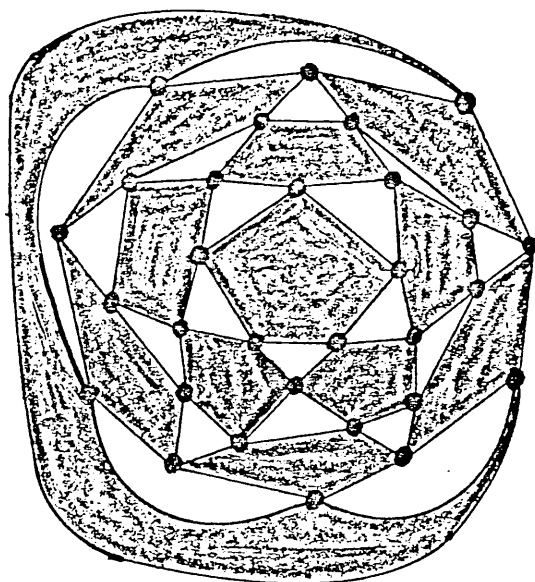
On the page after that are representations of two planar, regular, uniform hypergraphs:

- (VII) A 2-graph of degree 5. (This is the graph of the icosahedron.)
- (VIII) A 5-graph of degree 2.





(VII)



(VIII)

6.7: The Face-chromatic Number.

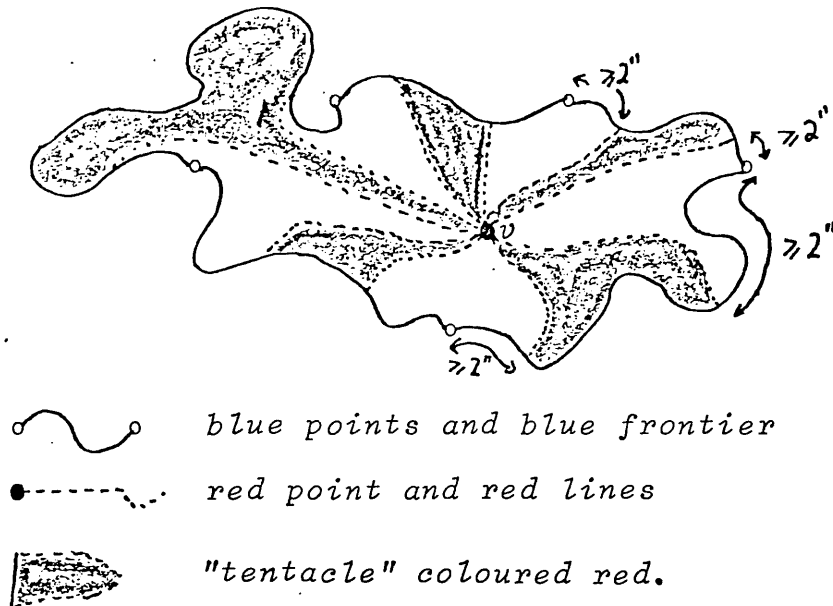
The edge e of the connected hypergraph H is called an *isthmus* if $H-e$ is not connected. We call by the name *hypermap* any plane imbedding of any hypergraph which is connected and without an isthmus.

A *face-colouring* of a hypermap is an assignment of colours, one to each of the faces of the hypermap, in such a way that no edge has all its incident faces coloured the same. (A face and an edge are incident if the intersection of the closure of the face and the disc representing the edge contains a curve of positive length.) The smallest number of colours needed for a face-colouring of the hypermap H is the *face-chromatic number* $\chi^*(H)$. (This number must exist since a hypermap has no isthmus.)

We shall show that the face-chromatic number $\chi^*(H)$ of the hypermap H is the same as the chromatic number $\chi(H^*)$ of an associated hypergraph H^* . Let us construct H^* :

A *side* of an edge of H is a part of its boundary joining two vertices of the edge and not meeting any other vertices. Without loss of generality, we may suppose that all the sides are Jordan arcs, and no two vertices are closer than five inches. Let us think of H as a set of blue points (vertices) and blue discs (edges) lying in a white plane.

Choose a face of H , and choose a point v lying in that face. Colour v red. The boundary of our face consists of a number of sides. Draw two red lines from v to each of these sides in such a way that no two red lines intersect except at v , and the distance (along the side) from a vertex to the point of intersection of a red line and a side is at least two inches. (See diagram.)



Each pair of red lines together with part of a side forms the boundary of a "tentacle" from v to that side. Colour the tentacle (and recolour its side) red. *

Do this for each of the faces of H . Let us recolour white those blue areas lying (strictly) within an inch of a vertex of H ; and, finally, let us recolour red the remaining blue areas.

*If an edge has more than one side on a face, we must delete all but one of the corresponding tentacles.

We are left with a red drawing on a white plane.
This is H^* .

Proposition 6.7.1: H^* is a plane imbedding of a
hypergraph.

Proof: The vertices of the hypergraph are represented
by those points which we chose, one lying in
each face of H . Consider the disc representing one
of the edges of H . In constructing H^* we removed (or
recoloured white) some parts of the disc lying within
neighbourhoods of its vertices; we also added
"tentacles" to the disc. These altered discs
represent the edges of our new hypergraph.

It remains to verify that no edge of this new
hypergraph contains fewer than two vertices: This
follows immediately from the fact that H contains no
isthmus. It is obvious that H^* is a plane imbedding. //

Theorem 6.7.2: Let H be a hypermap, and let H^* be
constructed as described above. Let
 H^* also denote the hypergraph represented by the
plane imbedding H^* . Then

$$\chi^*(H) = \chi(H^*).$$

Proof: There are obvious one-to-one correspondences
both between the faces of H and the vertices

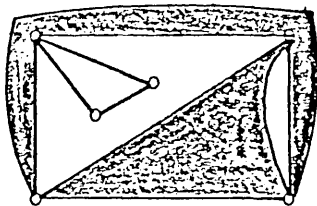
of H^* , and between the edges of H and the edges of H^* .

What we must check is that the vertices of any edge of H^* correspond with the faces surrounding the corresponding edge of H .

This is evident from our construction of the "tentacles". //

To translate Burstein's theorem (Theorem 6.3.1) into a result concerning the face-chromatic number of a hypermap, we must know what features of H will give rise to 2-edges in H^* .

From the definition of hypermap, we know that H contains no vertices of degree 1 (for the edge containing any such vertex would be an isthmus). It follows that any edge containing at least three vertices must be incident with at least three distinct faces. (It is not, however, true that each r -edge ($r > 3$) is incident with r distinct faces. The diagram



alongside is a hypermap with a 4-edge incident with only three faces.)

Thus we see that any 2-edge in H^* must arise from a 2-edge in H . We may deduce:

Corollary 6.7.3: If H is a hypermap with at most two 2-edges, $\chi^*(H) \leq 2$. //

CHAPTER 7

NOTES ON PREVIOUS CHAPTERS

"Afterthoughts" would not be an apt title for this chapter. Rather in the nature of an appendix, our final chapter will comprise results and references not directly relevant to our main thesis.

Chapter 4: We characterized those hypergraphs H whose chromatic numbers attained the bound $1+D(H)$. A tighter (in general) upper bound for the chromatic number is $1+\Delta(H)$; but it seems to be a difficult problem to establish when this bound is attained.

One approach to this problem (Lovász (1968) and Gardner (1975)) is to restrict the class of hypergraphs under consideration. A suitable restriction is to insist that the vertices of maximum valency be *normal*. (A vertex v of valency δ is *normal* if, whenever the set $\{e_1, e_2, \dots, e_\delta\}$ is a v -star, then

$$\bigcup_{i=1}^k e_i = \bigcup_{v \in e} e.$$

That is, the union of any v -star contains all the vertices adjacent to v .)

Theorem (Gardner (1975)): Let H be a connected hypergraph with the property that every vertex of valency $\Delta(H)$ is normal. Then

$$\chi(H) = 1 + \Delta(H) \text{ if, and only if,}$$

- i) $\Delta(H) = 2$, and H is an odd cycle graph with perhaps some multiple edges,

or

- ii) H contains a subhypergraph whose edges are all the k -subsets of a $(\Delta(k-1) + 1)$ -set of vertices. (Δ is $\Delta(H)$ and k is the minimum cardinality of an edge.) //

We mentioned no lower bounds for $\chi(H)$ in Chapter 4; let us remedy that here. The following bound was derived for graphs by Bondy (1969); and, as Mitchem (1974) has pointed out, Bondy's proof holds also for hypergraphs.

Theorem (Bondy 1969), (Mitchem 1974):

Let H be a hypergraph with n vertices of valencies $\delta_1 \geq \delta_2 \geq \dots \geq \delta_n$.

Define σ_j recursively by:

$$\sigma_1 = n - \delta_1,$$

$$\sigma_j = n - \delta_r, \text{ where } r = r(j) = \sum_{i=1}^{j-1} \sigma_i + 1.$$

If k satisfies $\sum_{j=1}^{k-1} \sigma_j < n$, then $\chi(H) \geq k$. //

Nordhaus and Gaddum (1956) found bounds for the sum and product of the chromatic numbers of a simple graph and its (graph theoretic) complement. Mitchem (1974) defined the *complement* \bar{H} of the hypergraph H to be the hypergraph with vertex set $V(\bar{H}) = V(H)$ and edge family (it is actually a set)

$$E(\bar{H}) = \{ e \subset V(H) : e \notin E(H) ; |e| > 2 \}.$$

(If G is a graph, it is not the case that \bar{G} is the (graph theoretic) complement of G .)

Theorem (Mitchem(1974)):

Let H be a hypergraph with n vertices:

$$\chi(H) \cdot \chi(\bar{H}) \geq n \quad \text{and}$$

$$\chi(H) + \chi(\bar{H}) \leq \left\{ \frac{3n}{2} \right\}. \quad //$$

Chapter 5:

In Section 5.6 we described an iterative algorithm for expressing the chromatic polynomial $\psi(H; \lambda)$ of a hypergraph as a sum of chromatic polynomials of graphs.

Our attention has since been drawn to a different procedure.

In his Ph.D. thesis, Václav Chvátal (1970) describes a procedure which, when iterated, enables us to express the chromatic polynomial of a hypergraph as a sum of chromatic polynomials of hypergraphs without edges. (The chromatic polynomial of such a hypergraph with n vertices is simply λ^n .)

Let $H = (V, E)$ be a hypergraph; let e be any edge, and let u be any vertex contained in e . For any subset $S \subset V$ let us write:

$$S|e = \begin{cases} (S-e) \cup \{u\} & \text{if } S \cap e \neq \emptyset \\ S & \text{if } S \cap e = \emptyset. \end{cases}$$

We also write: $E|e = \{f|e : f \in E - \{e\}\}$. Now let:

$$\begin{aligned} H|e &= (V|e, E|e) \quad \text{and} \\ H-e &= (V, E - \{e\}). \end{aligned}$$

We remark that $H|e$ may not be a hypergraph: Some elements of $E|e$ may have cardinality less than 2. In such a case, we formally define $\psi(H|e; \lambda) = 0$.

Theorem (Chvátal (1970)): With the above notation:

$$\psi(H; \lambda) = \psi(H-e; \lambda) - \psi(H|e; \lambda).$$

Chapter 6: In Section 6.7, we described the construction of a plane imbedding of a hypergraph H^* from a hypermap H . (Since there is no danger of confusion, we shall henceforth refer to both a hypergraph and its plane imbedding - when it is clear which imbedding is intended - by the same name.)

Two hypergraphs, (V,E) and (U,F) , are *isomorphic* if there are one-to-one and onto functions

$f : V \rightarrow U$ and $g : E \rightarrow F$ such that

$v \in e$ if, and only if, $f(v) \in g(e)$ for each $v \in V$ and for each $e \in E$.

When G is a *map* (a hypermap is a map if it is a plane imbedding of a graph) our construction yields a graph G^* which is isomorphic to the *geometric-dual* of G . (See Wilson (1972), page 72.)

When H is a hypermap, *no edge of which has more than one side on any single face*, some of the relationships between maps G and G^* , hold also between H and H^* :

- i) The number of faces of H = the number of vertices of H^* .
- ii) The number of vertices of H = the number of faces of H^* .
- iii) The number of edges of H = the number of edges of H^* .

(Moreover, there is a natural correspondence between the edges of H and H^* under which corresponding edges

contain the same number of vertices.)

The hypergraph H^* cannot contain an isthmus. (This may be proved by noting that the removal of an edge from H^* is, in a sense, equivalent to the contraction of the corresponding edge in H to a single point - this involves identification of vertices - and its subsequent deletion. Removal of an edge cannot disconnect H^* since the contraction process cannot disconnect H .) It follows that H^* is a hypermap, and so we may construct $(H^*)^*$; denote it H^{**} .

iv) H^{**} is isomorphic to H .

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