UNITARY ρ -DILATIONS AND THE HOLBROOK RADIUS FOR BOUNDED OPERATORS ON HILBERT SPACE

By

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ABSTRACT

In this thesis we deal with the theory of unitary ρ-dilations of bounded operators on a Hilbert space H, as developed by Sz.Nagy and Foias, and a related functional on B(H), the algebra of bounded linear operators on H.

In the first Chapter we consider the classes $\mathcal{C}_\rho$, $\rho>0$, of operators possessing a unitary ρ-dilation, and obtain their basic properties , using an approach which adapts itself to a unified treatment. Next in Chapter 2, we examine the behaviour of the sequence of powers of an element T of $\mathcal{C}_\rho$, ρ arbitrary and positive, and we show that the sequence

$$\{\|T^n x\|\}_{n=1}^{\infty}$$

converges to a non-negative limit, less than or equal to

$$\rho^{\frac{1}{2}}\|x\| ,$$

for all x in H.

This is a generalization of a result by M.J.Crabb in which he considers the special case $\rho=2$. We then give an intrinsic characterization of the elements x in H, for which

$$\lim \|T^n x\| = \rho^{\frac{1}{2}}\|x\| ,$$

and obtain various results concerning the structure of operators $T\varepsilon\mathcal{C}_\rho$ which satisfy

$$\|T^N x\| = \rho\|x\| ,$$

for some $N \varepsilon \mathbb{N}$, $x \varepsilon H$.

For every $\rho>0$, the classes $\mathcal{C}_\rho$, turn out to be balanced, absorbing sets of operators which contain the zero operator, and hence a generalized Minkowski functional may be unambiguously defined on them by

$$w_\rho(T) = \inf\{\alpha > 0: \frac{1}{\alpha}T \in \mathcal{C}_\rho\} .$$

This functional, usually referred to in the literature as the Holbrook radius of T, plays a very important role in the study of unitary ρ-dilations, since the elements T of $\mathcal{C}_\rho$ are characterized by

$$w_\rho(T) \leq 1.$$

The basic properties of the Holbrook radius for a bounded operator are studied in Chapter 3. A number of new results concerning the Holbrook radius of nilpotent operators of arbitrary index greater than 2 are obtained which enable us to have a clearer view of the general structure of the $\mathcal{C}_\rho$ classes, in a unified framework.

ORIGINAL RESULTS

(i) Theorem 1.1. The equivalence of (iii) and (iv) is not in print.

(ii) Theorem 1.5. Part (iv) is new.

(iii) Corollary 1.6 is new.

(iv) Theorem 2.1. Most of this result is contained in [8] and [27]. The proof given is new and was prepared without knowledge of the work of Eckstein and Mlak.

(v) Corollary 2.2. The case $\rho = 2$ is known.

(vi) Corollary 2.4 is new.

(vii) Corollary 2.5. New proof.

(viii) Corollary 2.6 is new.

(ix) Theorem 2.7 is new.

(x) Corollary 2.9 is new.

(xi) Theorem 2.10 is new.

(xii) Theorem 3.1. New proof.

(xiii) Remark I on p.82 is new

(xiv) The proof of Lemma 1.3 given on p.83 is new.

(xv) Theorems 3.4 and its generalization Theorem 3.5 are both new.

(xvi) Proposition 3.6 is new.

(xvii) Theorem 3.7 is new.

(xviii) The computational results in the Appendix are all new.

NOTATION AND TERMINOLOGY

Throughout this thesis, the letters $\mathcal{H}$, $\mathcal{K}$, etc. will stand for infinite dimensional separable complex Hilbert spaces, unless otherwise stated. As usual, $(. , .)_{\mathcal{H}}$ or $(. , .)$ will denote the inner product on the Hilbert space $\mathcal{H}$. Vectors will be denoted by x , f , g , etc..

In what follows we shall be concerned with operators, that is bounded linear transformations from a Hilbert space into itself. Operators will be denoted by capital letters T , S , etc. , while $\mathcal{B}(\mathcal{H})$ will stand for the algebra of operators on a Hilbert space $\mathcal{H}$.

For an arbitrary operator T in $\mathcal{B}(\mathcal{H})$ we use the standard notation and terminology , as well as the following : spectrum spT, approximate point spectrum apspT, spectral radius $r(T) = \sup\{|\lambda| : \lambda \in spT\}$, numerical range $W(T) = \{(Tx , x) : \|x\| = 1\}$ and numerical radius $w(T) = \{\sup|\lambda| : \lambda \in W(T)\}$.

Following Sz. Nagy and Foias we introduce the important concept of a unitary ρ - dilation. We say that an operator T in $\mathcal{B}(\mathcal{H})$ possesses a <u>unitary</u> <u>ρ - dilation</u> for some $\rho > 0$, if on $\mathcal{H}$

$$T^n = \rho P U^n , \quad n = 1,2,\dots ,$$

where U denotes a unitary operator on a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ as a subspace, and P is the orthogonal projection from $\mathcal{K}$ onto $\mathcal{H}$. We indicate this by $T \in \mathcal{C}_\rho(\mathcal{H})$ or simply $T \in \mathcal{C}_\rho$, when there is no need for reference to the underlying Hilbert space.

The theory of unitary ρ - dilations for operators was developed to provide a unified framework for two classical dilation theorems which may be stated as follows :

1. $T \in \mathcal{C}_1$ iff T is a contraction (i.e. $\|T\| \leq 1$),

and

2. $T \in \mathcal{C}_2$ iff T is a numerical radius contraction (i.e $w(T) \leq 1$).

The first of these results is due to Sz. Nagy and Foias ("Sur les contractions de l' espace de Hilbert ", Acta Sci. Math. (Szeged) 19 (1958), pp. 26-46.) and the second to C.A. Berger ("A strange dilation theorem", Notices A.M.S. 12, 590 (1965).)

CHAPTER 1

THE BASIC PROPERTIES OF THE $\mathcal{C}_\rho$ CLASSES

In this Chapter we present a unified treatment of the $\mathcal{C}_\rho$ classes $\rho>0$, consisting of operators possessing a unitary ρ-dilation. Theorem 1.1, gives necessary and sufficient conditions for an operator T to belong to a $\mathcal{C}_\rho$ class for some $\rho>0$. The properties of such operators are then shown to follow immediately from the definitions and Theorem 1.1. We then obtain some of the " topological " properties of the $\mathcal{C}_\rho$ classes and we conclude this Chapter by proving that an operator possessing a unitary ρ-dilation is similar to a contraction.

Let T be a bounded operator on a Hilbert space H and let $\rho > 0$. The following theorem gives necessary and sufficient conditions for T to belong to $\mathcal{C}_\rho(H)$.

THEOREM 1.1 [35], [42]

The following are equivalent :

(i) $T \in \mathcal{C}_\rho(H)$

(ii) $(\rho-2)\|zTh\|^2 - 2(\rho-1)\operatorname{Re} z(Th,h) + \rho\|h\|^2 \geq 0$, $h \in H$, $|z| \leq 1$.

(iii) $\begin{cases} (\rho-2)\|Th\|^2 - 2(\rho-1) \operatorname{Re} z(Th,h) + \rho\|h\|^2 \geq 0 , & h \in H, |z| = 1 . \\ r(T) \leq \min\{1,\rho\} \end{cases}$

(iv) $\| T\{(\rho-1)T - \rho zI\}^{-1} \| \leq 1$, $|z| \geq 1$.

(v) $\begin{cases} (\rho-2)\|Th\|^2 - 2|\rho-1| \, |(Th,h)| + \rho\|h\|^2 \geq 0 \\ r(T) \leq \min\{1,\rho\} \end{cases}$

PROOF

The proof of the equivalence of (i) and (ii) can be found in [42], Chapter I, section 11, Theorem 11.1 .

(ii) => (iii)

It is clear that (ii) implies the first part of (iii) . It thus remains to prove that when (ii) is satisfied, then

$$r(T) \leq \min\{1,\rho\} .$$

We in fact show that (ii) implies :

$$r(T) \leq 1 , \quad \rho > 0$$

and

$$r(T) \leq \rho , \quad 0 < \rho < 1$$

and this is clearly equivalent to

$$r(T) \leq \min\{1,\rho\} , \quad \rho > 0 .$$

Rewrite (ii) in the equivalent form

$$\operatorname{Re}((I - zT)h,h) \geq (1 - \frac{\rho}{2}) \,\| (I - zT)h \|^2, \; h \in H \,, \; |z| \leq 1 \,, \tag{1.1}$$

and assume first that

$$r(T) > 1 \,.$$

Then we can find $\mu \in spT$, such that

$$|\mu| = r(T) > 1 \,. \tag{1.2}$$

Since μ is in the boundary of spT, μ must be an approximate eigenvalue of T, ([17] , problem 63) that is, there are unit vectors h_n in H, $n = 1,2,\ldots$, such that

$$g_n = (T - \mu I)h_n \to 0 \quad \text{as } n \to \infty \,. \tag{1.3}$$

Let now

$$0 < \varepsilon \leq |\mu| - 1$$

and

$$z = \frac{1+\varepsilon}{\mu} \,.$$

Then, clearly

$$|z| \leq 1$$

and

$$(I - zT)h_n = -\varepsilon h_n - z g_n \,.$$

(1.1) with $h = h_n$, $n = 1,2,\ldots$, then reads :

$$-\varepsilon - \operatorname{Re}(h_n, zg_n) \geq (1 - \frac{\rho}{2}) \{\varepsilon^2 + 2\operatorname{Re}(h_n, zg_n) + |z|^2 \,\|g_n\|^2\} \,. \tag{1.4}$$

Letting now $n \to \infty$ in (1.4), it follows that

$$-\varepsilon \geq (1 - \frac{\rho}{2})\,\varepsilon^2 \,,$$

or equivalently, that

$$(1 - \frac{\rho}{2}) \leq - \varepsilon^{-1} \tag{1.5}$$

and this inequality becomes false if we choose ε sufficiently small if $\rho > 2$, and is never true when $0 < \rho \leq 2$, since $\varepsilon > 0$. Thus

$$r(T) \leq 1, \quad \rho > 0 . \tag{1.6}$$

To prove the second assertion, namely that

$$r(T) \leq \rho, \quad 0 < \rho < 1,$$

assume as before that

$$r(T) > \rho, \quad 0 < \rho < 1 .$$

Then there exist a sequence $\{h_n\}_{n=1}^{\infty}$ of unit vectors in H, and μ in spT, such that

$$|\mu| = r(T) > \rho, \tag{1.7}$$

and

$$g_n = (T - \mu I)h_n \to 0 \quad \text{as} \quad n \to \infty .$$

Let now ε be such that

$$0 < \varepsilon \leq |\mu| - \rho \tag{1.8}$$

Then

$$\varepsilon + \rho - 1 \leq |\mu| - 1 = r(T) - 1 \leq 0 \tag{1.9}$$

as in any case, $r(T) \leq 1$ by (1.6) .

Put

$$z = \frac{\rho + \varepsilon}{\mu} . \tag{1.10}$$

. Then,

$$|z| \leq 1 \quad \text{by } (1.8),$$

and

$$(I - zT)h_n = -(\varepsilon + \rho - 1)h_n - zg_n, \quad n = 1,2,\ldots,$$

(1.1) now with $h = h_n$, $n = 1,2,\ldots$, and z as in (1.10) reads :

$$-(\varepsilon + \rho - 1) - \mathrm{Re}(h_n, zg_n) \geq (1 - \tfrac{\rho}{2})\{(\varepsilon + \rho - 1)^2 + 2(\varepsilon + \rho - 1)\,\mathrm{Re}(h_n, zg_n) + |z|^2 \, \|g_n\|^2\} \qquad (1.11)$$

Letting $n \to \infty$ in (1.11), it follows that

$$-(\varepsilon + \rho - 1) \geq (1 - \tfrac{\rho}{2})(\varepsilon + \rho - 1)^2$$

or equivalently , since $\varepsilon + \rho - 1 \leq 0$ by (1.9), that

$$(1 - \tfrac{\rho}{2})(\varepsilon + \rho - 1) \geq 1 \qquad (1.12)$$

or what is the same thing, that

$$\varepsilon + \rho - 1 \geq \frac{2}{2 - \rho} \,. \qquad (1.13)$$

But, as $0 < \rho < 1$,

$$\frac{2}{2 - \rho} > 1,$$

and hence

$$\varepsilon + \rho - 1 > 1 \,. \qquad (1.14)$$

The inequality given by (1.14) is thus seen to contradict (1.9), whence

$$r(T) \leq \rho, \quad 0 < \rho < 1,$$

as required.

The implication (ii) => (iii) has thus been established.

(iii) => (iv)

Rewrite (iii) in the equivalent form

$$\begin{cases} \| \{(\rho-1)T - \rho z I\}h \| \geq \| Th \| , \ h \in H, \ |z| = 1 \\ r(T) \leq \min \{1,\rho\} \end{cases} \quad (1.15)$$

We have :

$$r\left(\frac{(\rho-1)T}{\rho z} \right) < 1 , \quad \rho > 0, \ |z| = 1 \quad (1.16)$$

as $1-\rho < 1$ if $0 < \rho < 1$

and $\frac{\rho-1}{\rho} < 1$ if $\rho \geq 1$

and $r(T) \leq \min \{1,\rho\}$.

Hence the operator

$$C(z) = T \{ (\rho-1)T - \rho z I\}^{-1}, \quad |z| = 1 ,$$

exists as a bounded operator and (iii) is seen to imply that

$$\| C(z) h \| \leq \|h\| , \ h \in H , \ |z| = 1 . \quad (1.17)$$

Also, as

$$r\left(\frac{(\rho-1)T}{\rho z} \right) < 1 \quad \text{if} \quad |z| \geq 1 , \quad (1.18)$$

$C(z)$ is an operator valued, analytic function of the complex variable z, for $|z| \geq 1$,

and $C(z) \to 0$ as $z \to \infty$.

Hence, an application of the maximum modulus principle for analytic functions of a complex variable, gives:

$$\| C(z)h \| = \| T\{(\rho-1)T - \rho zI\}^{-1} h \| \leq \| h \|, \; h \in H, \; |z| \geq 1. \tag{1.19}$$

Thus, (iii) implies that

$$\| T\{(\rho-1)T - \rho zI\}^{-1} h \| \leq \| h \|, \; h \in H, \; |z| \geq 1,$$

and consequently that

$$\| T\{(\rho-1)T - \rho zI\}^{-1} \| \leq 1, \; |z| \geq 1. \tag{1.20}$$

Hence

(iii) => (iv).

(iv) => (ii)

(iv) is easily seen to be equivalent to

$$\| zT\{(\rho-1)zT - \rho I\}^{-1} g \| \leq \| g \|, \; |z| \leq 1, \; g \in H, \tag{1.21}$$

and hence, since the operator

$$\{(\rho-1)zT - \rho I\}^{-1}$$

is in particular 'onto' H, it follows that

$$\| zTh \| \leq \| (\rho-1)zTh - \rho h \|, \; |z| \leq 1, \; h \in H, \tag{1.22}$$

and (ii) is easily recovered from (1.22) after some very obvious calculations.

Thus, (iv) implies (ii).

(iii) <=> (v)

(iii) => (v)

Let $\bar{z} = \frac{(Th\,,h)}{|(Th,h)|}$ if $\rho \geq 1$

and $\bar{z} = \frac{-(Th,h)}{|(Th,h)|}$ if $0 < \rho < 1$,

in (iii) if $Th \not\perp h$.

(If $(Th,h) = 0$, then clearly (iii) and (v) are equivalent)

(v) => (iii)

Observe that if $|z| = 1$,

$$-|(Th,h)| \leq \text{Re}\,(zTh,h) \leq |(Th,h)|\,,\quad h \in H \tag{1.23}$$

and hence, if $\rho \geq 1$ and (v) holds, that is

$$\begin{cases} (\rho-2)\,\|Th\|^2 - 2(\rho-1)|(Th,h)| + \rho\|h\|^2 \geq 0\,, & h \in H \\ r(T) \leq 1 \end{cases}$$

then also,

$$\begin{cases} (\rho-2\;\|Th\,\|^2 - 2(\rho-1)\text{Re}(z\,Th,h) + \rho\,\|\,h\,\|^2 \geq 0\,, & h \in H\,,\ |z| = 1 \\ r(T) \leq 1 \end{cases}$$

holds.

On the other hand, if $0 < \rho < 1$ and (v) is true, that is

$$\begin{cases} (\rho-2)\,\|Th\|^2 - 2(1-\rho)|(Th,h)| + \rho\|h\|^2 \geq 0\,, & h \in H \\ r(T) \leq \rho\,, \end{cases}$$

then as

$$- 2(\rho-1)\ \mathrm{Re}\, z\,(Th,h) + 2(1-\rho)|(Th,h)|$$

$$= 2(1-\rho)\ [\mathrm{Re}\, z\,(Th,h) + |(Th,h)|] \geq 0\ ,$$

$$\begin{cases} (\rho-2)\|Th\|^2 - 2(\rho-1)\mathrm{Re}\, z\,(Th,h) + \rho\|h\|^2 \geq 0\ , h \in H\ , |z| = 1 \\ r(T) \leq \rho \end{cases}$$

is also true.

Hence, (v) => (iii), and the proof is seen to be complete.

OBSERVATIONS

1. Let T be an operator of class $\mathcal{C}_\rho$, for some $\rho > 0$, on a Hilbert space H.

Then with the usual notation:

$$T^n = \rho\, PU^n\ ,\quad n = 1,2,\ldots\ .$$

Thus, for any $h, h' \in H$ and every $n = 1,2,\ldots$,

$$(T^n h, h') = (\rho\, PU^n h, h') = \rho(U^n h, Ph')$$

$$= (U^n h, \rho h') = (h, \rho\, U^{*n} h')$$

$$= (Ph, \rho U^{*n} h) = (h, \rho P\, U^{*n} h')\ .$$

On the other hand, as

$$(T^n h, h') = (h, T^{*n} h')\ ,\ h, h' \in H\ ,\quad n = 1,2,\ldots\ ,$$

it follows that

$$T^{*n} = \rho\, P\, U^{*n}\ ,\quad n = 1,2,\ldots\ .$$

Hence, the $\mathcal{C}_\rho$ classes, for $\rho > 0$, are $*$-closed. More precisely, if $T \in \mathcal{C}_\rho(H)$, for some $\rho > 0$, then $T^* \in \mathcal{C}_\rho(H)$ also.

2. If $T \in \mathcal{C}_\rho(H)$ for some $\rho > 0$, then since any power of a unitary operator is again unitary,

$$T^k \in \mathcal{C}_\rho(H) \quad \text{for any positive integer } k.$$

3. In the course of the proof of theorem 1.1, we obtained that for an operator T of class $\mathcal{C}_\rho(H)$,

$$r(T) \leq 1.$$

This result can also be obtained as follows ; if $T \in \mathcal{C}_\rho(H)$ for some $\rho > 0$, then, with the usual notation

$$T^n = \rho P U^n, \quad n = 1,2,\ldots \quad , \text{ and hence}$$

$$\| T^n \| = \rho \| P U^n \| \leq \rho, \qquad n = 1,2,\ldots \quad .$$

That is, an operator of class $\mathcal{C}_\rho$, is

POWER BOUNDED.

In particular,

$$\| T^n \|^{1/n} \leq \rho^{1/n}, \quad n = 1,2,\ldots ,$$

and consequently

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} \leq \lim_{n\to\infty} \rho^{1/n} = 1 .$$

Thus, an operator possessing a unitary ρ-dilation for some $\rho > 0$, has spectral radius not greater than 1. Later on we shall show by means of a counter example, that there exist power bounded operators not belonging to any $\mathcal{C}_\rho$ class, but that if T is an operator with spectral radius strictly less than 1, then T belongs to $\mathcal{C}_\rho$, for some ρ sufficiently large.

4. If $T \in \mathcal{C}_\rho(H)$ for some $\rho > 0$, then, any restriction of T to an invariant subspace M of H is also of $\mathcal{C}_\rho$ class. The validity of this statement follows immediately from proposition (ii) of theorem 1.1., for if (ii) holds for any $h \in H$, it will in particular hold for any h belonging to an invariant subspace M of T. Thus, if $T \in \mathcal{C}_\rho(H)$ and M is a subspace of H invariant under T, then

$$T\big|_M \in \mathcal{C}_\rho = \mathcal{C}_\rho(M) .$$

5. Proposition (iv) of theorem 1.1. is due to Chandler Davis [6] and will be referred to as the 'C. Davis criterion for membership in a $\mathcal{C}_\rho$ - class'.

6. The classes $\mathcal{C}_\rho$, $\rho > 0$ are closed under unitary equivalence. More precisely, if S and T are a pair of unitarily equivalent operators, then

$$S \in \mathcal{C}_\rho \quad \text{implies} \quad T \in \mathcal{C}_\rho .$$

PROOF

Let $S = U^{-1} T U$ for some unitary operator U, and suppose $S \in \mathcal{C}_\rho$, for some $\rho > 0$.

Then, according to proposition (iv) of theorem 1.1.,

$$\| S\{(\rho-1) S - \rho z I\}^{-1} \| \leq 1 , \quad |z| \geq 1 .$$

But, $\quad S = U^{-1} T U$, and hence,

$$\| U^{-1} T U\{(\rho-1) U^{-1} T U - \rho z I\}^{-1} \| \leq 1, \quad |z| \geq 1 ,$$

or equivalently,

$$\| U^{-1}[T\{(\rho-1)T-\rho zI\}^{-1}]U \| \leq 1, \quad |z| \geq 1,$$

and as

$\| A \| = \| B \|$, for any pair A, B of unitarily equivalent operators ,

$$\| T\{(\rho-1)T-\rho zI\}^{-1} \| \leq 1, \quad |z| \geq 1,$$

and hence ,

$$T \varepsilon \mathcal{C}_\rho .$$

7. The substitution $\rho = 1$ in (iv) of theorem 1.1. yields

$$\frac{\| T \|}{|z|} \leq 1, \quad |z| \geq 1 .$$

which is clearly equivalent to

$$\| T \| \leq 1 .$$

Thus, the class $\mathcal{C}_1$ admits the following very simple characterization:

$$T \varepsilon \mathcal{C}_1 \iff \| T \| \leq 1.$$

That is $\mathcal{C}_1$ consists precisely of contractions.

On the other hand, substituting $\rho = 2$ in (ii), it follows that

$$T \varepsilon \mathcal{C}_2 \iff \operatorname{Re} z(Th,h) \leq \|h\|^2, h \varepsilon H, \quad |z| \leq 1,$$

and as the last inequality is clearly equivalent to

$$|(Th,h)| \leq \|h\|^2, \quad h \varepsilon H,$$

it follows that the class $\mathcal{C}_2$ consists precisely of all numerical radius contractions, that is operators of numerical radius at most one.

Thus,

$$T \in \mathcal{C}_2 \iff w(T) \leq 1 .$$

8. Substituting T by uT, where u is any complex number of modulus less than or equal to one, in (ii) of theorem 1.1., it follows that if $T \in \mathcal{C}_\rho(H)$ for some $\rho > 0$, then so does zT for any $z \in \mathbf{C}$ with $|z| \leq 1$.

9. The condition.

$$r(T) \leq \min \{1,\rho\}$$

in (iii) and (v) is indispensable.

For, the operator

$$T = \frac{\rho}{\rho-2} I, \ \rho > 2$$

can easily be shown to satisfy the first part of (iii) and (v) but clearly not the second, as $r(T) > 1$ for $\rho > 2$. On the other hand, T cannot belong to any $\mathcal{C}_\rho$ class for $\rho > 2$ as it is not in particular power bounded.

10. Proposition (ii) of theorem 1.1. may be rewritten as follows :

$$\rho \, [\, \|zTh\|^2 - 2\text{Re}\, z(Th,h) + \|h\|^2 \,] + 2 \, [\text{Re}\, z(Th,h) - \|Th\|^2 \,] > 0 \, , h \in H, \ |z| \leq 1 .$$

As the first bracket is clearly non-negative, being equal to

$$\| \, z\,Th - h \, \|^2 ,$$

and the second bracket is independent of ρ, we deduce that

$$\mathcal{C}_\rho \subseteq \mathcal{C}_{\rho'} \quad \text{if} \quad \rho < \rho' .$$

That is the classes $\mathcal{C}_\rho$, $\rho > 0$, form a monotonically increasing family with respect to set theoretic inclusion as ρ increases.

Letting now T be the operator

$$T = \begin{bmatrix} 0 & \rho \\ 0 & 0 \end{bmatrix}$$

on some 2-dimensional Hilbert space, it follows from the C. Davis criterion, for example, that

$$T \in \mathcal{C}_\rho \quad \text{for all} \quad \rho > 0 .$$

But, as $\| T \| = \rho$, T cannot belong to any $\mathcal{C}_\sigma$-class with $\sigma < \rho$. (cf. (iv) of theorem 1.1. again).

Thus, if the underlying Hilbert space is 2-dimensional the inclusion

$$\mathcal{C}_\rho \subseteq \mathcal{C}_{\rho'} \quad \text{for} \quad \rho < \rho'$$

is actually strict.

Later on we shall show that this inclusion is in fact strict in every Hilbert space of dimension at least 2.

We now return to the remark made in observation 3, and we show that if T is an operator with spectrum situated in the interior of the unit disc in the complex plane, then T belongs to $\mathcal{C}_\rho$ for some $\rho > 0$ sufficiently large.

<u>PROPOSITION</u> 1.2 [22]

Let T be an operator on a Hilbert space H, with

$$\mathrm{sp}\, T \subseteq D_1 = \{z \in \mathbb{C} \ : \ |z| < 1\}$$

then

$$T \in \mathcal{C}_\rho \quad \text{for} \quad \rho \quad \text{large enough}$$

<u>PROOF</u>

Observe that if $\mathrm{sp}\, T \subseteq D_1$, then

$$\| (I - zT)\, h \| \geq \varepsilon \ \| h \|$$

for some $\varepsilon > 0$, all $z \in \mathbb{C}$ with $|z| \leq 1$ and all $h \in H$.

But then,

$$(\rho-2) \|(I - zT)h\|^2 + 2\,\mathrm{Re}\,((I - zT)h,h) \geq 0, \quad |z| \leq 1, h \in H,$$

if ρ is sufficiently large.

As the last inequality is clearly equivalent to

$$(\rho-2) \|zTh\|^2 - 2(\rho-1)\mathrm{Re}\, z\,(Th,h) + \rho \|h\|^2 \geq 0, \quad |z| \leq 1, h \in H$$

which is in turn equivalent to

$$T \in \mathcal{C}_\rho,$$

it follows that $T \in \mathcal{C}_\rho$ for ρ large enough .

Thus, if $\mathrm{sp}\, T \subseteq D_1$,

then

$$T \in \mathcal{C}_\infty = \bigcup_{\rho > 0} \mathcal{C}_\rho .$$

Let now T be a bounded, invertible operator of norm greater than 1 , on a Hilbert space H, and consider the operator matrix S on $H \oplus H$ given by

$$S = \begin{bmatrix} 0 & T^{-1} \\ T & 0 \end{bmatrix}.$$

It is a trivial matter to verify, that

$$S^2 = I \quad \text{on} \quad H \oplus H,$$

and $\|S\| > 1$.

We show that S cannot belong to any $\mathcal{C}_\rho$ class with $0 < \rho < \infty$.

<u>PROOF</u>

The following lemma, is essential for our purposes.

Lemma 1.3. [35]

Let T be an involutive operator of class $\mathcal{C}_\rho$, for some $\rho > 0$, on a Hilbert space H.

Then, T is a symmetry, ie. $T = T^* = T^{-1}$.

Proof of Lemma

Observe first that as a result of $T^2 = I$, we have $\|T\| \geq 1$ and hence $\rho \geq 1$, in view of the monotonicity of the $\mathcal{C}_\rho$ classes.

Secondly, we note that, as

$$r(T) = 1 ,$$

the operator

$$I + zT$$

is invertible for any $z \in \mathbb{C}$ with $|z| < 1$.

Thus, if h runs over H, then so does h', where $(I+zT)h' = h$.

Now, since $T \in \mathcal{C}_\rho$,

$$(\rho-2)\|(I-zT)h\|^2 + 2\mathrm{Re}((I-zT)h,h) \geq 0, \ h \in H, \ |z| \leq 1$$

Hence, with $h = (I+zT)h'$, $|z| < 1$, we have

$$(\rho-2)\|(I-z^2T^2)h'\|^2 + 2\mathrm{Re}((I-z^2T^2)h',(I+zT)h') \geq 0, \ h' \in H, \ |z| < 1$$

and by continuity

$$(\rho-2)\|(I-z^2T^2)h'\|^2 + 2\mathrm{Re}((I-z^2T^2)h',(I+zT)h') \geq 0$$

for all $h' \in H$ and all $z \in \mathbb{C}$ with $|z| \leq 1$.

But $T^2 = I$, and hence

$$(\rho-2)|1-z^2|\,\|h'\|^2 + 2\mathrm{Re}(1-z^2)\|h'\|^2 +$$

$$2\mathrm{Re}(1-z^2)(h', zTh') \geq 0, \ h' \in H, \ |z| \leq 1.$$

Taking $z = e^{i\theta}$, $\theta \in \mathbb{R}$, $\|h'\| = 1$, it follows that

$$(\rho-2)\,[\,(1-\cos 2\theta)^2 + \sin^2 2\theta] + 2(1-\cos 2\theta) + 2\mathrm{Re}\,(e^{-i\theta} - e^{i\theta})(h',Th) \geq 0\,,$$

or equivalently ,

$$4(\rho-1)\sin^2\theta + 4\,\mathrm{Im}\,(h', Th')\sin\theta \geq 0$$

for all θ and so T is Hermitian .

The result now follows.

Returning to the proof of the theorem, observe that if S were to belong to some $\mathcal{C}_\rho$ class, $(\rho \geq 1)$ then $\|S\|$ would have to equal to 1 in view of the lemma.

As this clearly contradicts the original assumption we conclude that

$$S \notin \mathcal{C}_\rho \quad \text{for any} \quad \rho > 0\,.$$

In particular, the matrix

$$T = \begin{bmatrix} 0 & 1/2 \\ 2 & 0 \end{bmatrix}$$

which satisfies the condition for the validity of proposition 1.2. cannot belong to $\mathcal{C}_\infty = \bigcup_{\rho>0} \mathcal{C}_\rho$.

On the other hand, it is a trivial matter to verify that T is power bounded.

In fact

$$\|T^n\| = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even}\,. \end{cases}$$

Thus, the matrix given above, provides us with an example of a power bounded operator not belonging to any $\mathcal{C}_\rho$ class, $0 < \rho < \infty$.

Nevertheless, the following theorem holds.

Theorem 1.4. [22]

$\mathcal{C}_\infty = \bigcup_{\rho>0} \mathcal{C}_\rho$, is dense (in the uniform operator topology) in the family of power bounded operators.

PROOF

Let T be a power bounded operator on a Hilbert space H.

Then, in particular

$$r(T) \leq 1 .$$

For any $0 < s < 1$ define the operator T_s on H, by

$$T_s = sT$$

It is then clear, that

$$r(T_s) = s\,r(T) \leq s < 1 ,$$

and hence

$$T_s \in \mathcal{C}_\infty = \bigcup_{\rho>0} \mathcal{C}_\rho \quad , \text{ by proposition 1.2.}$$

On the other hand, it is equally clear that

$$T_s \to T \text{ uniformly , as } s \to 1^- .$$

The proof is thus complete.

We now justify the claim made in observation 10, namely that the classes $\mathcal{C}_\rho$, $\rho > 0$, form a strictly increasing, with respect to set theoretic inclusion, family of ρ , provided that the underlying Hilbert spaces are at least 2-dimensional.

The proof of this statement is proposition 11.3. Chapter I Section 11 of [42] and is included here for the sake of completeness.

To this end, let H be a Hilbert space with

$$\dim H \geq 2 .$$

For each $\rho > 0$, we construct an operator T_ρ on H, such that $T_\rho \in \mathcal{C}_\rho$, and $\| T_\rho \| = \rho$.

Let

$$\{\phi_1, \phi_2, \psi_\nu \ (\nu \in \Omega)\}$$

be an orthonormal basis for H, where Ω is an arbitrary indexing set that could be empty, and define the operator T_ρ on H as follows:

$$T_\rho \phi_1 = \phi_2 \ , \ T_\rho \phi_2 = 0 \ , \ T_\rho \psi_\nu = 0 \ , \ \nu \in \Omega .$$

Thus, T_ρ is the direct sum of the operator

$$\begin{bmatrix} 0 & 0 \\ \rho & 0 \end{bmatrix} \quad \text{on} \quad M = \text{lin}[\phi_1, \phi_2]$$

and the zero operator on $N = \text{lin}[\psi_\nu; \nu \in \Omega]$.

It is evident that we have :

$$\| T_\rho \| = \rho \quad \text{and} \quad T_\rho^n = 0 \ , \quad n \geq 2 .$$

Let K be a Hilbert space with dimension

$$\aleph_0 \cdot \dim H$$

and choose an orthonormal basis for K.

Its elements can be arranged in the following way :

$$\{ \phi'_m \ (m = 0, \pm 1, \ldots) \ ; \ \psi'_{\nu m} \ (\nu \in \Omega, \ m = 0, \pm 1, \ldots)\} .$$

We identify

$$\phi_1' \text{ with } \phi_1,\ \phi_2' \text{ with } \phi_2 \text{ and } \psi_{\nu_0}' \text{ with } \psi_\nu,\ \nu \in \Omega .$$

This identification defines a canonical injection of H into K as a subspace.

We next define a unitary operator U on K as follows ;

$$U\phi_m' = \phi_{m+1}' ,\quad U\psi_{\nu m}' = \psi_{\nu,m+1} ,\quad \nu \in \Omega ,\quad m = 0, \pm 1, \ldots .$$

If as usual P denotes the orthogonal projection of K into H, then

$$\rho P U \phi_1 = \rho P \phi_2 = \rho \phi_2 = T_\rho \phi_1 ,$$

$$\rho P U \phi_2 = \rho P \phi_3' = 0 ,\ \rho P U \psi_\nu = \rho P \psi_{\nu,1}' = 0 ,\ \nu \in \Omega ,$$

and for $n \geq 2$,

$$\rho P U^n \phi_i = \rho P \phi_{i+n}' = 0 ,\quad i = 1,2,\ldots ,$$

$$\rho P U^n \psi_\nu = \rho P \psi_{\nu,n}' = 0 ,\quad \nu \in \Omega .$$

Thus

$$\rho P U^n h = T_\rho^{\,n} h ,\quad n \geq 1 ,\quad h = \phi_1, \phi_2, \psi_\nu ,\ \nu \in \Omega .$$

But, as $\phi_1, \phi_2, \psi_\nu, \nu \in \Omega$, form an orthonormal basis for H, the same relation must hold for any $h \in H$. Hence, U as defined above, is a unitary ρ-dilation of T_ρ on K .

On the other hand, as $\| T_\rho \| = \rho$, it is quite straightforward to show, using the C. Davis criterion for example, that T_ρ cannot belong to any $\mathcal{C}_\sigma$ - class with $\sigma < \rho$.

The next theorem gives some information about the 'topological' properties of the $\mathcal{C}_\rho$ - classes

Theorem 1.5.

(i) The classes $\mathcal{C}_\rho$ are closed in the strong operator topology.

(ii) The classes $\mathcal{C}_\rho$ with $0 < \rho \leq 2$ are convex.

(iii) $\mathcal{C}_\infty = \bigcup_{\rho > 0} \mathcal{C}_\rho$ is not convex.

(iv) For each $\rho > 0$, $\mathcal{C}_\rho$ is a balanced, absorbing set, and

(v) When $T \in \mathcal{C}_\infty$, then

$$E = \{\rho > 0 ; T \in \mathcal{C}_\rho\}$$

is a closed subset of $(0,\infty)$ with respect to the usual topology of $\mathbb{R}$.

PROOF

(i) follows immediately from proposition (ii) of theorem 1.1.

To prove (ii) let $T, S \in \mathcal{C}_\rho(H)$ for some ρ, $0 < \rho \leq 2$.

Then

$$\text{Re}\ ((I - zT)h,h) \geq (1 - \tfrac{\rho}{2}) \|(I - zT)h\|^2 , \quad \text{and}$$

$$\text{Re}\ ((I - zS)h,h) \geq (1 - \tfrac{\rho}{2}) \|(I - zS)h\|^2 ,$$

for all $h \in H$ and all $z \in \mathbb{C}$ with $|z| \leq 1$.

For $\alpha \in (0,1)$ consider the operator

$$\alpha T + (1 - \alpha)S .$$

Then, if $h \in H$ and $|z| \leq 1$, we have :

$$\text{Re}\ [(I - z(\alpha T + (1 - \alpha)S)h,h)] \geq$$

$$(1 - \tfrac{\rho}{2}) [\alpha \|(I - zT)h\|^2 + (1 - \alpha) \|(I - zS)h\|^2] \geq$$

$$(1 - \frac{\rho}{2}) \; \| \alpha(I - zT)h + (1-\alpha)(I - zS)h \|^2 =$$

$$(1 - \frac{\rho}{2}) \; \| (I - z(\alpha T + (1-\alpha)S))h \|^2 ,$$

where we have used the fact, that if

$\lambda , \mu \geq 0$ with $\lambda + \mu = 1$, then

$$\lambda \| x \|^2 + \mu \| y \|^2 \geq \| \lambda x + \mu y \|^2 ,$$

for any pair of vectors x and y in a Hilbert space. Hence, if T and S belong to $\mathcal{C}_\rho(H)$ for some $0 < \rho \leq 2$, then so does $\alpha T + (1-\alpha)S$, for any $\alpha \in (0,1)$.

For the proof of (iii) observe that the operator

$$T = \begin{bmatrix} 0 & 0 \\ \rho & 0 \end{bmatrix}$$

of observation 10, and its adjoint, both belong to $\mathcal{C}_\rho$, for any $\rho > 0$, and hence if $\mathcal{C}_\infty$ were to be convex, then

$$\operatorname{Re} T = \frac{1}{2}(T + T^*) = \frac{1}{2}\begin{bmatrix} 0 & \rho \\ \rho & 0 \end{bmatrix}$$

would have to be in $\mathcal{C}_\infty$ also.

But this is clearly a contradiction, the operator

$\frac{1}{2}\begin{bmatrix} 0 & \rho \\ \rho & 0 \end{bmatrix}$ not being, in particular, power bounded.

This establishes the non-convexity of $\mathcal{C}_\infty$

To prove (iv), observe that for any $\rho > 0$, $\mathcal{C}_\rho$ is clearly a balanced set, in view of observation 7.

On the other hand, if T is a non-zero, bounded operator on a

Hilbert space H, then, since $\frac{T}{\|T\|}$ has norm 1, $\frac{T}{\|T\|} \varepsilon \mathcal{C}_1$ by observation 1, and hence $\frac{T}{\|T\|} \varepsilon \mathcal{C}_\rho$ for any $\rho \geq 1$ by virtue of the monotonicity of the $\mathcal{C}_\rho$-classes. If now $0 < \rho < 1$, we show that

$$\frac{\rho}{(2-\rho)\|T\|} T \varepsilon \mathcal{C}_\rho .$$

To this end, observe that for any $z \varepsilon \mathbb{C}$ with $|z| \geq 1$

$$r\left(\frac{\rho-1}{2-\rho} \cdot \frac{T}{z\|T\|}\right) = \frac{1-\rho}{2-\rho} \frac{r(T)}{|z|\ \|T\|} \leq \frac{1-\rho}{2-\rho} < 1$$

and hence the operator

$$\frac{(\rho-1)T}{(2-\rho)\|T\|} - z\, T ,$$

is boundedly invertible, with

$$\left(\frac{(\rho-1)T}{(2-\rho)\|T\|} - z\, T\right)^{-1} = -\frac{1}{z} \sum_{n=0}^{\infty} \left[\frac{(\ -1)T}{(2-\rho)z\|T\|}\right]^n$$

the latter series converging in norm.

In particular

$$\left\| \frac{\rho T}{(2-\rho)\|T\|} \left(\frac{(\rho-1)\rho T}{(2-\rho)\|T\|} - \rho z I\right)^{-1} \right\|$$

$$= \frac{1}{(2-\rho)\|T\|} \left\| \frac{T}{z} \sum_{n=0}^{\infty} \left[\frac{(\rho-1)T}{(2-\rho)z\|T\|}\right]^n \right\|$$

$$\leq \frac{1}{(2-\rho)\|T\|} \sum_{n=0}^{\infty} \left(\frac{1-\rho}{2-\rho}\right)^n \frac{\|T^{n+1}\|}{\|T\|^n} \leq \frac{1}{2-\rho} \sum_{n=0}^{\infty} \left(\frac{1-\rho}{2-\rho}\right)^n$$

$$= \frac{1}{2-\rho} \cdot \frac{1}{1-\frac{1-\rho}{2-\rho}} = \frac{1}{2-\rho(1-\rho)} = 1 .$$

Thus, for any $0 < \rho < 1$, the operator

$\frac{\rho}{(2-\rho)\|T\|} \cdot T$ satisfies the C. Davis criterion for

membership in a $\mathcal{C}_\rho$ - class, and the proof is seen to be complete.

Finally, to prove (v) let $\rho \notin E$.

Then

$$(\rho-2) \| (I - zT) h_o \|^2 + 2\text{Re}\, ((I - zT)h_o, h_o) < 0$$

for some $h_o \in H$, $|z| \leq 1$, and this inequality remains unchanged if ρ is replaced by $\rho + \varepsilon$ with $\varepsilon > 0$ sufficiently small. Whence, the complement of E is open and consequently E is closed in $(0, \infty)$.

Associating now with every non-zero bounded operator T on a Hilbert space H and every $\rho > 0$ the set

$$W_{\rho,T} = \{\alpha > 0 : \frac{T}{\alpha} \in \mathcal{C}_\rho\}$$

we have the following very important corollary to the theorem.

COROLLARY 1.6.

For an arbitrary but bounded non-zero operator on a Hilbert space H, and any $\rho > 0$.

$$W_{\rho,T} \neq \emptyset .$$

PROOF

We distinguish the two cases :

(a) $0 < \rho < 1$

and (b) $1 \leq \rho < \infty$.

In case (a), as it was shown during the course of the proof of Theorem 1.5., $\frac{\rho}{2-\rho} \cdot \frac{T}{\|T\|} \in \mathcal{C}_\rho$, and hence

$$\frac{2-\rho}{\rho} \|T\| \in W_{\rho,T}$$

and similarly in case (b)

$$\|T\| \in W_{\rho,T}$$

We now prove the following :

Theorem 1.7.

Every operator T in $\mathcal{C}_\infty(H)$ is similar to a contraction.

PROOF

The following lemma is crucial for our purposes.

LEMMA 1.8. [34]

Let T be a bounded operator on a Hilbert space H. Then, a necessary and sufficient condition for T to be similar to a contraction is that there exists a contraction C on a Hilbert space K and bounded operators $A \in \mathcal{B}(H,K)$ and $B \in \mathcal{B}(K,H)$ such that

$$\sum_{n=0}^{\infty} \| T^n - BC^nA \|^2 < \infty \quad . \tag{I}$$

Proof of the lemma

Sufficiency

If $T \in \mathcal{B}(H)$ is similar to a contraction C on a Hilbert space K, then there exists an invertible operator $X \in \mathcal{B}(H,K)$ such that

$$T = X^{-1} C X \ .$$

But then

$$T^n = X^{-1} C^n X \ , \quad n = 0,1,2,\ldots \ ,$$

and hence, letting

$$B = X^{-1} \text{ and } A = X \ ,$$

the condition is seen to be satisfied.

<u>Necessity</u>

Suppose (I) holds.

It will be sufficient to show, that there exists a norm on H equivalent to the original norm, with respect to which T is a contraction.

Define for all $h \varepsilon H$,

$$|h|^2 = \inf\{\|\sum_{n\geq 0} C^n A h_n\|^2 + \sum_{n\geq 0} \|h_n\|^2 : \sum_{n\geq 0} T^n h_n = h\}$$

the infimum being taken over all sequences of vectors $\{h_n\}_{n\geq 0}$ where all the $h_n \varepsilon H$ are zero, except a finite number of them.

It is clear that

$$|h + h'| \leq |h| + |h'| \ , \ h,h' \varepsilon H \ , \text{ and}$$

$$|\alpha h| = |\alpha| \ |h| \ , \ \alpha \varepsilon \mathbb{C} \ , \ h \varepsilon H \ .$$

Thus, the application

$$h \mapsto |h|$$

is a seminorm on H .

Moreover, as

$$h = h + T \cdot 0 + \ldots \ ,$$

we have

$$|h|^2 \leq \| Ah \|^2 + \| h \|^2 ,$$

and hence

$$|h| \leq (\| A \|^2 + 1)^{1/2} \| h \| .$$

Also, if

$$h = \sum_{n \geq 0} T^n h_n ,$$

the Schwarz inequality gives ;

$$\| h \| = \| \sum_{n \geq 0} T^n h_n \| = \| \sum_{n \geq 0} BC^n Ah_n + \sum_{n \geq 0} (T^n - BC^n A) h_n \|$$

$$\leq \| B \| \, \| \sum_{n \geq 0} C^n Ah_n \| + \sum_{n \geq 0} \| T^n - BC^n A \| \, \| h \|$$

$$\leq (\| B \|^2 + \sum_{n \geq 0} \| T^n - BC^n A \|^2)^{1/2} (\| \sum_{n \geq 0} C^n Ah_n \|^2 + \sum_{n > 0} \| h_n \|^2)^{1/2}$$

and hence, taking the infimum of both sides over all finitely non-zero h_n, $n = 0,1,2,\ldots$, it follows that

$$\| h \| \leq (\| B \|^2 + \sum_{n \geq 0} \| T^n - BC^n A \|^2)^{1/2} \, |h| .$$

Thus,

$$h \mapsto |h|$$

is a norm on H, equivalent to the original one, and so, H is complete in the new norm.

Furthermore, if $h = \sum_{n\geq 0} T^n h_n$, it follows that $Th = \sum_{n\geq 1} T^n h_{n-1}$,

and hence, since C is a contraction, we have

$$|Th|^2 \leq \|\sum_{n\geq 1} C^n A h_{n-1}\|^2 + \sum_{n\geq 1} \|h_{n-1}\|^2$$

$$= \|C \sum_{n\geq 0} C^n A h_n\|^2 + \sum_{n\geq 0} \|h_n\|^2$$

$$\leq \|\sum_{n\geq 0} C^n A h_n\|^2 + \sum_{n\geq 0} \|h_n\|^2 \leq |h|^2 .$$

Thus, T is in fact a contraction with respect to the new norm.

We conclude the proof of the Lemma by showing that $|\cdot|$ is a Hilbert space norm, that is $|\cdot|$ satisfies the parallelogram identity.

To this end, let $\varepsilon > 0$ be given and $h, g \in \mathcal{H}$.

We prove that

$$|h+g|^2 + |h-g|^2 = 2\,(|h| + |g|^2) .$$

There exist finitely non-zero sequences $\{h_n\}_{n\geq 0}$ and $\{g_n\}_{n\geq 0}$ for which

$$\sum_{n\geq 0} \|h_n\|^2 + \|\sum_{n\geq 0} C^n A h_n\|^2 < |h|^2 + \varepsilon$$

and

$$\sum_{n\geq 0} \|g_n\|^2 + \|\sum_{n\geq 0} C^n A g_n\|^2 < |g|^2 + \varepsilon$$

where of course

$$\sum_{n\geq 0} Th_n = h \quad \text{and} \quad \sum_{n\geq 0} Tg_n = g .$$

Hence,

$$\sum_{n\geq 0} (\| h_n \|^2 + \| g_n \|^2) + \| \sum_{n\geq 0} C^n A h_n \|^2 + \| \sum_{n\geq 0} C^n A g_n \|^2$$

$$= \frac{1}{2} \sum_{n\geq 0} \| h_n + g_n \|^2 + \frac{1}{2} \sum_{n\geq 0} \| h_n - g_n \|^2$$

$$+ \frac{1}{2} \| \sum_{n>0} C^n A (h_n + g_n) \|^2 + \frac{1}{2} \| \sum_{n\geq 0} C^n A (h_n - g_n) \|^2$$

$$< |h|^2 + |g|^2 + 2\varepsilon \quad .$$

Moreover, as

$$h \pm g = \sum_{n\geq 0} T(h_n \pm g_n) \ ,$$

it follows that

$$|h + g|^2 + |h - g|^2 \leq 2(|h|^2 + |g|^2) \ .$$

The reverse inequality being easily obtained replacing $h + g$ by h and $h - g$ by g, we conclude that H is an inner product space with respect to the new norm and the proof of the lemma is seen to be complete. Returning to the proof of the theorem now, we see that since $T \in \mathcal{C}\rho$, there exists a Hilbert space K containing H and a unitary operator U on K such that

$$T^n = \rho P_H U^n \ , \quad n = 1, 2, \ldots \ ,$$

P_H denoting as usual the orthogonal projection from K onto H. If we now let

$C = U$, A be the canonical injection of H into K and $B = \rho P_H$, then all the summands in (I), except the one corresponding to $n=0$ vanish, and as this non-vanishing term contributes a finite number to this sum, we conclude that T is similar to a contraction in accordance with the lemma.

As a result of theorem 1.7. we have the following :

$$\mathcal{C}_\infty \subsetneq \{\text{ Operators similar to contractions }\}$$

$$\subsetneq \{\text{ Power bounded operators }\}$$

the inclusions being strict, for the operator

$$T = \begin{bmatrix} 0 & 1/2 \\ 2 & 0 \end{bmatrix}$$

is similar to a contraction and is power bounded but (cf. Lemma 1.3), $T \notin \mathcal{C}_\infty$.

On the other hand, Foguel's operator provides us with an example of a power bounded operator, not similar to any contraction.

NOTE

Foguel's operator is defined as follows :

Let H_o be a Hilbertspace with an orthonormal basis $\{e_o, e_1, e_2, \ldots\}$, and let J be an infinite subset of the natural numbers $\mathbb{N}$, which is 'Sparse' in $\mathbb{N}$, in the sense that if i and $j \in J$ with $i < j$, then

$$2i < j \text{ also .}$$

(e.g. $J = \{ 3^n : n \in \mathbb{N} \}$)

Let Q denote the orthogonal projection from H_o onto the span of the e_j's with $j \in J$, and consider the operator matrix A on $H_o \oplus H_o$ given by,

$$\underset{\sim}{A} = \begin{bmatrix} U_+^* & Q \\ 0 & U_+ \end{bmatrix} ,$$

U_+ denoting the unilateral shift on H_o . A trivial induction argument reveals that

$$A^n = \begin{bmatrix} U_+^{*n} & Q_n \\ 0 & U_+^n \end{bmatrix} , \quad n = 0,1,2,\ldots ,$$

where $Q_o = 0$ and $Q_{n+1} = \sum_{i=0}^{n} U_+^{*n-i} \, Q \, U_+^i$.

It can be shown that A as defined above is a power bounded operator not similar to any contraction. For details we refer to [13], [34] and [18] for a very elegant proof.

CHAPTER 2

CONVERGENCE PROPERTIES OF OPERATORS OF CLASS $\mathcal{C}_\rho$.

In this Chapter we are mainly concerned with the convergence properties of the iterates of an operator T of class $\mathcal{C}_\rho(H)$. More precisely, in Theorem 2.1 we prove that if T is an operator of class $\mathcal{C}_\rho(H)$ for some $\rho > 0$, then the sequence $\{\|T^n x\|\}_{n=1}^{\infty}$ is convergent for any x in H and moreover, $\lim_{n\to\infty} \|T^n x\| = L_x \|x\|$ with $0 \leq L_x \leq \rho^{\frac{1}{2}}$. An intrinsic characterization of the elements x of for which $L_x = \rho^{\frac{1}{2}}$ then follows, and, various Corollaries to Theorem 2.1 are discussed. We then obtain some results concerning the structure of operators T of class $\mathcal{C}_\rho(H)$ satisfying

$$\|T^N x\| = \rho\|x\|, \quad N \in Z^+ \quad x \in H,$$

and conclude this Chapter by exhibiting an operator T of class $\mathcal{C}_\rho(H)$, $\rho \geq 1$, and vectors x in the underlying Hilbert space for which $\lim_{n\to\infty} \|T^n x\|$ assumes any value between 0 and $\rho^{\frac{1}{2}}\|x\|$ inclusive.

THEOREM 2.1

Let T be an operator of class $\mathcal{C}_\rho$, for some $\rho > 0$, on a Hilbert space H.

Then, for any $x \in H$, the sequence

$$\{\| T^n x \|\}_{n=1}^{\infty}$$

is convergent, and moreover

$$\lim_{n\to\infty} \| T^n x \| = \begin{cases} 0 \text{ if } 0 < \rho < 1 \\ L_x \| x \| \quad \text{if } 1 \leq \rho < \infty \end{cases}$$

with

$$0 \leq L_x \leq \rho^{1/2} .$$

Furthermore, for $\rho \geq 1$,

$$L_x = \rho^{1/2} \iff \| T^n x \| = \rho^{1/2} \|x\| , n = 1,2,\ldots ,$$

$$\iff T^{*n} T^n x = \rho x , n = 1,2,\ldots .$$

PROOF

Let x be an arbitrary element of H, which without loss of generality we may take to be of unit norm. Our method of proof extends the technique used by M.J. Crabb in [5] for proving the special case of the theorem corresponding to $\rho = 2$.

We distinguish the following cases :

(a) $0 < \rho < 1$

and

(b) $1 \leq \rho < \infty$.

If $0 < \rho < 1$ and $T \in \mathcal{C}_\rho$, then as

$$\| T^n \| \leq \rho < 1, \quad n = 1, 2, \dots, \qquad \text{(cf. observation 3)}$$

we have

$$\| T^{n+1} x \| = \| T(T^n x) \| \leq \| T \| \, \| T^n x \| \leq \rho \, \| T^n x \| < \| T^n x \|$$

and hence

$$\lim_{n\to\infty} \| T^n x \| = 0, \tag{2.1}$$

the sequence $\{\| T^n x \|\}_{n=1}^{\infty}$ being strictly decreasing and consisting of non-negative terms.

If $\rho = 1$, then by a similar argument, $\{\| T^n x \|\}_{n=1}^{\infty}$ is a non-increasing sequence and

$$\lim_{n\to\infty} \| T^n x \| = L_x \quad \text{with} \quad 0 \leq L_x \leq 1. \tag{2.2}$$

So, assume $\rho > 1$.

Then, since $T \in \mathcal{C}_\rho$, by Theorem 1.1 we have:

$$(\rho-2) \| z T h \|^2 - 2(\rho-1) \operatorname{Re} z(Th,h) + \rho \| h \|^2 \geq 0, \quad h \in H, \ |z| \leq 1 \tag{2.3}$$

In particular with $z = 1$,

$$(\rho-2) \| Th \|^2 - 2(\rho-1) \operatorname{Re}(Th,h) + \rho \| h \|^2 \geq 0, \quad h \in H. \tag{2.4}$$

Taking

$$h = \alpha_0 x + \alpha_1 Tx + \dots + \alpha_n T^n x$$

with

$\alpha_i \in \mathbb{R}$, $i = 0, 1, 2, \dots, n$ in (2.4), it follows that

$$(\rho-2)\|\alpha_o Tx + \dots + \alpha_n T^{n+1}x\|^2 -$$

$$2(\rho-1)\ \mathrm{Re}\,(\alpha_o Tx + \dots + \alpha_n T^{n+1}x,\ \alpha_o x + \dots + \alpha_n T^n x) +$$

$$\rho\ \|\alpha_o x + \dots + \alpha_n T^n x\|^2 \geq 0,\ \alpha_i \in \mathbb{R},\ 0 \leq i \leq n. \qquad (2.5)$$

But, by observation 8,

$T \in \mathcal{C}_\rho \iff e^{i\theta} T \in \mathcal{C}_\rho$, for any $\theta \in \mathbb{R}$ and every $\rho > 0$, and hence we may replace T by $e^{i\theta}T$ and (2.5) still holds.. Integrating the resulting inequality with respect to θ over $[0, 2\pi)$ now, it follows that

$$(\rho-2)[\alpha_o^2\ \|Tx\|^2 + \dots + \alpha_n^2\ \|T^{n+1}x\|^2]$$

$$- 2(\rho-1)[\alpha_o\alpha_1\ \|Tx\|^2 + \dots + \alpha_{n-1}\alpha_n\|T^n x\|^2] + \qquad (2.6)$$

$$\rho[\alpha_o^2 + \alpha_1^2\ \|Tx\|^2 + \dots + \alpha_n^2\ \|T^n x\|^2] \geq 0,\ \alpha_i \in \mathbb{R},\ 0 \leq i \leq n,$$

and hence after some rearrangement ,

$$[(\rho-2)m_1 + \rho]\ \alpha_o^2 + [(\rho-2)m_2 + \rho m_1]\alpha_1^2 + \dots + [(\rho-2)m_{n+1} + \rho m_n]\alpha_n^2 -$$

$$2(\rho-1)(\alpha_o\alpha_1 m_1 + \dots + \alpha_{n-1}\alpha_n m_n) \geq 0,\ \alpha_i \in \mathbb{R},\ 0 \leq i \leq n \qquad (2.7)$$

where we have denoted

$$\|T^i x\|^2 \text{ by } m_i, \quad \text{for } i = 1, 2, \dots, n+1.$$

But, the left hand side of (2.7) is the quadratic form associated with the real, $(n+1) \times (n+1)$ symmetric matrix :

$$\begin{bmatrix} \rho + (\rho-2)m_1 & -(\rho-1)m_1 & 0 & & & & \bigcirc \\ -(\rho-1)m_1 & \rho m_1 + (\rho-2)m_2 & -(\rho-1)m_2 & \cdot & & & \\ 0 & -(\rho-1)m_2 & \rho m_2 + (\rho-2)m_3 & \cdot & & & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \rho m_{n-2} + (\rho-2)m_{n-1} & -(\rho-1)m_{n-1} & 0 \\ & & & -(\rho-1)m_{n-1} & \rho m_{n-1} + (\rho-2)m_n & -(\rho-1)m_n \\ \bigcirc & & & 0 & -(\rho-1)m_n & \rho m_n + (\rho-2)m_{n+1} \end{bmatrix}$$

and as this quadratic form is positive, semi-definite, D_n, the determinant of the above matrix must be non-negative.

Expanding now along the last row, we obtain the following recurrence relation for D_n :

$$D_n = [\rho m_n + (\rho-2)m_{n+1}]\, D_{n-1} - (\rho-1)^2 m_n^2 D_{n-2}\ , \quad n \geqq 2$$

$$D_1 = [\rho + (\rho-2)m_1]\,[\rho m_1 + (\rho-2)m_2] - (\rho-1)^2 m_1^2\ ,$$

$$D_o = \rho + (\rho-2)m_1\ .$$

Observe also, that

$$D_o \neq 0\ .$$

For, if we assume $D_o = 0$, we would then have

$$\rho + (\rho-2)\ \| Tx \|^2 = 0$$

and hence, as this is only possible if $\rho < 2$, it follows that :

$$\| Tx \| = \left(\frac{\rho}{2-\rho} \right)^{1/2}, \quad \rho < 2.$$

But, $T \in \mathcal{C}_\rho$, $\rho > 1$, and hence with the usual notation

$$T^n = \rho . P U^n , \quad n = 1,2,\dots ,$$

so that in particular,

$$\| Tx \| = \rho \| P Ux \| = \left(\frac{\rho}{2-\rho}\right)^{1/2}$$

giving

$$\| P Ux \| = \frac{1}{(\rho(2-\rho))^{1/2}} > 1 , \quad \text{for} \quad 1 < \rho < 2 .$$

As this is clearly a contradiction, U being a unitary operator and P an orthogonal projection, we conclude that our original supposition is false, and hence

$$D_o > 0 . \tag{2.10}$$

If now $D_n = 0$ for some n, let k be the smallest suffix for which $D_k = 0$.

Thus, $k > 0$ and

$$D_{k+1} = - (\rho-1)^2 m_{k+1}^2 D_{k-1} . \tag{2.11}$$

But, $D_{k+1} \geq 0$, and the right hand side of (2.11) is strictly negative, unless $m_{k+1} = 0$.

Hence, $m_{k+1} = 0$ and consequently

$$m_{k+i} = 0 , \quad i = 1,2,\dots , \tag{2.12}$$

in which case, of course,

$$\| T^n x \| \to 0 \quad \text{as} \quad n \to \infty \tag{2.13}$$

and we are done .

We may therefore assume from now on, that

$$D_n > 0 \qquad \text{for all } n = 1,2,\dots .$$

Then, if

$$\alpha_n = \frac{D_n}{D_{n-1}}, \quad n = 1,2,\dots ,$$

we have :

$$\alpha_n = \rho m_n + (\rho-2)m_{n+1} - (\rho-1)^2 m_n^2 \frac{1}{\alpha_{n-1}}, \ n \geq 2 \tag{2.14}$$

and

$$\alpha_1 = \frac{[\rho + (\rho-2)m_1]\,[\rho m_1 + (\rho-2)m_2] - (\rho-1)^2 m_1{}^2}{\rho + (\rho-2)m_1}, \tag{2.15}$$

so that :

$$\alpha_1 - (\rho-2)m_2 = \rho - \frac{(\rho - m_1)^2}{\rho+(\rho-2)m_1} . \tag{2.16}$$

Furthermore, if

$$\beta_n = \alpha_n - (\rho-2)m_{n+1}, \quad n = 1,2,\dots , \tag{2.17}$$

then

$$\beta_{n+1} \leq \beta_n . \tag{2.18}$$

For,

$$\begin{aligned}\beta_{n+1} = \alpha_{n+1} - (\rho-2)m_{n+2} &= \rho m_{n+1} - (\rho-1)^2 \frac{m_{n+1}^2}{\alpha_n} \\ &= 2(\rho-1)m_{n+1} - \frac{(\rho-1)^2 m_{n+1}^2}{\alpha_n} - \alpha_n + \alpha_n - (\rho-2)m_{n+1} \\ &= \alpha_n - (\rho-2)m_{n+1} - \frac{1}{\alpha_n}\,[\alpha_n - (\rho-1)m_{n+1}]^2 \\ &\leq \alpha_n - (\rho-2)m_{n+1} = \beta_n .\end{aligned}$$

Now, $\{m_n\}_{n=1}^{\infty}$ is bounded by definition, and

$$\alpha_n < \rho m_n + (\rho-2)m_{n+1}, \quad \text{by} \quad (2.14)$$

so that $\{\alpha_n\}_{n=1}^{\infty}$ is also bounded.

Hence, as $\{\beta_n\}_{n=1}^{\infty}$ is a bounded non-increasing sequence of real numbers ,

$$\beta_n \to L' \quad \text{as} \quad n \to \infty \tag{2.19}$$

with

$$L' \leq \beta_1 = \rho - \frac{(\rho-m_1)^2}{\rho+(\rho-2)m_1} . \tag{2.20}$$

It thus follows from (2.17) and (2.19), that

$$\alpha_n - (\rho-2)m_{n+1} - L' \to 0 \quad \text{with} \quad n$$

or, what amounts to the same thing, that

$$\rho m_n - (\rho-1)^2 \frac{m_n^{\ 2}}{\alpha_{n-1}} - L' \to 0 \quad \text{as} \quad n \to \infty \tag{2.21}$$

But the sequence $\{\alpha_n\}_{n=1}^{\infty}$ is bounded, and hence

$$(\rho m_n - L')\,\alpha_{n-1} - (\rho-1)^2 m_n^{\ 2} \to 0 \quad \text{with} \quad n . \tag{2.22}$$

On the other hand,

$$(\rho m_n - L')\alpha_{n-1} - (\rho-1)^2 m_n^{\ 2} =$$

$$= (\rho m_n - L')[\alpha_{n-1} - (\rho-2)m_n - L'] +$$

$$(\rho m_n - L')[(\rho-2)m_n + L'] - (\rho-1)^2 m_n^{\ 2} =$$

$$(\rho m_n - L')(\beta_{n-1} - L') - (m_n - L')^2 ,$$

so that

$$(\rho m_n - L')(\beta_{n-1} - L') - (m_n - L')^2 \to 0 \tag{2.22a}$$

in view of (2.22) .

Consequently, as $\beta_n \to L'$ and $\rho m_n - L'$ is bounded for all n, it follows that

$$m_n \to L' \quad \text{as} \quad n \to \infty \quad . \tag{2.24}$$

Observe also, that

$$L' \geq 0$$

being the limit of a non-negative sequence, and $L' \leq \rho$ from (2.20) .

So,

$$\| T^n x \| = m_n^{1/2} \to L'^{1/2} = L_x$$

with

$$0 \leq L_x \leq \rho^{1/2}$$

and the proof is seen to be complete.

If now, $L_x = \rho^{1/2}$, that is $L' = \rho$; then,

$$\beta_1 = \rho \quad \text{in view of (2.20)} ,$$

and hence, as the sequence $\{\beta_n\}_{n=1}^{\infty}$ is non-increasing and moreover converges to $\beta_1 = \rho$, we have

$$\beta_n = \alpha_n - (\rho-2)m_{n+1} = \beta_1 = \rho \text{ , } n = 1,2,\ldots \text{ . } \tag{2.23}$$

On the other hand, as

$$\alpha_n - (\rho-2)m_{n+1} = \rho m_n - (\rho-1)^2 \frac{m_n^2}{\alpha_{n-1}} \text{ ,}$$

it follows that,

$$\rho m_n - (\rho-1)^2 \frac{m_n^2}{\alpha_{n-1}} = \rho , \quad n = 1,2,\ldots . \qquad (2.24\,b)$$

Substituting for α_{n-1} from (2.23) in (2.24)

we get :

$$(\rho m_n - \rho)[\rho + (\rho-2)m_n] = (\rho-1)^2 m_n^2 , \quad n = 1.2.,,, ,$$

that is ,

$$m_n = \rho , \qquad n = 1,2,\ldots . \qquad (2.25)$$

Thus,

$$\lim_{n\to\infty} \| T^n x \| = \rho^{1/2} \iff \lim_{n\to\infty} m_n = \rho$$

$$\iff m_n = \rho , \quad n = 1.2.\ldots . \quad (2.26)$$

We now establish the equivalence of

(A) $\| T^n x \| = \rho^{1/2} \|x\| , \qquad n = 1,2,\ldots ,$

and

(B) $T^{*n} T^n x = \rho x , \qquad n = 1,2,\ldots .$

The 'only if' part follows immediately.

To prove the 'if' part.

To this end, we refer to [8], where a slight modification of the argument used by the author, shows that for any $x \in H$

$$\| T^n x \| \to \rho^{1/2} \| Q x \| ,$$

where Q is the strong limit of the orthogonal projections Q_n from K onto $\mathfrak{M}_n = \bigvee_{k=0}^{\infty} U^{-(n+1+k)} H$, K denoting a Hilbert space containing H and U a unitary ρ - dilation of T on K.

Thus, in particular, Q is an orthogonal projection with range

$$\bigcap_n \mathfrak{M}_n \quad .$$

Also, as $T = \rho P U^n$, $n = 1,2,\ldots$, where P denotes as usual the orthogonal projection from K onto H, we have: if

$$\| T^n x \| = \rho^{1/2} \| x \| , \quad n = 1,2,\ldots ,$$

that

$$Qx = x .$$

Hence, $x \in \bigcap_n \mathfrak{M}_n$, and consequently

$$x \in \mathfrak{M}_n , \quad \text{for all } n.$$

But then, as (see [8])

$$Pf = T^{*n} P U^n f , \quad \text{for all } f \in \mathfrak{M}_n ,$$

It follows that

$$x = Px = PQ_n x = T^{*n} P U^n Q_n x = T^{*n} P U^n x = \frac{1}{\rho} T^{*n} T^n x$$

for all n, and the proof is seen to be complete.

COROLLARY 2.2

Let T be an operator of class $\mathcal{C}_\rho$ for some $\rho \neq 1$, on a Hilbert space H and assume that there exists a unit vector x in H, such that

$$\| T^N x \| = \rho$$

for some positive integer $N \geq 1$.

Then :

(a) $\| T^{N-k}x \| = \rho^{1/2}$, $k = 1,2,\ldots, N-1$,

and (b) T is locally nilpotent of index $N+1$ at x, that is

$$T^{N+1}x = 0 \text{ and hence } T^{N+k}x = 0, \quad k = 1,2,\ldots \ .$$

PROOF

With the same notation as in Theorem 2.1 ,

$$m_N = \rho^2$$

and we wish to show that :

(a) $m_{N-k} = \rho$; $k = 1,2,\ldots,N-1$, and

(b) $m_{N+k} = 0$, $k = 1,2,\ldots$.

To this end, observe that as the sequence $\{\beta_n\}_{n=1}^{\infty}$ of the theorem is non-increasing and moreover converges to a non-negative limit ,

$$\beta_n \geqq 0 \quad , \quad n = 1,2,\ldots , \tag{2.27}$$

and hence, as

$$\beta_n = \alpha_n - (\rho-2)m_{n+1} = \rho m_n - (\rho-1)^2 \frac{m_n^2}{\alpha_{n-1}}$$

we must have :

$$\rho m_n - (\rho-1)^2 \frac{m_n^2}{\alpha_{n-1}} \geqq 0 , \quad n = 2,3,\ldots . \tag{2.28}$$

In particular,

$$\rho\, m_N - (\rho-1)^2 \frac{m_N^2}{\alpha_{N-1}} \geqq 0$$

and hence, since

$$m_N = \rho^2 ,$$

$$\alpha_{N-1} \geqq \rho(\rho-1)^2 \quad , \tag{2.29}$$

so that

$$\alpha_{N-1} - (\rho-2)m_N \geq \rho(\rho-1)^2 - (\rho-2)\rho^2$$

$$= \rho \qquad (2.30)$$

Consequently, as

$$\beta_1 = \alpha_1 - (\rho-2)m_2 \leq \rho \qquad \text{by } (2.20)$$

and the sequence $\{\beta_n\}_{n=1}^{\infty}$ is non-increasing we have in view of (2.30),

$$\rho \leq \alpha_{N-1} - (\rho-2)m_N \leq \alpha_{N-2} - (\rho-2)m_{N-1} \leq \cdots$$

$$\leq \alpha_1 - (\rho-2)m_2 = \beta_1 \leq \rho \qquad (2.31)$$

and hence

$$\alpha_{N-k} - (\rho-2)m_{N-k+1} = \rho\ , \qquad k = 1,2,\ldots,N-1\ . \qquad (2.32)$$

On the other hand, as

$$\alpha_{N-k} - (\rho-2)m_{N-k+1} = \rho m_{N-k} - (\rho-1)^2 \frac{m_{N-k}^2}{\alpha_{N-k-1}}$$

by (2.14), and

$$\alpha_{N-k-1} = \rho + (\rho-2)m_{N-k} \qquad \text{by } (2.32)\ ,$$

we have :

$$\rho = \rho m_{N-k} - (\rho-1)^2 \frac{m_{N-k}^2}{\rho+(\rho-2)m_{N-k}}\ ,$$

which gives

$$m_{N-k} = \rho\ , \quad k = 1,2,\ldots,N-1. \qquad (2.33)$$

Furthermore, as

$$\alpha_N - (\rho-2)m_{N+1} = \rho m_N - (\rho-1)^2 \frac{m_N^2}{\alpha_{N-1}}$$

$$= \rho m_N - (\rho-1)^2 \frac{m_N^2}{\rho+(\rho-2)m_N}$$

$$= \rho^3 - (\rho-1)^2 \frac{\rho^4}{\rho+(\rho-2)\rho^2}$$

$$= \rho^3 - \rho^3 = 0$$

it follows that

$$\beta_N = \alpha_N - (\rho-2)m_{N+1} = 0 \tag{2.34}$$

and consequently,

$$\beta_{N+k} = \alpha_{N+k} - (\rho-2)m_{N+k+1} = 0\,, \quad k = 1,2,\ldots\,, \tag{2.35}$$

by virtue of the monotonicity and non-negativity of the sequence $\{\beta_n\}_{n=1}^{\infty}$.

In particular ,

$$\beta_{N+1} = \alpha_{N+1} - (\rho-2)m_{N+2} = 0$$

and hence

$$\rho m_{N+1} - (\rho-1)^2 \frac{m_{N+1}^2}{\alpha_N} = 0\ . \tag{2.36}$$

Substituting now for α_N from (2.34) in (2.36), it follows that

$$\rho m_{N+1} - (\rho-1)^2 \frac{m_{N+1}^2}{(\rho-2)m_{N+1}} = 0\ . \qquad (\rho \neq 2) \tag{2.37}$$

So, if $\rho \neq 2$,

$$m_{N+1} = 0 \tag{2.38}$$

and therefore,

$$m_{N+k} = 0\,, \qquad k = 1,2,\ldots\,. \tag{2.39}$$

As the case $\rho = 2$, has been completely examined by Crabb in [5].

where he obtains exactly the same results, namely that

$$m_{N-k} = 2 \ , \quad k = 1,2,\ldots, N-1$$

and

$$m_{N+k} = 0 \ , \quad k = 1,2,\ldots \ ,$$

the proof is seen to be complete.

REMARKS

1. The case $\rho = 1$ was excluded from our considerations for there exist operators of class $\mathcal{C}_1$, satisfying the condition set in the corollary for some unit vector x and some $N \in \mathbb{Z}^+$, and yet these operators are not locally nilpotent at x. E.g. the identity operator.
2. If the integer N in the corollary is strictly greater than 1, then ρ is necessarily greater than or equal to 1, as the following argument shows:

   Assume that there exists an operator T of class $\mathcal{C}_\rho$, $0 < \rho < 1$, on a Hilbert space H, which is such that

   $$\| T^N x \| = \rho ,$$

   for some unit vector x in H and some integer $N > 1$. Then, $\| T^N \| \geq \rho$, and since T is power bounded with $\| T^k \| \leq \rho$, $k = 1,2,\ldots$, it follows that

   $$\| T^N \| = \rho .$$

   But then, as $\| T^N \| \leq \| T \|^N$, we would have

   $$\rho = \| T^N \| \leq \| T \|^N \leq \rho^N$$

   that is

   $$\rho \leq \rho^N \quad \text{with} \quad 0 < \rho < 1 \quad \text{and} \quad N > 1 .$$

   As this is clearly a contradiction, the result follows. On the other hand, the operator on $\mathcal{B}(\mathbb{R}^2)$ given by:

   $$T = \begin{bmatrix} 0 & 0 \\ \rho & 0 \end{bmatrix}$$

   clearly belongs to $\mathcal{C}_\rho$ for any $\rho > 0$, and in particular for $0 < \rho < 1$, and there exists $x \in \mathbb{R}^2$, $\|x\| = 1$, namely $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, for which

$$\| Tx \| = \rho .$$

That is, the integer N can actually equal 1.

<u>THEOREM</u> 2.3

Let T be an operator of $\mathcal{C}_\rho$ class for some $\rho \neq 1$, on a Hilbert space H, and assume that there exists an element x in H, $\| x \| = 1$, which is such that

$$\| T^N x \| = \rho$$

then:

(i) T is locally nilpotent of index $N+1$ at x

(ii) $\|T^k x\| = \rho^{1/2}$, $k = 1,2,\ldots, N-1$,

(iii) The vectors

$$x, Tx, \ldots, T^N x$$

are mutually orthogonal (and hence linearly independent)

and

(iv) $M_x = \text{lin}[x, Tx,\ldots,T^N x]$, the linear span of the vectors $x, Tx,\ldots,T^N x$, is a reducing subspace of T.

<u>REMARKS</u>

1. Although assertions (i) and (ii) have been proved in Corollary 2.2 it is our aim here to give an alternative proof using a technique based on an idea of Berger and Stampfli, [3], Theorem 6, which leans much more heavily on the theory of unitary ρ-dilations.
2. The special case of this theorem corresponding to $\rho = 2$, is theorem 2 of [5].

Since $T \in \mathcal{C}_\rho$, we have with the usual notation,

$$T^n \doteq \rho P U^n, \quad n = 1,2,\ldots \qquad \text{(I)}$$

In particular

$$T^N x = \rho P U^N x$$

and hence,

$$\rho = \| T^N x \| = \rho \| P U^N x \| .$$

Thus $$\| P U^N x \| = 1 ,$$

and this clearly implies that

$$P U^N x = U^N x \quad , \qquad (*)$$

U being a unitary operator and P an orthogonal projection.

Consequently,

$$T^N x = \rho U^N x \quad ,$$

and hence

$$T^{N+1} x = T(T^N x) = \rho P U(\rho U^N x) = \rho^2 P U^{N+1} x \qquad \text{(II)}$$

Directly, though, it follows from (I) that

$$T^{N+1} x = \rho P U^{N+1} x \qquad \text{(III)}$$

so that since $\rho \neq 1$,

$$T^{N+1} x = 0 ,$$

if (II) and (III) are to be consistent .

Thus, T is locally nilpotent of index $N+1$ at x and (i) has been proved.

Next observe that, if

$$1 \leq k \leq N-1 ,$$

we have:

$$\|T^k x\|^2 = (T^k x, T^k x) = (\rho P U^k x, \rho P U^k x)$$

$$= \rho^2 (P U^k x, U^k x) = \rho^2 (U^{N-k} P U^k x, U^N x)$$

$$= \rho^2 (U^{N-k} P U^k x, P U^N x)$$

$$= \rho^2 (P U^{N-k} P U^k x, U^N x)$$

$$= \frac{1}{\rho} (\rho P U^{N-k}(\rho P U^k x), \rho U^N x)$$

$$= \frac{1}{\rho} (T^{N-k}(T^k x), T^N x) = \frac{1}{\rho} (T^N x, T^N x)$$

$$= \frac{1}{\rho} \|T^N x\|^2 = \frac{1}{\rho} . \rho^2$$

$$= \rho .$$

Thus, if $1 \leq k \leq N-1$

$$\|T^k x\| = \rho^{1/2}$$

and hence (ii) is established.

On the other hand, if

$$1 \leq i < j \leq N$$

we have :

$$(T^j x, T^i x) = (\rho P U^j x, \rho P U^i x) = \rho^2 (P U^j x, U^i x)$$

$$= \rho^2 (U^{N-i} P U^j x, U^N x)$$

$$= \rho^2 (U^{N-i} P U^j x, P U^N x)$$

$$= \frac{1}{\rho} (\rho P U^{N-i}(\rho P U^j x), \rho U^N x)$$

$$= \frac{1}{\rho} (T^{N-i}(T^j x), T^N x)$$

$$= \frac{1}{\rho} (T^{N+j-i} x, T^N x)$$

$$= \frac{1}{\rho} (0, T^N x) = 0 .$$

since $j-i \geq 1$ and T is locally nilpotent of index $N+1$ at x, while a similar argument shows that

$$(T^k x, x) = 0 \quad \text{for any} \quad k = 1,2,\ldots, N .$$

The vectors

$$x, Tx, \ldots, T^N x$$

are thus seen to be mutually orthogonal (and hence linearly independent). If we now let M_x be the linear span of these vectors, it is obvious that M_x is an invariant subspace of T, by virtue of the local nilpotency of T at x. To show that it is in fact reducing for T, let α be a vector in H perpendicular to M_x.
Thus,

$$(T^k x, \alpha) = 0 \quad \text{for} \quad k = 0,1,2,\ldots, N$$

and hence, if $1 \leq k \leq N$

$$\begin{aligned}
(T^k x, T\alpha) &= \rho^2(P U^k x, P U\alpha) = \rho^2(U^k x, P U\alpha) \\
&= \rho^2(U^N x, U^{N-k} P U\alpha) \\
&= \rho^2(U^N x, P U^{N-k} P U\alpha) \quad \text{by} \quad (*) \\
&= (U^N x, T^{N-k+1}\alpha) \\
&= \rho\,(U^N x, U^{N-k+1}\alpha) \\
&= \rho\,(U^{k-1} x, \alpha) = \rho(U^{k-1} x, P\alpha) \\
&= (\rho P U^{k-1} x, \alpha) = (T^{k-1} x, \alpha) \\
&= 0, \quad \text{since} \quad 0 \leq k-1 \leq N-1 .
\end{aligned}$$

On the other hand, by a similar argument to the one used for proving the local nilpotency of T at x, and the fact that

$$\|T^{*N} U^N x\| = \|\rho P U^{*N} U^N x\| = \rho \|Px\| = \rho \|x\| = \rho ,$$

it follows that

$$0 = T^{*N+1} U^N x = \rho P U^{*N+1} U^N x = \rho P U^* x = T^* x .$$

Thus,

$$(T\alpha, x) = (\alpha, T^* x) = 0 .$$

So, if α is a vector in H, perpendicular to M_x, then $T\alpha$ is also perpendicular to M_x and this establishes that M_x is reducing. The proof of the theorem is thus seen to be complete. Note also, that one can easily show that for the same x as in the Theorem

$$T^* T^k x = \begin{cases} 0 & \text{if } k = 0 \\ T^{k-1} x & \text{if } k = 1,2,\ldots, N-1 \\ \rho T^{N-1} x & \text{if } k = N . \end{cases}$$

COROLLARY 2.4

Let T be an operator on a Hilbert space H, which is such that

$w(T) = \frac{1}{2^{1/N}} \|T^N\|^{1/N}$ for some positive integer N and assume that there exists a unit vector x in H such that

$$\|T^N x\| = \|T^N\| .$$

Then : (i) T is locally nilpotent of index N+1 at x

(ii) $\|T^k x\| = 2^{\frac{1}{2} - \frac{k}{N}} \|T^N\|^{k/N}$, k=1,2,...,N-1

(iii) The vectors

x, Tx,..., $T^N x$ are mutually orthogonal

and (iv) $M_x = \text{lin}[x, Tx,\ldots, T^N x]$ is a reducing subspace for T .

PROOF

Let $$S = \frac{2^{1/N}T}{\|T^N\|^{1/N}} .$$

Then

$$w(S) = 1$$

and

$$\|S^N x\| = 2 .$$

Thus, the operator S given above, satisfies all the conditions set in Theorem 2.3 , for $\rho = 2$ and hence

(a) S is locally nilpotent of index N+1 at x

(b) $\|S^k x\| = 2^{1/2}$, $k = 1,2,\dots,N-1$

(c) The vectors

$x, Sx, \dots, S^N x$ are mutually orthogonal [and hence linearly independent.]

and

(d) $M_x = \text{lin}[x, Sx, \dots, S^N x]$ is a reducing subspace for S.

But, T is a scalar multiple of S, and hence the same conclusions must hold true with S replaced by T. The proof is thus complete.

The following corollaries follow on immediately from Corollary 2.4

COROLLARY 2.5 [43]

The unique (up to unitary equivalence) irreducible operator T which attains its norm and whose numerical radius w(T) satisfies

$$w(T) = \frac{1}{2} \|T\| ,$$

is

$$T = a \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} , \quad a \in \mathbb{C} .$$

COROLLARY 2.6

The unique (up to unitary equivalence) irreducible operator T which is such that T^2 attains its norm and whose numerical radius w(T), satisfies

$$w(T) = \frac{1}{2^{1/2}} \, \| T^2 \|^{1/2}$$

is

$$T = a \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} , \quad a \in \mathbb{C} .$$

THEOREM 2.7

Let T be a nilpotent operator of index $N+1$, $N \geq 2$, on a Hilbert space H, satisfying

$$\| T^N \| = \| T \|^N$$

and assume moreover that T^N attains its norm that is, there exists a unit vector x in H, for which

$$\| T^N x \| = \| T^N \| \, \| x \| = \| T^N \| .$$

Then :

(i) $\|T^i x\| = \|T^i\| \, \|x\| = \|T^i\|$, $i = 1,2,\ldots,N$

(ii) The vectors

$$x \, , \frac{Tx}{\|T\|} \, , \ldots , \frac{T^i x}{\|T^i\|} \, , \ldots , \frac{T^N x}{\|T^N\|}$$

form an orthonormal set, and

(iii) $M_x = \mathrm{lin}[x \, , \frac{Tx}{\|T\|} \, , \ldots , \frac{T^i x}{\|T^i x\|} \, , \ldots , \frac{T^N x}{\|T^N\|}]$

is a reducing subspace for T .

PROOF

The following lemma is essential for our purposes :

LEMMA 2.8

If A is a bounded operator on a Hilbert space, with

$$\|A^k\| = \|A\|^k$$

for some positive integer k, then

$$\|A^i\| = \|A\|^i, \quad i = 1,2,\dots,k .$$

Proof of the Lemma

Let

$$\alpha_i = \frac{\|A^i\|}{\|A\|^i}, \quad i = 1,2,\dots$$

Then by assumption,

$$\alpha_k = 1 ,$$

and trivially,

$$\alpha_1 = 1 .$$

On the other hand, as

$$\frac{\alpha_{i+1}}{\alpha_i} = \frac{\|A^{i+1}\|}{\|A\|^{i+1}} \cdot \frac{\|A\|^i}{\|A^i\|} = \frac{\|A^{i+1}\|}{\|A^i\|\,\|A\|} \le \frac{\|A^i\|\,\|A\|}{\|A^i\|\,\|A\|} = 1 ,$$

the sequence $\{\alpha_i\}_{i=1}^{\infty}$ is non-increasing, and consequently, if

$2 \le i \le N-1$,

$1 = \alpha_k \le \alpha_i \le \alpha_1 = 1$,

and hence,

$$\alpha_i = 1 , \quad i = 1,2,\dots,k ,$$

so that

$$\|A^i\| = \|A\|^i , \quad i = 1,2,\dots,k$$

and the proof is complete.

Returning to the proof of the theorem now, observe that if

$$S = \frac{T}{\|T\|} ,$$

then

$$\|S^N\| = \|S\|^N = 1 ,$$

and

$$\|S^N x\| = \frac{\|T^N x\|}{\|T^N\|} = 1 .$$

Consequently, as $S \in \mathcal{C}_1$,

$$S^i = P U^i , \quad i = 1,2,\dots ,$$

where U is a unitary operator on a Hilbert space $K \supseteq H$, P denoting , as usual, the orthogonal projection from K onto H. Therefore, since

$$1 = \|S^N x\| = \|PU^N x\| ,$$

we conclude as in Theorem 2.3 that

$$S^N x = U^N x .$$

If now

$$1 \le i \le N ,$$

then

$$\begin{aligned} \|S^i x\|^2 &= (S^i x , S^i x) = (P U^i x , P U^i x) \\ &= (P U^i x , U^i x) = (U^{N-i} P U^i x , U^N x) \\ &= (U^{N-i} P U^i x , P U^N x) = (P U^{N-i} P U^i x , U^N x) \\ &= (S^{N-i}(S^i x), S^N x) \quad = (S^N x , S^N x) \\ &= \|S^N x\|^2 = 1 . \end{aligned}$$

Thus,

$$\| S^i x \| = 1 , \qquad i = 1,2,\ldots,N ,$$

and hence

$$S^i x = U^i x , \qquad i = 1,2,\ldots,N .$$

On the other hand, for $i = 1,2,\ldots,N$,

$$1 = \| S^i x \| \doteq \frac{\| T^i x \|}{\| T \|^i}$$

and hence

$$\| T^i x \| = \| T \|^i = \| T^i \| , \qquad i = 1,2,\ldots,N ,$$

since the operator T satisfies the assumptions for the validity of the lemma.

Assertion (i) has thus been proved.

Next note that, if

$$1 \leq i < j \leq N ,$$

we have

$$(S^j x , S^i x) = (U^j x , U^i x) = (U^{N+j-i} x , U^N x)$$

$$= (U^{N+j-i} x , P U^N x) = (P U^{N+j-i} x , U^N x)$$

$$= (S^{N+j-i} x , S^N x) = 0$$

since $N + j - i \geq N + 1$, and S is by definition, nilpotent of index $N + 1$.

Similarly,

$$(S^i x , x) = 0 \text{ for any } i = 1,2,\ldots,N$$

and hence the vectors

$$x , Sx , \ldots , S^N x$$

and consequently the vectors

$$x\,,\ \frac{Tx}{\|T\|}\,,\dots,\ \frac{T^i x}{\|T^i\|}\,,\dots,\ \frac{T^N x}{\|T^N\|}$$

form an orthonormal set of vectors whose span M_x, is clearly an invariant subspace for T due to the nilpotency of the operator. Finally observe that since

$$S^{*i} = P\,U^{*i}\,, \qquad i = 1,2,\dots$$

we have if

$$1 \leq i \leq N\,,$$

$$S^* S^i x = P\,U^*(U^i x) = P\,U^{i-1}x = S^{i-1}x$$

while

$$S^* x = 0\,,$$

as the following argument clearly demonstrates:

Since

$$S^{*N+1} S^N x = 0$$

in view of the nilpotency of S^*, we have

$$0 = S^{*N+1} S^N x = P\,U^{*N+1} U^N x = P\,U^* x = S^* x\,.$$

Thus, the linear span of

$$x\,,\ Sx\,,\dots,\ S^N x$$

is also invariant under S^*.

Replacing S by $\frac{T}{\|T\|}$ now, the result follows. The proof of the theorem is therefore complete.

The following corollary follows immediately.

COROLLARY 2.9

$$T = a \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{N+1} \Big\} N+1 \ , \quad a \in \mathbb{C} \ ,$$

is the unique (up to unitary equivalence) irreducible nilpotent operator of index $N+1$, satisfying

$$\| T^N \| = \| T \|^N$$

and which is such that T^N attains its norm.

Let now H be a complex Hilbert space with a countable orthonormal basis

$$\{e_k\}_{k=1}^{\infty} \ .$$

and let T be the (unique) operator on H, given by

$$Te_k = \omega_k \, e_{k+1} \ , \quad k = 1,2,\ldots,$$

where

$$\omega_1 = \rho^{1/2} \ , \quad \omega_k = 1, \ k = 2,3,\ldots \ , \quad \rho \geq 1 \ .$$

That is, T is the unilateral weighted shift with weight sequence

$$\{\omega_k\}_{k=1}^{\infty} \ .$$

THEOREM 2.10

With these assumptions we show that $T \in \mathcal{C}_\rho(H)$ for any $\rho \geq 1$ (and not for $0 < \rho < 1$) , $T = T(\rho)$.

PROOF

Let h be an arbitrary element of H.

Then ,

$$h = \sum_{k=1}^{\infty} \alpha_k e_k$$

for suitable $\alpha_k \in \mathbb{C}$, $k = 1,2,\ldots,$ with $\|h\|^2 = \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty$.

We have :

$$T \in \mathcal{C}_\rho \,,\ \rho \geq 1 \iff \begin{cases} (\rho-2)\|Th\|^2 - 2(\rho-1)|(Th,h)| + \rho\|h\|^2 \geq 0\,, & h \in H \\ r(T) \leq 1 \end{cases} \qquad (I)$$

(cf. Thm. I.1 part (v)) .

Using the formula for the spectral radius of a (unilateral) weighted shift ([17]) ; it is easily established that for this particular weighted shift, $r(T) = 1$ and hence, the second condition in the right hand side of (I) is seen to be satisfied.

Thus, with T as above,

$$T \in \mathcal{C}_\rho \,, \rho \geq 1 \iff (\rho-2)\|Th\|^2 - 2(\rho-1)|(Th,h)| + \rho\|h\|^2 \geq 0\,,\ h \in H\,.$$

But, if $h = \sum_{k=1}^{\infty} \alpha_k e_k$, then

$$Th = \sum_{k=1}^{\infty} \alpha_k \, Te_k = \sum_{k=1}^{\infty} \alpha_k \omega_k e_{k+1}$$

and

$$(Th,h) = \Big(\sum_{k=1}^{\infty} \alpha_k \omega_k e_{k+1} \,,\ \sum_{j=1}^{\infty} \alpha_j e_j \Big) = \sum_{k=1}^{\infty} \alpha_k \alpha_{k+1} \omega_k\,,$$

and hence,

$$T \in \mathcal{C}_\rho \,, \rho \geq 1 \iff (\rho-2) \sum_{k=1}^{\infty} |\alpha_k|^2 \omega_k^2 - 2(\rho-1)\Big|\sum_{k=1}^{\infty} \alpha_k \alpha_{k+1} \omega_k\Big| + \rho \sum_{k=1}^{\infty} |\alpha_k|^2 \geq 0,$$

$$\alpha_k \in \mathbb{C}\,,\quad k = 1,2,\ldots\,,\ \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty\,,$$

or equivalently

$$T \in \mathcal{C}_\rho \quad \rho \geq 1 \iff (\rho-2)\sum_{k=1}^{\infty}|\alpha_k|^2\omega_k^2 - 2(\rho-1)\sum_{k=1}^{\infty}|\alpha_k||\alpha_{k+1}|\omega_k + \rho\sum_{k=1}^{\infty}|\alpha_k|^2 \geq 0 ,$$

$$\alpha_k \in \mathbb{C}, \quad k = 1,2,\ldots , \quad \sum_{k=1}^{\infty}|\alpha_k|^2 < \infty .$$

But, with

$$\omega_1 = \rho^{1/2}, \quad \omega_k = 1, \quad k = 1,2,\ldots ,$$

$$(\rho-2)\sum_{k=1}^{\infty}|\alpha_k|^2\omega_k^2 - 2(\rho-1)\sum_{k=1}^{\infty}|\alpha_k||\alpha_{k+1}|\omega_k + \rho\sum_{k=1}^{\infty}|\alpha_k|^2 =$$

$$\rho(\rho-1)|\alpha_1|^2 + 2(\rho-1)\sum_{k=2}^{\infty}|\alpha_k|^2 - 2(\rho-1)\sum_{k=2}^{\infty}|\alpha_k||\alpha_{k+1}| - 2\rho^{1/2}(\rho-1)|\alpha_1||\alpha_2|$$

$$= (\rho-1)[\rho|\alpha_1|^2 - 2\rho^{1/2}|\alpha_1||\alpha_2| + |\alpha_2|^2]$$

$$+ (\rho-1)[\sum_{k=2}^{\infty}|\alpha_k|^2 - 2\sum_{k=2}^{\infty}|\alpha_k||\alpha_{k+1}| + \sum_{k=2}^{\infty}|\alpha_{k+1}|^2]$$

$$= (\rho-1)\,\{(\rho^{1/2}|\alpha_1| - |\alpha_2|)^2 + \sum_{k=2}^{\infty}(|\alpha_k| - |\alpha_{k+1}|)^2\}$$

which is clearly a non-negative quantity, since $\rho \geq 1$, for all possible choices of complex numbers

$$\alpha_k, \quad k = 1,2,\ldots ,$$

Thus, the weighted shift given above, belongs to $\mathcal{C}_\rho(H)$ for $\rho \geq 1$ proving the theorem.

On the other hand, it is a trivial matter to verify that for a (unilateral) weighted shift with weight sequence $\{\omega_k\}_{k=1}^{\infty}$,

$$T^n e_k = \omega_k\,\omega_{k+1}\cdots\omega_{k+n-1}\,e_{k+n} ,$$

and

$$T^{*n} e_k = \begin{cases} 0 & \text{if } k \leq n \\ \omega_{k-1}\omega_{k-2}\cdots\omega_{k-n}e_{k-n} & \text{if } k \geq n+1 \end{cases}$$

for any positive integer n and $k = 1,2,\ldots$.

Hence, if

$$h = \sum_{k=1}^{\infty} \alpha_k e_k, \qquad \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty \qquad \alpha_k \in \mathbb{C} ,$$

then

$$T^n h = \sum_{k=1}^{\infty} \alpha_k T^n e_k = \sum_{k=1}^{\infty} \alpha_k \omega_k \omega_{k+1}\cdots\omega_{k+n-1} e_{k+n}$$

and

$$T^{*n} h = \sum_{k=1}^{\infty} \alpha_k T^{*n} e_k = \sum_{k=n+1}^{\infty} \alpha_k \omega_{k-1}\omega_{k-2}\cdots\omega_{k-n} e_{k-n}$$

for any $n \in \mathbb{Z}^+$.

For the particular weighted shift of the Theorem,

$$T^n h = \rho^{1/2} \alpha_1 e_{n+1} + \sum_{k=2}^{\infty} \alpha_k e_{k+n}$$

and

$$T^{*n} h = \rho^{1/2} \alpha_{n+1} e_1 + \sum_{k=2}^{\infty} \alpha_{k+n} e_k$$

$$= \rho^{1/2} \alpha_{n+1} e_1 + \sum_{k=n+2}^{\infty} \alpha_k e_{k-n} \quad .$$

Hence,

$$\|T^n h\|^2 = \rho |\alpha_1|^2 + \sum_{k=2}^{\infty} |\alpha_k|^2$$

$$= (\rho-1) |\alpha_1|^2 + \sum_{k=1}^{\infty} |\alpha_k|^2$$

$$= (\rho-1) |\alpha_1|^2 + \|h\|^2$$

and

$$\| T^{*n} h \|^2 = \rho |\alpha_{n+1}|^2 + \sum_{k=n+2}^{\infty} |\alpha_k|^2 .$$

from which it easily follows that ,

$\| T^{*n} h \| \to 0$ as $n \to \infty$, in view of the convergence of $\sum_{k=1}^{\infty} |\alpha_k|^2$.

On the other hand, as $\| T^n h \|$ is independent of n, for any $h \in H$,

$$\lim_{n\to\infty} \| T^n h \| = \rho^{1/2} \| h \| \iff \| T^n h \| = \rho^{1/2} \| h \| , \quad n = 1,2,\dots ,$$

(Also by Theorem 2.1)

$$\iff (\rho-1)|\alpha_1|^2 + \| h \|^2 = \rho \| h \|^2$$

$$\iff (\rho-1) [|\alpha_1|^2 - \| h \|^2] = 0 ,$$

and hence if $\rho \underset{\neq}{>} 1$,

$$\lim_{n\to\infty} \| T^n h \| = \rho^{1/2} \| h \| \iff |\alpha_1| = \| h \| \iff h = \lambda e_1 , \quad \lambda \in \mathbb{C} .$$

Similarly,

$$\lim_{n\to\infty} \| T^n h \| = 0 \iff h = 0 , \quad (\text{for any } \rho \geq 1) .$$

Furthermore, if

$$1 < M < \rho^{1/2} ,$$

$$\lim_{n\to\infty} \| T^n h \| = M \| h \| \iff (M^2 - 1) \| h \|^2 = (\rho - 1) |\alpha_1|^2$$

$$\iff h = \sum_{k=1}^{\infty} \alpha_k e_k \quad \text{with}$$

$$(M^2 - 1) \sum_{k=1}^{\infty} |\alpha_k|^2 = (\rho - 1) |\alpha_1|^2 .$$

Similarly, if

$$0 < M < 1 \leq \rho^{1/2} ,$$

$$\lim_{n\to\infty} \| T^n h \| = M \|h\| \iff (\rho-1)|\alpha_1|^2 + (1-M^2)\|h\|^2 = 0$$

that is

$$\lim_{n\to\infty} \| T^n h \| = M \|h\| \iff h = 0 .$$

While if

$$M = 1 < \rho^{1/2} ,$$

$$\lim_{n\to\infty} \| T^n h \| = \|h\| \iff h = \sum_{k=2}^{\infty} \alpha_k e_k .$$

Thus, the unilateral weighted shift with weight sequence

$$\{ \rho^{1/2}, 1, 1, \dots \}$$

provides us with an example of an operator of $\mathcal{C}_\rho$ - class for $\rho \geq 1$, and examples of vectors in the underlying Hilbert space for which

$$\lim_{n\to\infty} \| T^n h \| = M \|h\| , \text{ for any } 0 \leq M \leq \rho^{1/2} .$$

CHAPTER 3

THE OPERATOR RADII OF HOLBROOK AND THEIR PROPERTIES

In Chapter 1 we have shown (Corollary 1.6) that if T is an arbitrary but bounded Hilbert space operator, then

$$W_{\rho,T} = \{\alpha > 0 : \frac{T}{\alpha} \varepsilon \mathcal{C}_\rho\} \neq \emptyset$$

for every $0 < \rho < \infty$. The following non-negative functional on $\mathcal{B}(H)$ defined by

$$\underline{w_\rho(T)} = \inf_\alpha W_{\rho,T}$$

is therefore well defined.

In this Chapter we shall show that the family $\{w_\rho(T) : \rho > 0\}$, $T \varepsilon \mathcal{B}(H)$, includes the familiar radii of operator theory associated with T, namely $\|T\| (= w_1(T))$, $w(T) (= w_2(T))$ and $r(T)(= w_\infty(T) = \lim_{\rho \to \infty} w_\rho(T))$. It thus seems natural to call w_ρ, $\rho > 0$, an operator radius. We shall also refer to $w_\rho(T)$ as the Holbrook radius of T, as it is J.A.R. Holbrook who first introduced this function and obtained its basic properties. ([22])

The importance of these radii lies in the fact that operators T of class $\mathcal{C}_\rho$ are characterized by $w_\rho(T) \leq 1$. We thus begin this Chapter, by proving this last statement and proceed to obtain some of the properties of w_ρ, in a natural way. The rest of this Chapter is devoted to the study of the Holbrook radii associated with nilpotent operators of index greater than or equal to 3.

The following theorem gives a characterization of the $\mathcal{C}_\rho$ classes associated with an arbitrary but bounded operator in terms of the operator's Holbrook radius.

THEOREM 3.1 [22]

Let $T \in \mathcal{B}(H)$.

Then, for any $\rho > 0$,

$$T \in \mathcal{C}_\rho \iff w_\rho(T) \leq 1 .$$

PROOF

Necessity follows immediately from the definition of $w_\rho(T)$. To establish the sufficiency of the condition assume that

$$w_\rho(T) \leq 1 \text{ , but } T \notin \mathcal{C}_\rho .$$

Then, there exist $0 < \alpha \leq 1$ and $z_o \in \mathbb{C}$, $|z_o| \geq 1$ such that

$$\text{(I)} \quad \| T \{(\rho-1)T - \rho \alpha z I\}^{-1} \| \leq 1 \text{ for all } z \in \mathbb{C} \text{ with } |z| \geq 1$$

and

$$\text{(II)} \quad \| T \{(\rho-1)T - \rho z_o I \}^{-1} \| > 1 ,$$

the first inequality corresponding to $w_\rho(T) \leq 1$, (and hence there exists $0 < \alpha \leq 1$ such that $\frac{T}{\alpha} \in \mathcal{C}_\rho$) and the second to $T \notin \mathcal{C}_\rho$.

Let now

$$z' = \frac{z_o}{\alpha} .$$

Clearly then,

$$|z'| = \frac{|z_o|}{\alpha} \geq |z_o| \geq 1 ,$$

and hence, taking $z = z'$ in (I) we get;

$$1 \geq \|T \{(\rho-1)T - \rho \alpha \frac{z_o}{\alpha} I\}^{-1}\| = \| T \{(\rho-1)T - \rho z_o I \}^{-1} \| > 1$$

in view of (II).

This being a contradiction, the required result follows.

Let now $T \in \mathcal{B}(H)$.

If $\alpha > 0$, is such that

$$\frac{1}{\alpha} T \in \mathcal{C}_\rho \quad \text{for some} \quad \rho > 0,$$

(such an α always exists in view of (iv), Thm. 1,5) then, since in particular $\frac{1}{\alpha}T$ is bounded in norm by ρ, that is

$$\left\| \frac{1}{\alpha} T \right\| \leq \rho,$$

it follows that

$$\alpha \geq \frac{1}{\rho} \| T \|,$$

and hence

$$w_\rho(T) = \inf\{\alpha > 0 \; ; \; \frac{1}{\alpha} T \in \mathcal{C}_\rho\} \geq \frac{1}{\rho} \| T \| . \tag{3.1}$$

Consequently, if for some $\rho > 0$,

$$w_\rho(T) = 0 \quad , \quad \text{then} \quad T \equiv 0 \quad . \tag{3.2}$$

On the other hand, as the zero operator 0, belongs to any $\mathcal{C}_\rho$ class, immediately from the definition, it follows that

$$w_\rho(0) = 0, \quad \rho > 0,$$

and hence as a result of this and (3.2) we have

$$w_\rho(T) = 0 \iff T \equiv 0 . \tag{3.3}$$

Observe also, that by virtue of Corollary 1,6 ,

$$w_\rho(T) < \infty, .$$

3.

for all $\rho > 0$, and every bounded Hilbert space operator T.
In fact referring to Theorem 1.5, we have

$$w_\rho(T) \leq (\frac{2}{\rho} - 1)\, \|T\|, \quad \text{if } 0 < \rho < 1$$

and

$$w_\rho(T) \leq \|T\|, \quad \text{if } \rho \geq 1. \tag{3.4}$$

Moreover, as for every $\rho > 0$, $\mathcal{C}_\rho$ is a balanced set, it is easy to show that

$$w_\rho(zT) = |z|\, w_\rho(T), \quad z \in \mathbb{C}, \tag{3.5}$$

that is, $w_\rho(\cdot)$ is a positively homogeneous functional on $\mathcal{B}(H)$ for any $\rho > 0$.
On the other hand, as by Theorem 1.5, $\mathcal{C}_\rho$ is a balanced, convex, absorbing set for any $0 < \rho \leq 2$, and $w_\rho(\cdot)$ is the associated Minkowski functional, it follows that for ρ in the prescribed range,

$$w_\rho(\cdot) \text{ is a norm on } \mathcal{B}(H).$$

Thus, for $0 < \rho \leq 2$,

$$w_\rho(\cdot) \text{ is a norm on } \mathcal{B}(H), \tag{3.6}$$

but not for $\rho > 2$, for as we shall see a little later on, if
$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then

$$w_\rho(T) = w_\rho(T^*) = \frac{1}{\rho}\|T\| = \frac{1}{\rho} \quad \text{for } \rho > 0,$$

while

$$w_\rho(T + T^*) = \|T + T^*\| = 1 \quad \text{for } \rho \geq 1,$$

and hence

$$1 = w_\rho(T + T^*) > w_\rho(T) + w_\rho(T^*) = \frac{2}{\rho} \quad \text{for } \rho > 2,$$

that is, $w_\rho(\cdot)$ fails to be be a subadditive functional on $\mathcal{B}(H)$ for $\rho > 2$, and thus cannot be a norm.

Observe also, that in view of Observation 1 and the homogeneity of $w_\rho(\cdot)$,

$w_\rho(\cdot)$ is a $*$-invariant of $\mathcal{B}(H)$, that is

$$w_\rho(T) = w_\rho(T^*) \quad \text{for all} \quad T \in \mathcal{B}(H), \tag{3.7}$$

while in view of Observation 6,

$w_\rho(\cdot)$ is also a unitary invariant of $\mathcal{B}(H)$, that is $w_\rho(T) = w_\rho(S)$ for all $\rho > 0$, and any pair of unitarily equivalent operators T and S. (3.8)

We next note that by virtue of theorem 3.1 and the homogeneity of $w_\rho(\cdot)$ and for an arbitrary operator $T \in \mathcal{B}(H)$,

$w_1(T)$ coincides with $\|T\|$, and

$w_2(T)$ with the numerical radius $w(T)$ of T.

Thus,

$$w_1(T) = \|T\| \tag{3.9}$$

and

$$w_2(T) = w(T) .$$

Furthermore, as

$$\mathcal{C}_\rho \subset \mathcal{C}_{\rho'} \quad \text{if} \quad \rho < \rho' ,$$

$w_\rho(\cdot)$ is a non-increasing function of ρ, for $\rho > 0$.

In other words, for $T \in \mathcal{B}(H)$,

$$w_{\rho'}(T) \leq w_\rho(T) , \quad \text{if} \quad \rho < \rho' \tag{3.10}$$

Moreover, as from (3.1) and (3.4) we have that

$$\frac{1}{\rho}\|T\| \leq w_\rho(T) \leq (\frac{2}{\rho}-1)\|T\|, \quad 0<\rho<1$$

and

$$\frac{1}{\rho}\|T\| \leq w_\rho(T) \leq \|T\|, \quad 1 \leq \rho$$

by (3.1) and (3.10), it follows that, for $0<\rho\leq 2$, $w_\rho(\cdot)$ is a norm on $\mathcal{B}(H)$ equivalent to the usual operator norm.

For a bounded but otherwise arbitrary operator T on a Hilbert space H, $w_\rho(T)$ turns out to be a convex function of ρ for $\rho \in (0,\infty)$, (for details we refer to [2]) and hence in particular, a continuous function of ρ for ρ in the same range.

Moreover, the other well known radius of operator theory, namely the spectral radius $r(T)$ of T, may be adjoined to the family $\{w_\rho(T) : \rho>0\}$ in a natural way, by

$$w_\infty(T) = \lim_{\rho\to\infty} w_\rho(T) = r(T) \tag{3.11}$$

To prove this statement, observe that the existence of a limit for $w_\rho(T)$ as $\rho\to\infty$ is guaranteed by the monotonicity and non-negativeness of the function in question.

It thus remains only to establish the actual value of this limit.

To this end, note that since

$$\frac{T}{w_\rho(T)} \in \mathcal{C}_\rho \quad \text{for all } \rho>0,$$

by virtue of the homogeneity of $w_\rho(\cdot)$ and Theorem 3.1, it follows that

$$r(\frac{T}{w_\rho(T)}) \leq 1, \ \rho>0, \qquad \text{(see Observation 3)}$$

and hence

$$r(T) \leq w_\rho(T) \;,\; \rho > 0 \,,$$

giving

$$r(T) \leq \lim_{\rho \to \infty} w_\rho(T) \,.$$

To obtain the inequality going the other way, we distinguish the following cases :

(a) $r(T) \neq 0$

and

(b) $r(T) = 0$.

If $r(T) \neq 0$, then for any $\varepsilon > 0$,

$$r\left(\frac{T}{(1+\varepsilon)r(T)}\right) = \frac{1}{1+\varepsilon} < 1 \,,$$

and hence

$$\frac{T}{(1+\varepsilon)r(T)} \in \mathcal{C}_{\rho_o} \,, \quad \text{for some } \rho_o \text{ sufficiently large} \,,$$

(see Proposition 1.2)

in other words,

$$w_{\rho_o}\left(\frac{T}{(1+\varepsilon)r(T)}\right) \leq 1 \,.$$

The monotonicity of $w_\rho(\cdot)$ now implies that

$$w_\rho\left(\frac{T}{(1+\varepsilon)r(T)}\right) \leq 1 \,, \quad \text{for all } \rho \geq \rho_o \,;$$

and hence

$$w_\rho(T) \leq (1+\varepsilon)r(T) \;,\; \rho \geq \rho_o \;,$$

from which the desired result follows .

If $r(T) = 0$, then for any $n > 0$,

$$r(nT) = 0 < 1 ,$$

and hence as before

$$w_\rho(nT) \leq 1 ,$$

for all ρ greater than or equal to some ρ_o, giving

$$w_\rho(T) \leq \frac{1}{n}, \quad \rho \geq \rho_o, \quad n > 0$$

and this clearly implies that

$$w_\infty(T) = \lim_{\rho\to 0} w_\rho(T) = 0 = r(T) ,$$

and the proof is seen to be complete.

One other very important result concerning the Holbrook radii of an arbitrary operator $T \in \mathcal{B}(H)$, is the so called "POWER INEQUALITY", namely that

$$w_\rho(T^k) \leq w_\rho(T)^k \quad \text{for all } \rho > 0 \text{ and } k \geq 1 . \tag{3.12}$$

This is an immediate consequence of Observation 2, the homogeneity of $w_\rho(\cdot)$ and Theorem 3.1 . Note also, that we have equality in (3.12) at $\rho = \infty$ in view of the spectral mapping theorem. Returning now to the C. Davis criterion for membership in a $\mathcal{C}_\rho$ class, namely that

$$\frac{T}{\alpha} \in \mathcal{C}_\rho \iff \|T\{(\rho-1)T - \rho\alpha z I\}^{-1}\| \leq 1 , |z| \geq 1 ,$$

and noting that

$$|\rho - 1| = |(2-\rho) - 1| \ ,$$

we obtain the following very useful "reciprocity law", due to T. Ando [2],

$$\rho \, w_\rho(T) = (2-\rho) \, w_{2-\rho}(T) \, , \quad 0 < \rho < 2 \ , \tag{3.13}$$

and hence, in particular, the graph of $\rho w_\rho(T)$ against ρ, for $0 < \rho < 2$, is symmetrical about the line $\rho = 1$.

Observe also, that as a consequence of Ando's reciprocity law, we have :

$$\lim_{\rho \to 0} \rho w_\rho(T) = 2 \, w_2(T) = 2w(T) \ , \tag{3.14}$$

from which the asymptotic behaviour of $w_\rho(T)$ near $\rho = 0$ is obtained.

Let now T be a nilpotent operator of index 2. Then, since the operator $\frac{T}{\|T\|}$ has norm 1,

$$\frac{T}{\|T\|} \in \mathcal{C}_1 \ ,$$

in view of Theorem 3.1 and (3.9), and hence with the usual notation,

$$\left(\frac{T}{\|T\|}\right)^n = P U^n \, , \quad n = 1,2,\ldots \ .$$

Consequently,

$$\left(\rho \frac{T}{\|T\|}\right)^n = \rho P U^n \, , \quad n = 1,2,\ldots, \text{ and all } \rho > 0 \ ,$$

due to the nilpotency of T, that is,

$$\rho \frac{T}{\|T\|} \in \mathcal{C}_\rho \, , \; \rho > 0 \; .$$

In particular,

$$w_\rho(\rho \frac{T}{\|T\|}) \leq 1 \, , \quad \rho > 0 \; ,$$

giving

$$w_\rho(T) \leq \frac{1}{\rho} \|T\| \qquad \rho > 0 \; ;$$

and since

$$w_\rho(T) \geq \frac{1}{\rho} \| T \| \, , \quad \text{always by } (3.1) \; ,$$

we have that for a nilpotent operator T of index 2,

$$w_\rho(T) = \frac{1}{\rho} \|T\| \, , \quad \rho > 0 \tag{3.15}$$

a result first obtained by J.A.R. Holbrook [22] .

Having obtained the basic properties of $w_\rho(\cdot)$, we now prove a Theorem concerning the shape of the curve representing $w_\rho(T)$ for an arbitrary operator $T \in \mathcal{B}(H)$. For ease of notation we shall write $w(\rho)$ instead of $w_\rho(T)$ when there is no danger of confusion.

<u>THEOREM 3.2</u> [24]

For an arbitrary but fixed operator $T \in \mathcal{B}(H)$,

EITHER

$w(\rho)$ is a strictly decreasing convex function of ρ on $(0, \infty]$

OR

there exists a value of ρ, ρ^* say, which is such that $w(\rho)$ is strictly decreasing and convex for all $\rho \varepsilon (0, \rho^*)$, and is constant and has the value

$$w(\rho^*) = r(T) \quad , \quad \text{for all} \quad \rho \varepsilon [\rho^*, \infty] .$$

PROOF

The following lemma, whose proof is included here because of its simplicity, is basic to our arguments.

Lemma 3.3 [22]

If for some ρ_1 and ρ_2, real and positive with $\rho_1 < \rho_2$ we have

$$w(\rho_1) = w(\rho_2) ,$$

then

$$w(\rho) = w(\rho_1) = r(T) \; (= w(\rho_2)) \quad \text{for all} \quad \rho \geq \rho_1 .$$

Proof of the lemma

If $\rho \varepsilon (\rho_1, \rho_2)$, then

$$w(\rho_1) = w(\rho_2) \leq w(\rho) \leq w(\rho_1) ,$$

and hence

$$w(\rho) = w(\rho_1) \quad \text{for all} \quad \rho \varepsilon [\rho_1, \rho_2] .$$

On the other hand, if $\infty > \rho > \rho_2$, then since $(\rho_2, 0)$ is an interior point of the line segment joining $(\rho_1, 0)$ to $(\rho, 0)$, there exists $\lambda \varepsilon (0, 1)$, such that

$$\rho_2 = \lambda \rho_1 + (1 - \lambda) \rho ,$$

and consequently

$$w(\rho_1) = w(\rho_2) = w(\lambda\rho_1 + (1-\lambda)\rho) \leq \lambda w(\rho_1) + (1-\lambda)w(\rho),$$

in view of the convexity of $w(\rho)$, and hence

$$w(\rho_1) \leq w(\rho) \quad \text{with} \quad \rho > \rho_2 .$$

But, by (3.1), $w(\rho) \leq w(\rho_1)$ also, and hence

$$w(\rho) = w(\rho_1) = w(\rho_2), \quad \rho \geq \rho_2.$$

Thus, for $\infty > \rho \geq \rho_1$, $w(\rho)$ is constant and equals $w(\rho_1)$, and hence being in particular a continuous function of ρ,

$$w(\rho) = w(\rho_1) = \lim_{\rho\to\infty} w(\rho) = r(T), \quad \rho \geq \rho_1,$$

and we are done.

Returning to the proof of the Theorem now, observe that in view of the monotonicity of $w(\rho)$, given any pair of positive real numbers σ_1 and σ_2 with $0 < \sigma_1 < \sigma_2 < \infty$, then

EITHER

$$w(\sigma_1) \underset{\neq}{<} w(\sigma_2),$$

OR

$$w(\sigma_1) = w(\sigma_2) .$$

In the second case, the lemma applies and the proof is seen to be complete.

Finally, note that, by virtue of Ando's reciprocity law, the real number ρ^* of the Theorem is necessarily greater than or equal to 1.

<u>COROLLARY</u> 3.4 [22]

If T is a normaloid operator, that is T has equal norm and spectral radius, then

$$w_\rho(T) = \frac{1+|\rho-1|}{\rho}\,\|T\| , \quad \rho > 0 . \tag{3.16}$$

<u>Proof</u>

$\|T\| = w_1(T) = w_\infty(T) = r(T)$, and the result follows from Lemma 3.3 and Ando's reciprocity law.

<u>FURTHER PROPERTIES OF THE HOLBROOK RADII</u>

For the reader's convenience we list below some further properties of the operator radii $\{w_\rho(T)\}_{\rho>0}$ associated with a non-zero operator $T \in \mathcal{B}(H)$.

We refer to [2], [22], [24], [25] and [26] for proofs and comments as well as to [44] for a treatment of these radii from an alternative viewpoint.

1. For $0 < \rho < \rho' \leq 1$,

$$\rho w_\rho(T) \leq (2\rho' - \rho)\, w_{\rho'}(T) , \tag{3.17}$$

while for $1 \leq \rho < \rho' < \infty$,

$$\rho w_\rho(T) \leq \rho' w_{\rho'}(T) . \tag{3.18}$$

Moreover, these inequalities are best possible. (See also Remark 1 immediately after this section)

2. For a pair T and S of doubly commuting operators,

$$w_{\rho\sigma}(TS) \leq w_\rho(T)\, w_\sigma(S) , \quad 0 < \rho \quad \sigma \leq \infty \tag{3.19}$$

<u>DEFINITION</u>

Two operators T and S are said to be doubly commuting, if

$$TS = ST \text{ and } T^*S = ST^* .$$

3. For an idempotent operator T (that is one for which $T^2 = T$),

$$w_\rho(T) = \frac{1}{\rho}\{w_1(T) + |\rho-1|\} = \frac{1}{\rho}\{\|T\| + |\rho-1|\} \quad \rho > 0 \tag{3.20}$$

while for an involutive operator S (ie. $S^2 = I$),

$$w_\rho(S) = \frac{1}{\rho}\{w_2(S) + \sqrt{w_2^2(S) + \rho(\rho-2)}\}$$

$$= \frac{1}{\rho}\{w(S) + \sqrt{w^2(S) + \rho(\rho-2)}\} \quad \rho > 0 . \tag{3.21}$$

Let now $\{T_n\}_{n=1}^{\infty}$ be a sequence of operators such that

$T_n \in \mathcal{B}(H_n)$, $n = 1,2,\ldots$, with $\sup_n \| T_n \| < \infty$.

We may define an operator T on $H = \overset{\infty}{\underset{1}{\oplus}} H_n$ as follows :

An element h of H is identified with a sequence

$\{h_n\}_{n=1}^{\infty}$ where $h_n \in H_n$, $n = 1,2,\ldots$, and $\|h\|^2 = \sum_1^{\infty} \|h_n\|^2 < \infty$.

We define Th to be $\{T_n h_n\}_{n=1}^{\infty}$.

It is then easy to see that T as defined above is a linear operator on H, and $\|T\| = \sup_n \|T_n\|$.

When T is constructed in this fashion we shall use the notation

$$T = \overset{\infty}{\underset{1}{\oplus}} T_n .$$

We then have :

4. $$w_\rho(\overset{\infty}{\underset{1}{\oplus}} T_n) = \sup_n w_\rho(T_n), \quad 0 < \rho < \infty \tag{3.22}$$

5. $$w_\infty(\overset{\infty}{\underset{1}{\oplus}} T_n) \geq \sup_n w_\infty(T_n) \tag{3.23}$$

while if $\sup_n \dim H_n < \infty$, also, we have

$$w_\rho(\overset{\infty}{\underset{1}{\oplus}} T_n) = \sup_n w_\infty(T_n) \ . \tag{3.24}$$

6. Denoting the restriction of an operator $T \in \mathcal{B}(H)$ to an invariant subspace M of H by $T\big|_M$ and recalling Observation 4 we obtain the following result:

$$w_\rho(T\big|_M) \leq w_\rho(T) \ , \qquad 0 < \rho \leq \infty \tag{3.25}$$

On the other hand, if H is spanned by a family $\{M_\alpha\}_{\alpha \in A}$ of subspaces invariant under T, we then have :

$$w_\rho(T) = \sup_{\alpha \in A} w_\rho(T\big|_{M_\alpha}) \ , \qquad 0 < \rho \leq \infty \ . \tag{3.26}$$

REMARKS

1. In [2.] it is stated by the authors that the inequality

$$\rho w_\rho(T) \leq (2\rho' - \rho) w_{\rho'}(T) ,$$

is best possible provided $0 < \rho < \rho' \leq 1$

It is our aim here, to prove that there is no non-zero operator T for which

$$\rho w_\rho(T) = (2\rho' - \rho) w_{\rho'}(T)$$

with $0 < \rho < \rho' < 1$, while if $\rho' = 1$ and there is equality in (3.17) for some $0 < \rho < 1$, then T is necessarily a normaloid operator.

Writing $w(\rho)$ instead of $w_\rho(T)$ we in fact show that there is no non-zero operator T and no $\varepsilon > 0$, however small, for which

$$(\rho-\varepsilon)w(\rho-\varepsilon) = (\rho+\varepsilon)w(\rho) \qquad 0 < \rho < 1 .$$

To this end, observe that, if this were the case, then by Ando's reciprocity law, we should also have that

$$(2 - \rho + \varepsilon)\, w(2 - \rho + \varepsilon) = (\rho+\varepsilon) \,.\, \frac{2-\rho}{\rho}\, w(2-\rho) ,$$

or equivalently, that

$$\frac{w(2-\rho+\varepsilon)}{w(2-\rho)} = \frac{(\rho+\varepsilon)(2-\rho)}{(2-\rho+\varepsilon)\rho} .$$

But

$$\frac{(\rho+\varepsilon)(2-\rho)}{(2-\rho+\varepsilon)\rho} - 1 = \frac{2\varepsilon(1-\rho)}{(2-\rho+\varepsilon)} > 0 ,$$

so that

$$\frac{w(2-\rho+\varepsilon)}{w(2-\rho)} > 1 .$$

This being a contradiction though, the result follows. On the other hand, if for some $0 < \rho < 1$,

$$\rho w(\rho) = (2-\rho)\ \|T\| ,$$

then, again by Ando's reciprocity law,

$$(2-\rho)w(2-\rho) = (2-\rho)\ \|T\| ,$$

and hence $w(2-\rho) = \|T\|$ $0 < \rho < 1$, in which case, Lemma 3.3 applies, and we are done.

2. Recall that in Lemma 1.3 of the first Chapter, it was shown that an involutive operator T of class $\mathcal{C}_\rho$, $\rho \geq 1$, is necessarily a symmetry, that is

$$T = T^* = T^{-1} .$$

The formula for the Holbrook radii for an involutive operator T, as given by (3.21), provides us with an alternative proof of this result, for if T is such an operator, then, since $w_\rho(T) \geq w_\infty(T) = r(T) = 1$, in addition to $w_\rho(T) \leq 1$, we have

$$w_2(T) + \sqrt{w_2^2(T) + \rho(\rho-2)} = \rho , \quad \rho \geq 1 ,$$

and hence

$$w_2(T) = 1 .$$

But, from (3.21) again,

$$w_1(T) = w_2(T) + \sqrt{w_2^2(T) - 1} ,$$

so that, $w_1(T) = \|T\| = 1$, also.

So, if x is any vector in the underlying Hilbert space, then

$$\begin{aligned} \| Tx - T^*x \|^2 &= \| Tx \|^2 + \| T^*x \|^2 - 2\mathrm{Re}(Tx, T^*x) \\ &= \| Tx \|^2 + \| T^*x \|^2 - 2\mathrm{Re}(T^2x, x) \\ &= \| Tx \|^2 + \| T^*x \|^2 - 2\|x\|^2 \\ &\leq 2\| x \|^2 - 2\| x \|^2 = 0 . \end{aligned}$$

Thus, $T = T^*$ and the proof is seen to be complete.

THE HOLBROOK RADII FOR NILPOTENT OPERATORS

Let T be a nilpotent operator of index 2.

Then, according to (3.15),

$$w_\rho(T) = \frac{1}{\rho}\,\|T\| \quad \rho > 0 .$$

Conversely now, we show that if T is a non-zero but otherwise arbitrary operator for which $w_\rho(T)$ is of the order of $\frac{1}{\rho}$, for $\rho > 0$ sufficiently large, then T is nilpotent of index 2.

More precisely we prove:

Theorem 3.4

If T is a non-zero, bounded Hilbert space operator, with

$$w_\rho(T) = O(\tfrac{1}{\rho}) ,$$

then T is nilpotent of index 2.

Proof

For some $A > 0$,

$$w_\rho(T) \leq \frac{A}{\rho}, \quad \text{for } \rho \geq \rho_o , \quad \text{say} .$$

Consequently, $\rho \frac{T}{A} \in \mathcal{C}_\rho$, for $\rho \geq \rho_o$.

In particular, $\rho \frac{T}{A}$ is power bounded by ρ, that is

$$\|(\rho \tfrac{T}{A})^n\| \leq \rho , \quad n = 1,2,\ldots, \quad \rho \geq \rho_o ,$$

and hence, taking $n = 2$ we have that.

$$\|T^2\| \leq \frac{A^2}{\rho}, \quad \rho \geq \rho_o ,$$

from which the desired result follows, upon letting $\rho \to \infty$.

The preceding Theorem admits the following generalization, in the form of

Theorem 3.5

Let k be a positive integer and assume that T is a bounded non-zero operator with

$$w_\rho(T) = O\left(\frac{1}{\rho^{1/k}}\right) .$$

Then, T is nilpotent of index at most $k+1$. Note that, if $k=1$, then the index of nilpotency of T, is exactly $k+1=2$ by the Theorem just proved.

Proof

Since, $w_\rho(T) = O\left(\frac{1}{\rho^{1/k}}\right)$, we conclude as before that

$$\left\| \left(\rho^{1/k} \frac{T}{A}\right)^n \right\| \leq \rho , \quad \text{for all } \rho \geq \rho_0 \text{, say, and all } n = 1,2,\ldots,$$

where A is a positive constant independent of ρ .

If we now let $n = k+1$ in the inequality above and then let $\rho \to \infty$ we obtain what we set out to prove.

Observe now, that as far as the theory of Holbrook radii for nilpotent operators is concerned, the case where the index of nilpotency is 2 presents, by virtue of Theorem 3.4 and 3.5, no more interest, and hence from now on we shall only consider nilpotent operators of index of nilpotency $k+1$, where k is an integer strictly greater than 1. Let therefore T be such a nilpotent operator.

Then, if $j = [\frac{(k+1)+1}{2}]$, ($[x]$ = the integer part of the real number x) we have:

$$j = \begin{cases} \frac{k+1}{2} & \text{if } k+1 \text{ is even} \\ \frac{k+2}{2} & \text{if } k+1 \text{ is odd} \end{cases} ,$$

and hence, as in both cases,

$$2j \geq k+1 \quad ,$$

but $$j \leq k \ ,$$

the operator T^j is nilpotent of index 2, and consequently

$$w_\rho(T^j) = \frac{1}{\rho} \|T^j\| \ , \ \rho > 0 \ . \tag{3.27}$$

An application of the 'power inequality' for the operator radii now yields

$$w_\rho(T) \geq \frac{1}{\rho^{1/j}} \|T^j\|^{1/j} \ , \quad \rho > 0 \ . \tag{3.28}$$

On the other hand, as $k > 1$,

$$2k > k+1 \ ,$$

and hence T^k is also nilpotent of index 2.

Thus, working as before, we conclude that

$$w_\rho(T) \geq \frac{1}{\rho^{1/k}} \|T^k\|^{1/k} \ , \quad \rho > 0 \tag{3.29}$$

If we now observe that

$$j = k \Longleftrightarrow k = 2 \ , \tag{3.30}$$

it follows that in general, the inequalities given by (3.28) and (3.29) will not be the same. We thus have the following important inequalities satisfied by the Holbrook radii of a nilpotent operator T of index $k+1$, $k > 2$.

$$w_\rho(T) \geq \frac{1}{\rho^{1/j}} \|T^j\|^{1/j} , \rho > 0 , \quad j = \left[\frac{(k+1)+1}{2}\right] \ , \tag{3.31}$$

and

$$w_\rho(T) \geq \frac{1}{\rho^{1/k}} \|T^k\|^{1/k} , \rho > 0 \quad . \tag{3.32}$$

We now restrict temporarily our attention to those nilpotent operators of index $k+1$, $k > 2$, which are such that

$$\|T^k\| = \|T\|^K$$

e.g.

$$T = a \cdot \underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}}_{k+1} \Bigg\} k+1 \quad , a \in \mathbb{C} ,$$

or for that matter any other operator unitarily equivalent to the one above.

Referring to Lemma 2.8 , we have that for these operators

$$\|T^m\| = \|T\|^m, \quad m = 1,2,\ldots, k ,$$

and hence, in particular,

$$\|T^j\| = \|T\|^j, \quad j = \left[\frac{(k+1)+1}{2}\right] .$$

For this special class of nilpotent operators, the inequalities (3.31) and (3.32) therefore reduce to

$$w_\rho(T) \geq \frac{1}{\rho^{1/j}} \|T\|, \quad j = \left[\frac{(k+1)+1}{2}\right] , \rho > 0 .$$

and

$$w_\rho(T) \geq \frac{1}{\rho^{1/k}} \|T\|, \quad \rho > 0 .$$

Furthermore, as $j < k$, the following proposition is easily obtained.

Proposition 3.6

If T is a nilpotent operator of index $k+1$, $k > 2$, with

$$\|T^k\| = \|T\|^k ,$$

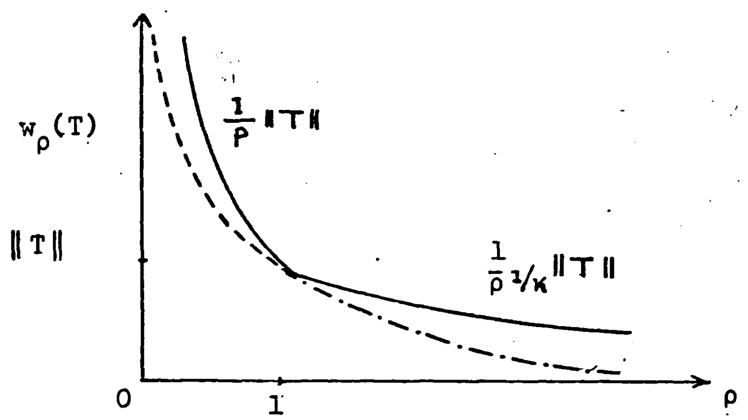
then

$$w_\rho(T) \geq \frac{1}{\rho} \|T\| \quad \text{for} \quad 0 < \rho < 1 ,$$

and

$$w_\rho(T) \geq \frac{1}{\rho^{1/k}} \|T\| \text{ for } 1 \leq \rho .$$

Thus, the curve representing $w_\rho(T)$ for such a nilpotent operator, will lie in the region of the plane on or above the thick lined curve in the accompanying diagram, passing of course, through the point $(1, \|T\|)$.



We shall return to this particular class of nilpotent operators later on.

One other important fact concerning the Holbrook radii associated with a nilpotent operator T, is that $w_\rho(T)$ is a strictly decreasing to $0 = r(T)$ convex function of ρ on $(0,\infty]$. This follows immediately from Theorem 3.2, since for any $\rho \in (0,\infty)$, $w_\rho(T) > r(T) = 0$.

Returning now to the inequalities given by (3.31) and (3.32), it is fairly straightforward to see that eventually, i.e. when ρ becomes greater than or equal to some ρ_0, the inequality (3.32) will be the dominant one.

The next theorem proves that for a nilpotent operator T of index $k+1$, $k>2$, $w_\rho(T)$ as a function of ρ tends to 0 as $\rho \to \infty$, like

$$\rho^{-1/k} \| T^k \|^{1/k} ,$$

that is, $w_\rho(T)$ tends asymptotically to $\rho^{-1/k} \| T^k \|^{1/k}$ as $\rho \to \infty$.

Theorem 3.7

For a nilpotent operator T of index $k+1$, $k \geq 2$,

$$\lim_{\rho \to \infty} \rho^{1/k} w_\rho(T) = \| T^k \|^{1/k} .$$

Proof

The following lemma is essential for our purposes.

LEMMA 3.8

Let T be an arbitrary, non-zero, bounded operator on a Hilbert space H.

Then, for any $\rho > 0$,

$$\max_{|\bar{z}| \geq 1} \| T \{(\rho-1)T - \rho z w_\rho(T) I\}^{-1} \| = 1 .$$

Proof of the Lemma

Observe firstly, that if $|z| \geq 1$, then

$$r\left(\frac{(\rho-1)T}{\rho z w_\rho(T)} \right) = \frac{|\rho-1| r(T)}{\rho |z| w_\rho(T)} \leq \frac{|\rho-1| r(T)}{\rho w_\rho(T)}$$

$$= \begin{cases} (1-\rho) \dfrac{r(T)}{\rho w_\rho(T)} , & 0 < \rho < 1 \\[2ex] \dfrac{(\rho-1)}{\rho} \cdot \dfrac{r(T)}{w_\rho(T)} , & 1 \leq \rho . \end{cases}$$

But, since for all $\rho > 0$,

$$r(T) \leq w_\rho(T) \quad \text{and}$$

$$\|T\| = w_1(T) \leq \rho\, w_\rho(T) ,$$

we have

$$r\left(\frac{(\rho-1)T}{\rho z w_\rho(T)} \right) \leq \begin{cases} (1-\rho)\frac{r(T)}{\|T\|} \\ 1 - \frac{1}{\rho} \end{cases} \leq \begin{cases} 1-\rho , & 0<\rho<1 \\ 1 - \frac{1}{\rho} , & 1 \leq \rho \end{cases} < 1 .$$

Thus

$$r\left(\frac{(\rho-1)T}{\rho z w_\rho(T)} \right) < 1, \quad \rho > 0, \; |z| \geq 1 ,$$

and consequently, the operator

$$\{ (\rho-1)T - \rho\, z\, w_\rho(T)\, I \}^{-1}$$

is well defined, ie. the operator $\{ (\rho-1)T - \rho z w_\rho(T) \}$ is boundedly invertible for any $|z| \geq 1$ and all $\rho > 0$.

Moreover, the operator valued function of the complex variable z, given by

$$f(z) = T\{ (\rho-1)T - \rho\, z\, w_\rho(T)\, I \}^{-1}$$

is analytic for $|z| \geq 1$, and since

$$\frac{T}{w_\rho(T)} \in \mathcal{C}_\rho \quad \text{for all} \quad \rho > 0 .$$

$$\| f(z) \| = \| T\{(\rho-1)T - \rho\, z\, w_\rho(T) I\}^{-1} \| \leq 1 , \quad |z| \geq 1.$$

Hence, an application of the maximum modulus principle for analytic functions of a complex variable, implies that

$$\max_{|z| \geq 1} \| T\{(\rho-1)\,T - \rho\, z\, w_\rho(T)\, I\}^{-1} \|$$

$$= \|T\{(\rho-1)T - \rho e^{i\theta} w_\rho(T) I\}^{-1}\|$$

for some $\theta \in [0, 2\pi)$, is at most one.

The lemma claims that this maximum is actually equal to one. To this end, consider the non-negatively valued function of the complex variable z, defined by

$$G(z) = \|T\{(\rho-1)T - \rho z I\}^{-1}\|, \quad z \in \mathbb{C}.$$

Since by the spectral mapping theorem,

$$sp\{(\rho-1)T - \rho z I\} = (\rho-1)\, spT - \rho z,$$

$$0 \in sp\{(\rho-1)T - \rho z I\} \text{ for suitable } z \in \mathbb{C},$$

and hence

$$\sup G(z) = +\infty.$$

Also, as

$$G(z) \to 0 \text{ when } z \to \infty, \text{ we have}$$

$$0 \leq G(z) < \infty.$$

So,

$$G_0(|z|) = \sup_\theta \|T\{(\rho-1)T - \rho|z|e^{i\theta}I\}^{-1}\|$$

which is a continuous function of $|z|$, takes all values in $[0,\infty)$ and in particular the value 1.

Now, for every $\rho > 0$ and any $\alpha \in \mathbb{R}^+$,

$$\frac{T}{\alpha} \in \mathcal{C}_\rho \iff \|T\{(\rho-1)T - \rho z I\}^{-1}\| \leq 1, \quad |z| \geq \alpha.$$

Suppose

$$\alpha_1 = \inf\{\alpha > 0 : \|T\{(\rho-1)T - \rho z I\}^{-1}\| = 1 \text{ for } |z| \geq \alpha\}.$$

So,

$$\| T\{(\rho-1)T-\rho zI\}^{-1} \| = 1 \ , \quad \text{for} \quad z = \alpha_1 e^{i\theta} \text{ for some } \theta \, .$$

If $\alpha_1 < w_\rho(T)$, a contradiction follows, since

$$\| T\{(\rho-1)T-\rho zI\}^{-1} \| = 1 \quad \text{on } |z| = \alpha_1$$

implies that

$$\frac{T}{\alpha_1} \in \mathscr{C}_\rho \, .$$

thus contradicting the definition of $w_\rho(T)$. If $\alpha_1 > w_\rho(T)$ a contradiction again follows, since by the maximum modulus principle.

$$1 = \max_{|z|=\alpha_1} \| T\{(\rho-1)T-\rho zI\}^{-1} \| < \max_{|z|=w_\rho(T)} \| T\{(\rho-1)T-\rho zI\}^{-1} \| \leq 1.$$

So, $\alpha_1 = w_\rho(T)$, and the result follows

Returning now to the proof of the Theorem, we have as a consequence of the Lemma, that

$$\text{(3.33)} \qquad \| T\{(\rho-1)T-\rho w_\rho(T) e^{i\theta} I\}^{-1} \| = 1 \quad \text{for some } \theta \in [0,2\pi) \, .$$

But, T is nilpotent (of index $k+1$) whenever $e^{-i\theta} T$ is, and hence we may without loss of generality assume that $\theta = 0$ in (3.33). Thus,

$$\| T \{(\rho-1)T-\rho w_\rho(T) I\}^{-1} \| = 1 \, . \tag{3.34}$$

On the other hand,

$$\{(\rho-1)T-\rho w_\rho(T) I\}^{-1} = -\frac{1}{\rho w_\rho(T)} \sum_{n=0}^{k} \left[\frac{(\rho-1)T}{\rho w_\rho(T)} \right]^n ,$$

and hence, substituting back in (3.34) and using the fact that T

is nilpotent of index $k+1$, it follows that

$$\left\| \frac{T}{\rho w_\rho(T)} + \left(\frac{\rho-1}{\rho}\right) \frac{T^2}{\rho w_\rho^2(T)} + \dots + \left(\frac{\rho-1}{\rho}\right)^{k-1} \frac{T^k}{\rho w_\rho^k(T)} \right\| = 1 ,$$

so that

$$(3.35) \quad \rho\, w_\rho^k(T) = \left\| w_\rho^{k-1}(T) \,.\, T + \left(\frac{\rho-1}{\rho}\right) w_\rho^{k-2}(T) . T^2 + \dots + \left(\frac{\rho-1}{\rho}\right)^{k-1} . T^k \right\| .$$

Letting now $\rho \to \infty$ in (3.35) we get :

$$\lim_{\rho \to \infty} \rho w_\rho^k(T) = \| T^k \| . \tag{3.36}$$

The desired result now follows upon taking the k-th root of both sides of (3.36).

APPENDIX

The formula given by (3.35) is especially useful in that it allows us, theoretically at least, to obtain a polynomial equation for the Holbrook radii of nilpotent matrices of index $k+1$, $k \geq 2$. Use of this equation has resulted in obtaining an expression for $w_\rho(T)$ in the special cases listed below. The calculations involved are sometimes tedious, but nevertheless straightforward.

1. $T = T(\rho) = \rho^{-1/k}$ subdiagonal $(\rho^{\frac{1}{2}}, \underbrace{1,\ldots,1}_{k}, \rho^{\frac{1}{2}})$, $\rho > 0$, $k \geq 3$.

$$w_\rho(T) = \frac{1}{\rho}(1 + |\rho - 1|)^{\frac{k-1}{k}} , \quad \rho > 0 .$$

N.B. Subdiagonal $(x_1, x_2, \ldots, x_n)$ stands for the $(n+1)\times(n+1)$ matrix whose only non-zero elements are those on the diagonal immediately below the main one, with

$$a_{i,\, i-1} = x_{i-1} , \quad i = 1,2,\ldots,n+1.$$

Note also that this result can be obtained directly from Theorem 2.3 .

2. $$T = \begin{bmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{bmatrix} , \quad a, c \in \mathbb{C} - \{0\} , b \in \mathbb{C} .$$

Here $w_\rho(T) = w(\rho)$ satisfies

$$\rho^4 w^4(\rho) - \rho^2 w^2(\rho)(|a|^2 + |b|^2 + |c|^2) + \rho(2-\rho)|ac|^2$$

$$- 2\rho\, |\rho-1| \rho w(\rho)[4w^3(2) - w(2)(|a|^2 + |b|^2 + |c|^2)] = 0 ,$$

where $w(2)$ is the largest positive solution of the cubic equation

$$4x^3 - x\,(|a|^2 + |b|^2 + |c|^2) = \mathrm{Re}\, b\,(\overline{ac})\ .$$

In particular if $b = 0$, $T =$ subdiagonal $(a\,, c)$ and

$$\rho^4\, w^4(\rho) - \rho^2 w^2(\rho)(|a|^2 + |c|^2) + |ac|^2 = (\rho - 1)^2\, |ac|^2\,,\ \rho > 0\,.$$

Hence, for $|a| = |c|$, that is for

$$T = a \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \end{bmatrix}\,, \qquad a \in \mathbb{C} - \{0\}\,, \theta \in \mathbb{R}\,,$$

$$w_\rho(T) = \frac{|a|}{\rho}\,(1 + |\rho - 1|)^{\frac{1}{2}}\,,\ \rho > 0\,.$$

On the other hand if $a = b = c = 1$, $w_\rho(T)$ is given by

$$w_\rho(T) = \frac{1}{2\rho}\,[1 + (5 + 4|\rho - 1|)^{\frac{1}{2}}]\,,\quad \rho > 0$$

while if $a = c = 1$, $b = -1$,

$$w_\rho(T) = \frac{1}{2\rho}\,[1 + (5 - 4|\rho - 1|)^{\frac{1}{2}}]\,,\quad \rho > 0\,.$$

In general, as every 3×3 nilpotent of index 3 matrix T is unitarily equivalent to a lower diagonal matrix, and $w_\rho(\cdot)$ is a unitary invariant of $\mathcal{B}(\mathcal{H})$, it follows that the Holbrook radius $w_\rho(T) = w(\rho)$ satisfies a polynomial equation of the form

$$\rho^4\, w^4(\rho) + A\rho^2\, w^2(\rho) + B\,|\rho - 1|\, w(\rho) + C\rho(2 - \rho) = 0,\quad \rho > 0,$$

for suitable constants A, B, C with A and C not equal to zero. Taking into account the fact that $\lim\limits_{\rho \to \infty} \rho\, w^2(\rho) = \|T^2\|$ (by (3.36)) and $\lim\limits_{\rho \to \infty} w(\rho) = 0$, it follows that

$$C = \|T^2\|^2\ .$$

On the other hand, since $w(1) = \|T\|$ and $w(2) = w(T)$, trivial calculations reveal that actually we have:

$$A = -(\|T\|^2 + \frac{\|T^2\|^2}{\|T\|^2}) \quad \text{and} \quad B = -2w(T)[4w^2(T) + A] .$$

Among nilpotent operators those with index of nilpotency $n = 2$ or $n = 3$ seem to play an important role in the theory of Holbrook radii, maybe because, in the notation of the previous Chapter, these are the only nilpotent operators for which $j = k$. On the other hand, as quite a lot is known about nilpotent operators of index 2, we restrict our attention to the case where the index of nilpotency is 3, and we make a conjecture concerning the Holbrook radii of nilpotent operators T of index 3 for which $\|T^2\| = \|T\|^2$.

More precisely,

CONJECTURE

Let T be a nilpotent operator of index 3 with $\|T^2\| = \|T\|^2$.

Then,

$$w_\rho(T) = \|T^2\|^{\frac{1}{2}} \frac{(1+|\rho-1|)^{\frac{1}{2}}}{\rho} \quad \rho > 0 .$$

One of the reasons we think the conjecture is true, is that if T is a 3×3 nilpotent matrix of index 3 with $\|T^2\| = \|T\|^2$, then as T is unitarily equivalent to a matrix of the form

$$a \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & e^{i\theta} & 0 \end{bmatrix} , \quad a \in C - \{0\} , \quad \theta \in R ,$$

$$w_\rho(T) = |a| \frac{(1+|\rho-1|)^{\frac{1}{2}}}{\rho} = \|T^2\|^{\frac{1}{2}} \frac{(1+|\rho-1|)^{\frac{1}{2}}}{\rho}, \rho > 0 .$$

Geometrical considerations also seem to support this conjecture, though this conjecture fails for nilpotent operators of index greater than 3, that is, it is not true that for a nilpotent operator T if index $k+1$, $k \geq 3$, with $\|T^k\| = \|T\|^k$,

$$w_\rho(T) = \|T^k\|^{1/k} \frac{(1+|\rho-1|)^{1/k}}{\rho}, \rho > 0 .$$

The following example clearly demonstrates this.
Let T = subdiagonal (1, 1, 1). Then clearly $T^4 \equiv 0$ and $\|T^3\| = \|T\|^3$. However, we can show that for this operator $w_\rho(T) = w(\rho)$ satisfies

$$\rho^6 w^6(\rho) - 3\rho^4 w^4(\rho) + [3-2(\rho-1)^2]\rho^2 w^2(\rho) - \rho^2(\rho-2)^2 = 0, \ \rho > 0 .$$

This polynomial equation for $\rho w(\rho)$ does not admit a solution of the form $\rho w(\rho) = \rho^{2/3}$, for any $\rho > 1$, which would have been the case if the generalized conjecture were true.
Finally, it also seems plausible to conjecture that for a nilpotent operator T of index 3, and for obvious reasons,

$$\rho^4 w^4(\rho) + A\rho^2 w^2(\rho) + B|\rho-1|\rho w(\rho) + C\rho(2-\rho) = 0 , \quad \rho > 0,$$

where

$$A = -(\|T\|^2 + \frac{\|T^2\|^2}{\|T\|^2}) \quad B = -2w(T)[4w^2(T) + A]$$

and $\qquad C = \|T^2\|^2$.

In conclusion note that, if T is a quasinilpotent operator, so that $\lim_{n\to\infty} \|T^n\|^{1/n} = 0$, it is reasonable to expect, taking into account (3.36), that for such an operator,

$$\lim_{n\to\infty} (\lim_{\rho\to\infty} \rho^{1/n} w_\rho(T)) = 0 .$$

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