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COVERING PROBLEMS IN EUCLIDEAN SPACE

CAROLE HATFIELD

ABSTRACT

Our purpose in these pages will be to develop a broad survey of some problems in covering which have been solved using the methods of convexity. The basic aim has been to show the flexibility of approach which is desirable, and to do this we shall discuss ten particular problems with widely differing methods of solution.

The problems to be discussed fall into three main categories: covering a single set by a single set with specified properties, covering a single set by many sets, and covering many sets by a single set.

The first chapter is a collection of some necessary preliminary definitions and theorems in convexity. In the second chapter we consider six particular problems of covering a single set by a single set: these include finding the convex covers of certain arcs in two and three dimensions, finding the smallest containing sphere of an n-dimensional set, and establishing properties of particular types of containing set.

The final chapter contains problems which fall into the other two categories outlined above, and includes finding a single set to cover an infinite class of sets, and covering one set by a finite class of sets. The last paper considered relates one set (the plane) and a given infinite class of sets: it is included for the sake of completeness and because of the methods of solution involved, but in fact can be considered as a problem in covering only if the latter term is not restricted to mean complete covering by a class of sets.

A full list of references to the original papers is given at the end.

CHAPTER I

PRELIMINARY RESULTS

The purpose of this first chapter is to establish a number of results which will be used later in the work, and also to state briefly what will be taken as common ground in subsequent discussion. We deal with the latter first.

It will be assumed that the most common terms used in set theory - for example, set union and intersection; complement; disjointness; the exterior, interior, frontier of a set; open and closed sets; discrete sets - are all known, and definitions are not given. Knowledge will be assumed of the meaning of statements such as

a set X contains a set Y ,

a set is symmetric in an interior point,

two sets X and Y are congruent,

a set is reflected in a point,

X circumscribes Y , or Y is inscribed in X .

We shall borrow without definition a number of terms from elementary mathematics, and in addition we shall use without proofs or definition of terms results from measure theory and elementary analysis, in particular that

(i) the uniform limit of a sequence of continuous functions is continuous,

(ii) a polygon of given perimeter length has maximum area when it is regular.

We shall make free use of set theory notation; of vector notation where it is convenient; shall denote the left-hand side (right-hand side) of an equation or inequation by LHS (RHS); and shall use as equivalent terms the words perimeter, boundary, frontier.

Since, however, the papers to be discussed deal mainly with problems involving convex sets and their properties, we set out in detail now a number of results from convexity which will be quoted or referred to in the other chapters. [Ref.]

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Euclidean space of n dimensions is denoted by E_n. A point is usually denoted by a small letter, and a set by a capital letter. The frontier of a set X is denoted by Fr(X), its interior by X°, and its closure by $\overline{X}$.

E_n is the class of all ordered sets of real numbers $(x_1, x_2, \ldots x_n)$, denoted by x, with a metric defined between two elements x, y, where $y = (y_1, y_2, \ldots y_n)$, of the class as

$$|x - y| = \left[\sum_{i=1}^{n} (x_i - y_i)^2\right]^{\frac{1}{2}},$$

and called the <u>distance</u> between x and y. This is the usual n-dimensional Euclidean space. We shall, where desirable, be free to regard the elements of the space as vectors, obeying the usual laws of a vector space.

The origin in E_n is denoted by o, and the line segment joining two distinct points x, y is denoted by xy.

A subset of E_n which is the empty set is denoted by ϕ.

The distance function has the following properties:

(i) $|x - y| = |y - x|$,

(ii) $|x - (y + z)| = |(x - y) - z|$,

(iii) $|x - y| \geq 0$, with equality if and only if x and y are coincident,

(iv) $|ax - ay| = a|x - y|$ for $a \geq 0$,

(v) $|x - y| + |y - z| \geq |x - z|$.

A subset X of E_n is <u>bounded</u> if there is a sphere of some finite radius r which contains X .

We define the <u>distance $\rho(x,Y)$ between a point x and a set Y</u> as

$$\rho(x,Y) = \inf_{y \in Y} |x - y| .$$

A <u>neighbourhood of a set</u> Y is denoted by $U(Y,\delta)$ and is defined as the set of points whose distance from the set Y is less than δ .

A <u>hyperplane</u> P is defined as the set of points x satisfying the equation $f(x) = 0$, where $f(x)$ is an expression of the form $a_1x_1 + a_2x_2 + \dots + a_nx_n - b$, where not all the a_i are zero. The inequations $f(x) > 0$, $f(x) < 0$, or $f(x) \geq 0$, $f(x) \leq 0$, define open or closed half-spaces respectively.

If two subsets X , Y of E_n are such that one of the closed half-spaces bounded by a hyperplane P contains X and the other closed half-space contains Y , we say that P <u>separates</u> X and Y . (If the same statement is true for the open half-spaces bounded by P , we say that P separates X and Y <u>strictly</u>.)

A <u>convex set</u> X is defined as a set of points such that, whenever x_1, x_2 are two points of X, the line segment containing every point of the form $\lambda x_1 + \mu x_2$, where $\lambda \geqslant 0$, $\mu \geqslant 0$ and $\lambda + \mu = 1$, is contained in X.

<u>Theorem 1</u>. If X is a convex set and if $x_1, x_2, \ldots, x_k$ are k points of X, then every point of the form

$$x = \mu_1 x_1 + \mu_2 x_2 + \ldots + \mu_k x_k ,$$

where $\mu_i \geqslant 0$ and $\mu_1 + \mu_2 + \ldots + \mu_k = 1$, also belongs to X.

<u>Proof</u>. For $k = 2$, this is just the definition of a convex set. Assume inductively that the assertion is true for $k = m$, and consider a point x of the form

$$x = \mu_1 x_1 + \ldots + \mu_m x_m + \mu_{m+1} x_{m+1}$$

where $\mu_i \geqslant 0$, $\mu_1 + \ldots + \mu_m + \mu_{m+1} = 1$.

If $\mu_{m+1} = 1$, then $x = x_{m+1} \in X$ and there is nothing to prove. So suppose $\mu_{m+1} < 1$, in which case

$$\mu_1 + \ldots + \mu_m = 1 - \mu_{m+1} > 0 .$$

Then we may write

$$x = (\mu_1 + \ldots + \mu_m)\left(\frac{\mu_1}{\mu_1 + \ldots + \mu_m} x_1 + \ldots + \frac{\mu_m}{\mu_1 + \ldots + \mu_m} x_m\right) + \mu_{m+1} x_{m+1} ,$$

and by the inductive hypothesis we know that the point y given by

$$y = \frac{\mu_1}{\mu_1 + \ldots + \mu_m} x_1 + \ldots + \frac{\mu_m}{\mu_1 + \ldots + \mu_m} x_m$$

is a point of X. Then

$$x = (\mu_1 + \ldots + \mu_m) y + \mu_{m+1} x_{m+1}$$

and since X is convex and contains both y and x_{m+1} , it follows from the definition of a convex set that, since $\mu_1 + \ldots + \mu_m + \mu_{m+1} = 1$, X must contain x also. Thus the inductive hypothesis is true for $k = m+1$, and hence the theorem is proved.

A set is <u>strictly convex</u> if it is convex and its frontier contains no straight line segments.

Two sets will be said to <u>overlap</u> if they have interior points in common (but not if they have only frontier points in common).

<u>Theorem 2.</u> The intersection of any number of convex sets, if it is non-empty, is convex.

<u>Proof.</u> Let I be an index set of elements i , and consider the family of convex sets X_i , $i \in I$. If x , y are two points of $\bigcap_{i\in I} X_i$, then x , y are points also of X_i . Since X_i is convex, this implies that the segment xy is contained in X_i . But X_i is an arbitrarily chosen set, and so it follows that the segment xy belongs to $\bigcap_{i\in I} X_i$ Thus $\bigcap_{i\in I} X_i$ is convex.

A hyperplane which intersects the closure of a convex set X but which does not have any point in common with an interior point of X is called a <u>support hyperplane</u> of X . We say that such a hyperplane supports X at the point or points at which it meets the closure of X .

Theorem 3. Every point of the frontier of a convex set X has at least one support hyperplane of X passing through it.

Proof. If X is not n-dimensional, any hyperplane containing X is a support hyperplane of X at all points of X .

If X is n-dimensional, we can show (lemma 1 - page 28) that there is a hyperplane P which separates X^o , the interior of X , from a given point p of its frontier. P does not have any point in common with X^o since this is an open set, but since X is convex we know (lemma 2 - page 30) that p belongs to the closure of X^o , and so it follows that P must contain p . Thus P is a support hyperplane of X at p .

The convex cover of a given point set X is defined as the intersection of all the convex sets which contain X , and is the smallest convex set containing X .

The diameter d(X) of a set X is defined as the least upper bound of the distances between any two points of the set.

Theorem 4. The diameter of the convex cover of a set is equal to the diameter of the set.

Proof. Consider any set X . Let d(X) denote its diameter, and let

$H(X)$ denote the convex cover of X. Since X is contained in $H(X)$, it follows immediately that $d(H(X)) \geqslant d(X)$.

Suppose that $d(H(X)) > d(X)$, and put $\delta = d(X)$. If we take any point x_2 of X we have then that $|x_1 - x_2| \leqslant \delta$ for all $x_1 \in X$, and it follows that the set X is contained in the (n-dimensional) sphere $S(x_2, \delta)$ with centre x_2 and radius δ. Since $S(x_2, \delta)$ is convex, the convex cover $H(X)$ of the set X is also contained in $S(x_2, \delta)$.

Now let y_1, y_2 be any two points of $H(X)$. In particular, we have that y_1 is a point of $S(x_2, \delta)$, and hence x_2 is a point of $S(y_1, \delta)$. Since x_2 was chosen arbitrarily, we have that the set X is contained in $S(y_1, \delta)$, and so, again since the sphere is convex, $H(X)$ is contained in $S(y_1, \delta)$. But then since y_2 is a point of $H(X)$, we have $|y_1 - y_2| \leqslant \delta$, and hence, since y_1, y_2 are arbitrary, $d(H(X)) \leqslant \delta = d(X)$. This is a contradiction with our assumption, and the theorem follows.

<u>Theorem 5. (Helly's theorem.)</u> Suppose that there is given in E_n a finite class of N convex sets, with $N \geqslant n+1$, such that for every subclass which contains $(n+1)$ members there is a point of E_n which is common to every member of the subclass. With these conditions holding, there is a point of E_n which is common to every member of the whole class of N sets.

Proof. Obviously the theorem is true when $N = n+1$, and we use this to prove the general result by induction.

Assume, then, that the theorem has been proved true for every class of $(N-1)$ sets, and consider the case when there are N sets, X_1 , X_2 , ..., X_N , say, in the class.

According to the induction hypothesis, if we take out the set X_i , then there is a point

$$\underline{x}_i = (x_{1i}, x_{2i}, \ldots, x_{ni})$$

which is a point of X_j for all $j \neq i$. Now consider the set of $(n+1)$ equations given by

$$\begin{cases} \mu_1 x_{11} + \mu_2 x_{12} + \cdots + \mu_N x_{1N} = 0 \\ \mu_1 x_{21} + \mu_2 x_{22} + \cdots + \mu_N x_{2N} = 0 \\ \cdots\cdots\cdots\cdots \\ \mu_1 x_{n1} + \mu_2 x_{n2} + \cdots + \mu_N x_{nN} = 0 \\ \mu_1 + \mu_2 + \cdots + \mu_N = 0 \end{cases} \qquad (1)$$

in the N unknowns μ_1 , μ_2 , ..., μ_N . Since we have assumed $N-1 \geq n+1$, that is $N > n+1$, we know that there are non-trivial solutions $(\mu_1, \mu_2, \ldots, \mu_N)$ of this set of equations. Consider one such set and denote by $\mu_{j_1}, \ldots, \mu_{j_k}$ those of the μ_i which are non-negative, and by $\mu_{h_1}, \ldots, \mu_{h_{N-k}}$ those which are strictly negative.

From these numbers define the point $\underline{y} = (y_1, \ldots, y_n)$ such that

$$y_v = \frac{\sum_{r=1}^{k} \mu_{j_r} x_{vj_r}}{\sum_{r=1}^{k} \mu_{j_r}} .$$

Now $\underline{x}_{j_r}$ is a point of X_j for all $j \neq j_r$, and it follows that the points $\underline{x}_{j_1}, \underline{x}_{j_2}, \ldots, \underline{x}_{j_k}$ all belong to the set X_j provided that $j \neq j_1, \ldots, j_k$. Then, since the u_{j_r} (page 8) are all non-negative, it follows by convexity (theorem 1, since X_j is convex) that the point $\underline{y}$ which we have defined also lies in the set X_j, $j \neq j_1, \ldots, j_k$. In other words, $\underline{y}$ belongs to each of the sets $X_{h_1}, \ldots, X_{h_{N-k}}$.

But also the equation (1) implies that the same point $\underline{y}$ can be defined by the equations

$$y_v = \frac{\sum_{s=1}^{N-k} \left(-\mu_{h_s}\right) x_{vh_s}}{\sum_{s=1}^{N-k} \left(-\mu_{h_s}\right)} ,$$

where now $-\mu_{h_s} > 0$ for each $s = 1, \ldots, N-k$, and it then follows as before that the point $\underline{y}$ belongs also to each of the sets X_{j_1}, $\ldots, X_{j_k}$. Thus $\underline{y}$ is a point common to all N sets of the class, the induction hypothesis is true for N, and hence the theorem follows.

<u>Theorem 6. (Carathéodory's theorem, weak form.)</u> If y is a point of

the convex cover of a set X, there is a set of s points x_1, $\ldots$, x_s, all belonging to X, where $s \leqslant n+1$, such that y is a point of the simplex whose vertices are $x_1, \ldots, x_s$.

Proof. It is shown below (lemma 3 - page 33) that for such a point y belonging to the convex cover of X, there is some finite integer k such that

$$y = \mu_1 x_1 + \ldots + \mu_k x_k \qquad (2)$$

where $x_i \in X$, $\mu_i \geqslant 0$ for each $i = 1, 2, \ldots, k$, and $\mu_1 + \ldots + \mu_k = 1$. We want to prove that it is possible to find such an expression for y with $k \leqslant n+1$.

If $k > n+1$, then since we are working in space of n dimensions the points $x_1, \ldots, x_k$ are linearly dependent and there exist numbers $\alpha_1, \ldots, \alpha_k$, not all zero, but with $\alpha_1 + \ldots + \alpha_k = 0$, such that

$$\alpha_1 x_1 + \ldots + \alpha_k x_k = 0 .$$

Now consider a point y of the form

$$y = (\mu_1 + \tau\alpha_1)x_1 + \ldots + (\mu_k + \tau\alpha_k)x_k$$

where $\mu_i + \tau\alpha_i \geqslant 0$ for all $i = 1, 2, \ldots, k$. With this restriction on the values that τ may take, we may note that the set of possible values of τ is closed (since the inequality is not strict), is non-empty (since $\tau = 0$ is certainly a possible value, because of the definition of the μ_i), and is not the whole of the real line (because at least one of the α_i is not zero). Hence there is a frontier point, say τ_0, of the set, and so for at least one integer i, $1 \leqslant i \leqslant k$, we must have $\mu_i + \tau_0\alpha_i = 0$.

Thus we have, putting τ_0 in place of τ above, an expression for y which is of the same form as (2) (since all the coefficients are non-negative and their sum is 1), but which has only $(k-1)$ significant terms.

Finally, this process may be repeated until the point y is expressed as a linear combination of not more than $(n+1)$ points of X , and the proof of the theorem is completed.

Consider some closed sphere $S(R)$ centre the origin and radius R in E_n , and suppose that X_1 , X_2 are two closed sets contained in $S(R)$. We define the distance between X_1 and X_2 as

$$\Delta(X_1,X_2) = \delta_1 + \delta_2$$

where δ_1 is the greatest lower bound of numbers δ such that $U(X_1,\delta)$ contains X_2 , and δ_2 is the greatest lower bound of numbers δ such that $U(X_2,\delta)$ contains X_1 .

$\Delta(X_1,X_2) \geqslant 0$ since it is the sum of two non-negative numbers. If X_1 and X_2 represent the same set, then clearly $\Delta(X_1,X_2) = 0$; conversely, if $\Delta(X_1,X_2) = 0$, then δ_1 and δ_2 as defined above are both zero, which implies that X_1 is X_2 . Clearly we have $\Delta(X_1,X_2) = \Delta(X_2,X_1)$ from the symmetry of the definition. Also, we have that

$$\Delta(X_1,X_2) + \Delta(X_2,X_3) \geqslant \Delta(X_1,X_3) .$$

For, let ε_1, ε_2, ε_3, ε_4 be four (non-negative) numbers such that

$$U(X_1,\varepsilon_1) \supset X_2 \quad , \quad U(X_2,\varepsilon_2) \supset X_3 \quad ,$$
$$U(X_2,\varepsilon_3) \supset X_1 \quad , \quad U(X_3,\varepsilon_4) \supset X_2 \quad .$$

Then $U(X_1,\varepsilon_1+\varepsilon_2) \supset U(X_2,\varepsilon_2) \supset X_3$

and $U(X_3,\varepsilon_3+\varepsilon_4) \supset U(X_2,\varepsilon_3) \supset X_1$.

Thus $\Delta(X_1,X_3) \leq (\varepsilon_1+\varepsilon_2) + (\varepsilon_3+\varepsilon_4)$

and taking lower bounds of the ε_i gives the result.

Theorem 7. (Blaschke Selection Theorem.) Every infinite collection of closed convex subsets of a bounded portion of E_n contains an infinite subsequence which converges to a closed non-empty convex subset.

Proof. Select from the collection and label an infinite sequence $\{X(i)\}$ of sets, $i = 1, 2, \ldots, n, \ldots$. Choose a positive number δ and define $Y(n)$ as the closure of the set $U(X(n), \delta)$. Let $S(R)$ be a sphere of radius R such that $Y(n) \subset S(R)$ for each n .

We prove the theorem in two stages.

(i) There is a subsequence of $\{Y(n)\}$ such that if $Y(i)$, $Y(j)$ are two members of the subsequence

$$\Delta(Y(i),Y(j)) \to 0 \quad \text{as} \quad i,j \to \infty \ . \qquad (3)$$

(ii) If a sequence Z_i of closed convex sets contained in a sphere $S(R)$ is such that

$$\Delta(Z_i,Z_j) \to 0 \quad \text{as} \quad i,j \to \infty \ ,$$

then there exists a closed non-empty convex set Z such that

$$\Delta(Z,Z_i) \to 0 \quad \text{as} \quad i \to \infty .$$

If we assume (i) and (ii) true, the theorem follows quickly. It is known (lemma 4 - page 36) that if X_1, X_2 are given closed convex sets, and Y_1, Y_2 respectively are defined in the same way as $Y(n)$ above, then

$$\Delta(Y_1,Y_2) = \Delta(X_1,X_2) . \tag{4}$$

(3) and (4) together imply that there is a subsequence of $\{X(n)\}$ such that if $X(i)$, $X(j)$ are members of the subsequence

$$\Delta(X(i),X(j)) \to 0 \quad \text{as} \quad i,j \to \infty .$$

But then (ii) applied to this subsequence gives the result.

Proof of (i). Take a sequence $p_1, p_2, \ldots$ of points dense in $S(R)$ and let $\{\varepsilon_i\}$ be a sequence of positive numbers decreasing to zero, and with $\varepsilon_1 < \frac{1}{2}\delta$. We assume inductively that a sequence of integers

$$i_1, i_2, i_3, \ldots$$

has been defined for $i = 0, 1, \ldots, k$, such that $\{(i+1)_j\}$ is a subsequence of $\{i_j\}$, $i_j = j$ for $i = 0$, and for $i > 0$

$$\Delta(Y(i_j),Y(i_h)) < \varepsilon_i , \quad j,h = 1, 2, \ldots .$$

To continue the sequence, we want to show that it is possible to choose a subsequence $\{(k+1)_j\}$ of $\{k_j\}$ such that

$$\Delta(Y[(k+1)_j], Y[(k+1)_h]) < \varepsilon_{k+1} .$$

Choose a large integer N such that every point of $S(R)$ is distant less than $\frac{1}{4}\varepsilon_{k+1}$ from some point p_j for which $j \leq N$. Denote by $P(n)$ the subset of the points p_j, $j \leq N$, which is contained in $Y(n)$. Now every point of $U(X(n), \delta - \frac{1}{4}\varepsilon_{k+1})$ is distant from one of the points p_j , $j \leq N$, by at most $\frac{1}{4}\varepsilon_{k+1}$ (since the set is contained in $S(R)$), and so each such p_j must be contained in $U(X(n), \delta)$, which is the interior of $Y(n)$. Thus each such p_j must belong to $P(n)$. Hence

$$U(P(n), \tfrac{1}{4}\varepsilon_{k+1}) \qquad U(X(n), \delta - \tfrac{1}{4}\varepsilon_{k+1}) \ .$$

(Note that it is not necessarily true by the definition of the set $P(n)$ that $P(n) \supset U(X(n), \delta - \frac{1}{4}\varepsilon_{k+1})$.)

Thus we have

$$U(P(n), \tfrac{1}{2}\varepsilon_{k+1}) \supset U(X(n), \delta) = (Y(n))^{\circ} \ ,$$

as we noted above. But by definition $P(n) \subset Y(n)$, and so we have

$$\Delta(P(n), Y(n)) < \tfrac{1}{2}\varepsilon_{k+1} \ . \qquad (5)$$

Now there is at most a finite number of ways in which the set of points p_1 , p_2, ..., p_N can be arranged into subsets to form the sets $P(n)$, and so it follows that we can select from the sequence $\{Y(k_j)\}$ an infinite subsequence $\{Y((k+1)_j)\}$, for each member of which the corresponding set $P((k+1)_j)$ is the same set. Then we have

$$\begin{aligned} \Delta[Y((k+1)_j), Y((k+1)_h)] &\leq \Delta[Y((k+1)_j), P((k+1)_j)] \\ &\qquad + \Delta[Y((k+1)_h), P((k+1)_j)] \\ &= \Delta[Y((k+1)_j), P((k+1)_j)] + \Delta[Y((k+1)_h), P((k+1)_h)] \\ &< \varepsilon_{k+1} \qquad \text{from (5)} \ . \end{aligned}$$

This completes the next step in the inductive definition of the sequence of integers $\{i_j\}$.

Now consider the sequence

$$1_1,\ 2_2,\ 3_3,\ \ldots,\ h_h,\ \ldots \quad .$$

For $h \geqslant i$, h_h is a member of the sequence $\{i_j\}$. Thus

$$\Delta(Y(h_h), Y(j_j)) \to 0 \quad \text{as} \quad h, j \to \infty$$

and this proves (i).

Proof of (ii). We put

$$Z = \bigcap_{i=1}^{\infty} \overline{\left[\bigcup_{j=i}^{\infty} Z_j \right]} \quad ,$$

and prove that Z is the limit of the sequence $\{Z_i\}$.

We are given that for arbitrary $\varepsilon > 0$ there is an integer N such that

$$\Delta(Z_i, Z_j) < \varepsilon \quad \text{for} \quad i, j \geqslant N \quad . \tag{6}$$

Then $\displaystyle\bigcup_{j=i}^{\infty} Z_j \subset U(Z_i, \varepsilon) \quad \text{for} \quad i \geqslant N$

and so $\displaystyle Z \subset \overline{\left[\bigcup_{j=i}^{\infty} Z_j \right]} \subset U(Z_i, 2\varepsilon)$, say, $i \geqslant N$. (7)

Now let z_0 be any point of Z_i , and put

$$\overline{(U(z_0, \varepsilon))} \cap Z_j = W_j \quad \text{for} \quad i, j \geqslant N \quad .$$

W_j is closed, and since from (6) the distance of z_0 from Z_j is less than ε it follows that W_j is non-empty for all $j \geqslant N$. Thus the set

$$T_N = \bigcap_{i=N}^{\infty} \overline{\left[\bigcup_{j=i}^{\infty} W_j \right]}$$

is also closed and non-empty. T_N is contained in $\overline{(U(z_0, \varepsilon))}$ (by the definition of the W_j) and also in Z . (For, if t were a point of T_N not in Z we should have

$$t \notin \overline{\left[\bigcup_{j=i}^{\infty} Z_j\right]} \qquad \text{for } i = \text{some } i_0 \text{, say, } i_0 \geq 1$$

so that also

$$t \notin \overline{\left[\bigcup_{j=i}^{\infty} Z_j\right]} \qquad \text{for } i \geq i_0 \text{ .}$$

But $W_j \subset Z_j$ for each $j \geq N$ and so

$$\overline{\left[\bigcup_{j=i}^{\infty} W_j\right]} \subset \overline{\left[\bigcup_{j=i}^{\infty} Z_j\right]} \qquad \text{for } i \geq N \text{ ,}$$

hence

$$t \notin \overline{\left[\bigcup_{j=i}^{\infty} W_j\right]} \qquad \text{for } i \geq \max(i_0, N)$$

which in turn implies that $t \notin T_N$, which is a contradiction.)

Thus the intersection of Z and the set $\overline{(U(z_0,\varepsilon))}$ is non-empty, which is to say that to any point z_0 of Z_i , $i \geq N$, there corresponds a point of Z whose distance from z_0 is at most ε . Hence we have

$$Z_i \subset U(Z, 2\varepsilon) \text{ , say, } i \geq N \text{ .}$$

This, together with (7), gives

$$\Delta(Z_i, Z) < 4\varepsilon \text{ , } i \geq N \text{ ,}$$

which proves (ii).

This completes the theorem.

Let B denote the class of all convex sets contained in the sphere S(R) as defined earlier. Corresponding to each member X of this class B , suppose that there is defined a real number f(X) . Then we say that f(X) is <u>continuous at X_0</u> if for every sequence $\{X_i\}$

for which $X_i \to X_0$ we have also $f(X_i) \to f(X_0)$. (By $X_i \to X_0$ we mean that there is a member X_0 of B such that $\Delta(X_i, X_0) \to 0$ as $i \to \infty$.)

If $f(X)$ is continuous for all X_0 of B we say merely that $f(X)$ is <u>continuous</u>.

Having thus defined continuity, we now define the <u>volume $V(X)$ of a convex set X</u> as its n-dimensional Lebesgue measure (and shall assume that it is known what is meant by this definition), and prove the following theorem.

<u>Theorem 8.</u> $V(X)$ is continuous.

<u>Proof.</u> In other words, we need to show that if $X_i \to X$ as $i \to \infty$, then $V(X_i) \to V(X)$.

Given some positive number ε, there is an integer M such that

$$U(X_i, \varepsilon) \supset X \text{ and } U(X, \varepsilon) \supset X_i \quad \text{for all } i \geq M .$$

It may be shown (lemma 5 - page 37) that if the interior of X is non-empty there is a positive number δ and a set Y similar to X but expanded in the ratio $1+\delta : 1$, such that $Y \supset U(X, \varepsilon)$, and such that $\delta \to 0$ as $\varepsilon \to 0$. Then for $i \geq M$ we have

$$V(X_i) \leq V(U(X, \varepsilon)) \leq V(Y) = (1 + \delta)^n V(X) . \qquad (8)$$

Now if $V(X) = 0$, the set X lies in an $(n-1)$-dimensional space, and if $\Delta(X_i, X) < \eta_i$, X_i must be contained in an n-dimensional cylinder of length $2\eta_i$ and cross-section an $(n-1)$-dimensional sphere of radius R (since all the X_i lie in the sphere $S(R)$), the volume of the sphere being K, say. Thus $V(X_i) < 2\eta_i K$ and so

$$\lim_{i\to\infty} V(X_i) = 0 .$$

If $V(X) > 0$, we need a lower bound for $V(X_i)$ (an upper bound being established already in (8)). This involves showing that the volume of the frontier of X is zero, which we may do by considering the sets $X(\mu)$, $X(-\mu)$ which are homothetic to X with an interior point of X as centre of similitude and with ratios of similitude $1+\mu : 1$, $1-\mu : 1$ respectively, $0 < \mu < 1$. Then the frontier of X is contained in the difference of these sets and so has volume not greater than

$$((1+\mu)^n - (1-\mu)^n)V(X) \quad .$$

Thus, since we may choose μ as near to zero as we please, the frontier of X has zero volume.

Now suppose that we have a sequence of points $\{p_i\}$ dense in the interior of X. If K_n denotes the convex cover of the first n of these points, the sequence $\{K_n\}$ increases to the interior of X, so that we have, according to one of the properties of a Lebesgue measure,

$$\lim_{n\to\infty} V(K_n) = V(X^\circ) = V(X) .$$

Thus given $\varepsilon > 0$, there is an integer N such that

$$V(K_N) > V(X) - \varepsilon .$$

Finally there is an integer M such that

$$X_i \supset K_N \qquad \text{for all } i \geqslant M ,$$

for if $\delta > 0$ is the smallest of the distances of points of K_N from the frontier of X, and if Y is a convex set which does not contain K_N, then we must have $\Delta(Y,X) > \delta$, but we know that the sets X_i tend to X as $i \to \infty$ so that eventually each X_i must contain K_N.

Thus we have

$$V(X_i) \geqslant V(K_N) > V(X) - \varepsilon \qquad \text{for } i \geqslant M$$

and this together with (8) implies that

$$V(X_i) \to V(X) \quad \text{as } i \to \infty .$$

If for a given set X there is a point p such that the reflexion of X in p coincides with X, then p is called the <u>centre</u> of X and X is said to be a <u>central set</u>.

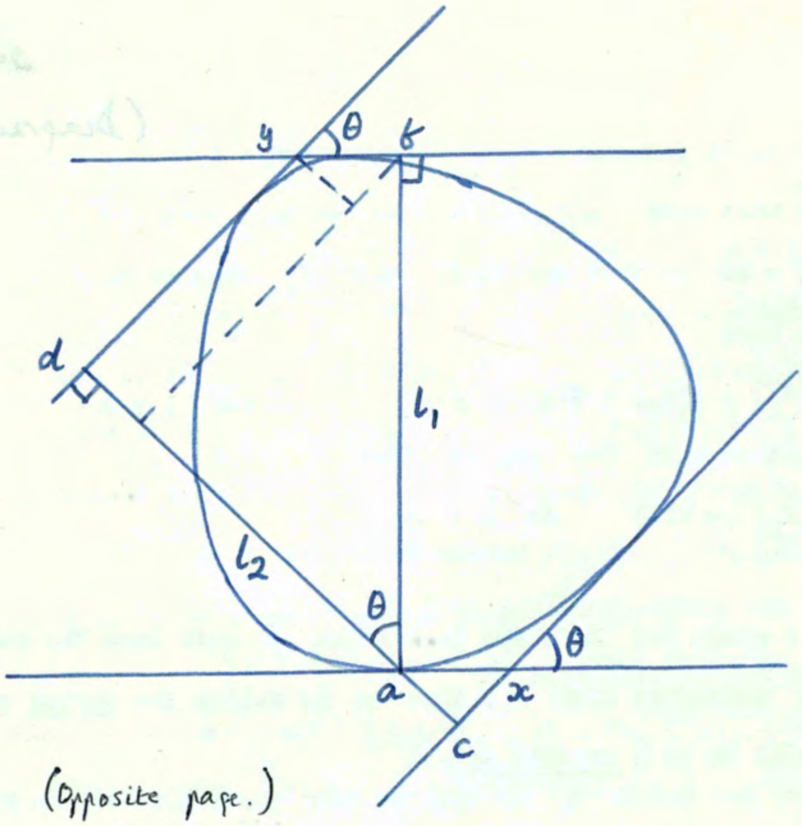
Perpendicular to any given direction θ, there are exactly two support hyperplanes of a bounded convex set X. The <u>width of X</u> in the direction θ is defined as the distance between these hyperplanes. The greatest lower bound of the widths of a set X is called the <u>minimal width</u> of X.

<u>Theorem 9</u>. The width in the direction θ of a fixed plane bounded convex set X is a continuous function of θ.

<u>Proof</u>. Consider two pairs of parallel support lines of X, the angle between the pairs being θ, where θ is small.

Denote by a one of the points of contact of the set X with the

(Diagram)



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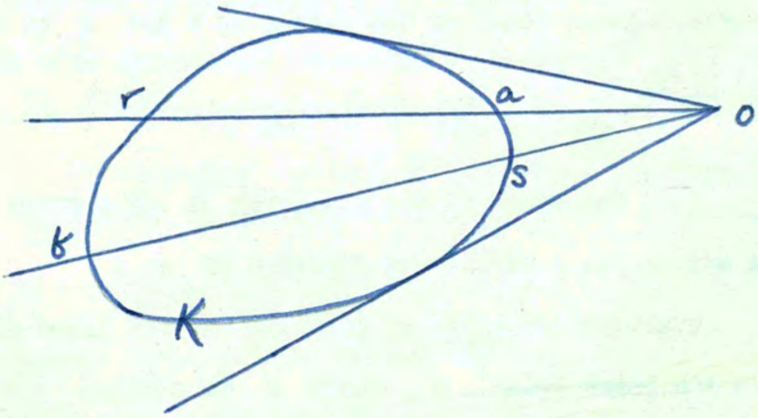


(Page 27.)

first pair of support lines, and draw ab , of length ℓ_1 say, perpendicular to the line pair to meet the other support line in b . Through the point a , as in the diagram, draw a line cd , of length ℓ_2 say, perpendicular to the other pair of support lines, with c , d lying on the lines. Denote by x , y the two points of intersection of the support lines which are nearest to the set X . (The points are well-defined since θ is small.)

Then from the diagram we have (see opposite page)

$$\ell_2 = \ell_1 \cos\theta + by \sin\theta + ax \sin\theta .$$

(Different expressions may result from different sets X and this construction on them, but the type of expression and its behaviour will be the same.)

The lengths of by and ax are bounded above certainly by the diameter of X for small values of θ , and so as θ decreases to zero, $\ell_2 \to \ell_1$. Thus the width of the set varies continuously with θ .

Suppose that K is a strictly convex set. Choose a point o and define M to be the set of points y such that

$$|o - y| = \delta |o - x|$$

where δ is a fixed positive number, not equal to 1 , x is a point of K , and such that y lies on the half-line through o and x terminating at o . We call M a similarity transform or homothetic translate of K (see also chapter III, section 4 - page 155).

(It will sometimes be convenient to widen the definition to include cases in which M is a translation of K .)

Theorem 10. Suppose that K , M are two distinct, strictly convex sets which are similarity transforms of each other, and consider any plane section of them which passes through their centre of similitude. Then the frontiers of K and M have at most two points of intersection lying in such a plane.

(The result could be made more general, to include any plane section of the sets, but in the cases where we shall need this theorem the restricted result is sufficient.)

Proof. Let o be the centre of similitude of K and M (o may be at infinity), and suppose that there is a plane section through o of K and M such that their frontiers intersect in at least three points a , b , c , ... Denote the set of such points by I .

The support lines of K which meet at o (the parallel common support lines of K and M if o is at infinity) in this plane section divide the frontier of K into two open arcs X , Y separated by the line joining the points of contact of the support lines with K . Neither of these points is common to the frontiers of K and M since this would contradict the strict convexity of M .

Suppose that the points a , b , c , ... are such that a lies in X and b in Y , and let r , s respectively be the points of intersection with the frontier of K of the lines through oa and ob . (See page 24, diagram.) Then r is a point of Y and s a point of X , and the points of M which coincide with a , b respectively are the points r′ corresponding to r in K and s′ corresponding to s in K .

But then since K and M are similar, we have

$$\frac{or'}{or} = \frac{os'}{os}$$

where or′ denotes the length of or′ , and so on. Thus

$$\frac{oa}{or} = \frac{ob}{os} \quad .$$

But by our assumption we have or < oa and os > ob , or else or > oa and os < ob , either of which gives a contradiction.

Hence all the points of the set I lie in one of the sets X , Y . Suppose that this set is X and that X lies nearer to o than Y does. There are at least three members of I , so consider three such points of X , x , y , z , say, and suppose that the point x lies between y and z on the arc X . Since x is a frontier point of M there is a support hyperplane of M at x , and M lies on the same side of this hyperplane as o . But from our choice of the point x , any hyperplane through x separates at least one of y and z from o (x , y , z are not collinear because K is strictly convex), and this gives a contradiction since all the points of I are points of M .

If X lies further from o than Y does, define points x′,

y', z' (again non-collinear) of the arc X in the same way as x, y, z were defined above, with x' between y' and z'. Then a support hyperplane of M at x' separates M from o, but again a contradiction follows, because any hyperplane through x' must have at least one of the points y' and z' on the same side as o.

Hence in either case there cannot be as many as three common points of the two frontiers in such a plane section, and the theorem is proved.

- - - - - - - - - - - - - - - -

Lemma 1. Given an open convex set X_1 and a convex set X_2 which does not meet X_1, there exists a hyperplane P which separates X_1 from X_2.

Proof. We consider first an open convex set X and a linear manifold L not meeting X. A linear manifold of dimension r, $0 \leqslant r < n$, is the set of points x defined by

$$x = \mu_1 x_1 + \dots + \mu_{r+1} x_{r+1}$$

where the vectors $x_2 - x_1, \dots, x_{r+1} - x_1$ form a linearly independent set, and $\mu_1, \dots, \mu_{r+1}$ are real numbers such that $\mu_1 + \dots + \mu_{r+1} = 1$. We prove that there exists a hyperplane P which contains L but which does not meet X.

Suppose then that P is a linear manifold of maximal dimension s , say, which contains L and does not meet X . Let Q be the linear subspace perpendicular to P , and project E_n onto Q . The dimension of Q is (n-s). Then P is projected onto a single point p and X onto an open convex subset X_1 of Q , and p does not belong to X_1 since the direction of the projection is parallel with P . Since we are assuming that the dimension of P is maximal, it follows that every line through p in Q meets X_1 . If Q is one-dimensional, then P is the hyperplane that we require. So suppose that Q has dimension greater than or equal to 2 . A plane through p in Q meets X_1 in a two-dimensional set X_2 which is convex and open relative to the plane. The half-lines through p meeting X_2 and terminating at p thus form an open sector, the angle of which is at most π since p does not belong to X_2 and X_2 is convex. But on the other hand, since <u>every</u> line through p meets X_2 and X_2 is open, the angle of the sector must be strictly greater than π , which is a contradiction. Hence Q must be one-dimensional, and P is the hyperplane whose existence we wanted to prove.

Now if instead of L we take a convex set X_2 not meeting X_1 , we consider the set $X_1 - X_2$ consisting of all points x of the form

$$x = x_1 - x_2 \quad \text{where} \quad x_1 \in X_1 \, , \, x_2 \in X_2 \, .$$

This set is convex and open, and the point o does not belong to the set since X_1 and X_2 have no point in common. Thus there is a hyperplane P which passes through o and does not meet $X_1 - X_2$.

Suppose that its equation is $\underline{a}.\underline{x} = 0$ ($\underline{a}$, $\underline{x}$ being vectors), and that the signs of the coefficients are such that $\underline{a}.\underline{x} > 0$ for $x \in X_1 - X_2$. Then we have

$$\underline{a}.\underline{x}_1 > \underline{a}.\underline{x}_2 \quad \text{for} \quad x_1 \in X_1 \ , \ x_2 \in X_2$$

and so there exists a finite number μ equal to inf ($\underline{a}.\underline{x}$, $x \in X_1$). It follows that the hyperplane defined by the equation $\underline{a}.\underline{x} = \mu$ separates X_1 from X_2.

Lemma 2. Given a convex set X with a non-empty interior X^o, the closure of X^o is identical with $\overline{X}$.

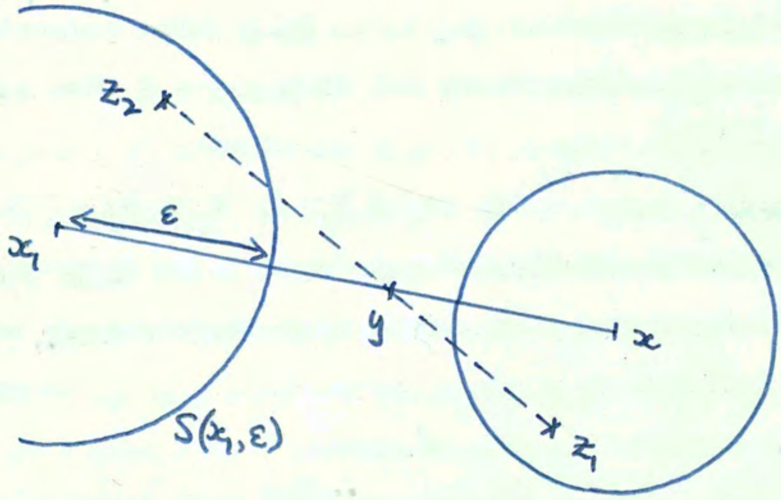
Proof. Since $X^o \subset X$, we have immediately that $\overline{X^o} \subset \overline{X}$. We want to show, then, that $\overline{X} \subset \overline{X^o}$.

Let $x \in \overline{X}$ and take any point x_1 of X^o. If we can prove that the whole of the segment xx_1 except possibly x is contained in X^o, then it follows that $x \in \overline{X^o}$, from which we have that $\overline{X} \subset \overline{X^o}$ and the lemma is proved.

We show, then, that apart from x the whole of xx_1 is contained in X^o. Since $x_1 \in X^o$, there is a spherical neighbourhood (in n dimensions) $S(x_1, \varepsilon)$ of x_1 for some positive number ε, such that $S(x_1, \varepsilon) \subset X^o$. Consider a point y of xx_1 which is not at either endpoint of the segment, and let z_1 be a point other than x of X which satisfies

$$\frac{|z_1 - x|}{|y - x|} < \frac{\varepsilon}{|y - x_1|} .$$

(Diagram)



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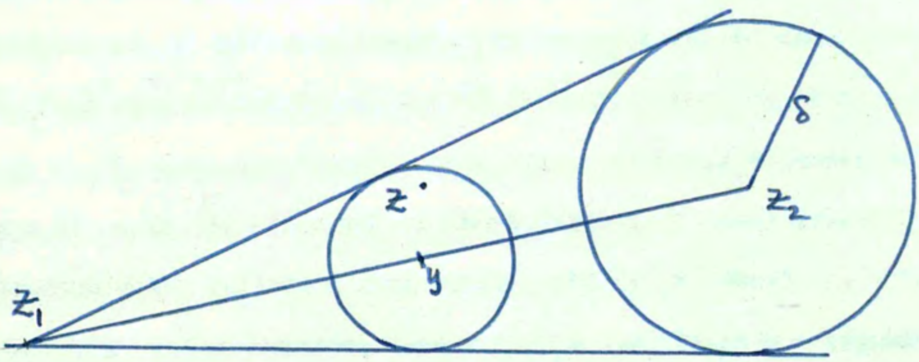


(Page 33.)

(The design of the construction given now, (see diagram opposite) which is in effect to select a different segment containing the point y, is to ensure that the endpoints of the segment we shall discuss lie one in X° and one in X, and not merely one in X° and one in $\overline{X}$ as we have at present. We cannot just shorten the segment xx_1 as y must be free to coincide with any interior point of it.)

Further, let z_2 be defined by

$$z_2 - x_1 = - \frac{|y - x_1|}{|y - x|}(z_1 - x) .$$

We note that $|z_2 - x_1| < \varepsilon$ so that $z_2 \in X^\circ$. With these definitions we have that

$$y = \frac{|y - x_1|\, x + |y - x|\, x_1}{|y - x_1| + |y - x|} \quad \text{since } y \text{ is a point of } xx_1$$

$$= \frac{|y - x_1|\, z_1 + |y - x|\, z_2}{|y - x_1| + |y - x|} \quad \text{by the definition of } z_2$$

so that y is also an interior point of the segment z_1z_2. Since y is any interior point of xx_1, it is sufficient now to show that the whole of the interior of z_1z_2 is contained in X°, and we shall have proved our result.

We have then that $z_1 \in X$ and $z_2 \in X^\circ$. Certainly every point of z_1z_2 belongs to X since X is convex; we need also to prove that every such point, apart from z_1, is an interior point of X. Since $z_2 \in X^\circ$, there is a positive number δ such that

$S(z_2, \delta) \subset X°$ (using the same notation as before). Consider a point y of z_1z_2 (not necessarily the same point as before) given by

$$y = \lambda z_1 + \mu z_2$$

where $\lambda + \mu = 1$, $\lambda \geqslant 0$, $\mu > 0$, so that y is not identical with z_1; and let z be a point of $S(y, \mu\delta)$ (which is the neighbourhood of y shown in the diagram (see page 31, diagram)). From the definition of y we have

$$|z - (\lambda z_1 + \mu z_2)| < \mu\delta$$

which implies, since $\mu > 0$, that

$$\left| \frac{z - \lambda z_1}{\mu} - z_2 \right| < \delta$$

so that the point $\dfrac{z - \lambda z_1}{\mu}$ belongs to $S(z_2, \delta)$ and hence to $X°$.

But z can be written in the form

$$z = \lambda z_1 + \mu\left[\frac{z - \lambda z_1}{\mu}\right], \quad \lambda + \mu = 1, \quad \lambda \geqslant 0, \mu > 0,$$

and X is convex, and so since z_1, $\dfrac{z - \lambda z_1}{\mu} \in X$ we have also that $z \in X$. Thus $S(y, \mu\delta) \subset X$ and so y and hence every point of z_1z_2 apart from z_1 is an interior point of X.

This proves the lemma.

<u>Lemma 3.</u> The set K of points of E_n which are expressible in the form

$$\mu_1 x_1 + \dots + \mu_k x_k ,$$

where k ranges over all positive integral values, $x_1, \dots, x_k$ are any points of a set X, and $\mu_1, \ldots, \mu_k$ are real numbers satisfying

$$\mu_1 + \cdots + \mu_k = 1 \quad , \quad \mu_i \geqslant 0 \quad , \quad i = 1, 2, \ldots, k,$$

is identical with the set of points which form the convex cover $H(X)$ of X.

Proof. Consider a particular point x of K which is expressible as, say,

$$\mu_1 x_1 + \cdots + \mu_k x_k$$

for some positive integer k and points $x_1, \ldots, x_k$, so that x belongs to the convex cover $H(x_1, \ldots, x_k)$ of the points $x_1, \ldots, x_k$. Since $x_i \in X$ for each i, $1 \leqslant i \leqslant k$, it follows that $H(x_1, \ldots, x_k) \subset H(X)$, and hence $x \in H(X)$. But x is an arbitrary point of K, and so we have that $K \subset H(X)$.

To show that $H(X) \subset K$, we show first that the set K is convex. For given two points y, z of K where

$$y = \lambda_1 y_1 + \cdots + \lambda_m y_m \quad ,$$

$$z = \mu_1 z_1 + \cdots + \mu_n z_n \quad ,$$

consider any point x of the segment yz,

$$x = \nu y + (1 - \nu) z \qquad (0 \leqslant \nu \leqslant 1) \quad .$$

Then we have

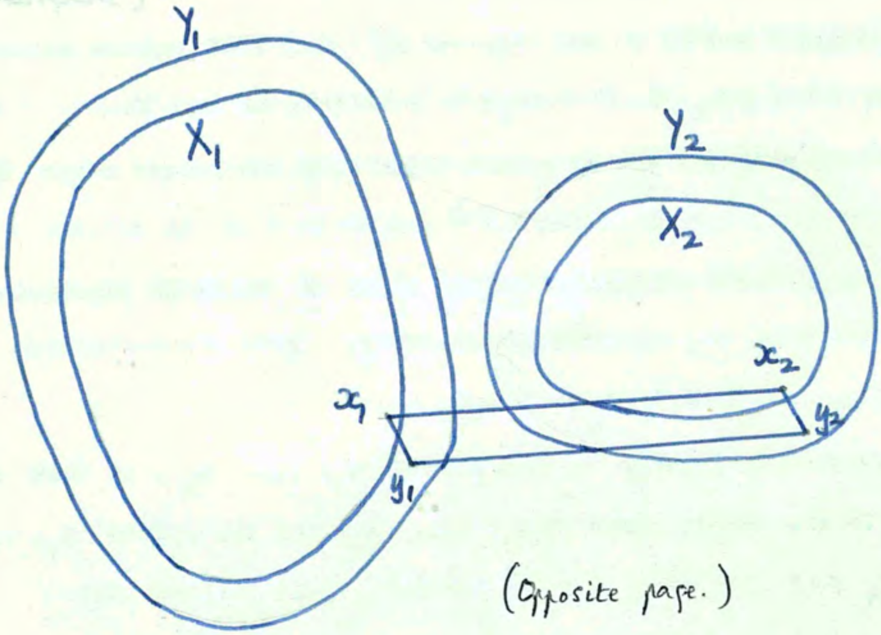
$$x = \nu\lambda_1 y_1 + \cdots + \nu\lambda_m y_m + (1-\nu)\mu_1 z_1 + \cdots + (1-\nu)\mu_n z_n$$

where $\nu\lambda_i \geqslant 0$, $(1 - \nu)\mu_j \geqslant 0$, and

$$\sum_{i=1}^{m} \nu\lambda_i + \sum_{j=1}^{n} (1 - \nu)\mu_j = \nu \sum_{i=1}^{m} \lambda_i + (1 - \nu) \sum_{j=1}^{n} \mu_j$$

$$= \nu + (1 - \nu) \qquad \text{since } y, z \in K$$

35
(Diagram)



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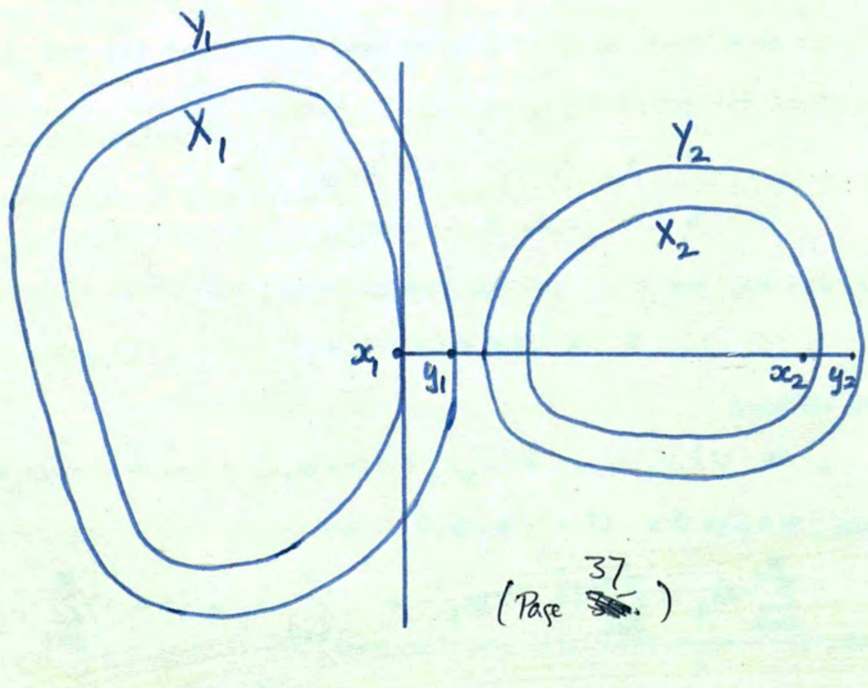

(Page 37.)

and so it follows that x is a member of K. Thus K is convex. Finally we note that the set X is clearly a subset of K, so that, since K is convex, the convex cover of X is also a subset of K. Thus $H(X) = K$ and this proves the lemma.

Lemma 4. If X_1, X_2 are given closed, convex sets, and

$$Y_1 = \overline{(U(X_1, \delta))} \quad , \quad Y_2 = \overline{(U(X_2, \delta))} \quad ,$$

then $$\Delta(Y_1, Y_2) = \Delta(X_1, X_2) \quad .$$

Proof. Suppose that R is chosen so that Y_1 and Y_2 both belong to the sphere $S(R)$ of radius R centred at the origin in a given coordinate system.

(See diagram opposite.)

We show first that if $U(X_1, \varepsilon) \supset X_2$, then $U(Y_1, \varepsilon) \supset Y_2$. To any point y_2 of Y_2 there corresponds a point x_2 of X_2 such that $|y_2 - x_2| \leqslant \delta$. If $U(X_1, \varepsilon) \supset X_2$, it follows that there is a point x_1 of X_1 such that $|x_1 - x_2| < \varepsilon$. Let y_1 be the point whose position is such that the four points x_1, x_2, y_2, y_1 are the four vertices of a parallelogram, as shown in the diagram. Then we have

$$|x_1 - y_1| = |x_2 - y_2| \leqslant \delta$$

so that, since $x_1 \in X_1$, $y_1 \in Y_1$. Also

$$|y_1 - y_2| = |x_1 - x_2| < \varepsilon$$

so that we have also $y_2 \in U(Y_1, \varepsilon)$, and hence $Y_2 \subset U(Y_1, \varepsilon)$.

To complete the proof, we show secondly that if $U(Y_1, \varepsilon) \supset Y_2$, then $\overline{(U(X_1, \varepsilon))} \supset X_2$, for then the lemma follows immediately from the definition of the metric δ . (see page 35, diagram)

If this second assertion is false, there is a point x_2 , say, of X_2 such that if x_1 is the point of X_1 nearest to x_2 then $|x_1 - x_2| > \varepsilon$. (Clearly there is such a point of X_1 .) With x_1 so defined, the hyperplane through x_1 perpendicular to x_1x_2 is a support hyperplane , P say, of X_1 . Now let y_2 be a point of Y_2 such that x_2 lies on the segment x_1y_2 and $|x_2 - y_2| = \delta$. (9)
Since x_1 is the point of X_1 nearest to x_2 , it follows that if we define y_1 to be the point of Y_1 on the segment x_1y_2 such that

$$|x_1 - y_1| = \delta \qquad (10)$$

then y_1 is the point of Y_1 which is nearest to Y_2 . Thus since $U(Y_1, \varepsilon) \supset Y_2$ we have $|y_1 - y_2| < \varepsilon$, and so from (10)

$$|x_1 - y_2| < \delta + \varepsilon .$$

But our assumption is that $|x_1 - x_2| > \varepsilon$, and this together with (9) gives

$$|x_1 - y_2| > \delta + \varepsilon .$$

This contradiction shows that our second assertion must be true, and the lemma follows.

<u>Lemma 5</u>. Given a positive number ε and a convex set X with a non-empty interior, there is a positive number δ and a set Y similar to X but expanded in the ratio $1+\delta : 1$ such that

$$Y(\delta) \supset U(X , \varepsilon)$$

and such that $\delta \to 0$ as $\varepsilon \to 0$.

Proof. The essence of the problem is not changed if we alter it slightly to read that given X and any positive δ , there is a set $Y(\delta)$ as defined such that, for some positive ε , $Y(\delta) \supset U(X, \varepsilon)$. If we prove this, then it follows that as we let ε tend to zero, δ is free to become as small as we please, so that this will satisfy the second requirement of the lemma.

The change in statement of the problem, or rather the consideration of a different but equivalent problem, makes the proof of it straightforward. For, we can select any interior point c of the given convex set X and, taking c as the centre of similitude, expand X about c in the ratio $1+\delta : 1$ to form a set $Y(\delta)$. Since c is in the interior of X it follows that the frontier of the new set Y does not meet the frontier of X in any point, and hence, since each frontier is a closed set, there is a positive number ε such that

$$|x - y| \geq \varepsilon$$

for any pair of points $x \in Fr(X)$ and $y \in Fr(Y(\delta))$.
Thus for ε so defined, since $U(X, \varepsilon)$ is an open set, we have that $Y(\delta) \supset U(X, \varepsilon)$ and the result follows. This, since δ tends to zero with ε , proves the lemma.

CHAPTER II

A COVERING OF A SINGLE SET BY A SINGLE SET WITH SPECIFIED PROPERTIES

We shall consider in this chapter six particular problems concerning single set covers for single sets. They are chosen partly to show the wide variety of approaches which may be made in the solution of covering problems, and are as follows.

(a) To establish an upper bound for the ratio of the greatest lower bound of the measure of the projection in any direction of a set X to its linear measure, where X is a measurable plane set of finite positive linear measure.

(b) To show that, given a plane curve of unit length, the area of its convex cover does not exceed $\frac{1}{2\pi}$.

(c) To find the maximum volume of the convex cover of a simple arc of space-curve of given length.

(d) To show that the minimal width of a triangle circumscribing a plane bounded convex set of given frontier length ℓ is at most $\frac{\ell}{\sqrt{3}}$, and is at most $\frac{1}{2}\ell$ if the convex set is central.

(e) To prove that there is a unique sphere of smallest radius r which contains a given bounded subset of E_n of diameter d , and

$$r \leqslant \left[\frac{n}{2(n+1)}\right]^{\frac{1}{2}} d \quad .$$

(f) To prove t at any set of diameter d may be contained in a set of constant width d .

Section 1 . To establish an upper bound for the ratio of the greatest lower bound of the measure of the projection in any direction of a set X to its linear measure, where X is a measurable plane set of finite positive linear measure. [3]

[Notation and Definitions: page 52.]

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Note first that the problem is stated for a plane set only. The method of proof to be used relies heavily on results involving a single (continuous) variable, namely the direction of projection of the set X , and does not lend itself to extension into three or more dimensions.

We need the following notation. $\Lambda(X)$ denotes the linear measure (page 53) of the set X , and $\Lambda(X_\theta)$ the linear measure of the projection of X onto a line perpendicular to the direction θ ; $\mu(X)$ denotes the greatest lower bound of $\Lambda(X_\theta)$ over all θ (and $\mu(X)$ exists since $\Lambda(X_\theta) \geqslant 0$ for all θ). Then the problem is to find the upper bound of the ratio $\frac{\mu(E)}{\Lambda(E)}$ for a measurable plane set E , with $\Lambda(E) > 0$, and we shall show that $\frac{\mu(E)}{\Lambda(E)} < \frac{2}{\pi}$.

Stated in this form, the problem is too general to be regarded as a problem in covering; but if we make the stricter condition that E is a connected set, then $\mu(E)$ is equivalent to the minimal width of the convex cover of E , so that the problem then becomes one of the relationship between a set and its convex cover.

By definition, $\mu(E) \leq \Lambda(E_\theta)$ for all θ, so we consider first $\frac{\Lambda(E_\theta)}{\Lambda(E)}$ for some value of θ.

We state here without proof a number of results necessary to begin the solution to the main problem; the proofs may be found in Besicovitch [1] and [2].

Any measurable plane set E is expressible as the union of a <u>regular set</u> E_1 (page 54) and an <u>irregular set</u> E_2 (page 54). Also, except for a set E_1'' of measure zero, E_1 is a measurable subset, E_1', of the union of an enumerable infinity of rectifiable arcs. The projection of an irregular set is of zero measure in almost all directions.

With this notation, we have immediately as a property of measure that

$$\Lambda(E_\theta) \leq \Lambda(E'_{1,\theta}) + \Lambda(E''_{1,\theta}) + \Lambda(E_{2,\theta}) \quad . \qquad (1)$$

Now $\Lambda(E_1'') = 0$ by definition, and hence the measure of the projection of E_1'' is zero in every direction,

i.e. $$\Lambda(E''_{1,\theta}) = 0 \quad \text{for all } \theta \; . \qquad (2)$$

We know that $$\Lambda(E_{2,\theta}) = 0 \quad \text{for almost all } \theta \; ; \qquad (3)$$

$\Lambda(E_2)$, however, may be zero or strictly positive. According to (1), therefore, we require an upper bound for $\Lambda(E'_{1,\theta})$.

It is proved below (lemma 2 - page 47) that

$$\int_0^{2\pi} \Lambda(E'_{1,\theta})d\theta \leq 4\Lambda(E_1') \quad . \qquad (4)$$

By writing E as the union of three sets (E_1', E_1'', E_2, which we may assume to be disjoint), we shall obtain a relation between the sum of measures of projections of the three sets and the sum of their linear measures.

Consider first the case in which $\Lambda(E_2) = \lambda > 0$. Given this value of λ, we have immediately from (4) that a value of θ may be chosen such that

$$\Lambda(E'_{1,\theta}) < \frac{2}{\pi}(\Lambda(E'_1) + \lambda) \quad . \qquad (5)$$

(For if not, the given integral is more than $4\Lambda(E'_1)$ by at least $4\lambda > 0$.)

It is proved below (lemma 1 - page 45) that $\Lambda(E'_{1,\theta})$ is a continuous function of θ. From this we may assume that (3) and (5) both hold for the same value of θ (since (3) holds for almost all θ). Then with (1) and (2) we have

$$\begin{aligned} \Lambda(E_\theta) &\leq \Lambda(E'_{1,\theta}) + \Lambda(E_{2,\theta}) && \text{for all } \theta \\ &< \frac{2}{\pi}(\Lambda(E'_1) + \lambda) && \text{for at least one value of } \theta \\ &= \frac{2}{\pi}\Lambda(E) && (6) \end{aligned}$$

since $\Lambda(E) = \Lambda(E'_1) + \Lambda(E''_1) + \Lambda(E_2)$ (since the sets are disjoint and all measurable), that is,

$$\Lambda(E) = \Lambda(E'_1) + \lambda \qquad \text{since } \Lambda(E''_1) = 0.$$

But $\mu(E) \leq \Lambda(E_\theta)$ for all θ

therefore (6) gives $\mu(E) < \frac{2}{\pi}\Lambda(E)$ for $\Lambda(E_2) = \lambda > 0$.

Now consider the case in which $\Lambda(E_2) = 0$. We can no longer use the strict inequality of (5), and look instead more closely at E'_1. It is convenient to split off a special case as follows (proof under lemma 3 - page 50). Either (a) almost all points of E_1' are collinear, or (b) there are two R-points of E_1', say p_1, p_2, such that p_2 does not lie on $t(p_1)$ and p_1 does not lie on $t(p_2)$, where $t(p_i)$ denotes the tangent at p_i. (See page 54.)

If $\Lambda(E_2) = 0$ and the special case (a) holds, then projecting parallel to the straight line containing almost all points of E_1' we have immediately that $\mu(E_1') = 0$ and hence $\mu(E) = 0$. The result $\mu(E) < \frac{2}{\pi}\Lambda(E)$ is thus trivially true in this case.

Finally, then, we must consider the case in which $\Lambda(E_2) = 0$ and (a) above is false.

Suppose that p_1 and p_2 are two R-points of E_1' for which (b) holds. It is proved below (lemma 4 - page 50) that if an R-point p of E_1' projects onto the point q of the set $E_{1,\theta}'$ and the direction of projection is not parallel to $t(p)$, then $E_{1,\theta}'$ has <u>unit density</u> at q (see pages 53-54). Suppose that the direction of the line joining p_1 and p_2 is ϕ. Let D_1 be the set of points of E_1' whose distance from p_1 is less than, say $\frac{1}{2}|p_1 - p_2|$, and D_2 be defined by $D_2 = E_1' - D_1$. Then $p_1 \in D_1$ and $p_2 \in D_2$ so that by lemma 4 $D_{1,\phi}$ and $D_{2,\phi}$ have a common point of unit density (since the direction ϕ of projection is not parallel to $t(p_1)$ or $t(p_2)$ by lemma 3). Hence

$$\Lambda(E_{1,\phi}') < \Lambda(D_{1,\phi}) + \Lambda(D_{2,\phi}) .$$

Applying lemma 2 to D_1 and D_2 (and using the continuity property established in lemma 1) we have then

$$\int_0^{2\pi} \Lambda(E'_{1,\theta})d\theta \quad < \quad 4\,\Lambda(D_1) + 4\,\Lambda(D_2) \quad = \quad 4\,\Lambda(E'_1) \quad .$$

Hence, since $\Lambda(E) = \Lambda(E'_1)$, we have that for some θ

$$\Lambda(E_\theta) \quad < \quad \frac{2}{\pi}\Lambda(E) \quad ,$$

which implies $\mu(E) < \frac{2}{\pi}\Lambda(E)$. Thus the inequality is proved for all cases.

- - - - - - - - - - - - - - -

<u>Lemma 1</u> . $\Lambda(E'_{1,\theta})$ depends continuously on θ .

<u>Proof</u>. Let $\{A_i\}$ be a sequence of rectifiable arcs of which E_1' is a measurable subset. The method of proof is to show that we can construct a finite set of arcs whose union C is a subset of $\cup A_i$ and such that the measure of C is arbitrarily close to that of E_1' . Then we can deduce continuity for $\Lambda(E'_{1,\theta})$ from the continuity of $\Lambda(C_\theta)$. (This follows immediately from the continuity of the width of the convex cover of an arc. See theorem 9 - page 23.)

Let $\{\varepsilon_i\}$ be a sequence of positive numbers decreasing strictly to zero. For each i there is a closed set $F_i \subset E_1'$ such that

$$\Lambda(F_i) > \Lambda(E_1') - \varepsilon_i \qquad (1)$$

and a positive integer N_i such that

$$\Lambda\Big(F_i \cap \bigcup_{j=1}^{N_i} A_j \Big) \quad > \quad \Lambda(F_i) - \varepsilon_i \quad . \qquad (2)$$

(We take F_i closed so that the set C we obtain will be a set of arcs.)

Since the complement in A_j of $A_j \cap F_i$, that is, $A_j - F_i$, is open and has finite measure, it must consist of an enumerable infinity or a finite number of open subintervals $\{B_{jk}\}$, and in either case we may arrange the B_{jk} so that for some chosen integer M_{ij}

$$\Lambda\Big(\bigcup_{k \geq M_{ij}} B_{jk} \Big) < \frac{1}{N_i} \varepsilon_i \quad . \tag{3}$$

The complement of $\bigcup_{1 \leq k < M_{ij}} B_{jk}$ in A_j, that is, $A_j - \bigcup_{1 \leq k < M_{ij}} B_{jk}$, consists of a finite number of arcs or points. Taking finally all the A_j with $1 \leq j \leq N_i$, let H_i be the set consisting of all the points and C_i the set of all the arcs, such

that $$C_i \cup H_i = \bigcup_{1 \leq j \leq N_i} \Big(A_j - \bigcup_{1 \leq k < M_{ij}} B_{jk} \Big) .$$

Note that $$C_i \cup H_i \supset \bigcup_{1 \leq j \leq N_i} \Big(A_j - \bigcup_{\text{all } k} B_{jk} \Big)$$

$$= \bigcup_{1 \leq j \leq N_i} (F_i \cap A_j)$$

i.e. $$C_i \cup H_i \supset F_i \cap \bigcup_{1 \leq j \leq N_i} A_j \tag{4}$$

and $$C_i - F_i \subset (C_i \cup H_i) - F_i$$

$$= \bigcup_{1 \leq j \leq N_i} \Big[(A_j - F_i) - \bigcup_{1 \leq k < M_{ij}} B_{jk} \Big]$$

since $F_i \cap B_{jk}$ is empty by definition

i.e. $$C_i - F_i \subset \bigcup_{1 \leq j \leq N_i} \bigcup_{k \geq M_{ij}} B_{jk} \quad . \tag{5}$$

C_i is the subset of $\bigcup A_j$ which we wanted. It is not necessarily a subset of E_1', nor does it necessarily contain E_1'. Thus to determine how close its measure is to that of E_1' we need an inequality for $\Lambda(E_1' - C) + \Lambda(C - E_1')$. Now we have

$$\Lambda(E_1' - C_i) + \Lambda(C_i - E_1')$$

$$= \Lambda(E_1') - \Lambda(E_1' \cap C_i) + \Lambda(C_i - F_i) - \Lambda(C_i \cap (E_1' - F_i))$$

since $F_i \subset E_1'$

$$\leqslant \Lambda(E_1') + \Lambda(C_i - F_i) - \Lambda(C_i \cap F_i)$$

$$\leqslant \Lambda(E_1') + \Lambda(C_i - F_i) - \Lambda((F_i \cap \bigcup_{1 \leqslant j \leqslant N_i} A_j) \cap F_i)$$

by (4), since $\Lambda(H_i) = 0$

$$= \Lambda(E_1') + \Lambda(C_i - F_i) - \Lambda(F_i \cap \bigcup_{1 \leqslant j \leqslant N_i} A_j)$$

$$< \Lambda(E_1') + \Lambda(C_i - F_i) - \Lambda(F_i) + \varepsilon_i \qquad \text{by (2)}$$

$$< 2\varepsilon_i + \Lambda(C_i - F_i) \qquad \text{by (1)}$$

$$< 3\varepsilon_i \qquad \text{by (5) and (3)}.$$

It follows that

$$|\Lambda(E'_{1,\theta}) - \Lambda(C_{i,\theta})| \leqslant \Lambda(E_1' - C_i) + \Lambda(C_i - E_1') < 3\varepsilon_i$$

so that, since $\Lambda(C_{i,\theta})$ is continuously dependent on θ, $\Lambda(E'_{1,\theta})$ is the uniform limit (ε_i is independent of θ) of a sequence of continuous functions. Thus $\Lambda(E'_{1,\theta})$ depends continuously on θ.

<u>Lemma 2.</u> $$\int_0^{2\pi} \Lambda(E'_{1,\theta})d\theta \leqslant 4\Lambda(E_1').$$

Proof. The proof comes easily from a series of simplifications of the initial problem. First of all we note that the result is true for any E_1' if, by the construction used in lemma 1, it is true for a union C_i of a finite number of rectifiable arcs, since we established above that $\Lambda(C_{i,\theta})$ tends uniformly in θ to $\Lambda(E'_{1,\theta})$ and $\Lambda(C_i)$ tends to $\Lambda(E_1')$. Also it is clear that the result follows for such a set C_i if it is established for a single arc.

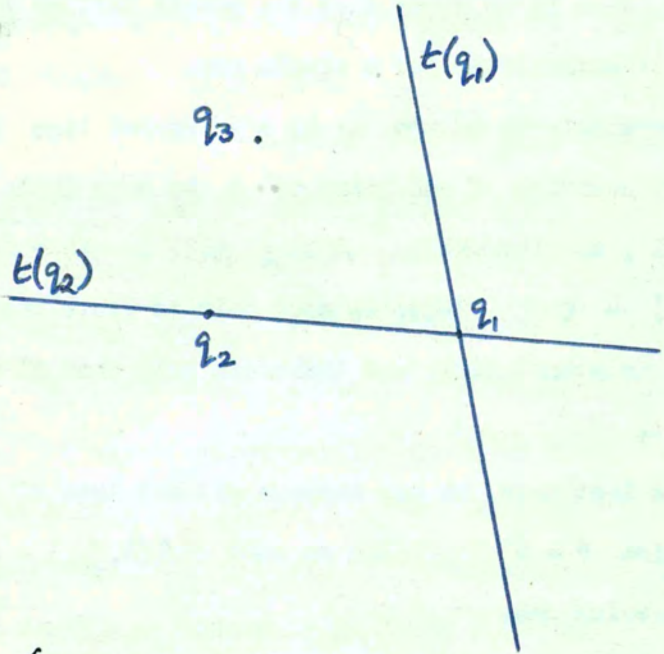
We may approximate to an arc A by a polygonal line L so that for an arbitrary positive ε no point of A is more than $\frac{1}{2}\varepsilon$ from some point of L, and for which $\Lambda(L) \leq \Lambda(A)$. Then for any θ, $\Lambda(L_\theta) \geq \Lambda(A_\theta) - \varepsilon$. Hence we need only to prove the inequality when E_1' is a polygonal line, and therefore only when it is a single straight segment.

But in this last case, we may suppose without loss of generality that the direction $\theta = 0$ is chosen so that $\Lambda(E'_{1,\theta}) = \Lambda(E_1')|\sin\theta|$, XXX, in which case

$$\int_0^{2\pi} \Lambda(E'_{1,\theta})d\theta = 2\Lambda(E_1')\int_0^{\pi} \sin\theta\, d\theta = 4\Lambda(E_1') .$$

Thus the result is true for a single segment, and so the lemma is proved. (Equality does not of course always hold, because of the approximations described in the proof, but at the same time we cannot improve on the general inequality because we must include the case when E and hence E_1' are single segments.)

(Diagram)



(Opposite page.)

Lemma 3. Either (a) almost all points of E_1' lie on one straight line, or

(b) there are two R-points of E_1', say p_1, p_2, such that p_2 does not lie on $t(p_1)$ and p_1 does not lie on $t(p_2)$.

Proof. If (a) is false, there are two R-points q_1, q_2 of E_1' for which q_2, say, does not lie on $t(q_1)$. If q_1 does not lie on $t(q_2)$ then condition (b) of the lemma is satisfied. So suppose q_1 does lie on $t(q_2)$. (See diagram opposite.)

Choose, if possible, a third R-point q_3 of E_1' which does not lie on either of $t(q_1)$, $t(q_2)$. Then $t(q_3)$ cannot contain both of q_1, q_2, for this would imply that q_3 lies on the line q_1q_2, which is $t(q_2)$ by our assumption. So one of the pairs (q_1, q_3), (q_2, q_3) must satisfy condition (b) of the lemma.

If it is not possible to choose such a q_3, almost all points of E_1' must lie on the union of $t(q_1)$ and $t(q_2)$. Since there must be R-points of E_1' other than q_1, q_2 on each of these lines, we can choose a point q_4, say, on $t(q_1)$ distinct from q_1. Then $t(q_4)$ is the same line as $t(q_1)$, so that q_2 does not lie on $t(q_4)$ (by assumption) and q_4 does not lie on $t(q_2)$ (by construction). So the points q_2, q_4 satisfy condition (b) of the lemma, and the lemma is proved.

Lemma 4. If an R-point p of E_1' projects onto the point q of

$E'_{1,\theta}$ and the direction of projection is not parallel to $t(p)$, then the set $E'_{1,\theta}$ has unit density at q.

Proof. For convenience, we make a change of notation so that for X_θ we shall write $P(X,\theta)$.

With the notation used in defining the density of a set at a point (see page 54), and because p is a point of unit density of A_i, say, and of $E_1' \cap A_i$, then given $\varepsilon > 0$ there is a positive number δ_0 such that

$$\left.\begin{array}{l} \Lambda(E_1' \cap A_i \cap c(p,\delta)) > (1-\varepsilon)2\delta \\ \text{and} \quad \Lambda(A_i \cap c(p,\delta)) < (1+\varepsilon)2\delta \end{array}\right\} \quad \text{for all } \delta < \delta_0 . \tag{1}$$

Put $A_i^* = A_i \cap c(p,\delta)$, and let $I(q,\delta)$ denote the closed linear interval perpendicular to θ whose midpoint is q and whose length is 2δ. Then since Λ is a measure,

$$\Lambda[P(A_i^*,\theta) \cap I(q,\delta)] \leqslant \Lambda[P(A_i^* - E_1',\theta) \cap I(q,\delta)] + \Lambda[P(A_i^* \cap E_1',\theta) \cap I(q,\delta)] . \tag{2}$$

$$\begin{aligned} \Lambda[P(A_i^* - E_1',\theta) \cap I(q,\delta)] &\leqslant \Lambda[(A_i^* - E_1') \cap c(p,\delta)] \\ &= \Lambda[A_i^* \cap c(p,\delta)] - \Lambda[A_i^* \cap E_1' \cap c(p,\delta)] \\ &\qquad \text{since } (A_i^* \cap E_1') \cap (A_i^* - E_1') = \phi \\ &\equiv \Lambda[A_i \cap c(p,\delta)] - \Lambda[A_i \cap E_1' \cap c(p,\delta)] \\ &< 4\varepsilon\delta \quad \text{for all } \delta < \delta_0 \text{ by (1)} . \end{aligned} \tag{3}$$

Also we can find a δ_1 such that if $\delta < \delta_1$

$$P(A_i^*,\theta) = I(q,\delta) . \tag{4}$$

The ratio we must consider in order to prove the lemma is

$$\frac{\Lambda[P(E_1' \cap A_i, \theta) \cap I(q,\delta)]}{2\delta} = \frac{\Lambda[P(E_1' \cap A_i^*, \theta) \cap I(q,\delta)]}{2\delta}$$

(equality of the two expressions following because $P(c(p,\delta),\theta) = I(q,\delta)$).

But from (2) , (3) and (4) ,

$$\Lambda[P(E_1' \cap A_i^*, \theta) \cap I(q,\delta)] > 2\delta - 4\varepsilon\delta$$
$$= (1 - 2\varepsilon)2\delta$$

for all $\delta < \min(\delta_0, \delta_1)$, and it is immediate that

$$\Lambda[P(E_1' \cap A_i^*, \theta) \cap I(q,\delta)] \leq 2\delta \quad .$$

Thus it follows from the identity given above that q is a point of unit density of $P(E_1', \theta)$, and the lemma is proved.

Section 1 - Notation and Definitions.

(The page number refers either to the first page on which a given notation is used, and therefore explained, or to the page on which a given term is defined in full.)

X_θ or $P(X,\theta)$	:	page 41
$\Lambda(X)$	:	page 41
$\mu(X)$	:	page 41
E_1'	:	page 42
$t(p)$	:	page 44

To define the terms left undefined in the main part of the section, it is convenient to use the following notation [1].
We are working in E_2.

$c(a,r)$	:	a closed circle with centre c , radius r .
$D(E)$	:	diameter of a point set E .
U	:	a convex area, including or excluding its frontier.
A	:	a finite or enumerably infinite set of convex areas U .
$A(\rho)$	:	a set A for which every member U has $D(U) \leq \rho$.
$A(E)$, $A(E,\rho)$	:	a set A , $A(\rho)$ respectively which contains E .

With this notation, the <u>linear measure</u> $\Lambda(E)$ of a plane set E is given by the equation

$$\Lambda(E) = \lim_{\rho \to 0} \inf \sum_{A(E,\rho)} D(U)$$

The lower density and upper density respectively of a set E at a point a (which may or may not be in E) are defined by

$$d_*(a,E) = \liminf_{r \to 0} \frac{\Lambda(E \cap c(a,r))}{2r} ,$$

$$d^*(a,E) = \limsup_{r \to 0} \frac{\Lambda(E \cap c(a,r))}{2r} .$$

If $d_*(a,E) = d^*(a,E)$, then $\lim_{r \to 0} \frac{\Lambda(E \cap c(a,r))}{2r}$ exists and is called the __density__ $d(a,E)$ of E at a .

A point of E at which the density exists and is equal to 1 is called a __regular point__ of E [2]; any other point of E is called an __irregular point__. If almost all points of E (that is, all except for a set of zero linear measure) are regular, E is called a __regular set__. If almost all points of E are irregular, E is called __an irregular set__.

E_1' is a measurable subset of the union of an enumerable infinity of rectifiable arcs. Select one such set of arcs A_1 , A_2 ,.... , and let p be a point of E_1' lying on an arc A_i of this set. (If p belongs to more than one arc A_i we may just choose one of these A_i and associate it with p throughout.) It is known ([1] pp. 303-4) that the density of A_i at almost all points p of A_i and the density of $E_1' \cap A_i$ at almost all points p of $E_1' \cap A_i$ both exist and are equal to 1 . Also A_i is rectifiable, so there is a __tangent__ to the arc at almost all points of it.

Thus at almost all points p of E_1' , the densities of A_i and $E_1' \cap A_i$ are equal to 1 and the tangent to A_i exists. Any point with these three properties is called an __R-point__ [3].

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Section 2. To show that, given a plane curve of unit length, the area of its convex cover K does not exceed $\frac{1}{2\pi}$.

[4]

- - - - - - - - - - - - [Notation and Definitions : page 64.]

(This section and section 3 following it are, of course, essentially the same problem, the one in two and the other in three dimensions. Although the proofs given are very dissimilar, it may be noted that the extremal curves obtained in the two results have, as would be expected, exactly parallel properties with regard to the curvature of the curve and the distance between the endpoints.)

Note first that the convex cover of a semi-circle of unit length has area $\frac{1}{2\pi}$, so this result is the best possible.

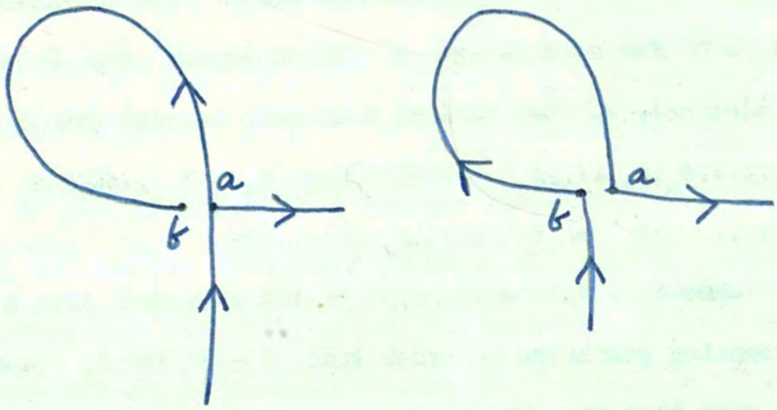
The length of a curve may be defined as the least upper bound of the length of inscribed polygonal lines. The intuitive way of approximating to this least upper bound is by choosing a polygonal line whose longest straight segment is arbitrarily short. In this way we can choose a polygonal line such that the area of its convex cover differs from the area of K by as little as we please. In other words, it is sufficient to consider only polygonal lines (of unit length). Let $P_0P_1 \ldots P_n$ be one such line, where the $P_{i-1}P_i$ are straight segments. It is convenient to call the P_i vertices, but three successive vertices may be collinear.

We associate with each such polygon a 2n-tuple $(\ell_1,\dots,\ell_n,\theta_1,\dots,\theta_n)$ where $\ell_i = L(P_{i-1}P_i)$, the length of $P_{i-1}P_i$, and θ_i is the angle it makes with some fixed directed line. For convenience we allow $\ell_i = 0$ for some of the i and we regard $\theta = 0$ and $\theta = 2\pi$ as distinct, so that we have a closed, bounded set S of points $(\ell_1,\dots,\ell_n,\theta_1,\dots,\theta_n)$ satisfying $\ell_i \geqslant 0$, $0 \leqslant \theta_i \leqslant 2\pi$ and $\ell_1 + \dots + \ell_n = 1$.

Associate with each point of the polygonal line a continuously increasing parameter s such that $s = 0$ at P_0 and $s = 1$ at P_n , and such that the parameters a , b of two points P , Q on the line satisfy $a < b$ if $L(P_0P_1\dots P)$ is strictly less than $L(P_0P_1\dots Q)$. Note that a point of the line may have two or more distinct parameters associated with it. We define a portion $[a,b]$ of the line as the set of those points of the line for which $a \leqslant s \leqslant b$ $(0 \leqslant a < b \leqslant 1)$. Two portions $[a,b]$, $[c,d]$, $b < c$, may have one or more portions in common, called double portions. (Generally, a multiple portion is a portion of the line which is common to two or more portions.) If two such portions have a point but not a portion in common and are such that the portion $[c,d]$ has points on each side of $[a,b]$ in a neighbourhood of the common point and vice versa, then we say that there is a crossover on the line.

Thus the most general polygonal line may contain both crossovers and multiple portions, and we show first that in fact these are

57
(Diagram)



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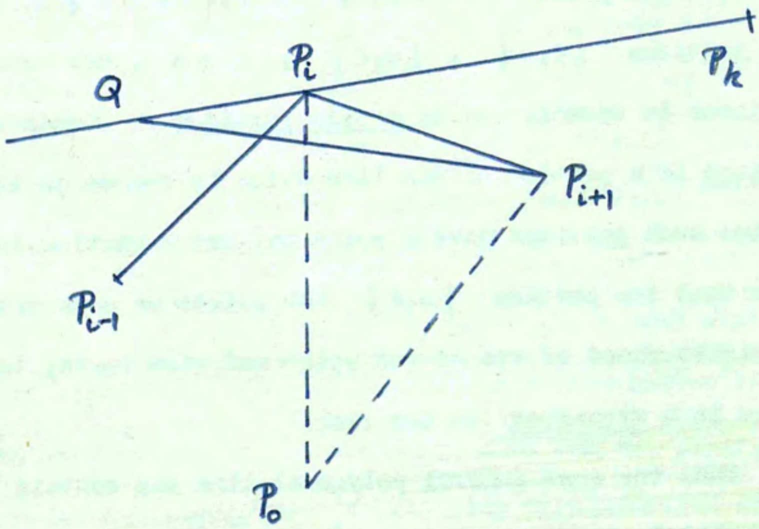


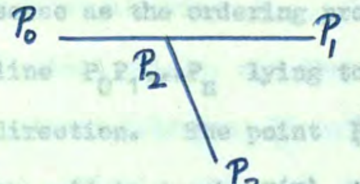
(Page 63.)

complications of the problem which can be eliminated.

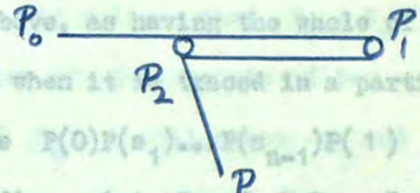
First, then, consider an n-sided polygonal line L which crosses itself but has no multiple portions. We may note immediately that it is sufficient to consider only those cases in which the crossover occurs at single points and where there are no points of the plane common to more than two portions of the curve, for any other cases can be approximated to by lines satisfying these conditions. Such a line can be replaced as follows by a polygonal line L' without crossovers but having the same length as L.

Suppose that, tracing the line in order from P_0, we reach the first crossover at I_1. If we continue to trace the line in order from I_1, we shall later come back to I_1, having traversed a portion $[a,b]$, say of the line. (See diagram opposite.)[1] Instead, then, from I_1 continue to trace the line along the portion from b towards a (identifying the parameter with the point of the line), and finally from I_1 to P_n, and redefine the line as ordered in this way. This eliminates the crossover at I_1 (although it is still a double point), without introducing any new crossovers. Now repeat this operation at the first crossover on the newly-defined line. As there can be only a finite number of crossovers on the line, this process will eventually yield a line L' of the same length as L and having no crossovers. (We may have introduced new vertices, of course, and also double points.) The area of K is unaltered.

So the result will be proved if it is shown to be true for a polygonal line of any number of sides which does not cross itself. Consider now the set of all such lines with a fixed number n of sides; these lines may have double or multiple points or lines. Since we are excluding lines which have crossovers, we can order the portions of a double or multiple segment in the following way. If, for example, P_1P_2 is wholly or partly coincident with P_0P_1, we may suppose it to lie on one side or the other of P_0P_1, and may choose our placing of it so that a crossover does not occur because of our choice. By ordering the line in this way, we are in effect able to think of coincident lines as if they were actually distinct but lying next to one another, as the diagram indicates.



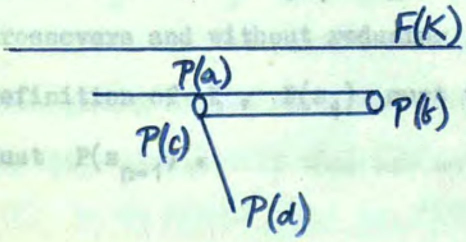
can be thought of as



where the small circles represent single points. As each line is now fully ordered in this way, we may associate with each a new parametric representation where the parameter increases steadily as the line is traced in order from P_0 to P_n.

The set S' of the 2n-tuples $(\ell_1,\ldots,\ell_n,\theta_1,\ldots,\theta_n)$ corresponding to such lines (without crossovers) is closed and bounded. (It is bounded because $S' \subset S$ and S is bounded; it is closed in S and hence closed because the set of lines with (non-degenerate) crossovers is clearly open, and S' is the complement of this set in S .) Therefore there is at least one line corresponding to a 2n-tuple in S' for which the (finite) upper bound of the area of K is attained.

Consider then a line $P_0P_1 \ldots P_n$ for which this is true, and suppose that it is fully ordered as we have described, so that there is a parametric representation s , $0 \leqslant s \leqslant 1$, and corresponding points $P(s)$ of the line so that $P(s)$ and $P(t)$ are regarded as distinct points whenever $s \neq t$, although they may represent coincident points on the line $P_0P_1 \ldots P_n$. The convex cover K of $P_0P_1 \ldots P_n$ has a frontier $F(K)$, a convex polygon, which may be regarded, in the same sense as the ordering process employed above, as having the whole of the line $P_0P_1 \ldots P_n$ lying to one side of it when it is traced in a particular direction. The point $P(s)$ of the line $P(0)P(s_1) \ldots P(s_{n-1})P(1)$ will be said to touch $F(K)$ if the corresponding point P of $P_0P_1 \ldots P_n$ lies on $F(K)$ and there is a portion $P(s_i)P(s_j)$ of the line, $s_i < s < s_j$, such that the part of $F(K)$ in some neighbourhood of $P(s)$ lies on the opposite side of $P(s_i)P(s_j)$ to any other points of the line in that neighbourhood.



$a < b < c < d$.

$P(a)$, $P(b)$ both touch $F(K)$, but $P(c)$ does not.

We now establish several properties of this line.

(a) $P(0)$, $P(1)$ (corresponding to P_0 , P_n) touch $F(K)$.

If not, suppose that the first vertex to touch $F(K)$ is $P(s_i)$. Then we can remove the portion $P(0)\ldots P(s_i)$ without altering K , and so the same upper bound for $A(K)$ is attained with a line of less than unit length. Since this is by definition impossible, $P(0)$ must touch $F(K)$, and so must $P(1)$ by similar reasoning.

(b) P_0 , P_n are distinct points (of the non-ordered line).

Suppose that they are not. There must be at least one other vertex of $P(0)P(s_1)\ldots P(s_{n-1})P(1)$ which touches $F(K)$ (since K is the convex cover of the line). If $P(s_i)$ is the first such vertex in order from P_0 , then again we can remove the portion $P(0)\ldots P(s_i)$ without altering K . Hence as in (a) , it follows that P_0 , P_n are distinct points, and therefore the frontier of K consists of two distinct polygonal lines between P_0 and P_n . Call these α and β .

(c) $P(s_1)$, $P(s_{n-1})$ touch $F(K)$.

If not, suppose $P(s_i)$, $i \neq 1$, is the first vertex in order from $P(0)$ which does touch $F(K)$. Then the portion $P(0)\ldots P(s_i)$ can be replaced with the straight segment $P(0)P(s_i)$, which is in fact part of $F(K)$, and thus $L(P_0P_1\ldots P_n)$ can be reduced without introducing any crossovers and without reducing $A(K)$. Since this contradicts the definition of K , $P(s_1)$ must touch $F(K)$, and in the same way so must $P(s_{n-1})$.

(d) If all the vertices do not lie in order on α or β , $P(0)P(s_1)\ldots P(s_{n-1})P(1)$ must have the following form: $P(s_1)$, $P(s_2)$,...,$P(s_h)$ lie in order on α (touching $F(K)$), $P(s_j)$, $P(s_{j+1})$, ..., $P(s_k)$ lie in order on β (touching $F(K)$), $P(s_l)$, $P(s_{l+1})$, ..., $P(s_m)$ lie on α , and so on, where $1 \leqslant h < j < \ldots \leqslant n-1$.

Suppose that all the vertices up to $P(s_h)$ touch $F(K)$ and lie on α , but that $P(s_{h+1})$ does not. We do not yet know that every vertex has to touch $F(K)$, so let $P(s_j)$, $j \geqslant h+2$, be the first vertex in order from $P(s_{h+1})$ to touch $F(K)$ on one of α and β ($P(s_{n-1})$ will do so, so $P(s_j)$ exists). By definition $P(s_j)$ cannot touch $F(K)$ on the part of α between $P(0)$ and $P(s_h)$, since we are assuming that all these vertices touch $F(K)$ and this would imply the existence of a crossover. If it touches $F(K)$ on the remaining part of α , then the straight line $P(s_h)P(s_j)$ must be part of $F(K)$ and replacing $P(s_h)P(s_{h+1})\ldots P(s_j)$ with the straight segment $P(s_h)P(s_j)$ reduces $L(P_0P_1\ldots P_n)$ without introducing any crossovers and without reducing $A(K)$, which contradicts the extremal property of the polygonal line with K . Thus for some $j > h+1$, $P(s_j)$ must lie on β if $P(s_{h+1})$ does not touch $F(K)$. This proves (d) .

(e) $P(s_h)\ldots P(s_j)$, $P(s_k)\ldots P(s_l)$, ... must be straight lines.

These are the polygonal lines between α and β in the arrangement of the vertices $P(0)$, $P(s_1)$, ..., $P(s_{n-1})$, $P(1)$ given in (d) above. If they are not straight, we can replace them simultaneously with straight segments without introducing any crossovers, but reducing $L(P_0P_1 \ldots P_n)$ without reducing $A(K)$. Since again this is not so, the lines must be straight.

With this result, we can assume that there are no vertices of $P(0)P(s_1) \ldots P(s_{n-1})P(1)$ which do not touch $F(K)$ (thus regarding $p(s_h) \ldots P(s_j)$ etc. as single segments). With this established, we can drop the ordering notation as it is no longer useful, and return to the original notation $P_0P_1 \ldots P_n$ for the line.

(f) If P_1 lies on α, all the vertices lie on α.

If not, suppose P_{i+1} is the first vertex which lies on β. (See page 57, diagram.) ($i+1 \neq n$.) Then the line P_0P_{i+1} is part of $F(K)$ and the area of triangle $P_0P_iP_{i+1}$ contributes to $A(K)$. Since K is maximal by definition, this means that $A(P_0P_iP_{i+1})$ must be as large as possible, keeping $L(P_0P_i)$ and $L(P_iP_{i+1})$ constant, and this so when the angle between P_0P_i and P_iP_{i+1} is $\frac{1}{2}\pi$.

Hence $L(P_0P_{i+1}) > L(P_iP_{i+1})$ so that we can now make the following construction. Supposing that P_kP_i ($k > i$) is on $F(K)$, extend it to Q (P_i lying between Q and P_k) so that

$$L(P_{i+1}Q) = L(P_{i+1}P_0) > L(P_iP_{i+1}) .$$

Now reflect $P_0P_1 \ldots P_{i+1}$ in P_0P_{i+1} and rotate P_0P_{i+1} rigidly about P_{i+1} until P_0 is coincident with Q. We have not altered the length of the polygonal line, but by putting the vertices $P_1, \ldots, P_i$ on the same polygonal arc from P_0 to P_n as P_{i+1} we have increased $A(K)$ by an amount equal to $A(QP_iP_{i+1})$. (Here it is assumed that the change in position of $P_0P_1...P_{i+1}$ leaves P_{i+1} still on the frontier of the new convex cover. If it does not, then the area of the convex cover is increased still further, but we know from property (e) that this is itself not maximal, since not all the vertices of the polygonal line then lie on its frontier.)

Thus we have finally that the area of the convex cover is greatest when all the vertices $P_1, ..., P_{n-1}$ of the polygonal line lie on one side of the straight segment P_0P_n. Reflecting $P_0P_1...P_n$ in P_0P_n, we obtain a 2n-sided polygon (not necessarily convex) of perimeter 2, and a polygon of given perimeter length has maximum area when it is regular. Hence the convex cover of $P_0P_1...P_n$ has, by elementary trigonometry, maximum area

$$A_n = \frac{1}{2}\left(\frac{1}{2n}\right)\cot\frac{\pi}{2n}$$

$$= \frac{1}{2\pi}\left(\frac{\pi/2n}{\tan \pi/2n}\right) < \frac{1}{2\pi}$$

and as n increases $A_n \to 1/2\pi$. This proves the result.

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Section 2 - Notation and Definitions.

(The page number refers either to the first page on which a given notation is used, and therefore explained, or to the page on which a given term is defined in full.)

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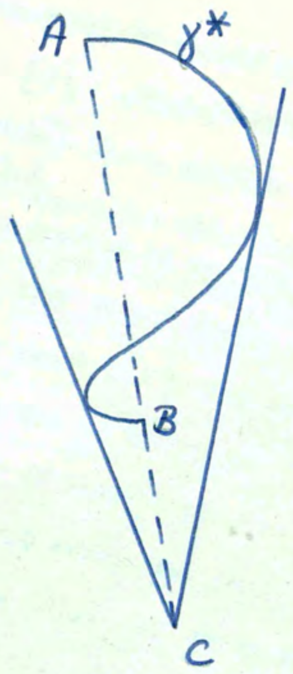
Section 3. To find the maximum volume of the convex cover of a simple arc of space-curve of given length. [5]

[Notation and definitions : page 80.]

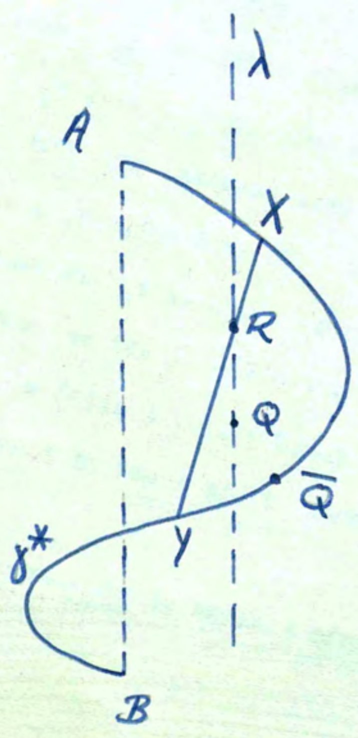
A bounded convex point set may be defined as a set which is identical to the set of its chords (using this term in its usual sense). Thus if $\omega(S)$ denotes the set of all those points contained in at least one chord of S , S is convex if and only if $\omega(S) = S$. With this definition of ω , it follows by Carathéodory's theorem (theorem 6 - page 13) that the convex cover $\Omega(S)$ of a bounded point set S in E_3 is given by $\Omega(S) = \omega(\omega(S))$. For, $\Omega(S) \supset \omega(\omega(S))$ trivially since $\Omega(S)$ is convex and contains S . On the other hand if a point $x \in \Omega(S)$, then by Carathéodory's theorem x is a point of a simplex with vertices $x_1, \ldots, x_4$, (in E_3), $x_i \in X$ for each i . The set $\omega(x_1, \ldots, x_4)$ defines the edges and interior diagonals of this simplex, and thus $\omega(\omega(x_1, \ldots, x_4))$ defines the whole simplex. So $x \in \omega(\omega(x_1, \ldots, x_4))$. But since the set of the points $x_1, \ldots, x_4$ is contained in S , we have $\omega(\omega(x_1, \ldots, x_4)) \subset \omega(\omega(S))$, and so $x \in \omega(\omega(S))$. Thus finally $\Omega(S) = \omega(\omega(S))$. ($\Omega(S) = \omega(S)$ is generally not true, as for example when S is a set of four non-collinear points in E_3 .)

Let $K(\gamma)$ denote a class of space-curves (in E_3) with the properties

(Diagram)



(Opposite page.)



(Page 72.)

(I) $\Omega(\gamma) = \omega(\gamma)$,

(II) every interior point of $\Omega(\gamma)$ is contained in one and only one chord of γ.

A simple arc γ^* of space-curve is defined as an arc which has no four points coplanar. The problem is to determine the maximum volume of the convex cover of all arcs of such space-curves γ^* which have a given length L, and we prove that the required maximum volume is $L^3 / (18\sqrt{3}\pi)$, and that it is obtained only for an arc of a helix given by the equations $x = \frac{L}{\sqrt{6}\pi} \cos t$, $y = \frac{L}{\sqrt{6}\pi} \sin t$, $z = \frac{L}{2\sqrt{3}\pi} t$, where $0 \leqslant t \leqslant 2\pi$.

We prove first an ancillary result which in effect enables us to use in the main proof two equivalent definitions for the class $K(\gamma^*)$.

A connexion between the classes $K(\gamma^*)$ and $K(\gamma)$ described above is immediately clear, since any arc which has four coplanar points cannot belong to the class $K(\gamma)$ because property (II) is contradicted. We show here that any simple arc γ^* is a member of the class $K(\gamma)$ and hence it will follow that $K(\gamma^*)$ and $K(\gamma)$ are in fact identical.

Suppose that γ^* is a simple arc with endpoints A and B. Consider the cone which projects γ^* from a point C which is collinear with A and B and is separated from A by B (or from B by A). (See diagram opposite.)

(i) The cone is closed, in the sense that its intersection with a sphere centred at C is a closed curve. This is clear since the projecting lines of A and B are coincident, and the arc must cut every other generator of the cone once and once only. For if it cut any generator more than once, the two points of intersection would be coplanar with A and B , which is a contradiction of the definition of γ^* ; and so since A and B share the same generator γ^* must meet every other generator exactly once.

(ii) The generator passing through A and B is the only multiple generator. As we have already noted in (i) , any other double generator (for example) would contain two points of γ^* and these points would then (since any two generators meet in C) be coplanar with A and B .

(iii) The cone is convex in the sense that if its intersection with a plane is a bounded curve then that curve is convex. For if it were not, we should be able to find four coplanar generators, which would imply four coplanar points of the arc. For, suppose there is a plane which cuts the cone in a bounded non-convex curve Γ . Then Γ has at least four collinear points, as follows.

Since the set X enclosed by Γ is non-convex, there are two points x , y of X such that at least one point z , say, of the segment xy lies in the exterior of the closure of X (that is, in the exterior of the union of X and Γ). Now consider the line through x , y and z . Since x is interior to X and z is exterior to X , there must be a point q not coincident with x or z on the segment xz such that q lies on the curve, and similarly there must be a point r of the curve on the segment zy , r not coincident with z or y . Finally, since the curve is bounded, there must be two points p , s of the curve which lie on xy , with x between p and y , y between x and s . Thus p , q , r , s are four collinear points of the curve. It follows that the generators of the cone through these four points are coplanar, and this in turn implies that there must be four coplanar points of the arc γ^* , which is a contradiction with its definition. Hence must be convex.

(iv) The cone is convex in the usual sense. For if it were not, we should be able to find two points x , y of the cone such that at least one point z , say, of the segment xy is not contained in the cone. If there is a plane through xy which cuts the cone in a bounded curve, then we have an immediate contradiction with (iii) . If on the other hand every plane through xy cuts the cone in an unbounded curve, we may note the following. There is a plane π through the endpoint B say of the arc γ^* such that γ^* lies entirely within one of the closed half-spaces bounded by π , and therefore there is a parallel plane π_1 through the vertex C of the cone such that except for C the cone lies entirely in one of the open half-spaces bounded by π_1 . Any plane parallel to π_1 which meets the cone meets it in a bounded curve since otherwise this would contradict the definition of π_1 . Project the points x , y , z from C onto points x , y_1 , z_1 say of a plane π_2 through x parallel to π_1 . Then by our assumption z_1 lies between x and y_1 and does not belong to the cone, whereas x and y_1 do belong to the cone. But by its construction π_2 meets the cone in a bounded curve, and this curve is convex (by property (iii)). It follows that z_1 must belong to the cone, and we have a contradiction. Hence the cone is convex in the usual sense.

By a similar argument it follows also that the projecting cylinder of γ^* which has its generators parallel to AB is a closed convex cylinder.

The cones which project the arc from its endpoints A and B may be defined as follows. The cone which has A as vertex, say, consists of (i) all the lines joining A to interior points of γ^* and (ii) the plane sector which lies in the angle between the line AB and the tangent to γ^* at A . As before this cone and the one at B are closed and convex, and in fact are the support cones of γ^* at A and B . (The definition of a support cone is, as its name implies, analogous to that of a support hyperplane. (See page 9.)) Let the point set which is the intersection of these two support cones be denoted by $I(\gamma^*)$. We now prove that $I(\gamma^*)$ is the convex cover of γ^* , i.e. that $I(\gamma^*) = \Omega(\gamma^*)$.

If $I(\gamma^*) = \omega(\gamma^*)$, the result follows easily. We know that $I(\gamma^*)$ is convex (see theorem 2 - page 9), so that

$\omega(I(\gamma^*)) = I(\gamma^*)$. But if $\omega(\gamma^*) = I(\gamma^*)$, then

$$\Omega(\gamma^*) = \omega(\omega(\gamma^*)) = \omega(I(\gamma^*)) = I(\gamma^*) . \qquad (1)$$

To prove $I(\gamma^*) = \omega(\gamma^*)$, consider any point Q in the interior of $I(\gamma^*)$. (See page 67, diagram.) The plane QAB cuts γ^* in a single point $\bar{Q}$ (since no four points of γ^* are coplanar). Denote by λ the line through Q which is parallel to AB , and suppose that the plane $\bar{Q}AB$ is rotated about λ through 180° . Call this rotating plane π . As π rotates, it has two well-defined points of intersection, say X and Y , with γ^*, so that in its initial position X is at A and Y at $\bar{Q}$, and in its final position X is at $\bar{Q}$ and Y at B . This is easy to see as follows.

The support cylinder of the arc γ^* has only one double generator, through A and B . Otherwise any generator contains a single point of γ^* . The plane π is rotating about a line in the interior of this cylinder parallel to the cylinder, and so since we have seen already that the cylinder is convex π cuts the surface of the cylinder in two generators, and hence π cuts γ^* in just two points X , Y (one on each generator), except in the case when one of the generators is AB , which occurs at the beginning of the rotation and then not again until the end of the rotation (when π has turned through 180°). Note also that from their construction the points X and Y move continuously along γ^* .

Now the chord XY and the line λ lie in the same plane π

and so have a point of intersection R , unless they are parallel. But they can never be parallel because if XY is ever parallel to λ then it is also parallel to AB , which implies that X , Y , A , B are four coplanar points of γ^* , thus contradicting the definition of γ^* . Thus R always exists. Since X and Y move continuously along γ^*, it follows that the point R moves continuously along λ . At the beginning of the rotation R is the point of intersection of the segment $A\bar{Q}$ and λ , and at the end of the rotation R is the point of intersection of $\bar{Q}B$ and λ . But the segments $A\bar{Q}$ and $\bar{Q}B$ lie completely in the frontier of $I(\gamma^*)$, and thus R starts and finishes on the frontier of $I(\gamma^*)$. Therefore since R moves continuously along λ it follows that during the rotation R passes through every point of λ which is interior to $I(\gamma^*)$. Thus there is a position of the chord XY such that it passes through the arbitrary point Q . Furthermore this is the only chord which passes through Q , since if there were two such chords, say X_1Y_1 and X_2Y_2 , the four points X_1 , X_2 , Y_1 , Y_2 of γ^* would be coplanar.

We have proved, then, that through an arbitrary point of the interior of $I(\gamma^*)$ there passes one and only one chord of γ^* , so that the arc γ^* possesses property (II) of the class $K(\gamma)$. Also, since this has proved that $I(\gamma^*) = \omega(\gamma^*)$, according to the definition of ω , we have

$$\Omega(\gamma^*) = I(\gamma^*) = \omega(\gamma^*) \qquad \text{from (1)}$$

which is property (I) of the class $K(\gamma)$. Thus every simple arc γ^* is a member of the class $K(\gamma)$, and hence, since we have already seen at the beginning of this section that an arc which is not simple is not a member of $K(\gamma)$, the two classes are identical.

Returning to the main problem, suppose then that γ is a simple arc of length $L \equiv \sqrt{3}\ell$ ($\ell > 0$), the volume of whose convex cover is maximal. (If such an arc is not simple, we can construct and consider instead a simple arc, still of length L, for which the convex cover differs in volume by an arbitrarily small amount from the maximum value that we want.) We may now restrict attention to those curves at each of whose points P there is a tangent which varies continuously with P. For, given a curve without this property, it would be possible to define a "smoothing" process for the arc which would increase the volume of the convex cover without increasing the length of the arc. In the lemma proved below, (page 77) it is shown that the volume V of the convex cover of a simple arc is given by

$$V = \frac{z(b) - z(a)}{6}\int_a^b \left[x(t)\dot{y}(t) - y(t)\dot{x}(t)\right] dt$$

where the functions $x(t)$, $y(t)$, $z(t)$ have continuous first order derivatives, and where $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \leqslant t \leqslant b$, are the equations of the arc in a given rectangular system of coordinates, chosen so that the z-axis lies along the chord AB joining the endpoints of the arc, such that $x(a) = x(b) = y(a) = y(b) = 0$, and $z(a) < z(b)$.

This formula implies

$$V = \frac{qT}{3}$$

where q is the length of the chord joining the endpoints A, B of the arc $(= z(b) - z(a))$; T is the area bounded by the projections of points of γ along lines parallel to the chord AB onto a plane perpendicular to AB. For the equations of the closed projected curve are $x = x(t)$, $y = y(t)$ as above, and the area T enclosed by this curve is given by $\int_a^b x(t)\dot{y}(t)dt$, integrating over strips parallel to the x-axis, or by $-\int_a^b y(t)\dot{x}(t)dt$ if the strips are taken parallel to the y-axis. (Signs depend, of course, on the orientation of the arc, and will be opposite to the ones given above it T would otherwise be negative.) Thus it follows trivially that T is also equal to the average of these two integrals, as in the formula for V.

We may obtain the largest value of V by considering the following inequalities.

(i) If p is the length of circumference of the cross-section of the projecting cylinder of γ projected parallel to AB, then since of all closed curves of given circumference the circle has largest area,

$$T \leq \pi\left(\frac{p}{2\pi}\right)^2 = \frac{p^2}{4\pi} .$$

(ii) If the projecting cylinder of γ parallel to AB is developed into a plane, then, since the endpoints of an arc of given length are furthest apart when the arc is a straight line, we have by Pythagoras' theorem that

$$p^2 + q^2 \leqslant 3\ell^2 \quad .$$

(iii) If $\ell > 0$, $q > 0$, then $(3\ell^2 - q^2)q \leqslant 2\ell^3$.

We have from (ii) that $p^2 + q^2 \leqslant 3\ell^2$ and we are trying to obtain an inequality for p^2q in order to reach a maximal value for V . If we put $p = \alpha\ell$, $q = \beta\ell$, we have $\alpha^2 + \beta^2 = 3$ in the extremal case, and $p^2q = \alpha^2\beta\ell^3$. By ordinary differentiation of $\alpha^2\beta$ with respect to β , say, we see that the maximum value of p^2q occurs when $\beta = 1$, so that $q = \ell$ and $(3\ell^2 - q^2)q = 2\ell^3$. Thus in general $(3\ell^2 - q^2)q \leqslant 2\ell^3$.

Hence V satisfies the inequalities

$$V = \frac{qT}{3} \leqslant \frac{p^2q}{12\pi} \leqslant \frac{(3\ell^2 - q^2)q}{12\pi} \leqslant \frac{\ell^3}{6\pi} = \frac{L^3}{2.3^{5/2}\pi}$$

and these are valid if and only if the projecting cylinder has circular cross-section (from (i)), γ is transformed into a straight line when the projecting cylinder is developed into a plane (from (ii)), and the length q of the chord AB is equal to $\ell \equiv \frac{L}{\sqrt{3}}$ (from (iii)). Thus the maximum value of V which we required is $L^3 \big/ (18\sqrt{3}\,\pi)$ and the only space-curve satisfying the conditions for it is an arc of a helix (because of (ii)) traced on a circular cylinder (because of (i)) of radius $L/(\sqrt{6}\pi)$ (because of (iii)), and corresponding to a rotation 2π (because the projecting cylinder is closed) and a translation $L/\sqrt{3}$ (being the length of AB) .

* * * * * *

<u>Lemma</u>. A simple arc γ in E_3 is defined in a given rectangular system of coordinates by the equations

$$x = x(t) \ , \quad y = y(t) \ , \quad z = z(t) \ , \quad a \leqslant t \leqslant b$$

where $x(t)$, $y(t)$, $z(t)$ have continuous first order derivatives with respect to t , and also

$$x(a) = x(b) = y(a) = y(b) = 0 \ , \quad z(a) < z(b) \quad . \qquad (1)$$

(The effect of these last restrictions is to make the line joining the endpoints of the arc coincident with the z-axis.) Then the volume V of the convex cover of γ is given by

$$V = \frac{z(b) - z(a)}{6} \int_a^b \left[x(t)\dot{y}(t) - y(t)\dot{x}(t) \right] dt \quad .$$

<u>Proof</u>. As we have shown in the main part of the section, a simple arc γ has the property that every interior point of its convex cover is contained in exactly one chord of γ . Consider two points of γ with parametric representations t_1 , t_2 . Any point of the chord

joining these may be given by the equations

$$\left.\begin{aligned} x &= \frac{1+t_3}{2}\,x(t_1) + \frac{1-t_3}{2}\,x(t_2) \\ y &= \frac{1+t_3}{2}\,y(t_1) + \frac{1-t_3}{2}\,y(t_2) \\ z &= \frac{1+t_3}{2}\,z(t_1) + \frac{1-t_3}{2}\,z(t_2) \end{aligned}\right\} \qquad (2)$$

where t_3 is variable, $-1 \leqslant t_3 \leqslant 1$; and if we specify that $a \leqslant t_1 < t_2 \leqslant b$, there is then a one-one correspondence between the points represented by the equations (2) and the interior and frontier points of $\Omega(\gamma)$, the convex cover of γ .

Hence it follows that the volume of $\Omega(\gamma)$ is given by

$$V = \iiint\limits_{\Omega(\gamma)} dx \; dy \; dz \;=\; \tfrac{1}{2}\int_{-1}^{1}\int_{a}^{b}\int_{a}^{b} \frac{\partial(x,y,z)}{\partial(t_1,t_2,t_3)}\, dt_1 dt_2 dt_3$$

where the factor $\frac{1}{2}$ arises because of the stipulation that $t_1 < t_2$.

From (2) , we have

$$\frac{\partial(x,y,z)}{\partial(t_1,t_2,t_3)} = \begin{vmatrix} \frac{1+t_3}{2}\,\dot{x}(t_1) & \frac{1-t_3}{2}\,\dot{x}(t_2) & \frac{1}{2}\,x(t_1) - \frac{1}{2}\,x(t_2) \\ \frac{1+t_3}{2}\,\dot{y}(t_1) & \frac{1-t_3}{2}\,\dot{y}(t_2) & \frac{1}{2}\,y(t_1) - \frac{1}{2}\,y(t_2) \\ \frac{1+t_3}{2}\,\dot{z}(t_1) & \frac{1-t_3}{2}\,\dot{z}(t_2) & \frac{1}{2}\,z(t_1) - \frac{1}{2}\,z(t_2) \end{vmatrix}$$

$$= \frac{(1+t_3)(1-t_3)}{8} \begin{vmatrix} \dot{x}(t_1) & \dot{x}(t_2) & x(t_1) - x(t_2) \\ \dot{y}(t_1) & \dot{y}(t_2) & y(t_1) - y(t_2) \\ \dot{z}(t_1) & \dot{z}(t_2) & z(t_1) - z(t_2) \end{vmatrix}$$

$$\equiv \frac{(1+t_3)(1-t_3)}{8}\,\big(\,\Delta(t_1) - \Delta(t_2)\,\big) \qquad \text{, say.}$$

Thus $V = \frac{1}{16}\int_a^b\int_a^b (\Delta(t_1) - \Delta(t_2))dt_1dt_2 \cdot \int_{-1}^{1}(1+t_3)(1-t_3)dt_3$

$$= \frac{4}{3}\cdot\frac{1}{16}\int_a^b\int_a^b (\Delta(t_1) - \Delta(t_2))dt_1dt_2$$

$$= \frac{1}{12}\int_a^b\int_a^b \begin{vmatrix} \dot{x}(t_1) & \dot{x}(t_2) & x(t_1) \\ \dot{y}(t_1) & \dot{y}(t_2) & y(t_1) \\ \dot{z}(t_1) & \dot{z}(t_2) & z(t_1) \end{vmatrix} dt_1dt_2$$

$$- \frac{1}{12}\int_a^b\int_a^b \begin{vmatrix} \dot{x}(t_1) & \dot{x}(t_2) & x(t_2) \\ \dot{y}(t_1) & \dot{y}(t_2) & y(t_2) \\ \dot{z}(t_1) & \dot{z}(t_2) & z(t_2) \end{vmatrix} dt_1dt_2 \quad .$$

As our choice to have $t_1 < t_2$ was arbitrary, we are free to interchange t_1 and t_2 in the second term, say, which gives

$$V = \frac{1}{6}\int_a^b\int_a^b \begin{vmatrix} \dot{x}(t_1) & \dot{x}(t_2) & x(t_1) \\ \dot{y}(t_1) & \dot{y}(t_2) & y(t_1) \\ \dot{z}(t_1) & \dot{z}(t_2) & z(t_1) \end{vmatrix} dt_1dt_2$$

$$= -\frac{1}{6}\int_a^b \dot{x}(t_2)dt_2 \int_a^b \begin{vmatrix} \dot{y}(t_1) & y(t_1) \\ \dot{z}(t_1) & z(t_1) \end{vmatrix} dt_1$$

$$+ \frac{1}{6}\int_a^b \dot{y}(t_2)dt_2 \int_a^b \begin{vmatrix} \dot{x}(t_1) & x(t_1) \\ \dot{z}(t_1) & z(t_1) \end{vmatrix} dt_1$$

$$- \frac{1}{6}\int_a^b \dot{z}(t_2)dt_2 \int_a^b \begin{vmatrix} \dot{x}(t_1) & x(t_1) \\ \dot{y}(t_1) & y(t_1) \end{vmatrix} dt_1 \quad .$$

Of these three terms, the first two go to zero because of (1) , and so we have finally

$$V = \frac{1}{6} \left(z(b) - z(a) \right) \int_a^b \begin{vmatrix} x(t) & \dot{x}(t) \\ y(t) & \dot{y}(t) \end{vmatrix} dt$$

Which is the required formula.

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Section 3 - Notation and Definitions.

(The page number refers either to the first page on which a given notation is used, and therefore explained, or to the page on which a given term is defined in full.)

$\omega(S)$	:	page 66
$\Omega(S)$	:	page 66
$K(\gamma)$	:	page 66
A , B	:	page 68
$I(\gamma^*)$	:	page 71

Space-curve	:	a curve of linear measure in E_3 .
Simple arc	:	page 68
Support cone	:	page 71

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Section 4. To show that the minimal width of a triangle circumscribing a plane bounded convex set of given frontier length ℓ is at most $\frac{\ell}{\sqrt{3}}$, and is at most $\frac{1}{2}\ell$ if the convex set is central. [6]

---------------- [Notation : page 89.]

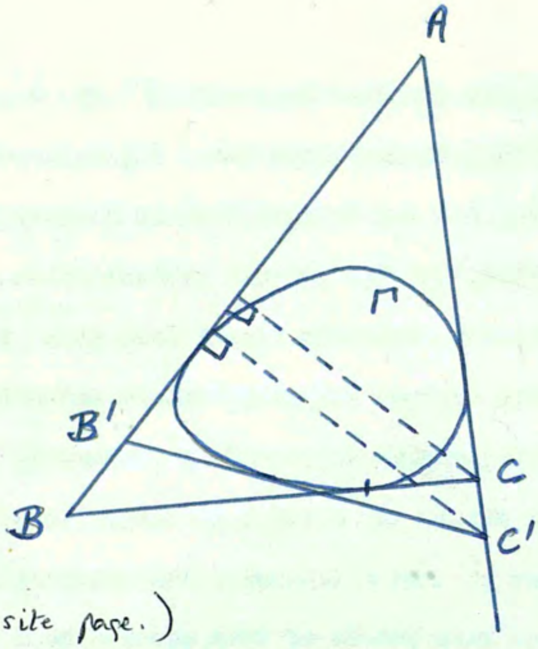
Γ denotes a plane bounded convex set whose frontier is of length ℓ, and ABC denotes a triangle which circumscribes Γ. One of the vertices of ABC may be at infinity, so that "triangles" with two sides parallel can be included. First note that if Γ is an equilateral triangle and ABC is a circumscribing equilateral triangle with sides parallel to those of Γ but oriented in the opposite sense, then the minimal width of ABC is $\frac{\ell}{\sqrt{3}}$.

There is one convex set Γ and one circumscribing triangle ABC for which the ratio of the minimal width of ABC to the frontier length of Γ attains the largest possible value. For, given any triangle A′B′C′ we can inscribe in A′B′C′ an infinity of convex sets of different frontier lengths, and from these sets it is always possible to pick an infinite sequence in which the frontier length either decreases strictly or becomes constant, and in either case by the Blaschke selection theorem (theorem 7 - page 16) (since the closures of the convex sets are all contained in A′B′C′) there is a convex set Γ' inscribed in A′B′C′ whose frontier length is at least as small as that of any other inscribed convex set. Denote the ratio of the minimal width of A'B'C' to the frontier length of Γ' by λ' . λ' must be non-negative, and is certainly less than 2 , say. (Every convex set of frontier length ℓ can be contained in a square of side length $\frac{1}{2}\ell$ (trivially), and the largest minimal width of a triangle which circumscribes such a square is itself less than ℓ - see [9].) The values of λ' thus form an infinite bounded set and so an infinite sequence of increasing values of λ' selected from this set must converge to some value λ , which is the ratio corresponding to some actual convex set Γ and a triangle ABC circumscribing Γ . Hence we may prove the main result of this section by a discussion of the extremal case, since we know that such a case exists.

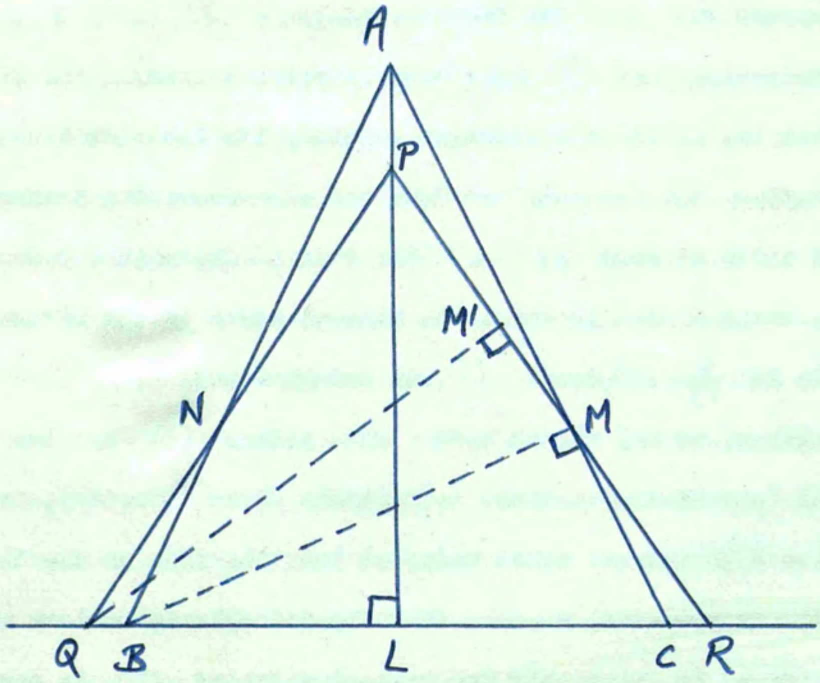
Suppose then that the frontier length of Γ is ℓ . We begin by establishing that Γ must have interior points. For if it does not, that is, if it is a straight segment, its frontier length is twice the length of the segment, and thus the circumscribing triangle has minimal width at most $\frac{1}{2}\ell$. But this is impossible because we have already noted a case in which the minimal width of the circumscribing triangle is $\frac{\ell}{\sqrt{3}}$. Hence Γ has interior points.

Further, we may assume that ABC either (i) has two sides parallel (one of the vertices being taken as at infinity), or (ii) is isosceles with the two equal sides at least as long as the third side. If (ii) is not true, suppose that the triangle is scalene with AB $>$ AC $\geqslant$ BC , so that the minimal width of ABC is equal to the

(Diagram)



(Opposite page.)



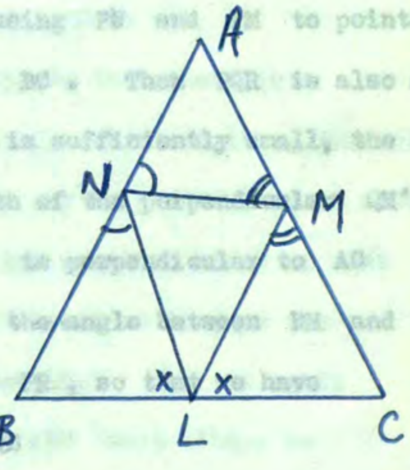
(Page 86.)

perpendicular length from C to AB . If there is a point of Γ other than at C on BC we can construct a second circumscribing triangle of Γ which has minimal width greater than that of ABC . (See diagram opposite.) For, let C′ be a point near C on AC produced (so that C lies between A and C′), and let C′B′ be a support line of Γ meeting AB in B′ . Since C is not the only point of Γ on BC , B′ tends to B as C′ tends to C , so that in this case there is a C′ near enough to C for which the minimal width of AB′C′ is greater than that of ABC . But by the definition of ABC this is impossible, and so if ABC is scalene the only point of Γ on BC is C . But in this case we can replace BC by the line through C parallel to AB , to obtain a circumscribed triangle whose minimal width is not less than that of ABC , which enables us to include it in (i) above. Thus our original assumption is valid.

(i) Consider then the case in which ABC has two sides parallel. Then the minimal width of ABC is not more than the diameter of Γ , and this is less than $\frac{1}{2}\ell$ since Γ is not a segment. This case cannot therefore be extremal because we have already noted a case for which the minimal width of the triangle is $\frac{\ell}{\sqrt{3}}$.

(ii) So ABC is isosceles with, say, AB = AC $\geq$ BC . Having established this much about the circumscribing triangle, we can go on to specify Γ in greater detail. First, there is an interior point of each side of ABC which is a point of Γ . For if, for instance, C is the only point of Γ on BC , and N is a point of Γ on AB , then the length of the perimeter of Γ is at least twice the length of CN , which is greater than or equal to the minimal width of ABC . This again cannot be an extremal case as we require, because of the case already noted for which the minimal width of the triangle is $\frac{\ell}{\sqrt{3}}$, whereas the minimal width in the present case is less than $\frac{1}{2}\ell$. Thus we can ignore the cases in which Γ does not pass through an interior point of one (or more) of the sides of ABC .

So let L , M , N be points of Γ which are interior to BC , CA , AB respectively. Then Γ is in fact the triangle LMN , since otherwise ABC circumscribes a convex set whose perimeter length is strictly less than ℓ , and this is impossible by the extremal property of Γ with ABC . Also, for the same reason, it follows that L , M , N are those points of BC , CA , AB for which the lengths of the broken lines NLM , LMN , LNM respectively are least. This gives us some information about the geometry of Γ and ABC , because some elementary trignometrical and algebraic manipulation shows that



$\hat{ANM} = \hat{BNL}$, $\hat{AMN} = \hat{CML}$, $\hat{BLN} = \hat{CLM}$.

Since ABC is isosceles it follows that triangles BLN and CLM are equiangular, so that $\hat{BNL} = \hat{CML}$, and thus we have that $AM = AN$ because triangle AMN is isosceles. Hence $BN = MC$ so that BLN and CLM are in fact congruent, and $BL = LC$. Also we have $\hat{LMC} = \hat{AMN}$ and since triangles AMN and ABC are proved equiangular $\hat{AMN} = \hat{LCM}$, and so $LC = LM$. Thus $BL = LC = LM$, and this implies that there is a circle centred at L which passes through the points B, C, M, and it follows, since BC is then a diameter of this circle, that the angle $\hat{BMC}$ is a right angle. Thus BM is perpendicular to AC.

Now suppose that ABC is not an equilateral triangle, so that $AB = AC > BC$. Then we can change the shape of ABC slightly by reducing the width AL of it. (See page 83, diagram.) Take a point P near A on AL (so that P is between A and L) and construct the circumscribing triangle of Γ with P as vertex by producing PN and PM to points Q, R respectively of the line containing BC. Then PQR is also a circumscribing triangle of Γ. If AP is sufficiently small, the minimal width of PQR is equal to the length of the perpendicular QM', say, from Q to PR. In the figure, BM is perpendicular to AC and QM' is perpendicular to PR. Thus the angle between BM and QM' is equal to the angle between AC and PR, so that we have

$$\begin{aligned} QM' &= BM \cos \hat{PMA} + QB \cos \hat{M'QB} \\ &= BM \cos \hat{PMA} + QB \sin \hat{PRQ} \qquad \text{(from triangle } M'RQ\text{)} \\ &= (\alpha + \beta) + \gamma \quad \text{, say,} \end{aligned}$$

where $\alpha = BM$, $\beta = O(PA^2)$ and is non-positive, and $\gamma = O(QB)$ and is positive. Then since $PA = O(QB)$, if PA is sufficiently small, $(\beta + \gamma)$ is positive, and so the minimal width of PQR is greater than that of ABC . But this contradicts the definition of ABC .

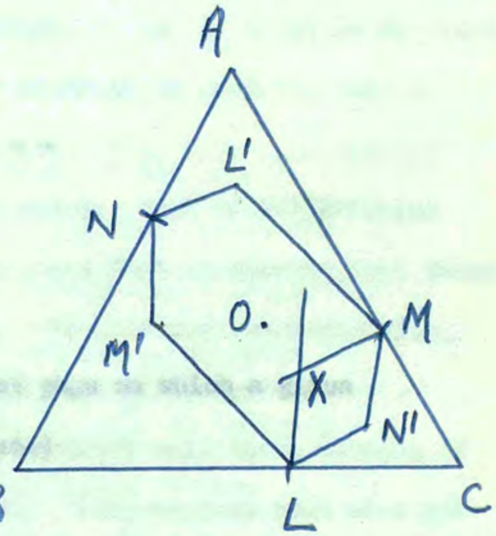
Hence the triangle ABC must be equilateral, with minimal width equal to AL , and $AL = LM\sqrt{3}$ which is the same as $\frac{(LM + MN + NL)}{\sqrt{3}}$, since LMN is equilateral. But $LM + MN + NL = \ell$, the frontier length of Γ , and so the theorem is complete.

To establish the corresponding result for the case in which Γ is a central set, we note first that if Γ is a regular hexagon (which is of course central) of frontier length ℓ , and ABC is an equilateral triangle such that a vertex of Γ is at the midpoint of each side of ABC , then the minimal width of ABC is $\frac{1}{3}\ell$.

As in the previous discussion we may suppose that there is a convex set Γ for which the minimal width of a circumscribing triangle ABC takes the largest possible value, and it may be proved just as above that this case occurs either when ABC has two parallel sides or when ABC is isosceles with the two equal sides at least as long as the third side. In the first case, since we know that Γ is again not a straight segment, the minimal width of ABC must always be strictly less than $\frac{1}{3}\ell$, so we need to consider only the second case.

In the second case, then, suppose again that Γ meets BC , CA , AB in interior points L , M , N respectively. Denoting by O the centre of Γ , let L' , M' , N' be the points of Γ which are the reflexions in O of L , M , N respectively. As Γ is not a straight segment, O must be an interior point of ABC .

We find now an expression for the length of the frontier of Γ , which in the extremal case is the polygon $LN'ML'NM'$. Suppose that X is the point of intersection of a line through M parallel to LN' and a line through L parallel to $N'M$. Then $XM = LN' = L'N$ and $XL = N'M = NM'$, and, since $XML'N$ and $XNM'L$ are therefore parallelograms, $XN = LM' = L'M$. So the length ℓ of the frontier of Γ is twice the sum of the lengths XL , XM , XN .



Consider then the expression $XL + XM + XN = \frac{1}{2}\ell$. When X is fixed, the sum is, since L , M , N lie on the sides of ABC , clearly least when XL , XM , XN are respectively perpendicular to BC , CA , AB . Fixing the perpendicularity and considering different positions of X we see that the sum is least when X is on BC . For, if the length of XL is denoted by h , and we put BC = a and $A\hat{B}C = A\hat{C}B = \alpha$, it follows by elementary trigonometry that

$$XL + XM + XN = h(1 - 2\cos\alpha) + a\sin\alpha \equiv f(h) \text{ , say.}$$

We know that $BC \leqslant AB$ so that $\cos\alpha \leqslant \frac{1}{2}$, and so $f(h)$ increases with h. It follows that the minimum value of $f(h)$ occurs when $h = 0$, which means that X lies on BC.

But if X is on BC, $XL + XM + XN = XM + XN$, which is equal to the length of the perpendicular from B to AC, say, and this, since ABC is isosceles, is the minimal width of ABC. Hence in this case the minimal width of ABC is equal to $\frac{1}{2}\ell$, and so we have proved for all cases that the minimal width of the circumscribing triangle of a central convex set is not more than half the boundary length of the set.

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Section 4 - Notation.

(The page number refers to the first page on which a given notation is used, and therefore explained.)

Γ	:	page 81
ℓ	:	page 81
ABC	:	page 81

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Section 5. To prove that there is a unique sphere of smallest radius r which contains a given bounded subset of E_n of diameter d, and

$$r \leqslant \left[\frac{n}{2(n+1)}\right]^{\frac{1}{2}} d . \quad [7]$$

- - - - - - - - - - - - [Notation : page 100.]

A sphere will be regarded in either of two ways, as having its position fixed, so that it is completely specified in any discussion, or as being free, so that the size of the sphere can be discussed without its position being fixed. In the latter case, such a sphere will be denoted by S_r, a sphere of radius r in E_n; if it is fixed, it will be denoted by S_r^*. Similar notation is used for the surfaces C_r and C_r^* of such spheres.

We have in essence two problems to solve: that of establishing the given inequality between r and d, and that of proving that there is a unique smallest sphere as defined. We consider the inequality first, and note that we can simplify this part of the problem immediately because it is sufficient to consider only those subsets M of E_n which consist of $(n+1)$ points. For, suppose that each set of $(n+1)$ points of a set M is containable by a sphere S_r of given radius r, and consider the family of spheres of radius r whose centres are at points of M. Then it follows immediately that every $(n+1)$ of these spheres have a point in common, and hence we know from

Helly's theorem (theorem 5 - page 11) that there is a point p, say, common to all the members of the family of spheres. As the centres of these spheres are at the points of M, we have in fact shown that the distance between p and any point of M is not more than r. Hence if each set of $(n+1)$ points of M is containable by S_r, then M is itself containable by S_r, and we need consider only sets M of $(n+1)$ points.

We note now that if $P = (p_1, p_2, \ldots, p_{n+1})$ is a set of $(n+1)$ points of E_n with diameter $d > 0$, there is a positive number r such that P is containable by S_r^* (whose position is now regarded as fixed), but is not containable by $S_{r'}$ for any $r' < r$. For since the class K_{R_0} of all the S_R^* with radius $R \leq R_0$ for some R_0, say, is a class of closed sets all of which contain P, then by the Blaschke selection theorem (theorem 7 - page 16) (since the S_R^* are certainly subsets of some sphere of radius $2R_0$) every infinite subclass of K_{R_0} contains a convergent sequence, and hence K_{R_0} is compact. Thus it follows that there is an S_r^* as defined.

Before we begin to establish the inequality between r and d, it is necessary to prove that this result for sets P of $(n+1)$ points is also, as in the first part of our discussion, true for general subsets M of E_n. We assert, then, that there is a smallest sphere S_r^* containing a bounded subset M of E_n. The assertion is clearly valid for $n = 1$, the required "sphere" being the shortest closed single segment (which certainly exists) containing M. Suppose that it is valid for every positive integer k less than n, and consider the case in n dimensions.

The set M may be a subset of E_n and not of any E_k with $k < n$, or it may be a subset of E_k, $1 \leq k < n$. In the latter case, there is by the induction hypothesis a smallest k-dimensional sphere containing M, and it follows at once that the n-dimensional sphere of the same radius, whose centre coincides with that of the k-dimensional sphere containing M, satisfies the conditions of the assertion.

In the former case, there is a non-empty class K_P consisting of all the sets P of $(n+1)$ points of M, and for any one such set P we have already seen above that there is a positive number $r(P)$, say, such that the (n-dimensional, fixed) sphere $S^{*}_{r(P)}$ contains P and no sphere of smaller radius than $r(P)$ contains P. Define r as the least upper bound of these numbers $r(P)$ over the whole class K_P. The number r is positive and finite since $0 < r(P) < d$ for all $P \in K_P$. Then we know that each set of $(n+1)$ points of M is containable by a sphere S_r (not regarded as fixed). If M is containable by a sphere $S_{r'}$, where $r' < r$, then by the definition of r there is a set P of $(n+1)$ points of M with $r \geq r(P) > r'$, so that P is not containable by $S_{r'}$, which leads at once to a contradiction. Hence S_r is the sphere of smallest radius containing M .

Having established, then, that the results which we need to be true for sets of $(n+1)$ points are true also for general subsets M of E_n , we can confine our attention to a set $P = (p_1, p_2, \ldots, p_n \quad)$ of $(n+1)$ points of E_n , and the smallest sphere S_r^* containing it. It is convenient here to note two properties of the centre c of this sphere, their proofs being given in lemmas 1 and 2 at the end of this section (pages 98, 99). Firstly, c is a point of the simplex whose vertices are the points of P . Secondly, if a point p_j of P is not on C_r^* (the surface of the sphere S_r^*), then c lies in the face of the simplex opposite p_j .

We assume inductively that $r \leqslant \left[\frac{k}{2(k+1)}\right]^{\frac{1}{2}} d$ for all positive integers $k < n$ (the inequality being obviously true for $n = 1$, when $r = \frac{1}{2}d$), and prove it true for n .

Relabelling the points of P if necessary, choose p_{n+1} so that the centre c of S_r^* does not lie in the face opposite p_{n+1} . Then by the second property of c that we noted, p_{n+1} is a point of C_r^* . Taking p_{n+1} as the origin, choose a coordinate system so that $\underline{p}_1$, $\underline{p}_2$, ..., $\underline{p}_n$, $\underline{0}$ are vectors corresponding to the points of P , and $\underline{c}$ is the vector corresponding to the centre c of S_r^* . Then the first property of c implies that there exist nonnegative constants k_1 , k_2 , ..., k_n , with $\sum_{k=1}^{n+1} k_i = 1$, such that

$$\underline{c} = \sum_{i=1}^{n+1} k_i \underline{p}_i \qquad \text{where} \qquad \underline{p}_{n+1} = \underline{0} .$$

We may assume that all the points of P lie on C_r^* ; for if, say, (n-k) such points do not, then there is a k-dimensional subset of P which is containable by a k-dimensional sphere of the same radius as S_r^* (but by no smaller sphere), in which case by the induction hypothesis we have that

$$r \leqslant \left[\frac{k}{2(k+1)}\right]^{\frac{1}{2}} d < \left[\frac{n}{2(n+1)}\right]^{\frac{1}{2}} d \qquad \text{for } k < n ,$$

so that the result is true for the case in which some of the points of P do not lie on C_r^* . Thus we may ignore the cases in which, by the second property of c , some of the k_i are zero.

Assume, then, that $k_i > 0$ for all i , so that all the points p_i lie on C_r^* . Now $|\underline{c}|$ is the radius of S_r^* and we can obtain an expression for $\underline{c}.\underline{c}$ by considering the scalar product $\underline{p}_i.\underline{c}$ for each i . We note that the point represented by the vector $2\underline{c}$ is diametrically opposite the origin on C_r^* , and hence it follows that the vector $2\underline{c} - \underline{p}_i$ is perpendicular to $\underline{p}_i$ for each i . Thus

$$k_i\underline{p}_i.(2\underline{c} - \underline{p}_i) = 0 \qquad \text{for each } i$$

and summing for i = 1 , 2, ..., n we have

$$\sum_{i=1}^{n} (k_i\underline{p}_i).\underline{c} = \tfrac{1}{2}\sum_{i=1}^{n} k_i\underline{p}_i.\underline{p}_i = \tfrac{1}{2}\sum_{i=1}^{n} k_i \left|\underline{p}_i\right|^2 .$$

Since $\underline{p}_{n+1} = \underline{0}$, $\sum_{i=1}^{n} k_i\underline{p}_i = \sum_{i=1}^{n+1} k_i\underline{p}_i = \underline{c}$, and since $|\underline{p}_i| \leqslant d$ for each i , this implies

$$|\underline{c}|^2 \leqslant \tfrac{1}{2}d^2 \sum_{i=1}^{n} k_i \quad .$$

But $\sum_{i=1}^{n} k_i = 1 - k_{n+1}$, and the inequality will therefore be best if we choose the labelling of the points p_i so that k_{n+1} takes the largest of the values of the k_i , that is, so that $k_{n+1} \geqslant k_i$ for $i = 1, \ldots, n$. Assuming this to have been done, then, we have that $k_{n+1} > \frac{1}{n}\sum_{i=1}^{n} k_i$, so that finally $\sum_{i=1}^{n} k_i \leqslant \frac{n}{n+1}$, and hence

$$r^2 = |\underline{c}|^2 \leqslant \frac{n}{2(n+1)} d^2 \quad \text{, that is} \quad r \leqslant \left[\frac{n}{2(n+1)}\right]^{\frac{1}{2}} d \quad .$$

Thus the required inequality is proved true for all n . To complete the proof we need finally to establish the uniqueness of the smallest sphere S_r^* that we have obtained for a given subset M of E_n . Suppose that it is not unique, so that there are two spheres $S_{r(p)}^*$ and $S_{r(q)}^*$ both of radius r but with centres at the points p , q respectively, and which both contain M . Then M is also contained in the common part of these two spheres and hence is contained in the smallest sphere containing their intersection. Unless p and q are coincident, this sphere has radius smaller than r , which contradicts the minimal property of r for M . Hence both the radius and the centre of the containing sphere of M are fixed, and uniqueness is established. This completes the proof of the main result.

We may note that the inequality between r and d is the best possible, since equality occurs for the n-dimensional sphere of smallest radius containing an equilateral simplex of edge d . This is easy to prove by an inductive process. For $n = 1$, it is obvious that $r = \frac{1}{2}d = \left[\frac{n}{2(n+1)}\right]^{\frac{1}{2}} d$. Assume then that for a simplex of $(n-1)$ vertices the radius of the circumscribing sphere satisfies the equality

$$r_{n-1} = \left(\frac{n-1}{2n}\right)^{\frac{1}{2}} d .$$

Now consider an n-dimensional equilateral simplex of edge d , Its faces are equivalent to the simplex of $(n-1)$ vertices, and the radius of its circumscribing sphere is $\frac{n}{n+1}$ of the distance between any such face and the opposite vertex. But this distance is given by

$$\left[x_{n-1}^2 - \frac{x_{n-1}^2}{n^2}\right]^{\frac{1}{2}}$$

where x_{n-1} satisfies the relation $r_{n-1} = \frac{n-1}{n} x_{n-1}$ (that is, x_{n-1} is the distance between a face of the simplex of $(n-1)$ vertices and the opposite vertex). Thus we have

$$r_n = \frac{n}{n+1} \sqrt{\frac{n^2-1}{n^2}} \cdot \frac{n}{n-1} \left(\frac{n-1}{2n}\right)^{\frac{1}{2}} d = \left[\frac{n}{2(n+1)}\right]^{\frac{1}{2}} d$$

and equality is proved for all n when M is an equilateral simplex of edge d .

Throughout this proof it will have been noted that, although we have been discussing the problem for a containing sphere only, we have used results which are true for more general convex sets also. In fact it may be seen that, as far as the point where we begin to establish the inequality between r and d, it is possible to obtain the same properties for closed central convex sets $K(r)$ of "radius" r, which are homothetic translates of one another, as we have obtained for spheres S_r. (See chapter III beginning of section 4 (page 155) for a possible definition of the number r. We could equally well use the diameter of the set K, as we need it only to distinguish sets of different sizes.) Thus we may deduce immediately from the previous discussion that there is a smallest set $K(r)$, for some "radius" r, the homothetic translate of a given closed central convex set K, which contains any given bounded subset M of E_n.

If the set K is closed, central and strictly convex, then we may show further that there is a <u>unique</u> smallest homothetic translate $K(r)$ of K containing M. For, suppose that K_1, K_2 are two such sets which contain M, K_2 being a translation of K_1 through a distance a, say. Then M is contained also in the intersection $K_1 \cap K_2$. For convenience, assume that a vector system has been defined in E_n, write $K_2 = K_1 + \underline{a}$, and consider any point $\underline{r}$ of $K_1 \cap K_2$. Then we have $\underline{r} \in K_1$ and $\underline{r} \in K_2$, which is equivalent to saying that

$$\underline{r} \in K_1 \quad \text{and} \quad \underline{r} - \underline{a} \in K_1 .$$

Since K_1 is strictly convex, it follows that $\underline{r} - \frac{1}{2}\underline{a}$ belongs to K_1^o, the interior of K_1. Thus

$$\underline{r} \in K_1^o + \tfrac{1}{2}\underline{a}$$

and so, since $\underline{r}$ is any point of $K_1 \cap K_2$,

$$K_1 \cap K_2 \subset K_1^o + \tfrac{1}{2}\underline{a} .$$

The set $K_1 \cap K_2$ is closed, and so the distance between the frontiers of $K_1 \cap K_2$ and $K_1^o + \frac{1}{2}\underline{a}$ is strictly positive. Thus it follows that there is a set concentric with $K_1^o + \frac{1}{2}\underline{a}$ which is homothetic to and smaller than K_1 ($K_1^o + \frac{1}{2}\underline{a}$ and K_1 being of equal size) and which contains $K_1 \cap K_2$ (and hence M). But this contradicts the definition of K_1 as a smallest set containing M. Thus K_1 must be unique.

The question of finding a relationship between the radius of K and the diameter of M becomes inappropriate when we consider a more general set than a sphere.

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Lemma 1. The centre c of the sphere S_r^* of smallest radius r containing P is a point of the simplex whose vertices are the points of P.

Proof. If not, there is a hyperplane which separates c from the simplex (e.g. one of the support hyperplanes of the simplex), and it follows immediately that the sphere $S_{r'}^*$, passing through the intersection of S_r^* with the hyperplane and having its centre in the hyperplane, contains P and has radius $r' < r$. This contradicts the definition of S_r^*, and the lemma follows.

Lemma 2. If a point p_j of P is not on C_r^*, the surface of the sphere S_r^*, then c lies in the face of the simplex opposite p_j.

Proof. Assume that c does not lie in the face of the simplex opposite p_j. Choose a rectangular coordinate system so that the (n-1)-dimensional hyperplane containing all the points of P except p_j has equation $x_n = 0$ and so that $x_n^{(j)}$, the n-th coordinate of P_j, is strictly positive. Then by lemma 1, c_n, the n-th coordinate of c, which by our assumption is not zero, is strictly positive. Let the equation of C_r^* be $f_r(x) = 0$. Consider the spherical surface C_s^* which contains the intersection of C_r^* with $x_n = 0$ and has equation

$$f_r(x) + 2tx_n = 0 \qquad (1)$$

say, for a positive value of t which will be chosen later. Now for each of the points p_i, $i \neq j$, $f_r(p_i) = 0$ since S_r^* is the smallest sphere which contains all the points p_i, and $x_n^{(i)} = 0$, so that each of these points is contained in C_s^*. For p_j, choose

t so that

$$f_r(p_j) + 2tx_n^{(j)} < f_r(p_j) + |f_r(p_j)|$$
$$= 0$$

since by hypothesis p_j is not on C_r^*, implying that $f_r(p_j) < 0$ (since p_j is certainly in S_r^*). Hence, for

$$t < \frac{|f_r(p_j)|}{2x_n^{(j)}} = t_0 \quad \text{, say,}$$

all the points of P belong to S_s^*. But if we choose $t < \min(t_0, c_n)$, then the centre of S_s^*, which is given by $(c_1, c_2, \ldots, c_{n-1}, c_n - t)$, (from equation (1)), is nearer the plane $x_n = 0$ than is c. Hence $s < r$ which contradicts the minimal property of r. It follows that our original assumption was false, and the lemma is proved.

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Section 5 - Notation.

(The page number refers to the first page on which a given notation is used, and therefore explained.)

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Section 6. To prove that any set of diameter d may be contained in a set of constant width d. [8]

------------- [Notation and Definition : page 109.]

A set of constant width is a bounded closed convex set for which every two parallel support hyperplanes are the same distance apart. Adopting again a method of proof used earlier in this chapter, we shall define a new class of sets which will be shown to be the same as the class of sets of constant width, and shall prove the main result using the new definition.

So we make first the following definition: a set X is complete if and only if, for any set Y containing X, either Y is equal to X or the diameter of Y is strictly greater than the diameter of X. In other words, the addition of any point to a complete set increases the diameter of the set.

With the given definitions of complete sets and sets of constant width, we may now show that the statements

(i) X is a set of constant width d,

(ii) X is a complete set of diameter d

are equivalent.

To prove that (i) implies (ii), note first that, since X is of constant width d, X has diameter d. For, there are two points x, $y \in X$ for which $|x - y| = D(X)$, where $D(X)$ denotes the diameter of X, and by the definition of the diameter of a set the hyperplanes through x, y perpendicular to the line xy are support hyperplanes of X. But the distance apart of any two parallel support hyperplanes of X is d, and so $D(X) = d$. Now let z be a point exterior to X and let x_1 be the point of X which is nearest to z. Then the hyperplane through x_1 perpendicular to x_1z is a support hyperplane of X, and there is a parallel support hyperplane of X which meets X in at least one point x_2, say. We may assume that x_1, x_2 and z lie in a straight line because if x_2 is not on zx_1 produced we have $D(X) \geq |x_1 - x_2| > d$, since d is the perpendicular distance between the hyperplanes. So since x_1, x_2 and z are collinear we have immediately

$$|z - x_2| > |x_1 - x_2| = d$$

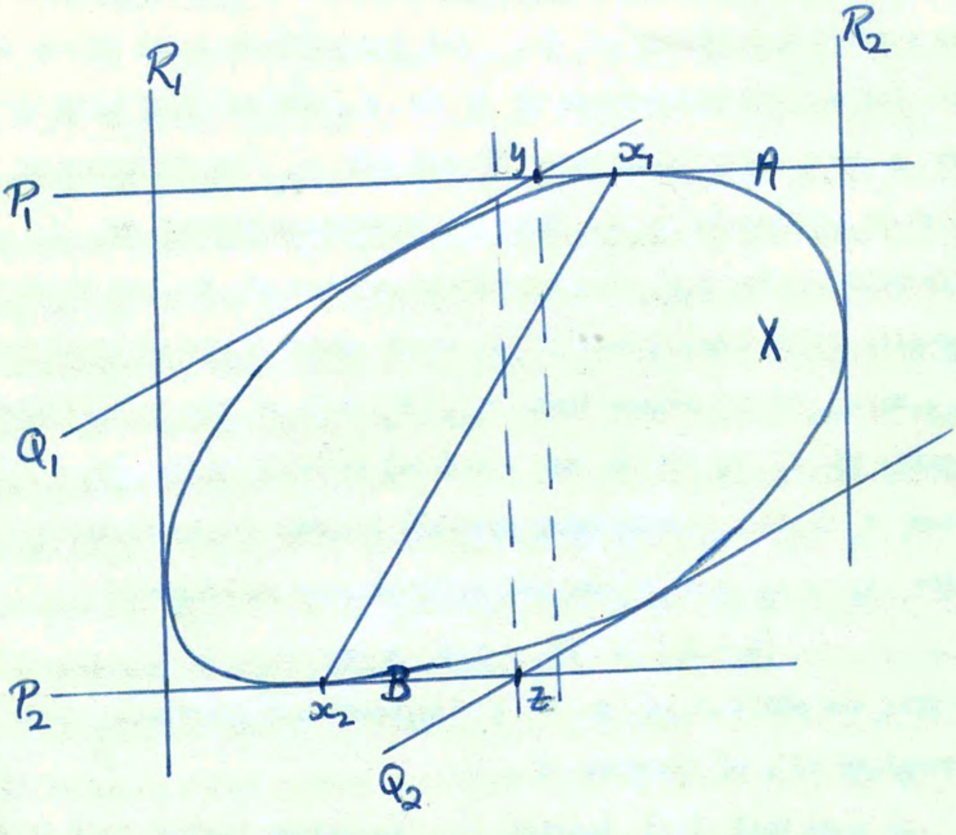
so that the addition of z to X increases its diameter. Thus X is a complete set, of diameter d.

To show that (ii) implies (i) we assume instead that it does not, so that there is a set X of diameter d which is complete but which has minimal width $w < d$. There are two parallel support hyperplanes of X, say P_1 and P_2, distance w apart. We assert that there are points x_1 and x_2 of X such that x_1 lies on P_1, x_2 lies on P_2, and x_1x_2 is perpendicular to P_1 and P_2. For if this is not so, suppose that x_1 and x_2 are two points of X such

(Diagram)

104

(Diagram)



(Opposite page.)

that x_1 lies on P_1, x_2 lies on P_2, and the distance between x_1 and x_2 is smaller than the distance between any other such pair.

Construct an infinite hexagonal prism as follows, such that it contains the set X, and has two of its faces, A, B say, lying in the hyperplanes P_1, P_2. (See diagram opposite.) Let the other four faces of the prism lie in support hyperplanes of X, say Q_1, Q_2, R_1, R_2, where the Q_i are parallel to each other and the R_i are parallel to each other. The points x_i lie each in one of the faces A and B of the prism, and by our definition of these points it is possible to construct the prism in such a way that, in the plane containing the points x_i and perpendicular to the hyperplanes P_i, there is no line perpendicular to the P_i which contains points of both A and B. (For if every such prism yielded such a line, with points y_1 in A and y_2 in B, say, it would follow that y_1 and y_2 must belong to the set X, which is a contradiction with the definition of x_1 and x_2.)

Since the prism is constructed from support hyperplanes of X, its minimal width is the perpendicular distance between the faces A and B (i.e. between the hyperplanes P_1 and P_2). Consider any perpendicular cross-section of the prism, and denote by y the point of A nearest to B and by z the point of B nearest to A. Then we may alter the prism slightly by rotating P_1 and P_2 through the same (directed) angle about y, z respectively, to obtain a new prism lying in the hyperplanes Q_i, R_i, P_i' say, which still contains the set X, the rotation being such that the acute angle between yz and the rotating plane is decreasing. But the minimal width of this new prism, provided that the rotation of the P_i is sufficiently small, is the perpendicular distance between the P_i', which may be verified to be smaller than the minimal width of the original prism. This is a contradiction since it implies that the minimal width of a set containing X is smaller than the minimal width of X itself. It follows that we cannot define two points x_1, x_2 as above unless the line joining them is perpendicular to P_1 and P_2, and this is what we wanted to establish.

Thus our assertion is true that there are two points x_1, x_2 of X such that x_1 is on P_1, x_2 is on P_2, and $|x_1 - x_2| = w$, the perpendicular distance between P_1 and P_2. Also there is a point y X such that $|y - x_1| = d$, because if $|y - x_1| \leq d_1$, say, for all $y \in X$, $d_1 < d$, then the set Y, which is the union of X and the points x for which $|x - x_1| \leq |d - d_1|$, properly contains X but is not of larger diameter than X, which is a contradiction since X is complete.

Now either of the minor circular arcs which are of radius d and have endpoints x_2 and y must be contained entirely in X. In fact it may be seen that, if $S(X)$ denotes the intersection of all spheres of radius d which contain X, then $S(X) = X$, and from this it follows easily that a minor circular arc of radius d which joins two points of X must itself lie in X . To show that $S(X) = X$, note first $S(X) \supset X$, trivially. If we do not have $X \supset S(X)$, then there is a point s of $S(X)$ which is not in X . But by the definition of $S(X)$ we have that for each $x \in X$, $|s - x| \leq d$, so the set formed by the union of s with X is still of diameter d . So X is not complete, which is a contradiction, and so we must have $X \supset S(X)$. Thus $S(X) = X$.

Now there is a minor circular arc of radius d joining x_2 and y which crosses the hyperplane P_2 , for if it only touches P_2 the centre of the arc must lie along x_1x_2 (since this is perpendicular to P_2), and since we know that $|y - x_1| = d$, the centre must be at x_1 . But this is impossible since $|x_1 - x_2| \neq d$, so the arc, which is contained in X , must cross P_2 . But P_2 is a support hyperplane of X , and so we have a contradiction. Hence our original assumption, that there is a complete set X of minimal width $w < d$, is false, and (ii) implies (i) .

This proves that the class of complete sets of diameter d and the class of sets of constant width d are identical. We are now able to prove our required result, that any set of diameter d may be contained in a set of constant width d .

Suppose that X is a set of diameter d . We may assume without loss of generality that X is closed and convex, since we know (see theorem 4 - page 10) that the convex cover of the closure of a set has the same diameter as the set itself. If X is of constant width, there is nothing to prove since it is containable by itself. If X is not of constant width, then it is not complete, and we shall prove our result by constructing a complete set which contains X, as follows:

Consider the set $Q(X)$ of all those points x for which the diameter of the union of x with X, $D(x \cup X)$, is equal to $D(X)$. Clearly there is at least one subset of $Q(X)$ which contains X and is complete. Select from $Q(X)$ one point x_1 whose distance from the set X is a maximum in $Q(X)$ ($Q(X)$ is closed and bounded, so x_1 certainly exists), and denote by Y_1 the set which is the convex cover of x_1 and X. Define a sequence of sets Y_i, $i = 1, 2, \ldots,$ inductively as follows: when Y_i has been defined, consider the set $Q(Y_i)$ as above, select x_{i+1} from it, and define Y_{i+1} as the convex cover of x_{i+1} and Y_i.

Now we have defined an infinite sequence of sets Y_i for which $Y_i \subset Y_{i+1}$ and $Y_i \subset Q(X)$ for all i. Thus the sequence $\{Y_i\}$ converges to a set Y, say. The closure of Y, $\overline{Y}$, is convex, by its construction, and $D(\overline{Y}) = D(X)$, since $\overline{Y} \subset Q(X)$ and $\overline{Y} \subset X$. We want to show that $\overline{Y}$ is complete. Suppose that it is not complete, and let y be a point not in $\overline{Y}$ such that $D(y \cup \overline{Y}) = D(\overline{Y})$. In other words, with the notation used above, $y \in Q(\overline{Y})$ and, since $Y_i \subset \overline{Y}$ for any i, $D(y \cup Y_i) \leqslant D(y \cup \overline{Y}) = D(\overline{Y}) = D(X)$.

But $D(y \cup Y_i) \geq D(Y_i) = D(X)$, so that we have $D(y \cup Y_i) = D(X)$ ~~d(X)~~ and hence $y \in Q(Y_i)$, that is (see page 7),

$$\rho(y, Y_i) \leq \rho(x_{i+1}, Y_i) \quad \text{by the definition of } x_{i+1} ,$$

for any i .

Since y is not in $\overline{Y}$, denote by δ the distance between the point and the set, where δ is strictly positive.

Now take two points x_i , x_j , $j > i$, using the same notation as above. $x_i \in Y_i$ and $Y_i \subset Y_{j-1}$, so that we have

$$| x_j - x_i | \geq \rho(x_j, Y_i) \geq \rho(x_j, Y_{j-1}) .$$

But we have noted that, in particular, $\rho(x_j, Y_{j-1}) \geq \rho(y, Y_{j-1})$, and $\rho(y, Y_{j-1}) \geq \rho(y, \overline{Y})$ since $Y_{j-1} \subset \overline{Y}$. So we have

$$|x_j - x_i| \geq \rho(y, \overline{Y}) = \delta > 0.$$

But $\{x_i\}$ is an infinite sequence of points and all the x_i lie in $Q(X)$ which is bounded, and thus it is impossible for every two of them to be at a distance at least δ apart.

Hence $\overline{Y}$ is complete, and this establishes the required result.

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Section 6 - Notation and Definition.

(The page number refers either to the first page on which the notation is used, or to the page on which the given term is defined in full.)

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CHAPTER III

A COVERING OF A SINGLE SET BY MANY SETS OR OF MANY SETS BY A SINGLE SET

In the previous chapter we considered some particular problems concerning the covering of a single set by a single set. In this second selection of papers we are to give consideration to finite or infinite classes of sets, and shall discuss four particular problems. The first one is that of finding a single set to cover an infinite class of sets, and the second and third involve covering one set by a finite class of sets. The fourth paper to be discussed relates one set (the plane) and a given infinite class of sets: it is included for the sake of completeness and because of the methods of solution involved, but in fact can be considered as a problem in covering only if the latter term is not restricted to mean complete covering by a class of sets. The statements of the problems are as follows.

(a) To prove that the diameter of any minimal universal cover of plane sets of unit constant width is less than 3, and to find such a minimal cover. To show that in three or more dimensions there is a compact minimal universal cover, of sets of unit constant width, whose diameter exceeds any specified length.

(b) To prove that, if K is a convex body of minimal width ℓ in E_n, and K is contained in the union of r parallel strips of widths

$h_1, h_2, \ldots, h_r$, then $h_1 + h_2 + \ldots + h_r \geqslant \ell$. (Tarski's plank problem.)

(c) To show that a small-circle of a unit sphere, of angular radius ρ, can be covered by a finite number of lunes only if the sum of their width is at least 2ρ, and, if the sum is exactly 2ρ, only when they all share the same vertices.

(d) To show that, for all maximal K-packings of positive minimal radius, the greatest lower bound of their lower densities is $\frac{\pi}{6\sqrt{3}}$, and is strictly less than $\frac{\pi}{6\sqrt{3}}$ if K-packings of ellipses are excluded.

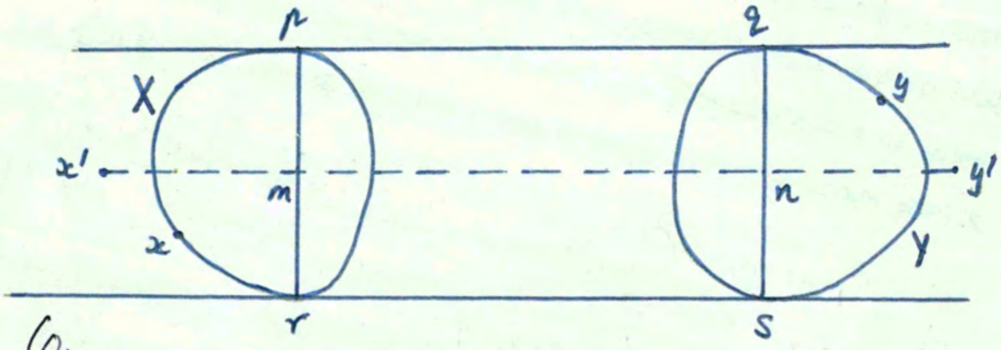
Section 1. To prove that the diameter of any minimal universal cover of plane sets of unit constant width is less than 3, and to find such a minimal cover. To show that in three or more dimensions there is a compact minimal universal cover, of sets of unit constant width, whose diameter exceeds any specified length. [10]

- - - - - - - - - - - - [Notation and Definitions : page 132.]

We denote by K_n the class of subsets of E_n which have unit constant width. A set $C \subset E_n$ is called a universal cover if it is closed, convex and such that for every set $A \subset E_n$ with diameter $d(A) \leqslant 1$ we can find a set $B \subset C$ such that B and A are congruent. We know from a result proved in the previous chapter (see page 102) that every such set A is contained in a member of K_n, and so it is sufficient in showing that a set C is a universal cover to show that for each $A \in K_n$ there is a congruent set $B \subset C$. A minimal universal cover is a universal cover of which no proper subset is a universal cover.

With these definitions we can ask the following question (see [11]): is there a finite upper bound, depending on n only, of the diameters of compact minimal universal covers in E_n? In the following discussion of this question we show that in the plane the diameter of any minimal cover is less than 3 and we show also that a

(Diagram)



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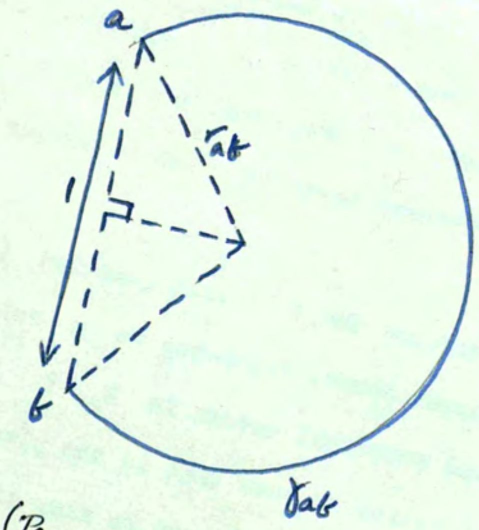

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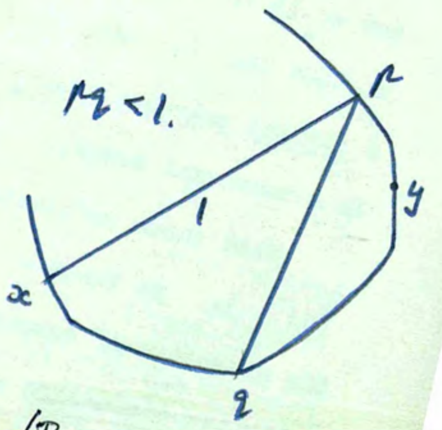


(Page 118, bottom.)

certain plane set, which will be defined, is a universal cover; however, in three or more dimensions we show that there is a compact minimal universal cover in E_n whose diameter exceeds any specified length.

The diameter of any minimal cover in the plane is less than 3.

Suppose that we are given a plane universal cover C of sets of unit constant width whose diameter is at least equal to 3. We shall show that such a cover is not minimal.

C contains two points x, y say for which $xy = 3$ (where xy denotes the length of xy), and there are subsets X, Y of C of unit constant width such that $x \in X$, $y \in Y$. (See diagram opposite.) Let pq, rs be the exterior common support lines of X and Y, meeting the frontiers of the sets in points p, $r \in X$ and q, $s \in Y$. The subset pqsr of C is a rectangle since X, Y are of constant width, $pr = qs = 1$, and both pr and qs are perpendicular to pq (and rs). Now construct points x', y' such that x' is distant 1 from p and r and not interior to C, and y' is distant 1 from q and s and again not interior to C. Then $x'y'$ is parallel to the common support lines of X, Y, meeting pr and qs at m, n, say.

Now since x', y' are not interior to C, we have $x'y' \geq 3$. For, the set X is contained in the intersection of the discs centred at the points p, r of X and having radius 1, and so the point

x (of X), and similarly the point y, lie within a figure which is bounded by the segments pq, rs, and arcs (of radius 1) px', $x'r$, qy', $y's$. It is clear that the diameter of this figure is the length of $x'y'$, and hence, since x and y are both contained by it, the length of $x'y'$ must be at least equal to the length of xy, which by hypothesis is equal to 3.

Also $x'm < 1$ and $ny' < 1$, so that $mn > 1$. Thus $pq > 1$, and hence C contains as a proper subset a square of side 1. But a square of side 1 is a universal cover for plane sets of constant width 1. Thus C is not a minimal universal cover.

Hence any minimal cover in the plane must have diameter less than 3.

A minimal universal cover in the plane.

We define Y to be the set which is the union of a circular disc and a Reuleaux triangle, both of constant width 1, the triangle being placed so that two of its vertices are at opposite ends of a diameter of the disc. (A disc is the union of a circle with its interior; a Reuleaux polygon of width 1 is the set bounded by a number of circular arcs, each of radius 1, the centre of each arc being the vertex of the polygon opposite that arc.)

If Y is a universal cover, it follows at once that it must be minimal since no proper subset of Y can contain both a disc and a Reuleaux triangle each of unit width. To prove that Y is a universal cover we note the following, which allows the problem to be discussed from an alternative standpoint.

In a loose sense it may be said that the "difficult" part of the set Y is the semi-circular part of its frontier, since this is of radius $\frac{1}{2}$ whereas a set of constant width 1 has its frontier consisting of circular arcs of radius 1. To take this into consideration, we assume (and shall then prove) the following property (P) of plane sets of unit constant width: if X is such a set, then on the frontier of X there are two points s, t say, at unit distance apart, such that of the two semi-circular arcs of radius $\frac{1}{2}$ which pass through both s and t, at least one does not meet the interior of X. Now with this property assumed, it follows that, since each point of X is at a distance at most 1 from s and t, X must lie entirely within a figure bounded by the semi-circular arc just described together with two arcs each of radius 1 centred at s and t respectively. But this figure is congruent to Y, and hence X is congruent to a subset of Y. But X is any set of unit constant width, and so it follows from our earlier remark that Y is a universal cover.

We now prove the property (P) which we assumed above, noting at the outset that it is sufficient to prove it when X is a Reuleaux polygon Z, say, because any other set X can be approximated to by means of circular arcs (which form the arcs of the frontiers of

Reuleaux polygons). We make the following definitions: (i) two vertices a , b of Z are opposite if and only if their distance apart is 1 , (ii) the line ab is defined to have a positive and a negative side, where the positive side is that on which the two circular arcs lie whose centres are a , b respectively. (These arcs cannot lie on opposite sides of ab , because then the distance between an interior point of one of the arcs and an interior point of the other would be greater than ab , which is equal to 1 , thus contradicting the definition of Z .)

Let γ_{ab} denote the circumcircle (considered as a disc) of the part of Z on the negative side of ab , and r_{ab} its radius. (See page 114, diagram.) $r_{ab} \geqslant \frac{1}{2}$ since any circumcircle γ_{ab} contains a chord (ab) of unit length. We assume, then, that there are no opposite vertices a , b for which $r_{ab} = \frac{1}{2}$, and show that this leads to a contradiction.

Let p , q be any two vertices on the frontier of Z . They divide the frontier into two parts, and if p , q are not opposite points one of the two parts contains no points opposite to p or q . (Otherwise suppose that x is a point on one of the parts of the frontier, opposite to p , say. (See page 114, diagram.) Then if y is a point on the other part, y cannot be opposite to p since there is only one circular arc corresponding to each vertex of the polygon. But if y is opposite to q , the distance between x and y is greater than the distance between q and y , which again contradicts the definition of Z .) The vertices of Z which lie on such an arc, containing no points opposite to p or q , are defined to lie between p and q . If p and q are opposite, the vertices defined to lie between p and q are those which lie on the negative side of pq .

We are assuming that, for any pair of opposite vertices p , q , $r_{pq} > \frac{1}{2}$. Then there must be at least one vertex between p and q which lies on γ_{pq} (because otherwise γ_{pq} would just be the semi-circle on pq , so that $r_{pq} \nleq \frac{1}{2}$). Suppose then that v , w are vertices lying on γ_{pq} such that there are no other vertices lying on γ_{pq} between p and v or between q and w . (v , w need not of course be distinct vertices.) Let V(p) , V(q) be the numbers of vertices lying between p , v and q , w respectively, and let

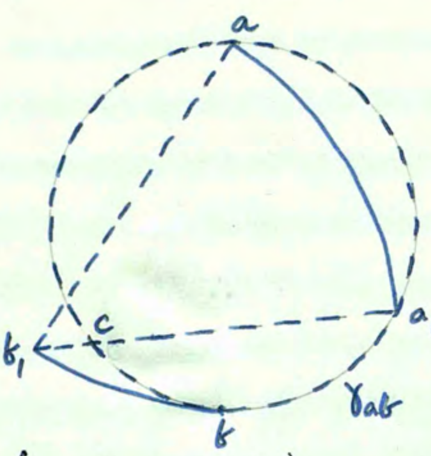
$$V(p,q) = \min(V(p),V(q)) \quad \text{and} \quad \tau = \min_{p,q} V(p,q) \quad .$$

Now choose opposite vertices a , b such that $\tau = V(a,b)$ and suppose for definiteness that $V(a,b) = V(a)$. Let b_1 be the vertex opposite a adjacent to b and a_1 be the vertex opposite b_1 adjacent to a . (Note that a_1 and b both lie to the same side of ab_1 since we have already seen that the circular arcs centres a and b_1 must lie on the same side of ab_1 .)

Consider first the case in which $\tau = 0$, so that a_1 lies on (the frontier of) γ_{ab} . b_1 lies outside γ_{ab} since $b_1 \neq b$

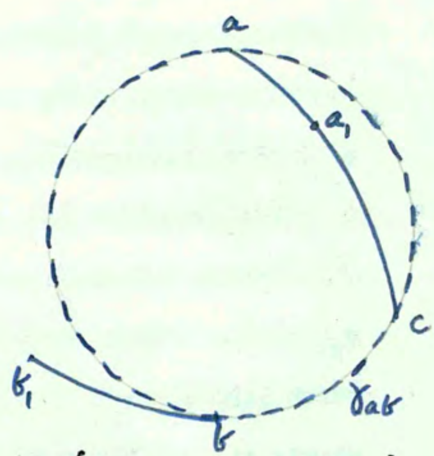
(Diagram)

$\tau = 0.$



(Opposite page, (i).)

$\tau > 0.$



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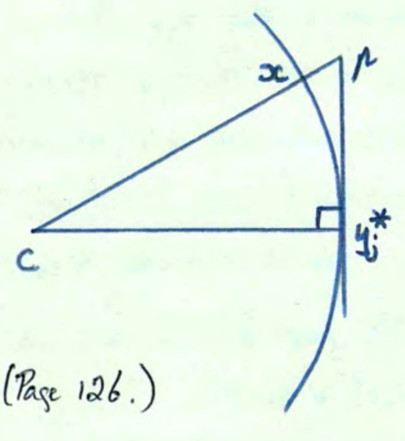

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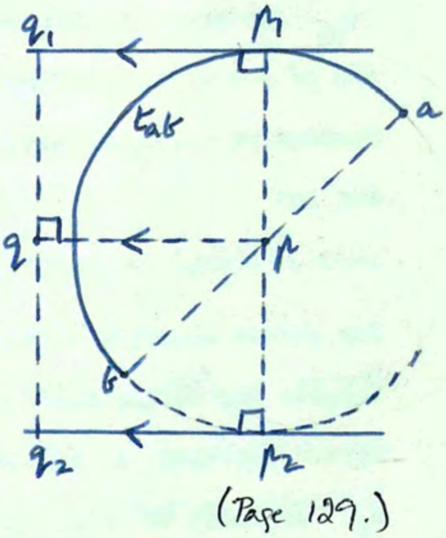

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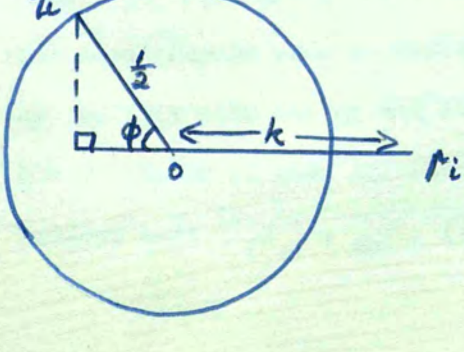

(Page 131.)

(so that ab_1 is distinct from but the same length as the diameter ab of γ_{ab}), and hence γ_{ab} cuts the segment a_1b_1 in two points, a_1 and c (say). (See diagram (i) opposite.) Since a and a_1 lie on γ_{ab} the line joining b_1 to the centre of γ_{ab} bisects internally the angle ab_1a_1 and so the centre of γ_{ab} lies on the positive side of a_1b_1 (being the side on which the arc aa_1 lies). Thus except for a_1 the part of γ_{ab} on the negative side of a_1b_1 lies interior to $\gamma_{a_1b_1}$, which implies that no vertex of Z on the negative side of a_1b_1 lies on $\gamma_{a_1b_1}$. But this would mean that $r_{a_1b_1} = \frac{1}{2}$, which by our assumption is impossible.

So we are left with the alternative that $\tau > 0$. In this case let c be the vertex of Z between a and b on γ_{ab} which is nearest to a. (See diagram (ii) opposite.) Because $\tau > 0$, b must be an interior point of $\gamma_{a_1b_1}$ (since b, b_1 are adjacent); and because of the extremal property of a, b and c with respect to τ, c and all the vertices of Z (if any) between a_1 and c must be interior points of $\gamma_{a_1b_1}$. Since the frontier of $\gamma_{a_1b_1}$ contains a_1 and a_1 is interior to γ_{ab}, and since b, c are on the frontier of γ_{ab} and are interior to $\gamma_{a_1b_1}$, it follows (since two distinct circles intersect in at most two points) that the part of γ_{ab} between b and c must lie interior to $\gamma_{a_1b_1}$. Thus all the vertices of Z between b and c are interior to $\gamma_{a_1b_1}$. But then all the vertices of Z between b_1 and a_1 are interior to $\gamma_{a_1 b_1}$, which is impossible by our assumption.

Hence our assumption is wrong, and there do exist two opposite vertices a, b for which $r_{ab} = \frac{1}{2}$. This completes the proof that Y is a universal cover.

It may be conjectured here, although no method of proof is suggested, that the cover Y which has been described is in fact the minimal universal cover of largest diameter in the plane.

A minimal universal cover of large diameter in E_n, $n \geqslant 3$.

We shall show that such a cover exists by constructing one of diameter at least equal to some given number D, however large. We begin with an obvious universal cover P, the semi-infinite prism $|x_i| \leqslant \frac{1}{2}$, $i = 1, 2, \ldots, n-1$, $x_n \geqslant 0$; and our minimal cover will be a subset of P.

Now let X_0 denote the perpendicular projection of any set X in E_n onto $x_n = 0$. If $Y \in K_n$ and $Y \subset P$, we have $Y_0 \in K_{n-1}$ and $Y_0 \subset P_0$, where P_0 is the $(n-1)$-dimensional cube whose faces lie in the intersections of the hyperplanes $x_i = \pm \frac{1}{2}$, $i = 1, 2, \ldots, n-1$, with $x_n = 0$. Denote by $F(P_0)$ the face of this cube which lies in $x_1 = \frac{1}{2}$. Y_0 must meet $F(P_0)$ in just one point since P_0 is a unit cube and Y_0 has unit constant width. Denote this point by $f(Y_0)$. Now consider the class $\mathcal{Y}$ of sets congruent to Y which

lie in P and are obtainable from Y by a combination of any rotation and/or reflexion together with a translation perpendicular to the x_n axis (so that the new set is contained in P again). Let $F(\mathcal{Y})$ or F be the set of points $f(Y)$ for all $Y \in \mathcal{Y}$, and let $F' \subset F$ be the set of points most distant from the centre $(\frac{1}{2}, 0, \ldots, 0)$ of $F(P_0)$ (the cube is symmetrical about $x_i = 0$ for each i, so all the coordinates except x_1 of the centre of $F(P_0)$ go to zero). If $(\frac{1}{2}, x_2, x_3, \ldots, x_{n-1}, 0) \in F$, then because of the reflexion and rotation referred to above, F contains all the 2^{n-2} points $(\frac{1}{2}, \pm x_2, \pm x_3, \ldots, \pm x_{n-1}, 0)$ and clearly they are all at the same distance from $(\frac{1}{2}, 0, \ldots, 0)$. It follows that there is a point of F', say $(\frac{1}{2}, x_2', x_3', \ldots, x_{n-1}', 0)$, for which $x_i' \geqslant 0$ XXX0 for all i. Choose one such point and let $Y' \in \mathcal{Y}$ be the set (or one of the sets) for which $f(Y') = (\frac{1}{2}, x_2', x_3', \ldots, x_{n-1}', 0)$. If $f(Y') = (\frac{1}{2}, 0, \ldots, 0)$, then every set belonging to $\mathcal{Y}$ has $f(Y) = (\frac{1}{2}, 0, \ldots, 0)$ (since $f(Y')$ is most distant from $(\frac{1}{2}, 0, \ldots, 0)$). XXXXX0XXXX Now the construction of the class $\mathcal{Y}$ ensures that the original set Y can be moved so that any given frontier point of Y_0 is a point of $F(P_0)$, so it is equivalent to say that if XXXX
f(Y) =
$(\frac{1}{2}, 0, \ldots, 0)$, then every face (because of the rotation property) of any $(n-1)$-dimensional unit cube containing Y_0 meets the frontier of Y_0 only at the centre of the face. Alternatively, if we take any two-dimensional section of Y_0 which is of unit constant

width, then every square circumscribing it has its sides bisected by points of Y_0. Then it may be proved (lemma 1 (page 128)) that every such section of Y_0 is a disc. Thus Y_0 is an $(n-1)$-dimensional ball (solid closed sphere), and hence so is the projection of Y in any direction (by the construction of $\mathcal{Y}$). It follows, then, that if $f(Y') = (\frac{1}{2}, 0, \ldots, 0)$, Y is a ball.

It may be shown (corollary to lemma 2 (page 132)) that there is at least one member of K_n such that the circumradius of its projection in any direction is strictly greater than $\frac{1}{2}$. Let Z denote that member for which the greatest lower bound of such circumradii has the largest possible value, and let this value be R. Now choose a large positive number M which satisfies the inequality $(2R - 1)M > D$. The reason for our choice will become clear at the end of the construction. Let P_M be the intersection with P of the cone defined by

$$x_1^2 + x_2^2 + \ldots + x_{n-1}^2 \leqslant \frac{(x_n + M)^2}{(2M+1)^2 - 1}, \qquad (1)$$

where this is chosen so that P_M just contains the n-dimensional unit ball B whose centre is at $(0, \ldots, 0, \frac{1}{2})$. (It may easily be verified that $B \subset P_M$ and that the frontiers touch at the points with $x_n = \dfrac{M}{(2M+1)}$.) Also the cone intersects $x_n = k$ in the ($(n-1)$-dimensional) ball centre $(0, \ldots, 0, k)$ and radius

$$r_k = \frac{(k + M)}{\sqrt{((2M+1)^2 - 1)}}.$$ Since r_k is large for large

values of k it follows that P_M is a universal cover (as the construction of P_M involves removing only a finite volume from the prism P).

Now for each set $Y \in K_n$, $Y \subset P$, take the corresponding set Y' as defined above, and translate Y' until it lies inside P_M but as near to $x_n = 0$ as possible. Let this new set be denoted by Y^* . Let Q be the union of all such sets Y^* , and let S be the convex cover of the closure of Q . Q is bounded, and therefore so is S . S is contained in P_M and meets the hyperplane $x_n = 0$ (since $Q \supset B$). Also, by considering the set Z^* corresponding to Z , we can see that S meets the hyperplane $x_n = D$. For if Z lies in $x_n < D$ then $R \leqslant r_D$ since R is the smallest value of the circumradii of S , and if $R > r_D$ Z would have to lie at least partly in $x_n \geqslant D$ for it still to be contained in the cone. Then

$$R \leqslant \frac{D + M}{((2M+1)^2 - 1)^{\frac{1}{2}}} = \frac{D+M}{2M}\left(1 + \frac{1}{M}\right)^{-\frac{1}{2}}$$

$$\leqslant \frac{D+M}{2M} \qquad \text{since } M > 0$$

and hence $(2R - 1)M \leqslant D$ which contradicts our choice of M . So Z^* and hence S must lie at least partly in $x_n \geqslant D$. Thus, since S also meets $x_n = 0$, the diameter of S is at least D .

S is a universal cover by its construction from each $Y \in K_n$, and is compact since any closed bounded subset of E_n is compact. Let $T \subset S$ be a minimal compact universal cover. By the same argument as above T contains points in $x_n \geqslant D$. We want to show that it contains a point in $x_n = 0$.

The line L defined by the equations

$$x_1 = \tfrac{1}{2} \quad , \qquad x_2 = x_3 = \dots = x_{n-1} = 0$$

meets Q in the point $q = (\tfrac{1}{2}, 0, \dots, 0, \tfrac{1}{2})$, since $B \subset Q$. Now q is the only point on L of the closure of Q. For, suppose that this is not so, so that there is a point p of $\overline{Q}$ on L, $p \neq q$. Then there is a sequence of points $p_j \in Q$ (not in general on L) such that $p_j \to p$ as $j \to \infty$. Then p_j (near p) belongs to one, say Y_j^*, of the sets whose union is Q. Y_j^* meets $x_1 = \tfrac{1}{2}$ in a single point (since Y_j^* is a set of constant width), say y_j^*. (See page 120, diagram) We need to know something about the length of py_j^*. Now Y_j^* lies wholly inside an n-dimensional ball of radius 1 which touches $x_1 = \tfrac{1}{2}$ at y_j^*. Call the centre of this ball c and denote the angle pcy_j^* by θ_j. Then $cy_j^* = 1$ and so $py_j^* = \tan \theta_j$. We know that $p_j \to p$ as $j \to \infty$, and $p_j \in Y_j$. Thus p_j belongs to the ball containing Y_j^*, and so $pp_j \geqslant px$ (with the notation of the diagram), where pxc is a straight line. Hence $pp_j \geqslant \sec \theta_j - 1$. But $p_j \to p$ as $j \to \infty$, which means that $\theta_j \to 0$ and so $py_j^* \to 0$ as $j \to \infty$. Thus if

$y_j^* \equiv (y_1^{(j)}, y_2^{(j)}, \ldots, y_n^{(j)})$, we must have $y_i^{(j)} \to 0$, $i = 2, 3, \ldots, n-1$, since $p \in L$ and L has $x_2 = x_3 = \ldots = x_{n-1} = 0$. Now of the sets $Y_j \in K_n$ corresponding to the Y_j^* , there is a subsequence Y_{ji} , converging to a set W , say, $W \in K_n$, and $f(W') = (\frac{1}{2}, 0 \ldots, 0)$ (since the first and last coordinates of $f(W')$ are fixed by the construction of f anyway). Then since the Y_j and hence W are arbitrarily placed in P_M , we may use lemma 1 again (page 128) to establish that W must be an n-dimensional ball. Thus W is congruent to B . But Y_{ji}^* is one of the sets whose union is Q and so it lies as near to $x_n = 0$ as possible while still being contained in P_M . If Y_{ji}^* touches $x_n = 0$, then so does W ; if not, then Y_{ji}^* touches the cone and hence so does W . Thus in either case W is in fact B , in which case Y_{ji}^* converges to B . Hence $\{y_{ji}^*\}$ converges to q and $p = q$. Thus we have a contradiction, and so the line L meets $\overline{Q}$ in the single point $q = (\frac{1}{2}, 0, \ldots, 0, \frac{1}{2})$.

Now S is the convex cover of $\overline{Q}$ (by definition) and $\overline{Q}$ meets L in the single point $q = (\frac{1}{2}, 0, \ldots, 0, \frac{1}{2})$, so it follows that S cannot meet L in any other point than q . But $S \supset \overline{Q}$ and so $q \in S$. T is a universal cover and hence contains an n-dimensional ball of radius $\frac{1}{2}$, and any n-dimensional ball of radius $\frac{1}{2}$ in P must touch the line L . Thus T must contain the point q, and the ball containing this point is B by our definition of B. Thus T contains B (since this is then the only such ball T can contain), and thus T contains the point $(0, \ldots, 0) \in B$. Hence T has points in both $x_n = D$ and $x_n = 0$, and so the diameter of T is at least D. This establishes the required result.

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Lemma 1. If $Z \in K_2$ and every square circumscribing Z has its sides bisected by the points of contact with Z then Z is a disc.

Proof. We require first the property that any member Z of K_2 contains a semi-circle of radius $\frac{1}{2}$, and this may be deduced from the discussion of the two-dimensional result proved in the first part of this section. For, we may, as before, take Z to be a Reuleaux polygon, and if, with the previous notation, $r_{ab} = \frac{1}{2}$, then all those vertices of Z on the negative side of ab lie within γ_{ab}. Then the frontier of Z on the positive side of ab must lie outside or on the frontier of γ_{ab}, since it consists of arcs of radius 1 centred at the vertices on the negative side of ab. Thus one of the semicircles forming the frontier of γ_{ab} lies inside Z (allowing that frontier points of Z may lie on the frontier of γ_{ab}).

So now let a, b be two opposite points of Z such that the

(See page 120, diagram.)

semicircle t_{ab} joining ab lies in Z . Let p denote the centre of the semicircle, and let q be a point of Z furthest from p , so that it follows that the line through q perpendicular to pq is a support line of Z , because otherwise there would be a point of Z further from p than q . There are two support lines of Z parallel to pq and since $Z \in K_2$ the distance between them is 1 , and hence by the condition of the lemma they must touch, in points p_1 , p_2 say, the circle of which t_{ab} is a part. Let the support line of Z through p_1 meet the support line of Z through q (perpendicular to pq) in the point q_1 . Then the two support lines through p_1 , p_2 and the line q_1q_2 form three sides of a square of side 1 circumscribing Z . But by hypothesis every such square has its sides bisected by the points of contact with Z , so that the segments p_1q_1 , p_2q_2 are each of length $\frac{1}{2}$. Thus the length of pq must also be equal to $\frac{1}{2}$, and so, since q is a furthest point from p , the whole of Z is contained in a circle centre p and radius $\frac{1}{2}$. Hence, since $Z \in K_2$, Z must be the circle itself.

Lemma 2. There is a member W of K_n , $n \geqslant 3$, whose projection in any direction is not an $(n-1)$-dimensional ball.

Proof. We prove the lemma by constructing a particular set W and showing that it satisfies the conditions of the lemma.

Let B_0 (in n dimensions) denote the ball of radius $\frac{1}{2}$ and with centre at $(0, \ldots, 0)$. Let B_i denote the ball of radius 1 and with centre at the point p_i where the i-th coordinate of p_i is k , say (to be chosen later), and every other coordinate is zero. Consider the intersection of B_0 with the n B_i , and let V denote the union of the resulting set and the points p_i . Then let W be a member of K_n which contains V . For this to be possible, we must have that the distance between any pair of most distant points of V , which is given by $k\sqrt{2}$, is at most 1 .

W denotes the projection of W in the direction θ . Now we can choose W so that there is a point f on the frontier of W_θ which is the projection of a point on the frontier of W that lies interior to B_0 and on the frontier of one of the B_i . For, this is equivalent to saying that every great-circle on B_0 (that is, every section of B_0 by a hyperplane through the centre of B_0) cuts properly - that, in a set X , say, where the interior of X is non-empty - the intersection of B_0 with one of the B_i ; for then the projection of W in a direction perpendicular to such a great-circle must contain the projection (in the same direction) of X , and except for this restriction we are free to choose W so that its frontier coincides with the frontier of B_i at a point of $X \div Fr(B_0)$, that is, at a point interior to B_0 . To show that it is possible to choose k so that every great-circle on B_0 cuts $B_0 \cap B_i$ properly, we note the following.

Consider the sets which are the intersections of $Fr(B_0)$ and $Fr(B_i)$, $i = 1, 2, \ldots, n$. We know that, since $n \geqslant 3$, there are more than two of these sets. Let q_i be a point of $Fr(B_0) \cap Fr(B_i)$, ~~XXXXXX~~ , and o the centre of B_0 . (See page 120, diagram.) The distance $p_i q_i$ is then, with the notation of the diagram, given by

$$p_i q_i^{\,2} = \frac{1}{4} + k^2 + k \cos\phi$$

where ϕ varies with the position of p_i . As we have not yet fixed the distance op_i ($= k$), choose ϕ to be $\frac{\pi}{4}$. Then it can be seen immediately, from the remark at the beginning of this paragraph, that every great-circle on B_0 must cut at least one of the $B_0 \cap B_i$ properly, as required.

Then we have

$$p_i q_i^{\,2} = \frac{1}{4} + k^2 + \tfrac{1}{2}\sqrt{2}k$$

and since q_i is on $Fr(B_i)$, $p_i q_i = 1$. So we have

$$4k^2 + 2\sqrt{2}k - 3 = 0 ,$$

which gives $k = \dfrac{\sqrt{7} - 1}{2\sqrt{2}}$. Note also that $k\sqrt{2} = \dfrac{\sqrt{7} - 1}{2}$, which is, as required, not bigger than 1 .

This completes consistently the construction of W , and we have noted above that there is then a point f on the frontier of W_θ which is the projection of a point on the frontier of W that lies interior to B_0 and on the frontier of one of the B_i . But this means that sufficiently near f the frontier of W_θ coincides with the frontier of an (n-1)-dimensional ball of radius 1 (i.e. B_1). Thus W_θ is not an (n-1)-dimensional ball of radius $\frac{1}{2}$. It follows, since W_θ has constant width 1 (trivially), that W_θ is not an (n-1)-dimensional ball.

Corollary. There is a member W of K_n such that the greatest lower bound of the circumradii of projections of W in all directions is greater than $\frac{1}{2}$.

Proof. Suppose that there is no such member of K_n . Then for every $W \in K_n$, the greatest lower bound of the circumradii of projections of W in all directions is $\frac{1}{2}$. But then by compactness it follows that there is a projection of W , for every W , for which the circumradius is equal to $\frac{1}{2}$, which contradicts the lemma. Hence the corollary follows.

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Section I - Notation and Definitions.

(The page number refers either to the first page on which a given notation is used, and therefore explained, or to the page on which a given term is defined in full.)

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<u>Section 2.</u> To prove that, if K is a convex body of minimal width ℓ in E_n, and K is contained in the union of r parallel strips of widths h_1, h_2, ..., h_r, then $h_1 + h_2 + \dots + h_r \geqslant \ell$. (Tarski's plank problem. [12]) [13]

------------ [Notation and Definition : page 141.]

A <u>parallel strip of width h</u> is defined as being that closed subset of E_n which lies between two parallel (n-1)-dimensional hyperplanes whose distance apart is h. (Note that the term "parallel strip" does not mean that the strips are parallel to one another.) Suppose that there is a system of vectors already defined in E_n with respect to some given origin and initial line. Then to a given parallel strip of width h there corresponds a pair $(\underline{u}, c)$, where $\underline{u}$ is a vector with $|\underline{u}| > 0$ and c is a constant, such that the equations of the hyperplanes enclosing the strip are given by $\underline{r}.\underline{u} + c = \pm \underline{u}^2$. If we call the hyperplane whose equation is $\underline{r}.\underline{u} + c = +\underline{u}^2$ the positive hyperplane, and the other the negative hyperplane, note that the vector $\underline{u}$ is perpendicular to these planes and directed from negative to positive; and put $2|\underline{u}| = h$. The constant c is then given by the equation $c + (\lambda + 1)\underline{u}^2 = 0$, where $|\lambda \underline{u}|$ is the distance of the negative hyperplane from the origin of vectors. If positive and negative are interchanged so that the direction of the strip is changed, the corresponding pair becomes $(-\underline{u}, -c)$.

Thus we can identify a parallel strip by its corresponding pair $(\underline{u}, c)$. Consider, then, r parallel strips $(\underline{u}_1, c_1)$, $(\underline{u}_2, c_2)$, ..., (u_r, c_r) . The part of E_n which is not contained in the union of these strips consists of (finite or infinite) polyhedra which may be denoted by $P_{\varepsilon_1,\dots,\varepsilon_r}$ where ε_i is equal to $+1$ or -1 according as the polyhedron is on the positive or negative side respectively of the strip $(\underline{u}_i, c_i)$. For convenience of notation, we shall write ε for the sequence $\varepsilon_1,\dots,\varepsilon_r$, and shall understand ε usually to run through all possible sequences formed by the numbers ± 1 , even if some of these 2^r sequences do not correspond to actual polyhedra P_ε . The sequence ε may thus be used in the same way as an r-dimensional vector in the discussion that follows.

Now suppose that we are given r parallel strips $(\underline{u}_1, c_1)$, $(\underline{u}_2, c_2)$, ..., $(\underline{u}_r, c_r)$ whose widths are denoted by $h_i = 2|\underline{u}_i|$, $i = 1,\dots,r$, and suppose that the h_i are such that their sum is less than ℓ , where ℓ is the minimal width of the convex body K we are considering. We want to prove that in this case K is not entirely contained in the union of the strips.

The part of K outside the union of the strips consists of the domains $K \cap P_\varepsilon$, which are all separated from one another by the strips. If we give to each $K \cap P_\varepsilon$ a translation $-\varepsilon\,\underline{u}$, where

$\varepsilon \underline{u}$ denotes the sum $\varepsilon_1\underline{u}_1 + \varepsilon_2\underline{u}_2 + \dots + \varepsilon_r\underline{u}_r$, we are in effect eliminating these separations. The translations may cause overlapping, but this makes no difference to the method of proof we are using. The total part of the space filled by these $K \cap P_\varepsilon$ after translation is

$$\bigcup_\varepsilon \left[(K - \varepsilon\underline{u}) \cap (P_\varepsilon - \varepsilon\underline{u})\right] .$$

Now the geometrical significance of $K - \varepsilon\underline{u}$ is merely that K is in a different position corresponding to a translation $-\varepsilon\underline{u}$ for one given ε; if then we consider the intersection of the sets $K - \varepsilon'\underline{u}$ where ε' ranges over all possible values of ε, this intersection must be contained in $K - \varepsilon\underline{u}$. Hence

$$\bigcup_\varepsilon \left[(K - \varepsilon\underline{u}) \cap (P_\varepsilon - \varepsilon\underline{u})\right] \supset \bigcup_\varepsilon \left[\left\{\bigcap_{\varepsilon'}(K - \varepsilon'\underline{u})\right\} \cap (P_\varepsilon - \varepsilon\underline{u})\right]$$
$$= \left\{\bigcap_{\varepsilon'}(K - \varepsilon'\underline{u})\right\} \cap \left[\bigcup_\varepsilon (P_\varepsilon - \varepsilon\underline{u})\right] , \quad (1)$$

for ε' ranges over all possible ε so that the union over all ε of $\bigcap_{\varepsilon'}(K - \varepsilon'\underline{u})$ leaves it the same.

Now we can show that in fact $\bigcup_\varepsilon(P_\varepsilon - \varepsilon\underline{u})$ is the whole space E_n, as follows. For a particular $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r)$, put $\delta^{(i)} = (0, \dots, \varepsilon_i, 0, \dots)$, where every term of the sequence except ε_i is zero. For a given sequence $\delta = (\delta_1, \delta_2, \dots, \delta_r)$, let H_δ denote the set of points $\underline{r}$ which satisfy $\underline{r}.(\delta\underline{u}) + \delta c \geqslant (\delta\underline{u})^2$, ~~XXXXX~~ , where as above we have $\delta\underline{u} = \delta_1\underline{u}_1 + \dots + \delta_r\underline{u}_r$, and $\delta c = \delta_1 c_1 + \dots + \delta_r c_r$; then for $|\delta\underline{u}| > 0$, H_δ is the half-space lying on the positive side of the strip $(\delta\underline{u}, \delta c)$. Then with this notation we see that P_ε is given by the equation

$$P_\varepsilon = \bigcap_i H_{\delta^{(i)}} .$$

Now define Q_ε by $Q_\varepsilon = \bigcap_{\varepsilon'} H_{(\varepsilon-\varepsilon')/2}$, where ε denotes one particular sequence of numbers ± 1, and ε' ranges over all possible such sequences. If we set ε' such that $\varepsilon'_i = -\varepsilon_i$ and $\varepsilon'_j = \varepsilon_j$ for all $j \neq i$, it is clear that each $\delta^{(i)}$ can be put in the form $(\varepsilon - \varepsilon')/2$. Hence $Q_\varepsilon \subset P_\varepsilon$, and we can now show that in fact $\bigcup_\varepsilon (Q_\varepsilon - \varepsilon \underline{u})$ is the whole space.

Consider any point $\underline{r}$ of E_n. $\underline{r}$ lies in $H_{(\varepsilon-\varepsilon')/2} - \varepsilon \underline{u}$ if and only if $\underline{r} + \varepsilon \underline{u}$ satisfies the inequation for $H_{(\varepsilon-\varepsilon')/2}$, that is,

$$(\underline{r} + \varepsilon \underline{u}) \cdot \left(\frac{\varepsilon - \varepsilon'}{2} \underline{u}\right) + \frac{\varepsilon - \varepsilon'}{2} c \geqslant \left(\frac{\varepsilon - \varepsilon'}{2} \underline{u}\right)^2$$

which gives

$$2\underline{r}\cdot(\varepsilon \underline{u}) + 2\varepsilon c + (\varepsilon \underline{u})^2 \geqslant 2\underline{r}\cdot(\varepsilon' \underline{u}) + 2\varepsilon' c + (\varepsilon' \underline{u})^2$$

where again ε is fixed and ε' is any sequence of the numbers ± 1. Then $\underline{r}$ lies in $Q_\varepsilon - \varepsilon \underline{u}$ when this inequality is satisfied for all ε'. This is true automatically for a value of ε (there may be more than one) which makes the left-hand side of the inequality maximal (since LHS and RHS have exactly the same form). Hence every point $\underline{r}$ lies in one of the $Q_\varepsilon - \varepsilon \underline{u}$, and this proves that

$$E_n = \bigcup_\varepsilon (Q_\varepsilon - \varepsilon \underline{u}) \subset \bigcup_\varepsilon (P_\varepsilon - \varepsilon \underline{u})$$

and hence $\bigcup_\varepsilon (P_\varepsilon - \varepsilon \underline{u}) = E_n$.

Thus we have from (1) that

$$\bigcup_{\varepsilon}\left[(K-\varepsilon\underline{u}) \cap (P_{\varepsilon}-\varepsilon\underline{u})\right] \supset \bigcap_{\varepsilon'}(K-\varepsilon'\underline{u})$$
$$= \bigcap(K \pm \underline{u}_1 \pm \underline{u}_2 \pm \dots \pm \underline{u}_r) \qquad (2)$$

where, by the definition of the ε', the last intersection ranges over all possible combinations of signs. To prove our result, we want to prove that this intersection is not empty.

It is sufficient to prove that for a convex body M of minimal width m, and $\underline{v}$ a vector whose length is $|\underline{v}| = \frac{1}{2}h$ where $h < m$, the intersection $\bigcap(M \pm \underline{v}) \equiv (M-\underline{v}) \cap (M+\underline{v})$, which is convex, is not empty and has minimal width not less than $(m-h)$. Then successive applications of this to (2) will establish the main result.

M has at least one chord (lying between parallel support hyperplance at its endpoints), of length k, say, $k \geqslant m$, in the direction of the vector $\underline{v}$. A length $k - 2|\underline{v}| = k - h > 0$ of this chord is contained in $\bigcap(M \pm \underline{v})$. Denote the endpoints of this length by A, on the boundary of $M+\underline{v}$, and B, on the boundary of $M-\underline{v}$. Now if we take A as the centre of similitude of the ratio $\frac{k-h}{k}$, which is less than 1, the set $M+\underline{v}$ may be carried into a subset M' of itself, and the minimal width of this subset is given by $m\left(\frac{k-h}{k}\right)$, which, since $h < k$, is not less than $(m-h)$. Similarly, if we take B as the centre of similitude of the same ratio, the set $M-\underline{v}$ is carried into exactly the same set M', which is therefore also a subset of $M-\underline{v}$. Hence $M' \subset \bigcap(M \pm \underline{v})$, and we have already seen that the minimal width of M' is not less than $(m - h)$.

Applying this result, then, to the intersection in (2) , we have that $\bigcap(K \pm \underline{u}_1 \pm \dots \pm \underline{u}_r)$ is not empty and in fact contains a convex set whose minimal width is not less than $\ell - h_1 - \dots - h_r$. Thus if $h_1 + \dots + h_r < \ell$, K is not entirely contained in the union of the strips, and so we have finally that if K is contained by such a union, the sum of the widths of the strips must be at least ℓ .

It may be noted, to conclude this section, that a much shorter and simpler proof of the same result is possible when K is a circle.

Suppose that K is a given circle of diameter ℓ which is covered by r parallel strips of widths h_1 , h_2 , ..., h_r . Consider the hemispherical surface H erected on K as its base, and denote by H_1 , H_2 , ..., H_r the zones on H which are bounded by planes perpendicular to K through the edges of the parallel strips covering K , so that the strips which cover K are the projections of the H_i perpendicular to K onto K . There is a one-one correspondence between the points of H and the points of K , so that it follows that H must also be completely covered by the zones H_1 , H_2, ..., H_r .

We may now make use of the property that the surface area of the zone of a sphere is a function only of the width of the zone (so long as both parallel edges of the zone meet the surface of the sphere) and not of its position relative to any fixed point or line. Thus we may replace any of the zones H_i with a zone Z_i, say, whose projection on K perpendicular to K is still of width h_i, and whose surface area (on the hemisphere) is equal to that of H_i. Choose, then, a set $Z_1, Z_2, \ldots, Z_r$ of such zones, such that they all lie parallel to one another on H and still cover H. (We know that this is possible because we have kept the same total area of the zones, and if by lying parallel to one another and not overlapping they do not cover H then there is no position of them for which H would be completely covered.)

Then, finally, the projections of the zones $Z_1, Z_2, \ldots, Z_r$ on K form a set of parallel strips, parallel to one another, of widths $h_1, h_2, \ldots, h_r$, which cover K since the Z_i cover H. Since K is covered, therefore, the sum of the widths of the strips in this position must be at least equal to the diameter of K, and thus in general, from the above argument, we have $h_1 + h_2 + \cdots + h_r \geqslant \ell$.

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Section 2 - Notation and Definition.

(The page number refers either to the first page on which a given notation is used, and therefore explained, or to the page on which the given term is defined in full.)

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Section 3. To show that a small-circle of a unit sphere, of angular radius ρ, can be covered by a finite number of lunes only if the sum of their widths is at least 2ρ, and if the sum is exactly 2ρ, only when they all share the same vertices. [14]

------------ [Notation and Definitions : page 154.]

Consider the class of all great-circles of a unit sphere. Then it is a well-known problem to find the shortest possible curve, with various restrictions on the definition of "curve", which cuts every member of the class at least once. If the curve must be continuous, the extremal is a semi-circle (of length π).

A variation of the problem is to find the shortest possible curve on a unit sphere which cuts every great-circle making an angle less than some fixed ρ with the equatorial plane, and we discuss this now. What we are considering in fact is not the problem as it stands, but an equivalent statement of it which is analogous to the "plank problem" of T. Bang.

In place of the zone of the sphere between two circles of latitude ρ we take the largest spherical cap which can be contained by such a zone; this has the advantage, as will be seen by the way the problem is treated, of having circular symmetry. Our equivalent problem is then to cover this cap by a set of lunes and to determine the least sum of the widths of the lunes for which such a covering is possible.

(A lune is that part of the surface of a sphere which is bounded by the arcs of two intersecting circles on the surface of the sphere.) We consider here covering by a finite number of lunes.

We need first some definitions. A small-circle Γ of a sphere is any circle whose centre does not coincide with the centre of the sphere. Denote by D the interior (on the sphere) and the frontier of Γ, and let C be the centre (on the sphere) of D. We define a sliver as a lune of small thickness δ (at its widest part) bounded by two great-semicircles; this will enable us to consider a large number of slivers in place of a smaller number of lunes covering D. Such a sliver is said to be centrally situated with respect to D if the great-circle joining its centre to C is perpendicular to the sliver. (If we call the points of intersection of the bounding semi-circles of the sliver its poles, the centre of the sliver is the point, of the sliver, which lies midway between these two poles and equally distant from both the bounding semi-circles.)

Now in moving to what we have noted is an equivalent problem, as it is set out at the head of this section, we have in fact moved from a statement involving linear measures to a problem whose solution demands the use of area. We want therefore to make sure that in considering the new problem we are not introducing more than the original problem contained. In order to do this, we show that we can attach at each point of D a "weight function" f such that the "weighted area" of the intersection with D of a centrally situated sliver S, that is, the integral of f over such a region, is proportional to the thickness of S and does not depend on the relative positions of S and D, provided that both edges of S actually intersect D. Also, since we are working with a cap of a sphere, we shall require that the function f has <u>circular symmetry</u> on D, that is, that it is a function only of the angular distance t from any given point to the centre C, on the sphere, of Γ. Lemma 1 (page 146) proves the existence of such a weight function, and also establishes certain smoothness properties which we need.

We have, then, a weight function f with a specific property related to centrally situated slivers, as we have described. We shall need to know something about the integral of f over the intersection with D of a sliver S which is not centrally situated, and it may be shown (lemma 2 (page 151)) that such an integral increases as S is slid, in the direction of its own length, towards a centrally situated position, and that the maximum value of the integral occurs only when S is centrally situated. With these properties established, it is easy to prove the main result.

As there is no restriction (except that we do not consider infinite numbers) on the number of lunes taken, we may take a large number n of slivers S_i, and consider the integral of the weight function f over the intersection with D of the union of the slivers. It follows that, using a condensed notation,

$$\int_{D\cap(\cup S_i)} f \leq \sum_{i=1}^{n} \int_{D\cap S_i} f$$

with strict inequality unless no two of the slivers overlap (although they may of course have frontier points in common). Let $S_i^{\prime}$ denote the sliver S_i when it is slid into the centrally situated position; then by lemma 2

$$\int_{D\cap S_i} f \leq \int_{D\cap S_i^{\prime}} f$$

and the inequality is strict unless $S_i = S_i^{\prime}$. Finally since f has circular symmetry, we may replace $S_i^{\prime}$ by $S_i^{\prime\prime}$, where $S_i^{\prime}$ and $S_i^{\prime\prime}$ are both the same distance from the centre C of D but all the $S_i^{\prime\prime}$ share the same vertices. Then we have

$$\int_{D\cap(\cup S_i)} f \leq \sum_{i=1}^{n} \int_{D\cap S_i^{\prime\prime}} f \quad , \text{and} \quad \sum_{i=1}^{n} \int_{D\cap S_i^{\prime\prime}} f \leq \int_D f$$

if the sum of the widths of the $S_i^{\prime\prime}$ is less than or equal to 2ρ. Equality in all three inequations occurs only if the lunes do not overlap, are centrally situated, share the same vertices, and the sum of their widths is equal to 2ρ. If Γ can be covered by the lunes S_i, we know that $\int_{D\cap(\cup S_i)} f = \int_D f$, and thus we have from above that the sum of the widths of the S_i must be at least 2ρ.

If their sum is exactly 2ρ , then equality must hold in all the inequations above, so that the lunes do not overlap, are centrally situated, and share the same vertices (which in fact implies the centrally situated property).

(From this follows the corresponding measure theory result for sets of great-circles, which we mentioned at the beginning of the section: the shortest continuous curve on the unit sphere which cuts every great-circle, making an angle of less than ρ with the equatorial plane, is part, of length ρ and bisected at the equator, of some circle of longitude.)

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Lemma 1 . The angular distance of a point P of D from the centre C is denoted by t . Let the angular radius of Γ (the frontier of D) be a . A function f is to have the following properties:

(i) f exists for every point P of D ,

(ii) f has circular symmetry, i.e. it is a function of t only,

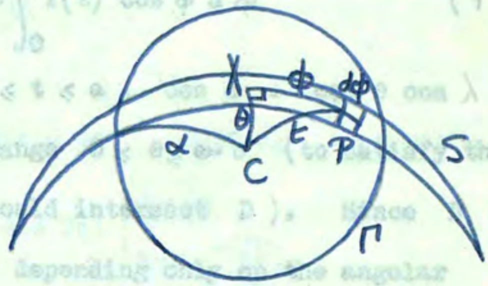
(iii) the integral of f over the intersection with D of a centrally situated sliver S of thickness δ , is independent, for fixed δ , of the relative positions of D and S , i.e. it is proportional to δ , provided that both edges of S actually intersect D .

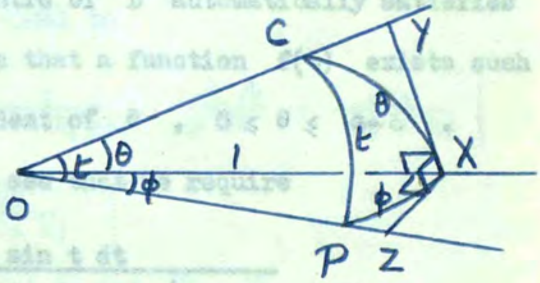
Then the (positive) weight function

$$f(t) = \frac{2c}{\pi}\left\{\cos^{-1}\left(\frac{\cos\alpha}{\cos t}\right) + \frac{2\cos\alpha}{\sqrt{(\cos^2 t - \cos^2\alpha)}}\right\},$$

where c is any positive constant, satisfies these requirements.

Proof. t is the angular distance from C of a small element of width $d\phi$ of the centrally situated sliver S, the angular distance of the element from the centre of the sliver being ϕ. θ denotes the angular distance from C of the centre of the sliver.





If CPX in the diagram represents the same spherical triangle, with sides of length θ, ϕ, t, and O denotes the centre of the sphere of which D is a cap (so that $OX = OP = OC = 1$), then XY and XZ, the tangents to the triangle's sides at X, are perpendicular to each other and to OX, and we have

$$YZ^2 = YX^2 + XZ^2 = \tan^2\theta + \tan^2\phi$$

and

$$YZ^2 = OY^2 + OZ^2 - 2.OY.OZ.\cos YOZ$$

$$= \sec^2\theta + \sec^2\phi - 2\sec\theta\sec\phi\cos t$$

which gives $\quad \cos t = \cos\theta\cos\phi$.

The area of the small slice of S to the first order is $\delta \cos\phi \, d\phi$. Then the property (iii) that we want for f is seen to be equivalent to saying that the integral

$$\delta \int_{-\lambda}^{\lambda} f(t) \cos\phi \, d\phi \equiv 2\delta \int_{0}^{\lambda} f(t) \cos\phi \, d\phi \qquad (1)$$

where $\cos t = \cos\theta \cos\phi$, $\theta \leqslant t \leqslant \alpha$, $\cos\alpha = \cos\theta \cos\lambda$, must be independent of θ in the range $0 \leqslant \theta \leqslant \alpha - \delta$ (to satisfy the condition that both edges of S should intersect D). Since D has circular symmetry, any function f depending only on the angular distance t of a point from the centre of D automatically satisfies (ii) , so that we need only to prove that a function f(t) exists such that the integral (1) is independent of θ , $0 \leqslant \theta \leqslant \alpha - \delta$.
Using $\cos\phi = \cos t / \cos\theta$ we see that we require

$$\int_{t=\theta}^{\alpha} f(t) \frac{\cos t}{\cos\theta} \cdot \frac{\sin t \, dt}{\cos\theta \sqrt{[1 - (\cos^2 t/\cos^2\theta)]}}$$

to be constant; that, putting $F(x) = f(t)$, $\cos^2 t = x$, $\cos^2\theta = y$, $\cos^2\alpha = k$, we require

$$\tfrac{1}{2} \int_{k}^{y} \frac{F(x)dx}{\sqrt{(y(y-x))}} = c$$

where c is constant.

This, put in the form

$$\int_{k}^{y} \frac{F(x)dx}{\sqrt{(y-x)}} = 2c\sqrt{y} \equiv g(y)$$

is a standard Abel type of integral equation, from which we want a solution for $F(x)$. (See ref. [15].) $g(y)$ is continuous in the closed interval $[k, y]$, and

$$\int_k^y \frac{g(\zeta)d\zeta}{\sqrt{(y-\zeta)}} \equiv \int_k^y \frac{2c\sqrt{\zeta}\,d\zeta}{\sqrt{(y-\zeta)}}$$

$$= 2cy\int_{\zeta=k}^{y} 2\sin^2 q\,dq \qquad \text{putting } \zeta = y\sin^2 q$$

$$= 2cy(q - \tfrac{1}{2}\sin 2q)\Big|_{q=\sin^{-1}\sqrt{(k/y)}}^{q=\frac{1}{2}\pi}$$

has derivative with respect to y equal to

$$2c\left[\frac{\pi}{2} - \sin^{-1}\sqrt{\left(\frac{k}{y}\right)} - \frac{y}{\sqrt{(1-k/y)}}\cdot\frac{(-\frac{1}{2})\sqrt{k}}{y^{3/2}} + \frac{\frac{1}{2}\sqrt{k}}{\sqrt{(y-k)}}\right]$$

$$= 2c\left[\cos^{-1}\sqrt{\left(\frac{k}{y}\right)} + \frac{\sqrt{k}}{\sqrt{(y-k)}}\right]$$

which is continuous in the half-open interval $(k, y]$, but $g(k) = 2c\sqrt{k} \neq 0$ since $\sqrt{k} = \cos\alpha$ and $\alpha \neq \frac{\pi}{2}$ since D is strictly less than a hemisphere (Γ was defined not to be a great-circle).

Hence the unique solution for $F(x)$ is given by

$$F(x) = \frac{\sin\frac{1}{2}\pi}{\pi}\frac{g(k)}{\sqrt{(x-k)}} + \frac{\sin\frac{1}{2}\pi}{\pi}\frac{d}{dx}\int_k^x \frac{g(\zeta)d\zeta}{\sqrt{(x-\zeta)}}$$

$$= \frac{2c\sqrt{k}}{\pi\sqrt{(x-k)}} + \frac{2c}{\pi}\left[\cos^{-1}\sqrt{\left(\frac{k}{x}\right)} + \frac{\sqrt{k}}{\sqrt{(x-k)}}\right]$$

$$= \frac{2c}{\pi}\left[\cos^{-1}\sqrt{\left(\frac{k}{x}\right)} + \frac{2\sqrt{k}}{\sqrt{(x-k)}}\right]$$

i.e. $$f(t) = \frac{2c}{\pi}\left[\cos^{-1}\left(\frac{\cos a}{\cos t}\right) + \frac{2\cos a}{\sqrt{(\cos^2 t - \cos^2 a)}}\right] ,$$

as in the statement of the lemma.

We may take $c = \frac{1}{2}\pi$ since it is arbitrary. We shall be concerned with integrals of f with t taking any value between 0 and a , and we need to know that these integrals will exist and converge throughout. It may easily be calculated that

$$f'(t) = \frac{\tan t \cos a\,(\cos^2 t + \cos^2 a)}{(\cos^2 t - \cos^2 a)^{3/2}} \qquad 0 \leqslant \theta \leqslant t < a$$

$$> 0 \quad \text{for all } t \text{ in the given range}$$

and therefore $f(t)$ increases monotonically, $\theta \leqslant t < a$. Since $f(0) = a + 2\cos a > 0$, $f(t) > 0$ for all t in the range we are considering. Also we have

$$f(a+h) = \cos^{-1}\left(\frac{\cos a}{\cos(a+h)}\right) + \frac{2\cos a}{\sqrt{(\cos^2(a+h) - \cos^2 a)}} .$$

Then for h negative and sufficiently small,

$$\cos^{-1}\left(\frac{\cos a}{\cos(a+h)}\right) \to 0 \qquad \text{as } h \to 0$$

and $$\frac{2\cos a}{\sqrt{(\cos^2(a+h) - \cos^2 a)}} = 2\cos a\,(h^2\sin^2 a - 2h\sin a\cos a)^{-\frac{1}{2}}$$

neglecting higher powers of h under the root sign

$$= (-h)^{-\frac{1}{2}} 2\cos\alpha\,(\sin 2\alpha + O(h))^{-\frac{1}{2}}$$

$$\to \infty \quad \text{in the same way as}$$

$$(2\cot\alpha)^{\frac{1}{2}}(\alpha - t)^{-\frac{1}{2}} \quad \text{as} \quad t \to \alpha \quad ,$$

and the integral of $(\alpha - t)^{-\frac{1}{2}}$ with respect to t converges over any range $[a, b]$ for which $0 \leqslant a \leqslant b \leqslant \alpha$. From these last considerations it will be seen that all the integrals to be considered do converge.

<u>Lemma 2</u>. If a sliver S is not centrally situated, then the integral of f over $S \cap D$ increases as S is slid, in the direction of its own length, towards a centrally situated position, and the maximum value of the integral occurs only when S is centrally situated.

<u>Proof</u>. We can ignore the case when $S \cap D = \phi$. Otherwise there are three possible cases to consider:

(i) S cuts Γ twice ;

(ii) S cuts Γ only once but cuts the perpendicular great-circle from C to S ;

(iii) S cuts Γ only once and does not cut the perpendicular great-circle from C to S.

Suppose that S is displaced from its centrally situated position through an angular distance η, and put $\lambda = \cos^{-1}(\cos\alpha / \cos\theta)$ as in lemma 1. Then the integrals over $S \cap D$ for the three cases are seen to be as follows:

(i) $$\int_0^{\lambda} f\cos(\phi+\eta)\,d\phi + \int_0^{\lambda} f\cos(\phi-\eta)\,d\phi \equiv I_1 ,$$

and $0 \leqslant \eta \leqslant \frac{\pi}{2} - \lambda$;

(ii) $$\int_0^{\lambda} f\cos(\phi-\eta)\,d\phi + \int_0^{\frac{1}{2}\pi-\eta} f\cos(\phi+\eta)\,d\phi \equiv I_2 ,$$

and $\frac{\pi}{2} - \lambda \leqslant \eta < \frac{\pi}{2}$;

(iii) $$\int_{\eta-\frac{1}{2}\pi}^{\lambda} f\cos(\phi-\eta)\,d\phi \equiv I_3 ,$$

and $\frac{\pi}{2} \leqslant \eta \leqslant \frac{\pi}{2} + \lambda$.

In these integrals ϕ denotes the angular distance of the element from the perpendicular great-circle from C to S ; ψ , if we denote the integrand by $f\cos\psi$ in each case, is the angular distance of the element from the centre of the sliver; the integrals are thus seen to be of the same form as the integral $\int_0^{\lambda} f(t)\cos\phi\,d\phi$ of lemma 1 .

To prove the increasing property of the integral of f as stated in the lemma, we need to show that each of the integrals I_1 , I_2 , I_3 increases as the value of η decreases. The two integrals whose sum is I_1 have the same range of integration, which is independent of η ; so we need only note that the sum of the integrands is XX $2f\cos\phi\cos\eta$, which certainly increases as η decreases. Hence so does I_1 . In passing from (i) to (ii) , note that $I_1 = I_2$

for $\eta = \frac{\pi}{2} - \lambda$. The range of integration of the first integral contributing to I_2 is independent of η , and the integrand is $f \cos(\phi - \eta)$ where $\eta - \phi > 0$ so that $f \cos(\phi - \eta)$ increases as η decreases; in the second integral both the integrand and the range of integration increase as η decreases. Hence so does I_2 . Note again that $I_2 = I_3$ for $\eta = \frac{\pi}{2}$. Finally $\eta - \phi > 0$ again for I_3 , so that the integrand, the range of integration, and thus I_3 , all increase as η decreases.

This proves the first assertion of the lemma. For the second we need only to note that, throughout the range of values for which the integral I_1 holds, the integrand $2f \cos\phi \cos\eta$ is in fact increasing strictly, so that the maximum value of I_1 occurs only at $\eta = 0$ in this range, and since the values of I_2 and I_3 do not, from the first part of the lemma, ever exceed any value of I_1 , the lemma is proved completely.

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Section 3 - Notation and Definitions.

(The page number refers either to the first page on which a given notation is used, and therefore explained, or to the page on which a given term is defined in full.)

Γ, ρ	:	page	143, 142
D	:	page	143
c , δ	:	page	143
f	:	page	143
S	:	page	144
t	:	page	146
ϕ , θ	:	page	147

Small-circle	:	page	143
Sliver	:	page	143
Centrally situated sliver	:	page	143
Centre of a sliver	:	page	143
Circular symmetry	:	page	144

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<u>Section 4</u>. To show that, for all maximal K-packings of positive minimal radius, the greatest lower bound of their lower densities is $\frac{\pi}{6\sqrt{3}}$, and is strictly less than $\frac{\pi}{6\sqrt{3}}$ if K-packings of ellipses are excluded. [16]

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The definitions of the terms of the problem are stated first. E_2 is the Euclidean plane with a prescribed cartesian system, with origin o . A convex body in E_2 is defined as a closed bounded strictly convex set of points of E_2 which is symmetric in an interior point of the set. Let K be a convex body which is symmetric in o . Associate with K the unique distance function $F(x)$, defined and continuous for all $x \in E_2$, such that

(i) $F(tx) = |t|F(x)$ for all real t ,

(ii) $F(x) > 0$ for all x other than o ,

(iii) $F(x + y) \leqslant F(x) + F(y)$ for all x , y , with equality only if x , y are collinear with o ,

(iv) K is the set of points x for which $F(x) \leqslant 1$.

Given $z \in E_2$ and R a positive real number, we denote by $K(z,r)$ the set of points x for which $F(x - z) \leqslant r$; the set $K(z,r)$ is known as the homothetic translate of K with centre z and radius r . We define a maximal K-packing of minimal radius r to be a collection M of homothetic translates of K , of possibly different radii but all at least r, such that

(a) no two distinct members of M have an interior point in common,

(b) any homothetic translate of K with radius r which is not in M has at least one interior point in common with a member of M.

Now given a K-collection N of bodies, let $V(N,k)$ denote the area of the point set union of the intersection of members of N with the circular disc centred at o and with radius $k > 0$. We define the lower density of N to be the number $\liminf_{k\to\infty} \frac{V(N,k)}{\pi k^2}$. Let $k^* = k^*(K)$ denote the greatest lower bound of the lower densities of all K-collections of sets of minimal radius $r > 0$. (The general definitions are given, although in practice most of what follows will refer to particular K-collections M which are maximal K-packings.)

It was conjectured by Fejes Tóth ([17]) that if K is a circle, there is a maximal K-packing of lower density k^*, every member of which is a circle of radius r. The conjecture was in fact made also for saturated systems of circles which may not form a packing. While not establishing this completely, we shall prove here a modification of it, which is the analogous result for packings when K is any strictly convex domain with a centre - it is obtained on the lines of the corresponding proof by Eggleston ([18]) for K a circle. The authors of the paper to be discussed have also ([19]) extended Eggleston's result to the case when the circles form a uniform saturated set, the method of proof following very similar lines to those which we shall be considering here.

(A set of closed circular discs of radii r_1 , r_2 ,... in the plane is a uniform saturated set if it has the properties:

(i) $r = \inf r_i > 0$,

(ii) any circle of radius r has at least one point in common with a member of the set.)

The main result to be proved is stated as follows.

Theorem A. There is a maximal K-packing of minimal radius r whose lower density is $k^* = k^*(K)$ and every member of which has radius r . If $t(K)$ denotes the area of the largest triangle contained in K , then

$$k^*(K) = \frac{V(K)}{8t(K)} .$$

Its proof is a consequence of the following theorems 1 and 2 .

Theorem 1 . Given a maximal K-packing M of minimal radius r , there is a triangulation of the plane with the properties that

(i) each vertex of each triangle is the centre of a member of M ,

(ii) if abc is any one of the triangles and $K(a)$, $K(b)$, $K(c)$, are those members of M with centres a , b , c respectively, then there is a point p such that the homothetic translate of K with centre p and radius r has a point in common with each of $K(a)$, $K(b)$, $K(c)$.

Theorem 2. Let $K(a)$, $K(b)$, $K(c)$ be non-overlapping homothetic translates of K , with centres at the vertices a , b , c of a non-degenerate triangle, each of radius at least r, and each having a point in common with a homothetic translate of K of radius r. Then

$$\frac{V[abc \cap (K(a) \cup K(b) \cup K(c))]}{V(abc)} \geq \frac{V(K)}{8t(K)}$$

where abc denotes the convex cover of the points a, b, c.

It is known (and is stated without full proof in, for example, Bambah and Rogers [20]) that $\frac{V(K)}{2t(K)} \leq \frac{2\pi}{3\sqrt{3}}$, with equality needed only for ellipses. Thus we have finally, using theorem A, the following result.

<u>Theorem 3.</u> $k^{*}(K) = \frac{V(K)}{8t(K)} \leq \frac{\pi}{6\sqrt{3}}$, with equality only when K is an ellipse.

<u>Proof of theorem 1.</u>

Let M be a maximal K-packing of minimal radius r. In order to begin the construction of a suitable triangulation of the plane, we make the following definitions.

Given $K(a,R(a)) \in M$ and an arbitrary point x of the plane, put

$$d_a(x) = F(x - a) - R(a),$$

and denote by $S(a)$ the set of all those points x of the plane which have the property $d_a(x) \leq d_b(x)$ for all $K(b,R(b)) \in M$.

We note that $d_a(x)$ can be negative and that therefore $S(a)$

always includes $K(a,R(a))$ itself. Also $d_a(x) \leq r$ for all a and $x \in S(a)$ since M is a maximal packing. We state here also three properties of the sets $S(a)$ which will be used later in the proof of this theorem; the proofs are given in lemmas 2 , 3 and 4 . (See page 193.)

S(i) If $x \in S(a)$, then every point except possibly x of the line segment ax is an interior point of $S(a)$.

S(ii) If a , b are distinct centres of members of M , then $S(a) \subset K(a,F(b-a))$.

S(iii) If a , b , c are non-collinear centres of members of M then $S(a) \cap S(b)$ cannot have any point in common with the closed set

$$H(a,c,b^-) \cap H(b,c,a^-)$$

(nor, similarly, $S(b) \cap S(c)$ in $H(b,a,c^-) \cap H(c,a,b^-)$, $S(c) \cap S(a)$ in $H(c,b,a^-) \cap H(a,b,c^-)$), where $H(x,y,z^+)$ denotes the closed half-plane, containing the point z , whose frontier passes through the points x , y ; and $H(x,y,z^-)$ denotes the closed half-plane, not containing the point z , whose frontier passes through the points x , y . Now we define a crosspoint of M to be a point v of the plane which lies in at least three distinct sets $S(a)$, $S(b)$, $S(c)$ where a , b , c are centres of distinct members of M . We say that a , b , c belong to v . Again we state three properties of the crosspoints of M which will be used later; their proofs are in lemmas 5, 6, 7 (page 197) of this section.

C(i) Any three centres belong to at most one crosspoint.

C(ii) The crosspoints of M form a discrete set.

C(iii) At most a finite number of centres of M belong to one crosspoint.

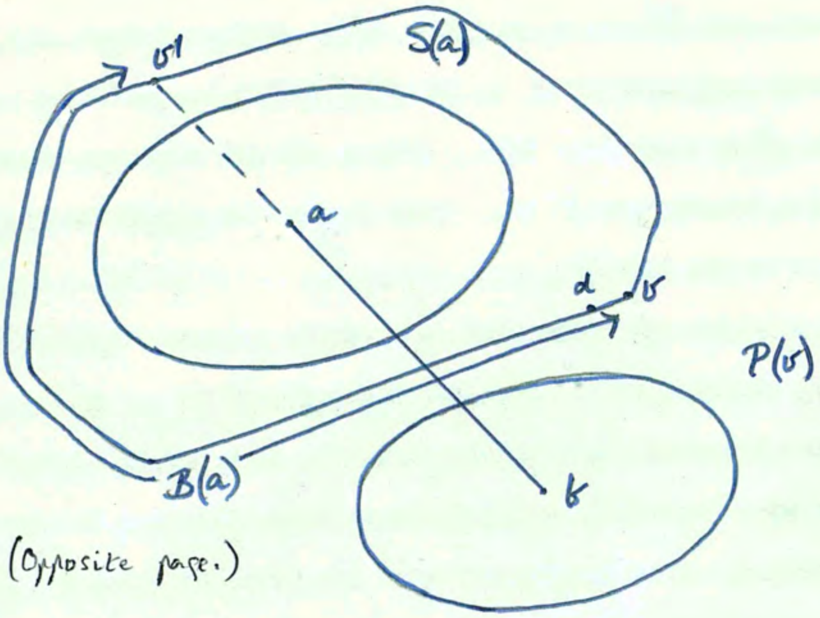
For each crosspoint v we denote by P(v) the convex cover of all the centres of M which belong to v. Since we are attempting to construct a triangulation of the whole plane, we prove first that crosspoints exist in the plane and that the sets P(v), which by property C(iii) are polygons, cover the whole plane.

By their definition the sets S(a) cover the plane without overlapping (although of course as they are closed they may have frontier points in common). Each S(a) is by its definition bounded and because of property S(i) a star body (that is, a body such that a segment which joins the point a to any point of the frontier is entirely contained in the body). Thus its frontier is a simple closed curve, and every point of the frontier belongs to at least one other such set S(b). S(a) must meet at least two such sets, because if its whole frontier belonged to S(b), for example, S(b) would not be a star body (since no interior points of S(a) belong to S(b)). But we know (lemma 1 (page 191)) that only finitely many members of M intersect any fixed disc in the plane, and hence S(a) can meet only finitely many sets S(b) . So the frontier of S(a) can be represented as the union of a finite number of (and at least two) closed sets each belonging to a set S(b) different from S(a) . Hence there must be a point v of S(a) which is contained in at least two other such sets S(b) (since all the sets are closed), and so v is a crosspoint of M . This proves the existence of a crosspoint in the plane.

Now consider any crosspoint v and the polygon P(v) which is the convex cover of all the centres belonging to v . Let a , b be two centres belonging to v such that ab is a proper edge of P(v) . P(v) has an interior since any three centres belonging to v cannot be collinear. (See the beginning of the proof of lemma 5 , page 197 .) In order to show that the P(v) cover the plane, we need to show that there is a crosspoint w of M different from v such that the set P(w) $\cup$ P(v) contains the segment ab as an interior diagonal. For then we may complete the proof as follows. The union of all the P(v) is closed, so that its complement is open. If this is not empty, it has a frontier which must consist of line segments each being on the frontier of a polygon P(v) . But if the existence of polygons P(v) and P(w) on either side of an arbitrary line segment ab joining centres of members of M has been proved, this is a contradiction and we shall have shown that the polygons P(v) cover the whole plane.

We want, then, the existence of w satisfying the conditions

(Diagram)



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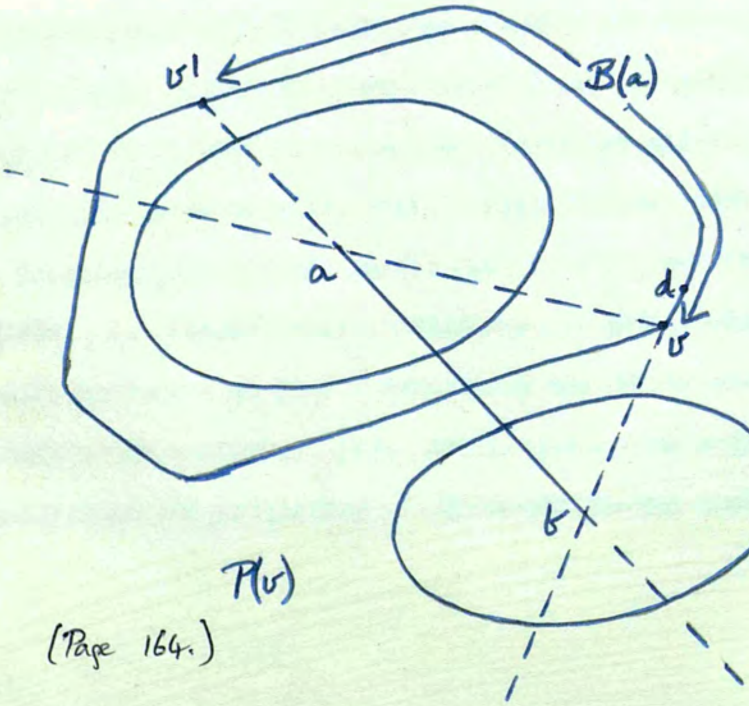


(Page 164.)

given above. We must obtain it for each of two cases, since we do not know if v and $P(v)$ are on the same side or opposite sides of ab, so suppose first that they are on the same side of ab. (See diagram opposite.) Denote by v' the point of the frontier of $S(a)$ which lies on the ray ba produced past a, and let $B(a)$ denote that part of the frontier of $S(a)$ which has endpoints v, v' and which passes, at least in part, through the side of ab opposite to that containing $P(v)$. Each point of $B(a)$ lies as we have seen in at least one set $S(c)$, say, other than $S(a)$. We assert now that there is a neighbourhood of v such that except for v the part of $B(a)$ lying in the neighbourhood belongs to $S(a)$ and $S(b)$ but to no other such set.

For if not, there is a set $S(c)$, different from $S(a)$ and $S(b)$, which has points other than v on $B(a)$ and arbitrarily close to v. It follows that v itself belongs to $S(c)$ since by definition $S(c)$ is closed, and so c is a centre belonging to v. Now consider the possible positions of the point c. Since a (See page 162, diagram), b, c are all centres belonging to v, c cannot be collinear and with a and b, and since c is a point of the polygon $P(v)$ it must lie on the same side of ab as $P(v)$. Also c cannot lie in the triangle abv. For, $v \in S(a) \cap S(b)$ so that we have $r \geqslant d_a(v) = F(v-a) - R(a)$ and $r \geqslant d_b(v) = F(v-b) - R(b)$; hence the triangle abv, except possibly for the point v, lies completely in the interior of the union of the sets $K(a,R(a)+r)$, $K(b,R(b)+r)$, and thus no centre c of M

can lie in abv since $R(c) \geq r$ for all $K(c,R(c)) \in M$. If we take any point d distinct from v in the set $B(a) \cap S(c)$, we know by property S(i) that the segment cd consists except for d of interior points of S(c). This implies that cd cannot cross either av (which except for v is interior to S(a)) or bv (which except for v is interior to S(b)), since by definition no two sets S(a), S(b) can have interior points in common, and we have already specified that d is distinct from v. So c can only lie collinear with d and v, with v lying between d and c. Thus v is an interior point of S(c), since both c and d belong to S(c), which is a contradiction since v is a crosspoint of M. Hence if v and P(v) are on the same side of ab, there is a neighbourhood of v such that except for v the part of B(a) lying in the neighbourhood belongs to S(a) and S(b) but to no other such set.

If, on the other hand, v and P(v) are on opposite sides of ab, let B(a) be defined exactly as before. (See page 162, diagram.) We make the same assertion and again assume the contrary to obtain a contradiction. Again we have a set S(c) as above and $v \in S(c)$ so that c is a centre belonging to v. The point c cannot be collinear with a and b, and again it must lie on the same side of ab as P(v). If we consider a point d of $B(a) \cap S(c)$ as before, then d can be chosen as close to v as we please. Now the segment cd cannot cross av, bv, or ab (since all points except one of ab are interior to one of $S(a)$, $S(b)$ and d is variable so we cannot ensure that cd always passes through this one particular point), and so it follows that c is confined to one of the regions

$$H(a,b,v^-) \cap H(a,v,b^-) ,$$

$$H(a,b,v^-) \cap H(b,v,a^-) .$$

But then v lies in one of the regions

$$H(a,c,b^-) \cap H(a,b,c^-) ,$$

$$H(b,c,a^-) \cap H(a,b,c^-) .$$

Either of these is a contradiction with property S(iii) , since $v \in S(b) \cap S(c)$ and $v \in S(a) \cap S(c)$.

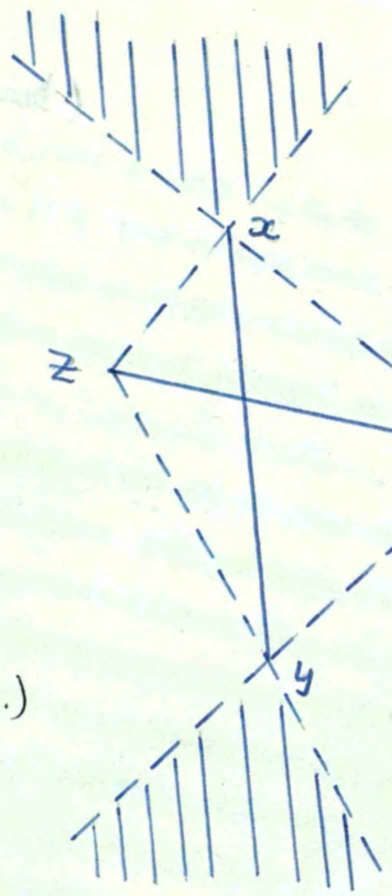
Thus in either case our original assertion is true, that there is a neighbourhood of v such that except for v the part of $B(a)$ in the neighbourhood belongs to $S(a)$ and $S(b)$ but to no other such set. Now the endpoint v' of $B(a)$ does not lie in $S(b)$ since the segment bv' lies partly in $S(a)$, thus contradicting property S(i) . Hence there is a first point, which will be seen to be the point w that we want, of $B(a)$ other than v which lies in some $S(c)$, say, other than $S(a)$ and $S(b)$. Then w is a crosspoint of M different from v and a , b , c are centres belonging to w . Since this is so, we have again that a , b , c cannot be collinear, and since c is a point of the polygon $P(w)$ it must lie on the same side of ab as $P(w)$. If c also lies on the same side of ab as $P(v)$, then this is the same as was true in each case above and leads to the same contradiction with the properties of v . Thus c must lie on the opposite side of ab to P(v) , which implies that P(v) , P(w) lie on opposite sides of ab and so contain ab as an interior diagonal (since each polygon P(v) has an interior, in other words, is not degenerate).

Hence, as we have already seen, we may deduce from the existence of the crosspoint w that the polygons P(v) cover the whole plane. We require finally that no two such polygons overlap, that is, have interior points in common, for then it is straightforward to divide each P(v) up into triangles having their vertices at the vertices of P(v) , and so to triangulate the whole plane according to the conditions of theorem 1 .

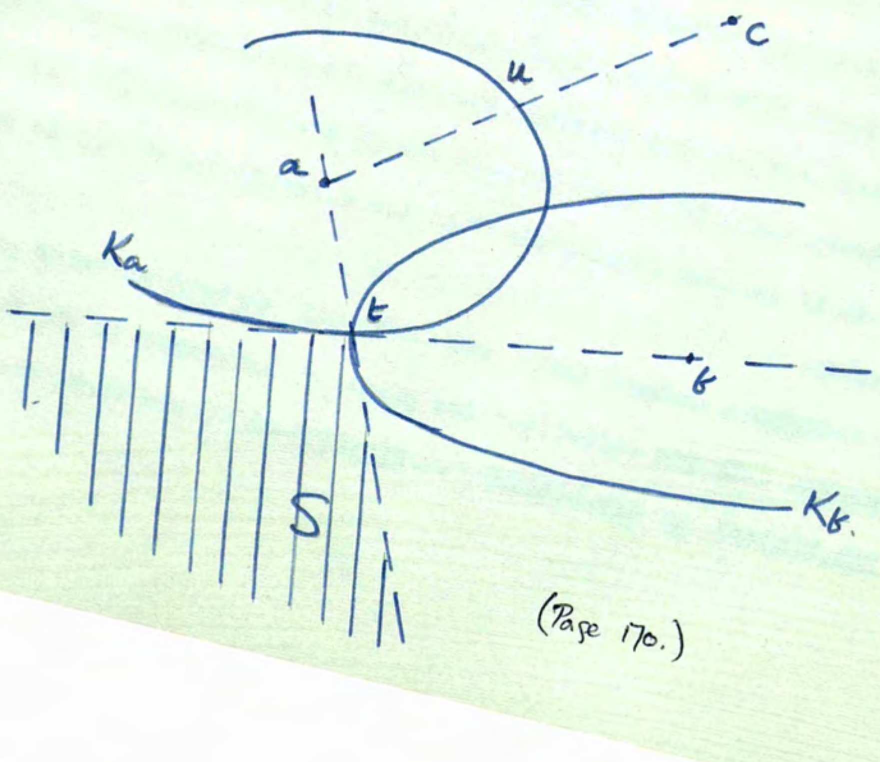
So suppose that we are given distinct crosspoints v , w such that the centres a , b , c of M belong to v and centres d , e , f to w . By property C(i) at least one of the centres d , e , f is distinct from a , b , c . If we can establish that the triangles abc and def do not overlap, then this is sufficient since each is an arbitrary triangle formed by vertices of the polygon P(v) or P(w) and so it follows immediately that the polygons themselves do not overlap.

Suppose instead that abc and def do have interior points in common. By its definition the point v is common to the three sets K(a,R(a)+r) , K(b,R(b)+r) , K(c,R(c)+r) , and since the sets are

(Diagram)



(Opposite page.)



(Page 170.)

strictly convex and a , b , c are their centres it follows, because of the packing properties of M , that apart from v (possibly) the triangle abc is contained in the union of their interiors; thus, abc and similarly def can contain no centre except at their vertices. Then abc and def must overlap in such a way that at least one edge of abc intersects an edge of def in a point which is interior to both edges. Suppose these edges are denoted by xy , zt respectively, (see diagram opposite) and let $C(x,y)$ denote the path consisting of the two line segments xv and vy , and $C(z,t)$ the path consisting of the segments zw and wt . Except for the point v the path $C(x,y)$ lies in the union of the interiors of $S(x)$ and $S(y)$ (by property S(i)), whereas except for w $C(z,t)$ lies in the union of the interiors of $S(z)$ and $S(t)$. Also $v \neq w$, so $C(x,y)$ and $C(z,t)$ have no point in common (by the definition of the sets $S(x)$). But, according to the property S(iii) , w cannot lie in the region $H(x,z,t^-) \cap H(x,t,z^-)$ (x , z , t being non-collinear), or in the region $H(y,z,t^-) \cap H(y,t,z^-)$ (y, z, t being non-collinear). Hence $C(z,t)$ cannot intersect either of these regions and so it must intersect the segment xy in an interior point (since the regions defined above are closed). Similarly $C(x,y)$ must intersect zt in an interior point. Hence $C(x,y)$ and $C(z,t)$ must have a point in common; but we have already shown that this is impossible. Thus the triangles abc and def cannot overlap.

With this result established, theorem 1 follows, as we have seen, at once, because the plane can now be triangulated as required.

Proof of theorem 2.

In accordance with the conditions of the theorem, suppose that we are given a triangle abc and bodies $K(a,R(a))$, $K(b,R(b))$, $K(c,R(c))$, no two of which overlap, but such that $K(a,R(a)+r)$, $K(b,R(b)+r)$, $K(c,R(c)+r)$ have a point in common. If we put

$$A = \frac{V\left[abc \cap \{K(a,R(a)) \cup K(b,R(b)) \cup K(c,R(c))\}\right]}{V(abc)} ,$$

the theorem will follow, as will be seen later, if we show that inf A is obtained when $R(a) = R(b) = R(c) = r$. Our method of proof of this subsidiary theorem is to begin by making a reduction of the problem and then to prove the result for a restricted class of configurations.

The reduction is as follows. Let N be the set of all non-negative numbers n for which

$$K(a,R(a)+r-n) \cap K(b,R(b)+r-n) \cap K(c,R(c)+r-n) \neq \phi .$$

N is non-empty, since by hypothesis $0 \in N$, and is bounded above by r , also by hypothesis, and so there is a least upper bound m which belongs to N since the homothetic translates of K are all closed. Since K is strictly convex, the set

$$K(a,R(a)+r-m) \cap K(b,R(b)+r-m) \cap K(c,R(c)+r-m)$$

consists of a single point t , say.

We shall need to know that t is a point of abc. For convenience of notation we write K_α for $K(\alpha, R(\alpha)+r-m)$, $\alpha = a, b, c$ respectively. Since the K_α are strictly convex, at least one pair, say K_a , K_b , overlap (with interior points in common). The set $K_a \cap K_b$ is convex (see ch. 1 , theorem 2, page 9) with two "vertices" at the points of intersection of the frontiers of K_a , K_b , and K_c is tangential to this set at the point t . (That there are just two points of intersection of the frontiers of K_a and K_b is proved in theorem 10 of ch. 1 - see page 26). If t is not at one of the vertices, then K_c is tangential to K_a , say, and in this case it follows that t is an interior point of the line segment ac (since ac must pass through the point of contact of K_a and K_c). Thus t belongs to abc . On the other hand, suppose that t is at a vertex of $K_a \cap K_b$. (See page 167, diagram.) Denote by S the region $H(a,t,b^-) \cap H(b,t,a^-)$. If c lies in S it follows at once that t belongs to abc . If c does not lie in S , then suppose it does not lie in $H(a,t,b^-)$. Let u denote the point of the frontier of K_a which lies in the segment ac . Then since the K_α have a point in common, u must belong to K_c (for if it did not, no point of K_a would belong to K_c) so that either the major arc ut or the minor arc ut must lie in K_c since t is in K_c . But the major arc cannot lie in K_c since then we should have the point a in K_c which is impossible. Thus the minor arc ut lies in K_c , and therefore K_c must overlap $K_a \cap K_b$, which is again impossible by the construction of the number m (by which the K_α are defined). We have therefore a contradiction, and so the point c must lie in $H(a,t,b^-)$. Similarly c must lie in $H(b,t,a^-)$ and therefore in S, and so it is proved that t is a point of the triangle abc.

Now take coordinates so that t is at the origin o. If we put $s = r - m \geqslant 0$ we have $s \neq 0$ since we have seen that at least two of the K_α must overlap. If o is an interior point of abc, it follows at once that the bodies $K(a,R(a))$, $K(b,R(b))$, $K(c,R(c))$ all touch $K(o,s)$ externally (and of course do not overlap, by hypothesis); for if $K(a,R(a))$, say, meets $K(o,s)$ in interior points, then there is a body $K(o',s')$ with $s' < s$ which touches all three bodies $K(a,R(a))$, $K(b,R(b))$, $K(c,R(c))$ externally, which in turn implies that the bodies $K(\alpha,R(\alpha)\ s')$, $\alpha = a, b, c$, have a common point of intersection o', and since $s' < s$ this is in contradiction with the definition of the number s. If o is not an interior point of abc, then it cannot be at a vertex since $s > 0$ implies that o is exterior to each of the bodies $K(\alpha,R(\alpha))$. If, then, o is an interior point of one of the sides of abc, say ab, we must have first that $K(o,s)$ touches each of $K(a,R(a))$, $K(b,R(b))$ externally, for if it meets either in interior points the minimal property of the number s is again contradicted. If it also touches $K(c,R(c))$ externally, then the reduction is complete.

If not, then $K(o,s)$ and $K(c,R(c))$ have interior points in common. In this case we reduce $R(c)$ until one of two things happens, either (i) $R(c)$ is reduced to r, the minimal radius of the packing, or (ii) $R(c)$ is reduced to $R'(c) \geqslant r$ and $K(c,R'(c))$ touches $K(o,s)$ externally. If (ii) holds, then again the reduction is complete. If (i) holds, then vary abc slightly by moving c perpendicularly away from ab to a point c′ through a distance small enough for $K(c',r)$ still to meet $K(o,s)$. Denote by X the set of points of abc which do not lie in

$$K(a,R(a)) \cup K(b,R(b)) \cup K(c,r)$$

and by X′ the set of points of abc which do not lie in

$$K(a,R(a)) \cup K(b,R(b)) \cup K(c',r) \quad .$$

Let $\delta > 1$ be the ratio of the length of oc′ to the length of oc . Then a line perpendicular to ab which meets X in a segment of length ℓ meets X′ in a segment of length ℓ' where $\ell' \geqslant \delta\ell$ always and $\ell' > \delta\ell$ for some segments. Thus we have

$$V(X') > \delta V(X) \quad .$$

Then since $\qquad V(abc') = \delta V(abc)$

we have $$\frac{V(abc') - V(X')}{V(abc')} < \frac{V(abc) - V(X)}{V(abc)} \quad ,$$

and so it is seen that the value of Λ (defined at the beginning of theorem 2 (page 169)) is reduced by altering abc slightly in this way. Since we want the value of $\inf \Lambda$, we can continue this process until

$K(c',r)$ just touches $K(o,s)$. This completes the reduction.

Now we change the class of configurations under discussion. Let I denote the class of triangles xyz such that the vertex x lies on the segment oa , x not being at o , the vertex y lies not at o on the segment ob , and the vertex z lies not at o on the segment oc . Define K_x , K_y , K_z now to be the bodies $K(x,R(x))$, $K(y,R(y))$, $K(z,R(z))$ respectively, where $R(x)$, $R(y)$, $R(z)$ are such that K_x , K_y , K_z , touch $K(o,s)$ externally - which we can do because of the reduction described above. Then theorem 2 will follow as shown below if we can prove the statement of theorem 4 .

<u>Theorem 4.</u> The infimum of the ratio

$$\frac{V[xyz \cap (K_x \cup K_y \cup K_z)]}{V(xyz)}$$

extended over all members of the subclass I_0 of I for which $R(x) \geq s$, ~~XXXXX~~ $R(y) \geq s$, $R(z) \geq s$ and no two of K_x , K_y , K_z overlap is attained by a member of I_0 for which $R(x) = R(y) = R(z)$.

<u>Proof.</u> Put $V(X(xyz)) = V[xyz \cap (K_x \cup K_y \cup K_z)]$. Among the members of I_0 there is a sequence of triangles $x_n y_n z_n$, $n = 1,2,\ldots,$ ~~XXXXXXX~~ such that

$$\frac{V(X(x_n y_n z_n))}{V(x_n y_n z_n)} \to \inf_{xyz \in I_0} \frac{V(X(xyz))}{V(xyz)} .$$

We have $F(x_n) \leq F(a)$ by the construction of the class I , and

$F(x_n) = R(x_n) + s \geqslant 2s > 0$, and similarly
$0 < 2s \leqslant F(y_n) \leqslant F(b)$, $0 < 2s \leqslant F(z_n) \leqslant F(c)$. Thus
by compactness the sequence $\{x_n y_n z_n\}$ contains a convergent subsequence, so that there is a triangle uvw in the class I_0 for which

$$\frac{V(X(uvw))}{V(uvw)} = \inf_{xyz \in I_0} \frac{V(X(xyz))}{V(xyz)} .$$

We may assume without loss of generality that $F(u) \geqslant F(v) \geqslant F(w)$. Theorem 4 will follow if we can prove $F(u) = 2s$, and to do this we assume that $F(u) > 2s$ and shall show that this leads to a contradiction.

Now put $X(uvw) = uvw \cap (K_u \cup K_v \cup K_w)$,

$Y(uvw) = uvw - X(uvw)$,

$Z(uvw) = Y(uvw) \cap (uv \cup uw)$.

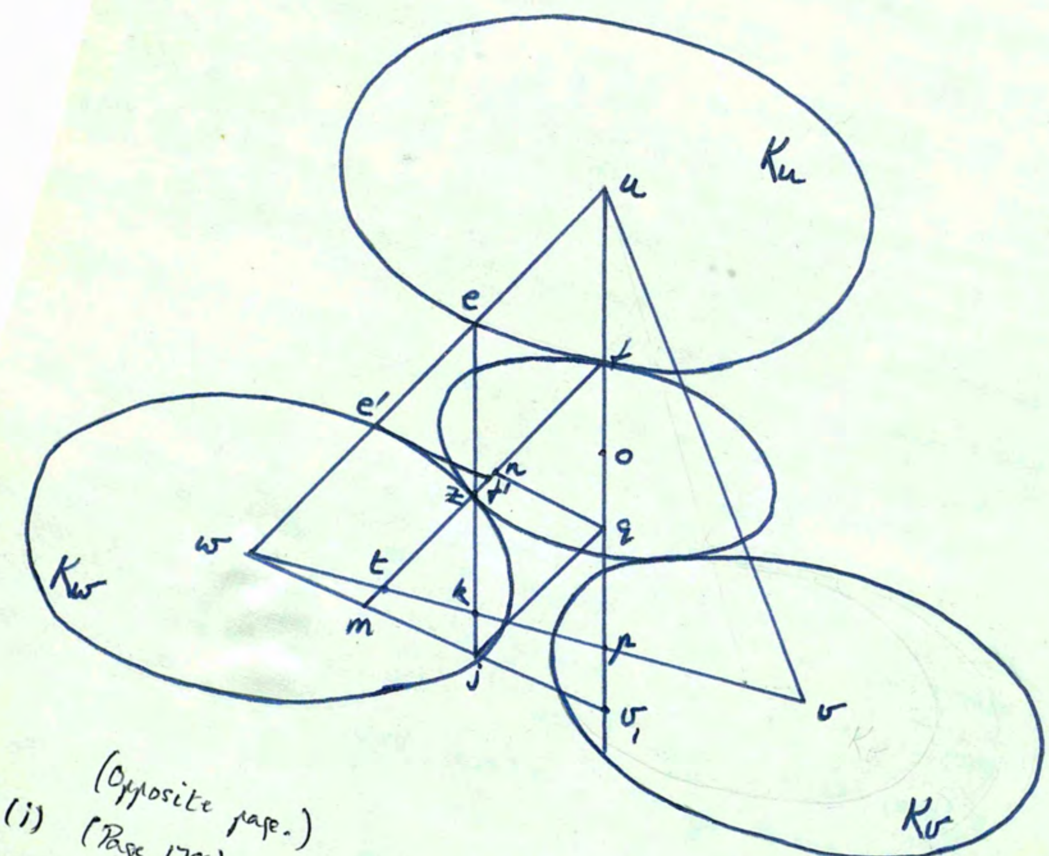
Also define $W(uvw)$ to be the part of uvw which lies on lines parallel to ou through points of $Z(uvw)$. We write as $Y_v(uvw)$, $W_v(uvw)$ those parts of Y and W which lie on the same side of ou as v , and as $Y_w(uvw)$, $W_w(uvw)$ those parts which lie on the same side as w .

The essence of the proof is to show that

$$V(W(uvw)) < V(Y(uvw)) ,$$

and clearly it is sufficient to show that $V(W_v) \leqslant V(Y_v)$ and $V(W_w) \leqslant V(Y_w)$ with strict inequality in at least one of these

(Diagram)



(Opposite page.)
(i) (Page 178.)

cases. The proofs are parallel and we shall in fact show that $V(W_w) \leq V(Y_w)$ with strict inequality provided that o is not on the line segment uw.

If $V(W_w(uvw)) = 0$, we have from the strict convexity of K that $V(Y_w(uvw)) \neq 0$, and so in this case the result is clear. So in what follows we assume that $V(W_w) \neq 0$. (See diagram opposite.) Now let K_u meet uw and ou in points e, f respectively, and let K_w meet uw in e′. Let f′ be the point such that ff′ is equal to ee′ and the direction of f to f′ is the same as that of e to e′. Denote by P the curvilinear parallelogram eff′e′ bounded by the segments ee′, ff′, the arc ef of the frontier K_u which lies in uvw, and the translation e′f′ of this arc.

Produce the line segment uo to a point v_1 such that o is the midpoint of uv_1, and let the line through uv_1 meet the segment vw in p. Then $F(u) = F(v_1)$, and $F(w) \leq F(u)$ by hypothesis, so that w is contained in the body $K(o,F(u))$. Also $F(v) \leq F(u)$, again by hypothesis, and so since p is on the line segment vw it follows that $F(p) \leq F(u)$. Hence $F(p) \leq F(v_1)$, which implies that v_1 is either outside uvw or on its frontier.

We may assume that o is not on the segment uw or $W_w(uvw)$ would be the empty set which is contrary to our hypothesis. $W_w(uvw)$ by its construction is bounded by two lines parallel to ou. Of these let the one through e meet wv_1 in j, and draw a line through j parallel to uw to meet uv_1 in the point q. We shall need to know something about the position of the point q.

The triangle uwv_1 is similar to qjv_1 with centre of similitude v. Hence

$$F(q - u)F(w - v_1) = F(j - w)F(v_1 - u) .$$

Also uwv_1 is similar to ewj with centre of similitude w, so that we have

$$F(j - w)F(u - w) = F(e - w)F(v_1 - w) .$$

Thus
$$F(q - u) = \frac{F(e - w).F(v_1 - u)}{F(u - w)} . \qquad (1)$$

We have
$$\begin{aligned} F(v_1 - u) &= 2F(u) && \text{by the construction of } v_1 \\ &= 2(s + F(u - f)) \\ &= 2(s + F(u - e)) . && (2) \end{aligned}$$

Thus

$$\begin{aligned} F(e - w)F(v_1 - u) &= F(e - w).2(s + F(u - e)) \\ &= F(e - w)F(u - e) + F(e - w)(2s + F(u - e)) . \qquad (3) \end{aligned}$$

$$F(e - w)F(u - e) = (F(u - w) - F(u - e))F(u - e)$$

and
$$\begin{aligned} F(u - w) &< F(u) + F(w) && \text{since } o \text{ is not on } uw \\ &\leqslant 2F(u) && \text{by hypothesis} \\ &= F(u - v_1) && \text{by the construction of } v_1 \end{aligned}$$

so that

$$F(e - w)F(u - e) < (F(u - v_1) - F(u - e))F(u - e)$$
$$= (2s + F(u - e))F(u - e) \qquad \text{by (2)}$$

and hence by (3)

$$F(e - w)F(v_1 - u) < (2s + F(u - e))(F(u - e) + F(e - w))$$
$$= (2s + F(u - e))F(u - w) \quad .$$

Thus from (1) we have

$$F(q - u) < 2s + F(u - e) \quad .$$

If then q is on the line segment ov_1 we have

$$2s + F(u - e) > F(q - u) = F(q) + F(u) = F(q) + s + F(u - e)$$

so that $F(q) < s$ and hence q is contained in $K(o,s)$. If q is not on the line segment ov_1 , then either q lies on the segment fo and so is in $K(o,s)$, or q lies on the segment uf . We deal separately with the cases

(i) q lies in $K(o,s)$,

(ii) q lies on the segment uf .

(i) See the figure (i) (page 175, diagram). Let the line through ff′ meet v_1w in m , and let the line through q parallel to v_1w meet fm in n . The triangle fqn is similar to uv_1w , and $F(f) \geq F(q)$ since q lies in $K(o,s)$ and f lies on its frontier. If we were to take a triangle fq′n′ , also similar to uv_1w but such that o is the midpoint of fq′ (as it is of uv_1), then we should have

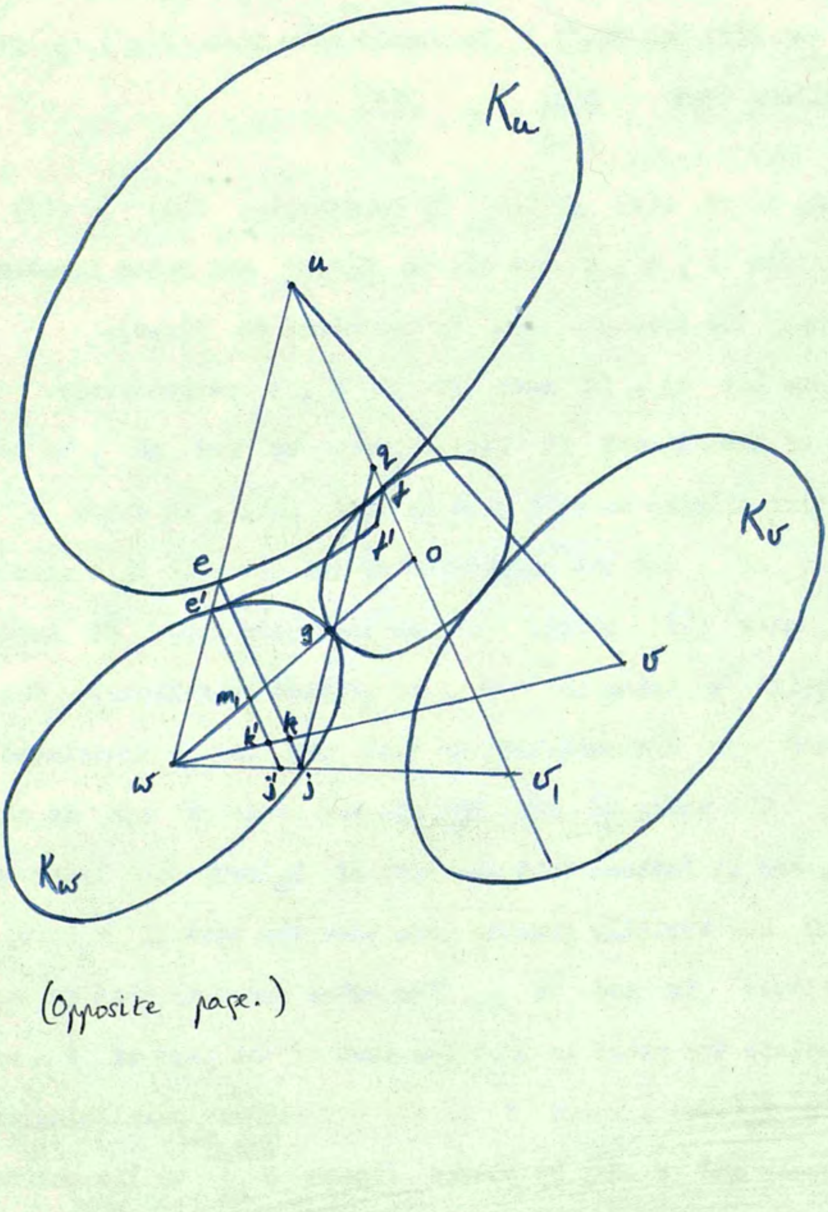
$$\frac{F(n')}{F(w)} = \frac{F(f)}{F(u)} .$$

But since such a triangle would be larger than fqn (since $F(q) \leqslant F(f) = F(q')$) we should have also $F(n') \geqslant F(n)$. Thus it follows that $\frac{F(n)}{F(w)} \leqslant \frac{F(f)}{F(u)}$

and so, since $F(w) \leqslant F(u)$ by hypothesis, $F(n) \leqslant F(f)$. Thus the points f , q , n are all in K(o,s) and hence immediately by convexity the triangle fqn is contained in K(o,s).

Now let ej , fm meet vw in k , t respectively. If the whole of the segment ft lies between up and ek , we have the same situation arising as will hold in case (ii) , in which q lies on the segment uf , and the completion of the proof in this case is therefore given under (ii) below. If, on the other hand, ft intersects ek in a point z lying in uvw , we proceed as follows. The triangles fqn and zjm are congruent so that fqn may be translated to cover zjm , The whole of fqn but not the whole of zjm is contained in uvw , and it follows that the part of $Y_w(uvw)$ not lying between fm and uw has strictly greater area than the part of $W_w(uvw)$ which is not between fm and uw . The other property that we can establish to complete the proof is that the area of the part of $P \cap uvw$ which lies in $Y_w(uvw)$, where P is the curvilinear parallelogram defined previously and P may be proved (lemma 8 (page 200)) to lie entirely within

(Diagram)



(ii) (Opposite page.)

$Y_w(uvw)$, is at least as great as the part of $W_w(uvw)$ which lies between fm and uw . This property will be used again in case (ii) , and the proof is given there. With these two results established, we have at once that

$$V(W_w(uvw)) < V(Y_w(uvw)) .$$

(ii) See the figure (ii) (opposite). It is proved in lemma 8 of this section (page 200) that the curvilinear parallelogram P is contained in $Y_w(uvw)$. It is possible to map P into a set which contains $W_w(uvw)$, by the following transformation which preserves its area. Consider P as the collection of line segments parallel to ee′ . Take any such line segment and map it onto the line segment collinear with it and with endpoints lying on ej and e′j′ , where j′ is the point of intersection with v_1w of a line through e′ parallel to ej . If the direction of translation of an individual segment is from u towards w , then any point of the image which lies in uvw must have a preimage which also lies in uvw . But we do not know at once that the direction of translation is from u towards w . If for any segment it were from w towards u , the arc e′f′ in the frontier of P must cross the ray e′j′ in a point k , distinct from e′ . The arc e′f′ is convex, so k is unique. To show that this is in fact impossible, it is sufficient to prove that the part of e′j′ which lies in the triangle ouw lies also in K_w , in which case no point of e′j′ can lie in the arc e′f′ , because according to lemma 8 no point of P (except e′) lies in K_w . We need only to consider the part of e′j′ inside ouw because again from lemma 8 we know that P is wholly contained within the triangle ouw and a defined sector of $K(o,s)$, and by convexity no point of the segment $e'j'$ outside ouw can lie in $K(o,s)$. Let m_1 be the point of intersection of $e'j'$ and ow. To prove that the segment $e'm_1$ lies in K_w, we note that by hypothesis

$$\begin{aligned} F(e' - u) \geqslant F(e - u) &= F(f - u) \\ &= F(u) - F(f) \\ &\geqslant F(w) - F(f) \\ &= F(w) - F(g) \end{aligned}$$

where g is the point of the frontier of K_w which lies on ow

$$= F(g - w) = F(e' - w)$$

since e', g are both on the frontier of K_w. Then since $e'm_1$ and uo are parallel it follows that

$$F(m_1) \geqslant F(m_1 - w) \quad .$$

If m_1 lies in K_w there is nothing to prove, so suppose that it does not. In this case m_1 must lie in $K(o,s)$ (for since the two bodies $K(o,s)$ and K_w touch and m_1 lies on the line of centres, it must lie in one or the other), in which case $F(m_1) \leqslant s$. Then $F(m_1 - w) \leqslant s \leqslant R(w)$ so that m_1 does lie in K_w, which is a contradiction. Thus the part of $e'j'$ which lies in ouw lies also in K_w. Hence the map we have described is an area-preserving map whose image contains $W_w(uvw)$ (because of the fact that in this case qj lies between ff' and ee') and such that any point of $W_w(uvw)$ has a preimage lying in $P \cap uvw$, and by lemma 8 $P \cap uvw$ is contained in $Y_w(uvw)$. Hence $V(W_w(uvw) \leqslant V(Y_w(uvw))$. Finally let ej , $e'j'$ meet vw in k , k' respectively. The trapezium $kk'j'j$ does not belong to $W_w(uvw)$ but is in the image of the above map, and at least a part, with positive area, of this trapezium must have a preimage lying in $P \cap uvw$, so that in fact the inequality is strict.

In all cases, then, $V(W_w(uvw)) < V(Y_w(uvw))$. We may now complete the proof of theorem 4 . Vary the triangle uvw by taking u_1 close to u on the segment ou so that $ou > u_1o > 2s$. Then u_1vw is a member of the class I_0 defined in the theorem. Put $uvw = T_0$, $u_1vw = T_1$, $W(uvw) = W_0$, $W(u_1vw) = W_1$, $Y(uvw) = Y_0$, ~~XXX~~ $U(u_1vw) = Y_1$, $W_w(uvw) = W_0^{(w)}$, $W_w(u_1vw) = W_1^{(w)}$, $Y_w(uvw) = Y_0^{(w)}$, and $Y_w(u_1vw) = Y_1^{(w)}$. Let the ratio of the distance up to the distance u_1p be δ (> 1) . Then

$$V(T_0) = \delta V(T_1) \quad . \tag{4}$$

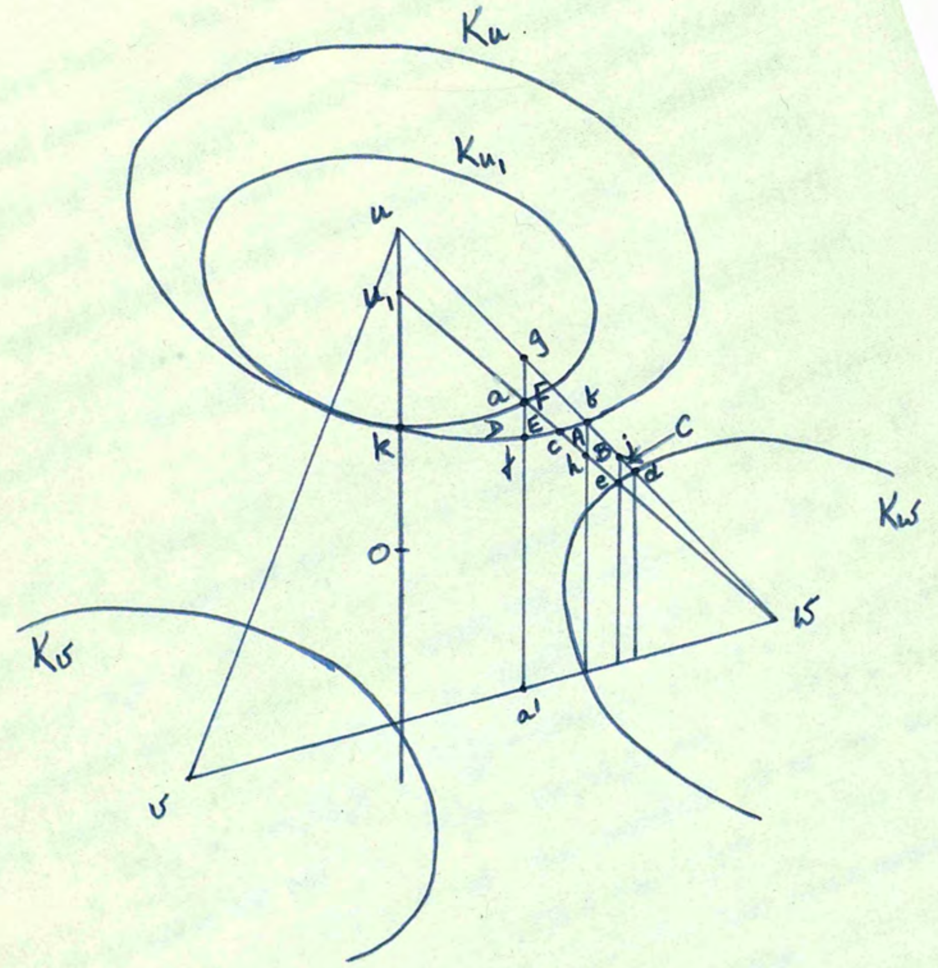
We have shown above that

$$V(W_0) < V(Y_0) \quad . \tag{5}$$

Thus $$\frac{V(Y_0)}{V(T_0)} - \frac{V(Y_1)}{V(T_1)} = \frac{V(Y_0) - \delta V(Y_1)}{V(T_0)} \text{ from (4).} \tag{6}$$

We now find an expression for $V(Y_0)$ in terms of $V(Y_1)$ and $V(W_1)$.

(Diagram)



(Opposite page.)

(The original paper contains an error here.)

Consider $V(Y_0^{(w)})$. Disregarding all previous notation for points in the configuration we are discussing, except for the points o , u , u_1 , v , w , define points as follows. (See diagram opposite.) Let a be the point of intersection of u_1w with the frontier of K_{u_1} . Let b , c respectively be the points of intersection of uw , u_1w with the frontier of K_u . Let d , e respectively be the points of intersection of uw , u_1w with the frontier of K_w . If a′ is the point of vw such that aa′ is parallel to uo , let f be the point of intersection, between a and a′ , of aa′ with the frontier of K_u , and let g be the point of intersection of a′a (produced) with uw . Let h be the point of u_1w such that bh is parallel to uo , and let j be the point of uw such that je is parallel to uo . Finally let k be the common point on uo of the frontiers of K_u and K_{u_1} .

Curves denoted by ka , kf , fc , cb , de always represent the arcs of the frontiers of the bodies K_u , K_{u_1} , K_w (as appropriate), and all others should be taken to denote line segments. Keeping this in mind, let A , C , D , E respectively denote the areas of the curvilinear triangles bch , dej , afk , acf , B the area of the trapezium bhej , and F the area of the curvilinear quadrilateral acbg .

Then we have

$$V(Y_0^{(w)}) = V(Y_1^{(w)}) + A + B + C - (D + E)$$

and $$\delta V(W_1^{(w)}) - V(W_1^{(w)}) = A + B + F$$

so that $$V(Y_0^{(w)}) = V(Y_1^{(w)}) + (\delta - 1)V(W_1^{(w)}) + C - (D + E + F) \quad .$$

It can easily be seen that because of their relative shapes and positions in the sector between wu and wu_1, the areas C and F satisfy the inequality $C < F$. [F is larger than a triangle which is an enlargement in ratio ga : je of C.] Thus

$$\beta \equiv D + E + F - C > 0 ,$$

and hence $V(Y_0) = V(Y_1) + (\delta - 1)V(W_1) - \gamma$ where $\gamma > 0$.

Putting this expression in (6), we have

$$\frac{V(Y_0)}{V(T_0)} - \frac{V(Y_1)}{V(T_1)} = \frac{(\delta - 1)(V(W_1) - V(Y_1)) - \gamma}{V(T_0)}$$

$$< \frac{(\delta - 1)(V(W_1) - V(Y_1))}{V(T_0)} \quad .$$

Now $\delta > 1$, and as $\delta \to 1+$, we have $V(W_1) \to V(W_0)$, $V(Y_1) \to V(Y_0)$, so that for δ sufficiently near 1, it follows from the inequality (5) that

$$\frac{V(Y_0)}{V(T_0)} - \frac{V(Y_1)}{V(T_1)} < 0 \quad .$$

Hence $$\frac{V(X_0)}{V(T_0)} - \frac{V(X_1)}{V(T_1)} > 0$$

(carrying through the same notation). This is a contradiction because T_0 was defined as a minimal configuration, and with this contradiction theorem 4 is proved.

To show how theorem 2 follows from this result, we note that an equivalent statement of the result is that the vertices of T_0 lie on the frontier of $K(o,2s)$. Also $V(X_0) = \frac{1}{2}V(K(o,s))$, since K_u, K_v, K_w all have the same radius s and so the parts of them that lie in T_0 have in sum an area equal to half of that of $K(o,s)$, since K is symmetric. Then we have

$$\frac{V(X_0)}{V(T_0)} \geqslant \frac{\frac{1}{2}V(K(o,s))}{t(K(o,2s))} = \frac{V(K(o,s))}{8t(K(o,s))} = \frac{V(K)}{8t(K)} ,$$

where as before $t(K)$ denotes the area of the largest triangle contained in K. Thus theorem 2 is proved.

<u>Proof of theorem A.</u>

Let M be a maximal K-packing of minimal radius r. It may be proved (lemma 9 - see page 204) that we can in fact replace M by a maximal packing M^* for which no body $K(a,R(a))$ has radius greater than some finite constant N. It follows that we can find a constant N', depending only on N, K and r, with the property that given any circle C with centre o and radius k, the union of those triangles of the configuration described in theorem 1 which lie completely in C contains a concentric circle of radius $(k - N')$.

Then we have

$$\frac{V(M^{*},K)}{\pi k^2} \geq \frac{\bigcup_{abc} V\left[abc \cap (K(a) \cup K(b) \cup K(c))\right]}{\pi k^2}$$

where the union is taken over all triangles abc contained completely in C

$$\geq \frac{V(K)}{8t(K)} \frac{\bigcup_{abc} V(abc)}{\pi k^2} \qquad \text{by theorem 2}$$

$$\geq \frac{V(K)}{8t(K)} \cdot \frac{\pi(k - N')^2}{\pi k^2}$$

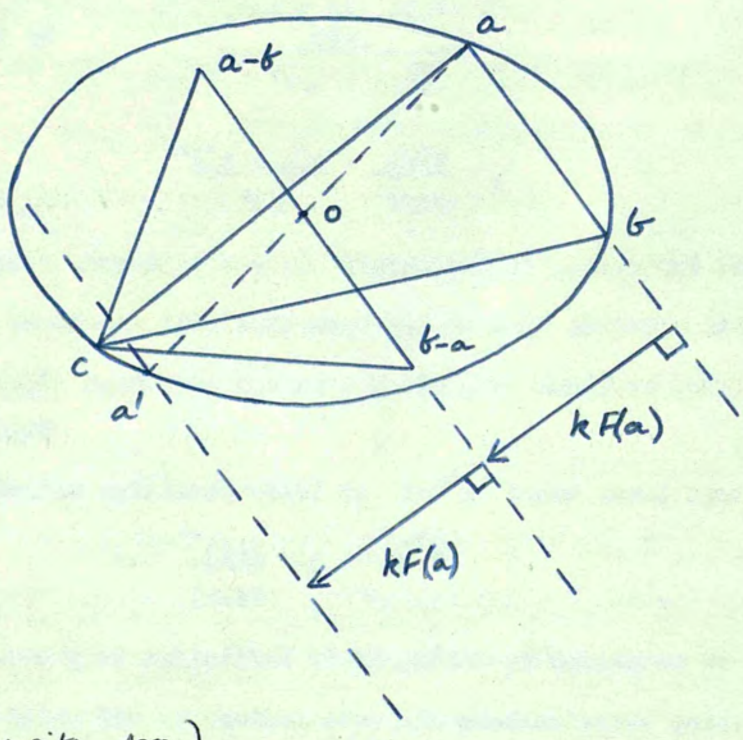
by the definition of the number $(k - N')$ above. Taking lower limits on both sides as $k \to \infty$ we have then that the lower density of M^{*}, and thus, by lemma 9, of M, is not less than $\frac{V(K)}{8t(K)}$. Hence the greatest lower bound $k^{*}(K)$ of lower densities satisfies

$$k^{*}(K) \geq \frac{V(K)}{8t(K)} .$$

To establish equality, it is sufficient to produce a maximal K-packing whose members all have radius r and which has lower density $V(K)/(8t(K))$. (It is known that such a lattice packing is a maximal packing.) Consider then a body $K(o,2r)$ and let abc be the triangle of largest area contained in it. We can establish certain properties of this triangle abc which we shall need to form the packing. It

(Diagram)

⊛ (For if this edge is taken as the base of the triangle, the new triangle has a base of the same length but further from the opposite vertex than before.)



(Opposite page.)

contains the point o , because if it did not we could obtain a larger triangle than abc by replacing an edge of abc nearest to o by its image in o . ~~containing a~~ (⊛ opposite page). Also, the lengths of the sides, $F(a - b)$, $F(b - c)$, $F(c - a)$, are all at least equal to $2r$. For, if say $F(a - b) < 2r$, then both of the points $a-b$, $b-a$ lie in the interior of $K(o,2r)$, and, if we consider the triangle in $K(o,2r)$ with vertices $a-b$, $b-a$ and the point c , we see that the length of the side opposite to c is $2(a - b)$ and since o is inside abc the height of the triangle from c to the side opposite is at least half of the height of abc , so that the area of this triangle is at least equal to that of abc . (See diagram opposite.) But two of its vertices are interior points of $K(o,2r)$, so that $K(o,2r)$ contains larger triangles - which by the definition of abc is impossible.

Thus $K(a,r)$, $K(b,r)$, $K(c,r)$ do not overlap, and each touches $K(o,r)$ externally. As $K(a,r)$, for example, cannot cross the broken segment boc , none of the bodies centred at a , b , c can intersect the side of abc opposite to the centre of the body. The sides of abc can thus be used to generate a lattice (by means of a series of reflexions in the sides) such that the bodies K of radius r centred at the points of this lattice give a maximal K-packing of bodies all with radius r , whose lower density is, from theorem 2 , equal to $V(K)/(8t(K))$. This completes the proof.

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<u>Lemma 1</u>. Given a maximal K-packing M of minimal radius r, and any fixed disc C in E_2, there are at most finitely many members of M which intersect C.

<u>Proof.</u> We assume that infinitely many members of M meet some C, and shall obtain a contradiction. Let $K(a,R(a)) \in M$ intersect C. If it meets C only in interior points of $K(a,R(a))$, then C is completely contained in the interior of $K(a,R(a))$. Then no other member of M can meet C since this would be inconsistent with the packing property (a) (given in main section (page 156)). So suppose that the frontier of $K(a,R(a))$ meets C in a point x, so that $F(x - a) = R(a)$. Our method of proof is to construct bodies $K(p,r)$ for points p to be defined, each of which is contained in one of the $K(a,R(a))$ which meet C. We note the following:

$$F[(1 - rR^{-1}(a))(x - a)] = [1 - rR^{-1}(a)]\, R(a) \qquad (\text{since } 1 - rR^{-1}(a) \geq 0\ )$$

$$= R(a) - r \quad ,$$

and so, if we put $p \equiv (1 - rR^{-1}(a))(x - a) + a$, any point y of $K(p,r)$, which automatically satisfies the inequation

$$F(y - p) \leq r$$

satisfies also $F(y - p) \geq F(y - a) - F[(1 - rR^{-1}(a))(x - a)]$

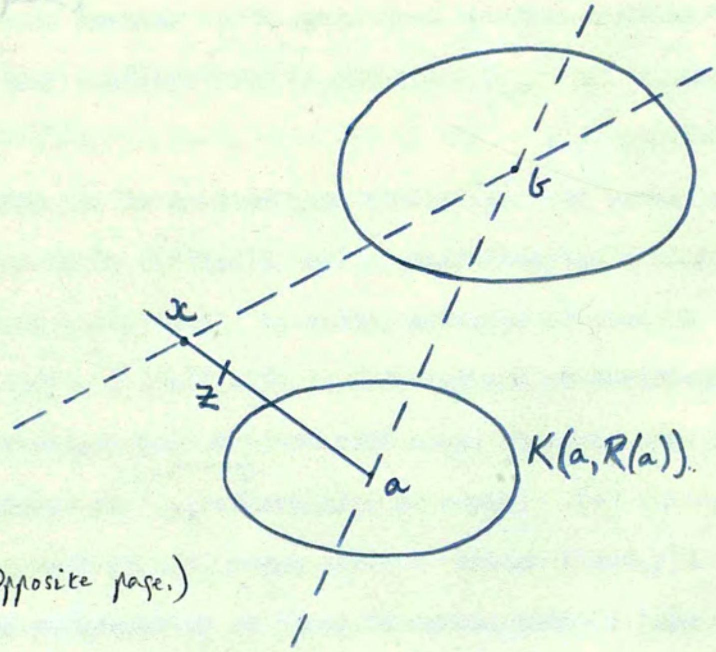
by the definition of p

which implies $F(y - a) \leq R(a)$.

Thus $K(p,r) \subset K(a,R(a))$, and also

192[redacted]

(Diagram)



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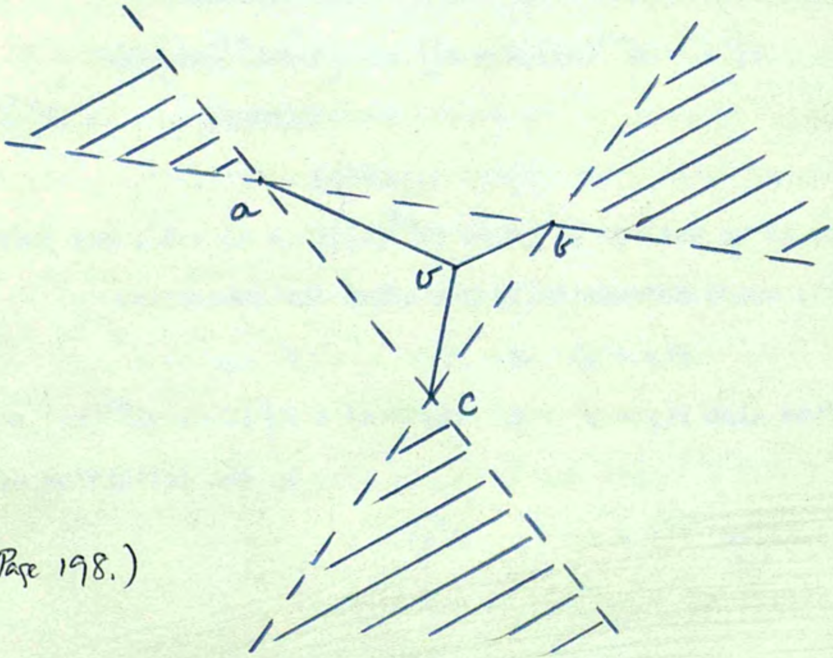


(Page 198.)

$$F(x-p) = F\left[rR^{-1}(a)(x-a)\right]$$
$$= rR^{-1}(a)F(x-a)$$
$$= r \quad \text{since } x \text{ is on the frontier of } K(a,R(a))$$

so that x is a point of $K(p,r)$. If we now make the same construction for each $K(a,R(a))$ which meets C we obtain infinitely many bodies $K(p,r)$ all having radius r and all intersecting C in such a way that no two overlap since $K(p,r) \subset K(a,R(a))$ and the $K(a,R(a))$ do not have interior points in common. So if the diameter of $K(p,r)$ is D we have constructed infinitely many $K(p,r)$ packed in the disc whose centre is the centre of C and whose radius is D plus the radius of C, a construction which is clearly impossible. Thus the lemma is proved.

Lemma 2. If $x \in S(a)$, then every point except possibly x of the line segment ax is an interior point of $S(a)$.

Proof. Let $z \neq x$ be a point of ax, and consider any body $K(b,R(b)) \in M$ with $a \neq b$. We show first that z belongs to $S(a)$, and prove that in fact $d_a(z) < d_b(z)$, and then prove that it is an interior point of $S(a)$. There are two cases.

If a, b, z are non-collinear, (see diagram opposite) then we have since $z \neq x$,

$$F(z-x) + F(z-b) > F(x-b) .$$

Thus we have

$$F(x - a) - [F(z - x) + F(z - b)] < F(x - a) - F(x - b)$$

which implies, since $F(x - a) - F(z - x) = F(z - a)$, that

$$F(z - a) - F(z - b) < F(x - a) - F(x - b) .$$

Thus, by definition, $d_a(z) - d_b(z) < d_a(x) - d_b(x)$, and we know that $d_a(x) \leqslant d_b(x)$ since $x \in S(a)$, and so $d_a(z) < d_b(z)$.

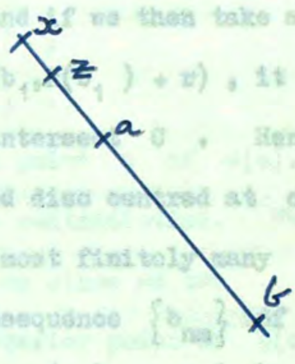
If a , b , z are collinear, then we have

$$\begin{aligned} d_a(z) &= d_a(x) - F(x - z) \\ &\leqslant d_b(x) - F(x - z) \quad \text{since } x \in S(a) \\ &\leqslant d_b(z) . \end{aligned}$$

Now for equality to hold, we must have in particular

$$d_b(x) - F(x - z) = d_b(z) ,$$

that is, $F(x - b) - F(x - z) = F(z - b)$



which implies that z must lie between x and b . But z is by hypothesis a point of ax , and so is not a point of the segment ab . If, then, z is outside the segment ab and on the side of a opposite to b , we have

$$\begin{aligned} F(z - b) &= F(z - a) + F(a - b) \\ &\geqslant F(z - a) + R(a) + R(b) \end{aligned}$$

since the members of M do not meet in interior points, and so

$$d_b(z) = F(z-b) - R(b) \geqslant F(z-a) + R(a)$$
$$= d_a(z) + 2R(a)$$
$$> d_a(z)$$

and so equality cannot hold. A similar argument holds if z is on the side of b opposite to a, and hence in all cases we have $d_a(z) < d_b(z)$.

To prove the remaining part of the lemma, we assume that z is not an interior point of $S(a)$ and shall obtain a contradiction. Then there is a sequence of points $\{z_n\}$, having z as the limit point, and a sequence of bodies $K(b_n,R(b_n))$ in M, all different from $K(a,R(a))$ and such that $z_n \in S(b_n)$ for each n. Since M is maximal, we have $S(b_n) \subset K(b_n,R(b_n) + r)$. We may order the b_n so that b_1 is furthest from the origin o, and if we then take a fixed disc C centred at o which contains $K(b_1,R(b_1) + r)$, it follows that all the bodies $K(b_n,R(b_n) + r)$ intersect C. Hence all the bodies $K(b_n,R(b_n))$ intersect some fixed disc centred at o, and it follows from lemma 1 that there are at most finitely many distinct b_n, which implies that there is a subsequence $\{b_{nm}\}$ of centres which are all equal to some c. Now $d_{b_n}(z_n) \leqslant d_a(z_n)$ for each n since $z_n \in S(b_n)$, and so $d_c(z_n) \leqslant d_a(z_n)$, and since F is a continuous function it follows that $d_c(z) \leqslant d_a(z)$. This is a contradiction, since we have already proved that $d_a(z) < d_b(z)$ for any b. Thus the lemma is proved completely.

<u>Lemma 3.</u> If a , b are distinct centres of members of M , then

$$S(a) \subset K(a, F(b - a)) .$$

<u>Proof.</u> Since a , b are distinct, $K(a,R(a))$ and $K(b,R(b))$ do not overlap, and so it follows that $F(b - a) \geqslant r + R(a)$ since $R(b) \geqslant r$ for all $K(b,R(b)) \in M$. Now if $x \in S(a)$ then $r \geqslant d_a(x)$ (by the maximal property of M), so that

$$r \geqslant F(x - a) - R(a)$$

which implies that $F(x - a) \leqslant r + R(a) \leqslant F(b - a)$.
Thus $x \in K(a,F(b - a))$ and the lemma follows at once.

<u>Lemma 4.</u> If a , b , c are non-collinear centres of members of M , then $S(a) \cap S(b)$ cannot have any point in common with the closed set

$$H(a,c,b^-) \cap H(b,c,a^-) ,$$

nor, similarly,

$$S(b) \cap S(c) \quad \text{in} \quad H(b,a,c^-) \cap H(c,a,b^-) ,$$

$$S(c) \cap S(a) \quad \text{in} \quad H(c,b,a^-) \cap H(a,b,c^-) .$$

<u>Proof.</u> We assume the contrary, that there is a point x of $S(a) \cap S(b)$ in $H(a,c,b^-) \cap H(b,c,a^-)$, and shall obtain a contradiction. The point c satisfies $d_c(c) = -R(c) < 0$ and so since no two members of M overlap we have that c is an interior point of $S(c)$. Thus $x \neq c$ since $x \in S(a) \cap S(b)$ and sets of the form $S(a)$ can by definition only have frontier points in common. So c is a point of the triangle abx and is not a vertex of the triangle.

By lemma 3 we have $S(a) \subset K(a, F(b - a))$ and $S(b) \subset K(b, F(a - b))$, and so since $x \in S(a) \cap S(b)$ it follows that any point except possibly x of abx is in the interior (since K is strictly convex) of either $K(a, F(x - a))$ or $K(b, F(x - b))$. Hence either $F(c - a) < F(x - a)$ or $F(c - b) < F(x - b)$.

In the first case,

$$F(c - a) < F(x - a) = d_a(x) + R(a) \leqslant r + R(a) \quad \text{since } x \in S(a) .$$

But this means that the distance between the centres of $K(a, R(a))$ and $K(c, R(c))$ is less than the sum of their radii, which implies that the bodies have interior points in common. This is impossible since M is a packing, and so we must have $F(c - b) < F(x - b)$. But this leads in the same way to a contradiction in making $K(c, R(c))$ and $K(b, R(b))$ overlap. Thus our first assumption was wrong, and the lemma is proved.

<u>Lemma 5</u>. Any three centres belong to at most one crosspoint.

<u>Proof</u>. Suppose that v and w are distinct crosspoints common to $S(a)$, $S(b)$, $S(c)$. By lemma 3, if a, b, c are collinear, with b, say, between a and c, then $S(a) \subset K(a, F(b - a))$ and $S(c) \subset K(c, F(b - c))$ and by strict convexity the only point common to $K(a, F(b - a))$ and $K(c, F(b - c))$ is the point b. But $b \notin S(a)$ or $S(c)$ and so $S(a)$ and $S(c)$ have no point in common. This is a contradiction, and so a, b, c cannot be collinear.

Then by lemma 4 neither v nor w can lie in the closed regions

$$H(a,c,b^-) \cap H(b,c,a^-) \quad , \quad H(b,a,c^-) \cap H(c,a,b^-) \quad ,$$

$$H(c,b,a^-) \cap H(a,b,c^-) \quad .$$

We note the following restrictions on the position of w. (See page 192, diagram.)

(i) Except for v , the line segments va , vb , vc lie in the interiors of S(a) , S(b) , S(c) respectively (lemma 2), so w cannot lie on any of these line segments since by definition it is a point of the frontiers of the S(a) .

(ii) Except for w , every point of the segment wc lies in the interior of S(c) , and so, since va and vb except for v lie in the interiors of S(a) and S(b) respectively, and since no two sets of the form S(a) have interior points in common (by definition), we cannot cross va or vb .

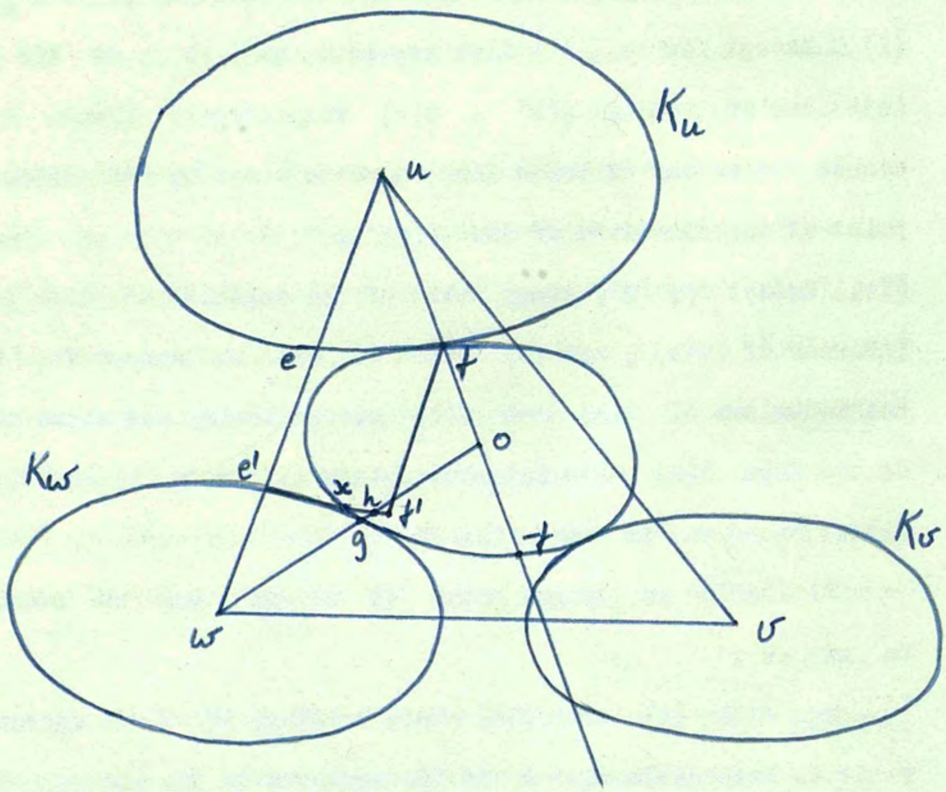
Similarly wa cannot cross vb or vc , and wb cannot cross va or vc .

But with (i) and (ii) both holding, if w is distinct from v it is impossible for w to lie anywhere in the plane. Thus we must have w = v and the lemma is proved.

<u>Lemma 6.</u> The crosspoints of M form a discrete set.

<u>Proof.</u> If not, then there is a limit point x say of crosspoints. From lemma 5 it follows that there must be an infinite number of the sets S(a) which intersect the disc with centre x and radius 1 , say.

199 [redacted]
(Diagram)



(Opposite page.)

Then there must also be an infinite number of members of M which intersect the disc with centre x and radius $(1+r)$, which contradicts lemma 1 (page 191). Thus the lemma is proved.

<u>Lemma 7.</u> At most a finite number of centres of M belong to one crosspoint.

<u>Proof.</u> Again, assume the contrary. Then there is a crosspoint v such that infinitely many centres of M belong to v . Then there are infinitely many members of M which intersect the disc with centre v and radius r , and this again contradicts lemma 1 . The lemma follows.

<u>Lemma 8.</u> The curvilinear parallelogram P defined in the main part of this section (page 176) does not overlap any one of the bodies K_u , K_v , K_w .

<u>Proof.</u> We use the same notation as before, so that K_u meets uw and ou in e , f respectively, K_w meets uw in e′ , and f′ is the point such that ff′ is equal and parallel (in the same direction) to ee′ . (See diagram opposite.) P is the curvilinear parallelogram eff′e′ bounded by the segments ee′ , ff′ and the arcs ef , e′f′ where e′f′ is a translation of the arc ef of the frontier of K_u .

Two of the cases of the lemma are easily dealt with. By its construction P cannot overlap K_u . Also since K is centrally symmetric and strictly convex the arc $e'f'$ meets K_w only in the point e' , so that P does not overlap K_w . Thus we have only to show that P does not overlap K_v .

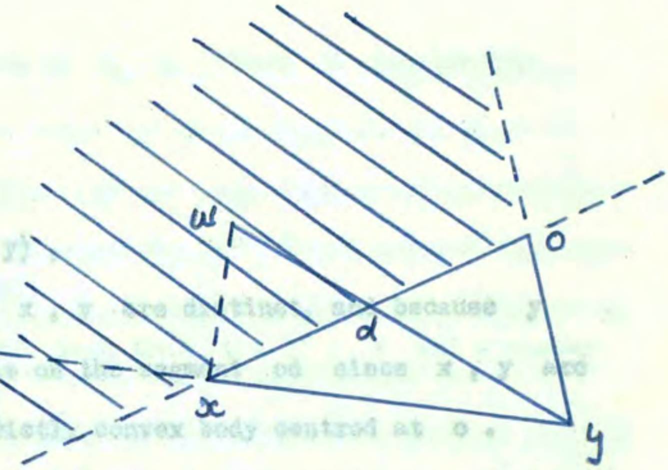
Let g be the point common to the frontier of K_w and the line segment ow . The frontier of $K(o,s)$ meets ou in the points f , $-f$ (using ordinary vector notation), and meets ow in the point g (since ow is the line of centres of the two bodies and thus g is their unique point of contact). Taking $-fg$ to denote the arc of the frontier of $K(o,s)$ which has endpoints $-f$, g and which lies in the sector $H(o,u,g^+) \cap H(o,w,f^-)$, let N denote the region bounded by the arc $-fg$ and the segments $-fu$, uw , wg . We shall show that N does not overlap K_v and that P is contained in N , thus proving the lemma.

Firstly, then, we assert that N does not overlap K_v . The sector bounded by $-fo$, og and the arc $-fg$, which together with the triangle ouw comprises N , lies entirely in $K(o,s)$ and so cannot overlap K_v . But also the segments ou , ow lie, except for the points f , g , in the interior of $K_u \cup K_v \cup K(o,s)$, and so K_v cannot cross these segments. Hence K_v cannot overlap the triangle ouw , and thus K_v does not overlap N .

Secondly, we assert that P is contained in N , and to prove this we assume the contrary and shall obtain a contradiction. K_u cannot cross ow , and so the arc ef and the segment ee' both lie in N . If f' lies in N then it lies either in the triangle ouw , in which case ff' is in N , or in K(o,s) and again ff' is in N by the convexity of K . Now we are assuming that P does not lie in N , so that if f' lies in N the arc $e'f'$ is not contained in N . Similarly if f' does not lie in N , this implies that $e'f'$ is not contained in N . Thus in either case it follows that $e'f'$ must cross the segment ow at least once. Let h be the point of ow such that the part of the arc $e'f'$ with endpoints e' and h lies in the triangle ouw . Since K is strictly convex, the whole except for the point g of the segment og , and hence the point h , must lie in the interior of K(o,s) . Thus since e' is exterior to K(o,s) , the arc $e'f'$ must cross the frontier of K(o,s) in a point x say, lying in the triangle ouw . Now $e'f'$ does not lie entirely in N , so it crosses the frontier of K(o,s) also in a point $y \neq x$ (not shown in the diagram) which lies outside the triangle ouw . The points x and y satisfy $F(x) = F(y)$ since they both lie on the frontier of K(o,s) . But the translation which takes e to e' maps u onto a point u' on the segment uw for which the arc $e'f'$ is part of the frontier of $K(u',R(u))$, the translation of K_u , so that we have also

$$F(x - u') = F(y - u') . \qquad (1)$$

But u' is not o since



we have eliminated this case (that is, in which $V(W_w(uvw)) = 0$), and u' lies in the region $H(o,x,y^-) \cap H(o,y,x^+)$. Given this, u' lies also either in $H(x,y,o^+)$, or in $H(x,y,o^-)$.

If $u' \in H(x,y,o^+)$, the segments $u'y$, ox meet in the point d say, so that $F(u'-y) = F(u'-d) + F(d-y)$.

By convexity, $F(u'-x) \leq F(u'-d) + F(d-x)$

and $F(y) = F(o-y) < F(o-d) + F(d-y)$

(with strict inequality since u' is not o).

Adding, we have

$$F(u'-x) + F(y) < F(u'-y) + F(x)$$

so that, since $F(x) = F(y)$, as we noted above,

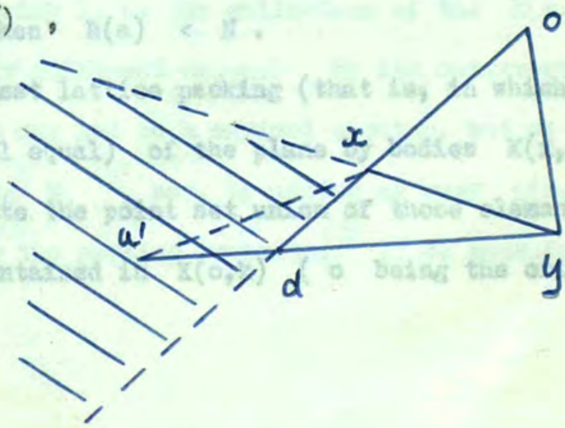
$$F(u'-x) < F(u'-y) ,$$

which is a contradiction with (1) .

If, then, $u' \in H(x,y,o^-)$, the segment $u'y$ meets ox produced in the point d say, such that

$F(u'-y)$

$= F(u'-d) + F(d-y)$.



By convexity,

$F(u' - x)$

$$\leq F(u' - d) + F(d - x)$$

and $F(d) < F(y) + F(d - y)$,

with strict inequality since x , y are distinct, and because y cannot by its construction lie on the segment od since x , y are both frontier points of a strictly convex body centred at o .

Adding, we have $F(u' - x) + F(d) < F(u' - y) + F(d - x) + F(y)$,

that is, $F(u' - x) + F(x) < F(u' - y) + F(y)$,

which again gives a contradiction with (1) .

Thus our original assumption, that the arc $e'f'$ crosses the segment ou at least once, is false, and hence P is contained in N . This proves the lemma.

Lemma 9. Given a maximal K-packing M of minimal radius r , there exists a positive constant N such that there is a second maximal K-packing M^* , of minimal radius r , with the properties

(i) the lower density of M^* does not exceed the lower density of M ,

(ii) if $K(a,R(a)) \in M^*$ then $R(a) < N$.

Proof. Let L denote a densest lattice packing (that is, in which the radii of the bodies are all equal) of the plane by bodies $K(x,r)$. Given $k > 0$, let L_k denote the point set union of those elements of L which are completely contained in $K(o,k)$ (o being the origin), and let $V(L_k)$ denote the area of L_k . Since K is strictly convex we know that

$$\limsup_{k\to\infty} \frac{V(L_k)}{V(K(o,k))} = t < 1 .$$

Then we may choose a number t' such that $t < t' < 1$ and a number N_1 such that

$$\frac{V(L_k)}{V(K(o,k))} \leq t' \qquad \text{for } k \geq N_1 . \qquad (1)$$

In what follows, we shall refer to a number N which we do not define at present but which we shall restrict through the proof so that it satisfies the requirements of the lemma. If M does not contain a body with radius at least N , there is nothing to prove, so we assume that there is a member $K(a,R)$ of M with $R \geq N$. If we remove this member from M we can restore the packing property by putting in its place all the bodies $K(x,r)$ which make up L_R , each translated by a . Thus we may form a packing P given by

$$P = (M - \{K(a,R)\}) \cup (L_R + a) .$$

(Using L_R here we are considering it as the collection of the $K(x,r)$ in $K(o,R)$ rather than as their point set union.) By its construction P has minimal radius r . It may not be a maximal packing, but we can complete it to a maximal packing M_1 , say, by adding as many $K(x,r)$ as we can while still retaining the packing property. If we consider any one of these $K(x,r)$ in $M_1 \div P$, we know that it cannot overlap any member of P, so that the point x is confined to the band $K(a,R + r) \div K(a,R - 3r)$. For, if x lies outside $K(a,R + r)$, then $K(x,r)$ lies completely outside $K(a,R)$, which implies that M is not maximal; and if x lies inside $K(a,R - 3r)$, then $K(x,r)$ is completely contained in $K(a,R - 2r)$, which implies that L is not maximal, again a contradiction since it is known that a lattice packing is a maximal packing. Thus the sets $K(x,r)$ of $M_1 \div P$ can cover at most the band $K(a,R + 2r) \div K(a,R - 4r)$.

Now outside $K(a,R + 2r)$ the packings M and M_1 agree. Inside $K(a,R + 2r)$, the packing M covers a set whose area is at least $V(K(a,R)) = R^2 V(K)$ (by the definition of K). M_1, on the other hand, covers a set inside $K(a,R + 2r)$ whose area is at most

$$\begin{aligned} A &= V\left[K(a,R + 2r) \div K(a,R - 4r)\right] + V(L_R) \\ &\leq V(K(a,R + 2r)) - V(K(a,R - 4r)) + t'V(K(o,R)) \\ &\qquad\qquad \text{by } (1) \text{ if } R \geq N_1 , \\ &= \left[(R + 2r)^2 - (R - 4r)^2 + t'R^2\right] V(K) . \end{aligned}$$

Now
$$\lim_{R\to\infty} \frac{(R + 2r)^2 - (R - 4r)^2}{R^2} = 0 ,$$

and so we may choose a number N_2 such that

$$\frac{(R + 2r)^2 - (R - 4r)^2}{R^2} < 1 - t' \quad \text{for } R \geq N_2 .$$

Then $A \leqslant R^2V(K)$ for $R \geqslant \max(N_1, N_2)$.

At this point define the number N with which we began as $N = \max(N_1, N_2)$. Then since we have already made the condition that $R \geqslant N$, it follows from the areas covered by the packings inside $K(a, R + 2r)$ that the lower density of M_1 does not exceed that of M.

This construction, then, has given us a means of obtaining a new packing by removing a body $K(a, R(a))$ for which $R(a) \geqslant N$. If M contains only finitely many such bodies, then clearly we can repeat the construction for each of them in turn until we obtain a maximal packing M^* which satisfies the conditions of the lemma. If, finally, M contains infinitely many such bodies, we choose $K(a,R) \in M$, $R \geqslant N$, to be a body for which the distance between a and o is smallest, and construct M_1 as above. Repeating this process we obtain a sequence of packings $M_1, M_2, \ldots$, all of which are maximal with minimal radius r, and have the property, following from lemma 1A (page 191), that given any disc D centred at o there corresponds a suffix n_0 such that for $m, n > n_0$ the members of M_m which intersect D are precisely the members of M_n which intersect D. Thus there is a limiting collection M^* which satisfies the conditions of the lemma. This completes the proof.

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