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Gauge-invariant quasi-free states on the algebra of the anyon commutation relations

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Abstract

Let $X = \mathbb{R}^2$ and let $q \in \mathbb{C}$, $|q| = 1$. For $x = (x^1, x^2)$ and $y = (y^1, y^2)$ from X^2, we define a function $Q(x, y)$ to be equal to q if $x^1 < y^1$, to $\bar{q}$ if $x^1 > y^1$, and to $\Re q$ if $x^1 = y^1$. Let ∂_x^+, ∂_x^- ($x \in X$) be operator-valued distributions such that ∂_x^+ is the adjoint of ∂_x^-. We say that ∂_x^+, ∂_x^- satisfy the anyon commutation relations (ACR) if $\partial_x^+ \partial_y^+ = Q(y, x) \partial_y^+ \partial_x^+$ for $x \neq y$ and $\partial_x^- \partial_y^+ = \delta(x - y) + Q(x, y) \partial_y^+ \partial_x^-$ for $(x, y) \in X^2$. In particular, for $q = 1$, the ACR become the canonical commutation relations and for $q = -1$, the ACR become the canonical anticommutation relations. We define the ACR algebra as the algebra generated by operator-valued integrals of ∂_x^+, ∂_x^-. We construct a class of gauge-invariant quasi-free states on the ACR algebra. Each state from this class is completely determined by a positive self-adjoint operator T on the real space $L^2(X, dx)$ which commutes with any operator of multiplication by a bounded function $\psi(x^1)$. In the case $\Re q < 0$, the operator T additionally satisfies $0 \leq T \leq -1/\Re q$. Further, for $T = \kappa^2 \mathbf{1}$ ($\kappa > 0$), we discuss the corresponding particle density $\rho(x) := \partial_x^+ \partial_x^-$. For $\Re q \in (0, 1]$, using a renormalization, we rigorously define a vacuum state on the commutative algebra generated by operator-valued integrals of $\rho(x)$. This state is given by a negative binomial point process. A scaling limit of these states as $\kappa \to \infty$ gives the gamma random measure, depending on parameter $\Re q$.

Keywords: Anyon commutation relations; gauge-invariant quasi-free state; particle density; negative binomial point process; gamma random measure.

2010 MSC: 47L10, 47L60, 47L90, 60G55, 60G57, 81R10

1 Preliminaries and introduction

The main aim of this paper is to construct a class of gauge-invariant quasi-free states on the algebra of the anyon commutation relations. Let us first recall the definition of the anyon commutation relations and their representation in the anyon Fock space.

1.1 Fock space representation of the anyon commutation relations

Let $X := \mathbb{R}^d$, let $\mathscr{B}(X)$ denote the Borel σ-algebra on X, and let m denote the Lebesgue measure on $(X, \mathscr{B}(X))$. We denote by

$$\mathscr{H} := L^2(X, m), \quad \mathscr{H}_{\mathbb{C}} := L^2(X \to \mathbb{C}, m)$$

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the L^2-space of real-valued, respectively complex-valued functions on X. (The scalar product in $\mathscr{H}_{\mathbb{C}}$ is supposed to be linear in the first dot and antilinear in the second dot.)

Consider a function $Q : X^2 \to \mathbb{C}$ satisfying $Q(x,y) = \overline{Q(y,x)}$ and $|Q(x,y)| = 1$ for all $(x,y) \in X^2$. In 1995, Liguori and Mintchev [34, 35] introduced the notion of a *generalized statistics corresponding to the function Q*. Heuristically, this is a family of creation operators ∂_x^+ and annihilation operators ∂_x^- at points $x \in X$ such that ∂_x^+ is the adjoint of ∂_x^- and these operators satisfy the following commutation relations:

$$\partial_x^+ \partial_y^+ = Q(y,x)\partial_y^+ \partial_x^+, \tag{1}$$

$$\partial_x^- \partial_y^- = Q(y,x)\partial_y^- \partial_x^-, \tag{2}$$

$$\partial_x^- \partial_y^+ = \delta(x-y) + Q(x,y)\partial_y^+ \partial_x^-. \tag{3}$$

(Formula (2) is, in fact, a consequence of (1).) A rigorous meaning of the operators ∂_x^+ and ∂_x^- and the commutation relations (1)–(3) is given by smearing these relations with functions from the space $\mathscr{H}_{\mathbb{C}}$. More precisely, for any $h \in \mathscr{H}_{\mathbb{C}}$, one defines linear operators

$$a^+(h) = \int_X m(dx)\, h(x)\, \partial_x^+, \quad a^-(h) = \int_X m(dx)\, \overline{h(x)}\, \partial_x^- \tag{4}$$

on a dense linear subspace Θ of a complex Hilbert space $\mathscr{G}$ such that the adjoint of $a^+(h)$ restricted to Θ is $a^-(h)$, and these operators satisfy the commutation relations:

$$a^+(g)a^+(h) = \int_{X^2} m^{\otimes 2}(dx\,dy)\, g(x)h(y)Q(y,x)\partial_y^+ \partial_x^+, \tag{5}$$

$$a^-(g)a^-(h) = \int_{X^2} m^{\otimes 2}(dx\,dy)\, \overline{g(x)h(y)}\, Q(y,x)\partial_y^- \partial_x^-, \tag{6}$$

$$a^-(g)a^+(h) = \int_X \overline{g(x)}\, h(x)\, m(dx) + \int_{X^2} m^{\otimes 2}(dx\,dy)\, \overline{g(x)}\, h(y)Q(x,y)\partial_y^+ \partial_x^- \tag{7}$$

for any $g, h \in \mathscr{H}_{\mathbb{C}}$. Of course, the linear operators on the right hand side of formulas (5)–(7) should be given a rigorous meaning. In the case $Q \equiv 1$, formulas (5)–(7) become the *canonical commutation relations* (CCR), describing bosons, while in the case $Q \equiv -1$, formulas (5)–(7) become the *canonical anticommutation relations* (CAR), describing fermions. In the general case, we will call (5)–(7) the *Q-commutation relations* (Q-CR).

Liguori and Mintchev [34, 35] derived a representation of the Q-CR in the Fock space of Q-symmetric functions. By using also [10], we will now briefly recall this construction.

A function $f^{(n)} : X^n \to \mathbb{C}$ is called *Q-symmetric* if for any $i \in \{1, \ldots, n-1\}$ and $(x_1, \ldots, x_n) \in X^n$.

$$f^{(n)}(x_1, \ldots, x_n) = Q(x_i, x_{i+1})f^{(n)}(x_1, \ldots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \ldots, x_n). \tag{8}$$

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For each $n \in \mathbb{N}$, we have $\mathscr{H}_{\mathbb{C}}^{\otimes n} = L^2(X^n \to \mathbb{C}, m^{\otimes n})$. We denote by $\mathscr{H}_{\mathbb{C}}^{\circledast n}$ the subspace of $\mathscr{H}_{\mathbb{C}}^{\otimes n}$ which consists of all ($m^{\otimes n}$-versions of) Q-symmetric functions from $\mathscr{H}_{\mathbb{C}}^{\otimes n}$. We call $\mathscr{H}_{\mathbb{C}}^{\circledast n}$ the n-th Q-symmetric tensor power of $\mathscr{H}_{\mathbb{C}}$.

Consider the group S_n of all permutations of $1, \ldots, n$. For each $\pi \in S_n$, we define a function $Q_\pi : X^n \to \mathbb{C}$ by

$$Q_\pi(x_1, \ldots, x_n) := \prod_{\substack{1 \le i < j \le n \\ \pi(i) > \pi(j)}} Q(x_i, x_j). \tag{9}$$

Note that, in the case $Q \equiv 1$, we get $Q_\pi \equiv 1$, while in the case $Q \equiv -1$, we get $Q_\pi \equiv (-1)^{|\pi|} = \operatorname{sgn} \pi$. Here $|\pi|$ is the number of inversions of π, i.e., the number of $i < j$ such that $\pi(i) > \pi(j)$.

For a function $f^{(n)} : X^n \to \mathbb{C}$, we define its Q-symmetrization by

$$(P_n f^{(n)})(x_1, \ldots, x_n) := \frac{1}{n!} \sum_{\pi \in S_n} Q_\pi(x_1, \ldots, x_n) f^{(n)}(x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}). \tag{10}$$

The operator P_n determines the orthogonal projection of $\mathscr{H}_{\mathbb{C}}^{\otimes n}$ onto $\mathscr{H}_{\mathbb{C}}^{\circledast n}$. Furthermore, for any $k, n \in \mathbb{N}$, $k < n$, we have

$$P_n(P_k \otimes P_{n-k}) = P_n. \tag{11}$$

Here P_1 denotes the identity operator in $\mathscr{H}_{\mathbb{C}}$. For any $f^{(n)} \in \mathscr{H}_{\mathbb{C}}^{\circledast n}$ and $g^{(m)} \in \mathscr{H}_{\mathbb{C}}^{\circledast m}$, we define the Q-symmetric tensor product of $f^{(n)}$ and $g^{(m)}$ by

$$f^{(n)} \circledast g^{(m)} := P_{n+m}(f^{(n)} \otimes g^{(m)}).$$

By (11), the tensor product $\circledast$ is associative.

For a Hilbert space H and a constant $c > 0$, we denote by Hc the Hilbert space which coincides with H as a set and the tensor product in Hc is equal to the tensor product in H times c. We define a Q-Fock space over $\mathscr{H}$ by

$$\mathscr{F}^Q(\mathscr{H}) := \bigoplus_{n=0}^{\infty} \mathscr{H}_{\mathbb{C}}^{\circledast n} n! \, .$$

Here $\mathscr{H}_{\mathbb{C}}^{\circledast 0} := \mathbb{C}$. The vector $\Omega := (1, 0, 0, \ldots) \in \mathscr{F}^Q(\mathscr{H})$ is called the *vacuum*. We denote by $\mathscr{F}_{\text{fin}}^Q(\mathscr{H})$ the subset of $\mathscr{F}^Q(\mathscr{H})$ consisting of all finite sequences

$$F = (f^{(0)}, f^{(1)}, \ldots, f^{(n)}, 0, 0, \ldots)$$

in which $f^{(i)} \in \mathscr{H}_{\mathbb{C}}^{\circledast i}$ for $i = 0, 1, \ldots, n$, $n \in \mathbb{N}$. This space can be endowed with the topology of the topological direct sum of the $\mathscr{H}_{\mathbb{C}}^{\circledast n}$ spaces. Thus, convergence in $\mathcal{F}_{\text{fin}}^Q(\mathcal{H})$

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means uniform finiteness of non-zero components and coordinate-wise convergence in $\mathscr{H}_{\mathbb{C}}^{\circledast n}$.

For each $h \in \mathscr{H}_{\mathbb{C}}$, we define a *creation operator* $a^+(h)$ and an *annihilation operator* $a^-(h)$ as linear operators acting on $\mathscr{F}_{\mathrm{fin}}^Q(\mathscr{H})$ that satisfy

$$a^+(h)f^{(n)} := h \circledast f^{(n)}, \quad f^{(n)} \in \mathscr{H}_{\mathbb{C}}^{\circledast n}, \quad a^-(h) := (a^+(h))^* \restriction_{\mathscr{F}_{\mathrm{fin}}^Q(\mathscr{H})}. \tag{12}$$

These operators act continuously on $\mathscr{F}_{\mathrm{fin}}^Q(\mathscr{H})$. Furthermore, for $h \in \mathscr{H}_{\mathbb{C}}$ and $f^{(n)} \in \mathscr{H}_{\mathbb{C}}^{\circledast n}$, we have:

$$(a^-(h)f^{(n)})(x_1, \ldots, x_{n-1}) = n \int_X \overline{h(y)}\, f^{(n)}(y, x_1, \ldots, x_{n-1})\, m(dy). \tag{13}$$

Thus, if we introduce informal operators ∂_x^+ and ∂_x^- by formulas (4), we get, for $f^{(n)} \in \mathscr{H}_{\mathbb{C}}^{\circledast n}$,

$$\partial_x^+ f^{(n)} = \delta_x \circledast f^{(n)}, \qquad \partial_x^- f^{(n)} = n f^{(n)}(x, \cdot).$$

where δ_x is the delta function at x. Now, one can easily give a rigorous meaning to the operators on the right hand side of formulas (5)–(7) and show that the Q-CR hold.

We note that, in the obtained representation of the Q-CR, we only used the values of the function Q $m^{\otimes 2}$-almost everywhere. Hence, for this representation, we could assume from the very beginning that there exists a set $\Delta \in \mathscr{B}(X^2)$ which is symmetric (i.e., if $(x, y) \in \Delta$, then $(y, x) \in \Delta$) and satisfies $m^{\otimes 2}(\Delta) = 0$, and the function Q is only defined on the set $\widetilde{X}^2 := X^2 \setminus \Delta$. Since the measure m is non-atomic, we may also assume that $D \subset \Delta$, where $D := \{(x, x) \mid x \in X\}$ is the diagonal in X^2.

In physics, intermediate statistics have been discussed since Leinass and Myrheim [32] conjectured their existence in 1977. The first mathematically rigorous prediction of intermediate statistics was done by Goldin, Menikoff and Sharp [20, 21] in 1980, 1981. The name *anyon* was given to such statistics by Wilczek [50, 51]. Anyon statistics were used, in particular, to describe the quantum Hall effect, see e.g. the review paper [46].

Liguori, Mintchev [34, 35] and Goldin, Sharp [22] showed that anyon statistics can be described by the Q-CR in which $X = \mathbb{R}^2$, the set Δ is chosen as

$$\Delta := \left\{ (x, y) \in X^2 \mid x^1 = y^1 \right\} \tag{14}$$

and

$$Q(x, y) = \begin{cases} q, & \text{if } x^1 < y^1, \\ \bar{q}, & \text{if } x^1 > y^1. \end{cases} \tag{15}$$

Here, $q \in \mathbb{C}$ with $|q| = 1$, and for $x \in X$ we denote by x^i the ith coordinate of x. With such a choice of the function Q, formulas (5)–(7) are called the *anyon commutation relations* (ACR). We note that Goldin, Sharp [22] realized the ACR by using operators acting on the space of functions of finite configurations in X (or, equivalently, in the symmetric Fock space).

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Goldin and Majid [19] showed that, in the case where q is a kth root of 1 and $q \neq 1$, the corresponding statistics satisfies the natural anyonic exclusion principle, which generalizes Pauli's exclusion principle for fermions:

$$a^+(f)^k = 0 \quad \text{for each } f \in \mathscr{H}_{\mathbb{C}}. \tag{16}$$

For further discussions of anyons in mathematical physics literature (including the discrete setting), see e.g. [13, 15, 17, 19, 33, 36–41] and the references therein. We also refer to the paper [8] which deals with a Fock representation of the commutations relations identified by a sequence of self-adjoint operators in a Hilbert space which have norm ≤ 1 and which satisfy the braid relations.

1.2 Gauge-invariant quasi-free states on the CCR and CAR algebras

In the theory of the CCR and CAR algebras, quasi-free states, in particular, gauge-invariant quasi-free states, play a fundamental role. We refer the reader to e.g. Sections 5.2.1–5.2.3 and Notes and Remarks to these sections in [11], and [16, Chapter 17], see also the pioneering book [6, Chapter II] and paper [7]. We note that gauge-invariant quasi-free states describe, in particular, the infinite free Bose gas at finite temperature [3] (see also [14] and Section 5.2.5 in [11]) and the infinite free Fermi gas at both finite and zero temperatures [4] (see also [14] and Section 5.2.4 in [11]). Free analogs of quasi-free states have been discussed in [45], see also [24].

Let us recall that the CCR algebra (or the CAR algebra), $\mathbf{A}$, is a complex algebra generated by linear operators $a^+(h)$, $a^-(h)$ ($h \in \mathscr{H}_{\mathbb{C}}$) satisfying the CCR (the CAR, respectively). Because of the commutation relations, each element of $\mathbf{A}$ can be represented as a finite sum of a constant and operators

$$a^{\sharp_1}(h_1) \cdots a^{\sharp_n}(h_k), \quad h_1, \ldots, h_k \in \mathscr{H}_{\mathbb{C}}, \ \sharp_1, \ldots, \sharp_k \in \{+, -\},$$

which are in the *Wick order*. The latter means that there is no $i \in \{1, \ldots, k-1\}$ such that $\sharp_i = -$ and $\sharp_{i+1} = +$, i.e., there is no creation operator acting before an annihilation operator.

Let τ be a state on the algebra $\mathbf{A}$. One defines *n-point functions* by

$$\mathbf{S}^{(k,n)}(g_k, \ldots, g_1, h_1, \ldots, h_n) := \tau\big(a^+(g_k) \cdots a^+(g_1)a^-(h_1) \cdots a^-(h_n)\big), \tag{17}$$

where $g_1, \ldots, g_k, h_1, \ldots, h_n \in \mathscr{H}_{\mathbb{C}}$ and $k, n \in \mathbb{N}$. One says that the state τ is *gauge-invariant* if it is invariant under the group of Bogoliubov transformations

$$a^+(h) \mapsto a^+(e^{i\theta}h) = e^{i\theta}a^+(h), \quad a^-(h) \mapsto a^-(e^{i\theta}h) = e^{-i\theta}a^-(h), \quad \theta \in [0, 2\pi).$$

By (17), τ is gauge-invariant if and only if $\mathbf{S}^{(k,n)} \equiv 0$ for $k \neq n$. Thus, a gauge-invariant state is completely determined by $\mathbf{S}^{(n,n)}$ ($n \in \mathbb{N}$).

A state τ is called a *gauge-invariant quasi-free state* if $\mathbf{S}^{(k,n)} \equiv 0$ for $k \neq n$ and the n-point functions $\mathbf{S}^{(n,n)}$ are completely determined by $\mathbf{S}^{(1,1)}$. More precisely, in the case of the CCR algebra, we have

$$\mathbf{S}^{(n,n)}(g_n, \ldots, g_1, h_1, \ldots, h_n) = \mathrm{per}\left[\mathbf{S}^{(1,1)}(g_i, h_j)\right] = \sum_{\pi \in S_n} \prod_{i=1}^{n} \mathbf{S}^{(1,1)}(g_i, h_{\pi(i)}), \qquad (18)$$

and in the case of the CAR algebra, we have

$$\mathbf{S}^{(n,n)}(g_n, \ldots, g_1, h_1, \ldots, h_n) = \det\left[\mathbf{S}^{(1,1)}(g_i, h_j)\right] = \sum_{\pi \in S_n} \mathrm{sgn}\,\pi \prod_{i=1}^{n} \mathbf{S}^{(1,1)}(g_i, h_{\pi(i)}). \quad (19)$$

A gauge-invariant quasi-free state on the CCR algebra is completely identified by a bounded linear operator T in $\mathscr{H}_{\mathbb{C}}$, with $T \geq 0$, which satisfies

$$\mathbf{S}^{(1,1)}(g, h) = (Tg, h)_{\mathscr{H}_{\mathbb{C}}}. \qquad (20)$$

Respectively, a gauge-invariant quasi-free state on the CAR algebra is completely identified by a bounded linear operator T in $\mathscr{H}_{\mathbb{C}}$, with $0 \leq T \leq 1$, which satisfies (20).

The corresponding representation of the CCR/CAR algebra can be given on the symmetric/antisymmetric Fock space over $\mathscr{H} \oplus \mathscr{H}$ by using the bounded linear operators $\sqrt{T}$ and $\sqrt{1 + T}$ in the CCR case and $\sqrt{T}$ and $\sqrt{1 - T}$ in the CAR case, see e.g. Examples 5.2.18 and 5.2.20 in [11].

1.3 A brief description of the results

While our main interest in this paper will be the ACR, we will actually deal with a slightly more general form of the Q-CR: we will assume that $X = \mathbb{R}^d$ with $d \geq 2$ and the function $Q : \widetilde{X}^2 \to \mathbb{C}$ satisfies $Q(x, y) = Q(x^1, y^1)$ (with an obvious abuse of notation). Here $\widetilde{X}^2 := X^2 \setminus \Delta$ with Δ being given by (14).

We saw in the Fock space representation that defining a function Q on $\widetilde{X}^2$ was enough. However, we will see below that, in the general case, this is not enough for relation (3) and we need to specify the values of $Q(x, x)$ for $x \in X$. In the case of the bose and fermi statistics, we take, of course, $Q(x, x) \equiv 1$ and $Q(x, x) \equiv -1$, respectively.

So from now on we will assume that, for some constant $\eta \in \mathbb{R}$, we have $Q(x, y) = \eta$ for all $(x, y) \in \Delta$, in particular, $Q(x, x) = \eta$ for all $x \in X$. We will define a Q-CR algebra so that the value η will matter for relation (3), but η will be of no importance to relations (1), (2) as they will still depend on the values of the function Q $m^{\otimes 2}$-almost everywhere.

We see that, in the anyon case (with $q \neq \pm 1$), the function Q cannot be extended to a continuous function on X^2, so there is a freedom in choosing the value of η. A natural choice for η seems to be $\eta = \Re(q) = (q + \bar{q})/2$.

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The form of the Q-CR means that it is not enough to consider a complex algebra generated by the operators (4). Instead, in Section 2, we consider a complex algebra $\mathbf{A}$ generated by operator-valued integrals

$$\int_{X^k} m^{\otimes k}(dx_1 \cdots dx_k)\, \varphi^{(k)}(x_1, \ldots, x_k)\, \partial_{x_1}^{\sharp_1} \cdots \partial_{x_k}^{\sharp_k}, \tag{21}$$

where the class of functions $\varphi^{(k)} : X^k \to \mathbb{C}$ appearing in the integral (21) will be specified. We show that the anyon exclusion principle (see (16)) holds in the general ACR algebra for q being a root of 1, $q \neq 1$.

If τ is a state on $\mathbf{A}$, then due to (3), τ is completely determined by the n-point functions

$$\mathbf{S}^{(k,n)}(\varphi^{(k+n)}) := \tau \bigg(\int_{X^{k+n}} m^{\otimes (k+n)}(dx_1 \cdots dx_{k+n})\, \varphi^{(k+n)}(x_1, \ldots, x_{k+n})$$

$$\times\, \partial_{x_1}^+ \cdots \partial_{x_k}^+ \partial_{x_{k+1}}^- \cdots \partial_{x_{k+n}}^- \bigg). \tag{22}$$

We say that the state τ is *gauge-invariant* if it is invariant under the group of Bogoliubov transformations $\partial_x^+ \mapsto e^{i\theta}\partial_x^+$, $\partial_x^- \mapsto e^{-i\theta}\partial_x^-$, with $\theta \in [0, 2\pi)$, or equivalently if $\mathbf{S}^{(k,n)} \equiv 0$ for $k \neq n$.

So it is intuitively clear what it should mean that τ is a *gauge-invariant quasi-free state*: we should have $\mathbf{S}^{(k,n)} \equiv 0$ if $k \neq n$ and the n-point functions $\mathbf{S}^{(n,n)}$ should be completely determined by $\mathbf{S}^{(1,1)}$. However, to write down a proper generalization of formulas (18), (19) is not straightforward: instead of the sign of a permutation π we should use the function Q_π (see (9)), and the functions $\varphi^{(k+n)}$ appearing in (22) do not necessarily factorize to separate their variables. We solve this problem in Section 2 by properly introducing certain measures on $\mathbb{R}^{2n}$ corresponding to the n-point functions. As a result, the definition of a gauge-invariant quasi-free state for the Q-CR generalizes the available definitions in the CCR and CAR cases.

In Section 3 we construct operator-valued integrals (21) in the Q-symmetric Fock space. The presentation in this section is given at a rather general level. In particular, in this section we assume that X is a locally compact Polish space, while m is a non-atomic Radon measure on X.

In Section 4, we construct a representation of the Q-CR algebra $\mathbf{A}$ and a class of gauge-invariant quasi-free states τ on it. This construction is done in a JQ-symmetric Fock space over $\mathscr{H} \oplus \mathscr{H}$. Here $JQ : Z^2 \to \mathbb{C}$ is a function on the space $Z := X_1 \sqcup X_2$, the disjoint union of two copies of X. Explicitly, the function JQ is defined through the function Q by formula (54) below.

The operator T, being defined analogously to formula (20), satisfies in our setting the following assumptions:

- T is a self-adjoint bounded linear operator in the real space $\mathscr{H}$ and is extended by linearity to $\mathscr{H}_{\mathbb{C}}$;

- T commutes with any operator of multiplication by a bounded function $\psi(x^1)$;

- in the case $\eta \geq 0$, we have $T \geq 0$, and in the case $\eta < 0$, we have $0 \leq T \leq -1/\eta$.

For
$$\varphi^{(2n)}(x_1, \ldots, x_{2n}) = g_n(x_1) \cdots g_1(x_n) h_1(x_{n+1}) \cdots h_n(x_{2n})$$
with $g_1, \ldots, g_n, h_1, \ldots, h_n \in \mathscr{H}_{\mathbb{C}}$, we obtain

$$\mathbf{S}^{(n,n)}(\varphi^{(2n)}) = \sum_{\pi \in S_n} \int_{X^n} \left(\prod_{i=1}^{n} g_i(x_i)(Th_{\pi(i)})(x_i) \right) Q_{\pi}(x_1, \ldots, x_n) \, m^{\otimes n}(dx_1 \cdots dx_n),$$

compare with (18)–(20).

Finally, in Section 5, we discuss the particle density associated with a gauge-invariant quasi-free state on the ACR algebra with $\eta = \Re q$. The particle density is informally defined by $\rho(x) := \partial_x^+ \partial_x^-$ for $x \in X$. It follows from the ACR that these operators commute, cf. [19, 22]. Hence, the state on the algebra generated by the commutative operators $\rho(f) := \int_X m(dx) \, f(x)\rho(x)$ (f running through a space of test functions on X) should be given by a probability measure μ.

In the case of the CCR and CAR algebras, it was shown in [42, 44] that, for T being a locally trace class operator, μ is a permanental (determinantal, respectively) point process on X. Note, however, that our assumptions on the operator T exclude locally trace class operators.

In this paper, we treat the case where T is a constant operator, $T = \kappa^2 \mathbf{1}$ with $\kappa > 0$. Under this assumption, it is not possible to give a rigorous meaning to $\rho(f)$ as a self-adjoint linear operator in the JQ-symmetric Fock space over $\mathscr{H} \oplus \mathscr{H}$. So a renormalization is needed. Similarly to the construction of the renormalized square of white noise algebra [1, 2], as renormalization we will use Ivanov's formula [25] which suggests that the square of the delta function, δ^2, can be interpreted as $c\delta$, where c is any positive constant. For our calculations, we choose $c = 1$, so that Ivanov's formula becomes $\delta^2 = \delta$. We note that a different choice of the constant c would lead to similar results in which the measure m is replaced by cm.

So, using this renormalization and the ACR, we rigorously define a functional τ on the algebra generated by commutative operators $\rho(f)$. However, due to renormalization, it is not *a priori* clear whether τ is a state, i.e., whether it is positive definite. We prove that τ is indeed a state if and only if $\eta \in [0, 1]$.

Furthermore, for $\eta = 0$, the state τ is given by the Poisson point process on X with intensity measure $\kappa^2 m$, while for $\eta > 0$, τ is given by a negative binomial point process on X, which depends on two parameters, η and κ. The latter process takes values in the space $\ddot{\Gamma}(X)$ of multiple configurations in X, i.e., Radon measures on X which take values in $\{0, 1, 2, \ldots, \infty\}$. Note also that the negative binomial point process is a measure-valued Lévy process on X whose Lévy measure is finite. Finally,

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we prove that, for a fixed $\eta > 0$, a (scaling) limit of the states depending on κ exists as $\kappa \to \infty$, and the limiting state is given by the gamma random measure, depending on parameter η. This random measure is known to have many distinguished properties, see e.g. [18, 29–31, 47–49]. We stress that the results of Section 5 are new even in the CCR case.

2 The Q-CR algebra and gauge-invariant quasi-free states on it

2.1 Preliminary definitions

In this section, we assume that $X = \mathbb{R}^d$ with $d \geq 2$, m is the Lebesgue measure on it, $Q(x, y) = Q(x^1, y^1)$ for $(x, y) \in \widetilde{X}^2$, and $Q(x, y) = \eta$ for $(x, y) \in \Delta$. To define the Q-CR algebra, we need an appropriate class of functions $\varphi^{(k)}$ appearing in (21).

Let $k \in \mathbb{N}$. We denote by $\Pi(k)$ the set of all partitions of the set $\{1, \ldots, k\}$, i.e., all collections of mutually disjoint sets whose union is $\{1, \ldots, k\}$. For each partition $\theta \in \Pi(k)$, we denote by $X_\theta^{(k)}$ the subset of X^k which consists of all $(x_1, \ldots, x_k) \in X^k$ such that, for all $1 \leq i < j \leq k$, $x_i = x_j$ if and only if i and j belong to the same element of the partition θ. Note that the sets $X_\theta^{(k)}$ with $\theta \in \Pi(k)$ form a partition of X^k. We denote $X^{(k)} := X_\theta^{(k)}$ for the minimal partition $\theta = \big\{\{1\}, \{2\}, \ldots, \{k\}\big\}$.

Let $\theta = \{\theta_1, \ldots, \theta_l\} \in \Pi(k)$ and assume that

$$\min \theta_1 < \min \theta_2 < \cdots < \min \theta_l.$$

We have $m^{\otimes l}(X^l \setminus X^{(l)}) = 0$, so we can consider $m^{\otimes l}$ as a measure on $X^{(l)}$. Consider the mapping $I_\theta : X^{(l)} \to X_\theta^{(k)}$ given by $I_\theta(x_1, \ldots, x_l) = (y_1, \ldots, y_k)$, where for each $i \in \theta_1$ we have $y_i = x_1$, for each $i \in \theta_2$ we have $y_i = x_2$, etc. We denote by $m_\theta^{(k)}$ the pushforward of the measure $m^{\otimes l}$ under I_θ. We extend the measure $m_\theta^{(k)}$ by zero to the whole space X^k. Note that $m^{(k)} = m_\theta^{\otimes k}$ for the minimal partition $\theta = \big\{\{1\}, \{2\}, \ldots, \{k\}\big\}$.

Let us fix any $\sharp_1, \ldots, \sharp_k \in \{+, -\}$. We denote by $\Pi(k, \sharp_1, \ldots, \sharp_k)$ the subset of $\Pi(k)$ which consists of all partitions $\theta = \{\theta_1, \ldots, \theta_l\}$ such that each set θ_i has at most two elements, and if $\theta_i = \{a, b\}$ has two elements then $\sharp_a \neq \sharp_b$. We define a measure on X^k by

$$m_{\sharp_1, \ldots, \sharp_k}^{(k)} := \sum_{\theta \in \Pi(k, \sharp_1, \ldots, \sharp_k)} m_\theta^{(k)}.$$

For example, for a measurable function $f : X^3 \to [0, \infty)$, we have

$$\int_{X^3} f(x_1, x_2, x_3) \, m_{-,-,+}^{(3)}(dx_1 \, dx_2 \, dx_3) = \int_{X^3} f(x_1, x_2, x_3) \, m^{\otimes 3}(dx_1 \, dx_2 \, dx_3)$$

$$+ \int_{X^2} f(x_1, x_2, x_1)\, m^{\otimes 2}(dx_1\, dx_2) + \int_{X^2} f(x_1, x_2, x_2)\, m^{\otimes 2}(dx_1\, dx_2). \tag{23}$$

Completely analogously, starting with the Lebesgue measure on $\mathbb{R}$ rather than X, we define a measure $\nu^{(k)}_{\sharp_1,\ldots,\sharp_k}$ on $\mathbb{R}^k$. For example, similarly to (23), for a measurable function $f : \mathbb{R}^3 \to [0, \infty)$, we have

$$\int_{\mathbb{R}^3} f(s_1, s_2, s_3)\, \nu^{(3)}_{-,-,+}(ds_1\, ds_2\, ds_3) = \int_{\mathbb{R}^3} f(s_1, s_2, s_3)\, ds_1\, ds_2\, ds_3$$

$$+ \int_{\mathbb{R}^2} f(s_1, s_2, s_1)\, ds_1\, ds_2 + \int_{\mathbb{R}^2} f(s_1, s_2, s_2)\, ds_1\, ds_2.$$

We denote by $L^0(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1,\ldots,\sharp_k})$ the linear space of classes of complex-valued measurable functions on X^k, with any two measurable functions $f, g : X^k \to \mathbb{C}$ being identified if $f = g$ $m^{(k)}_{\sharp_1,\ldots,\sharp_k}$-a.e. We define a linear mapping

$$\Phi^{(k)}_{\sharp_1,\ldots,\sharp_k} : \mathscr{H}_{\mathbb{C}}^k \times L^\infty(\mathbb{R}^k \to \mathbb{C}, \nu^{(k)}_{\sharp_1,\ldots,\sharp_k}) \to L^0(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1,\ldots,\sharp_k}) \tag{24}$$

by

$$\Phi^{(k)}_{\sharp_1,\ldots,\sharp_k}\big[h_1, \ldots, h_k, v^{(k)}\big](x_1, \ldots, x_k) := h_1(x_1) \cdots h_k(x_k) v^{(k)}(x_1^1, \ldots, x_k^1), \tag{25}$$

where $h_1, \ldots, h_k \in \mathscr{H}_{\mathbb{C}}$ and $v^{(k)} \in L^\infty(\mathbb{R}^k \to \mathbb{C}, \nu^{(k)}_{\sharp_1,\ldots,\sharp_k})$. We denote by $\mathbb{F}^{(k)}_{\sharp_1,\ldots,\sharp_k}$ the range of $\Phi^{(k)}_{\sharp_1,\ldots,\sharp_k}$, which is a subspace of $L^0(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1,\ldots,\sharp_k})$.

Remark 1. It should be noted that the mapping $\Phi^{(k)}_{\sharp_1,\ldots,\sharp_k}$ is not injective. For example, if $h_1, \ldots, h_k \in \mathscr{H}_{\mathbb{C}}$, $v^{(k)} \in L^\infty(\mathbb{R}^k \to \mathbb{C}, \nu^{(k)}_{\sharp_1,\ldots,\sharp_k})$, and $\alpha \in L^\infty(\mathbb{R}, ds)$, we have

$$\Phi^{(k)}_{\sharp_1,\ldots,\sharp_k}\big[g, h_2 \ldots, h_k, v^{(k)}\big] = \Phi^{(k)}_{\sharp_1,\ldots,\sharp_k}\big[h_1, \ldots, h_k, w^{(k)}\big],$$

where $g(x) := h_1(x)\alpha(x^1) \in \mathscr{H}_{\mathbb{C}}$ and $w^{(k)}(s_1, \ldots, s_k) := v^{(k)}(s_1, \ldots, s_k)\alpha(s_1) \in L^\infty(\mathbb{R}^k \to \mathbb{C}, \nu^{(k)}_{\sharp_1,\ldots,\sharp_k})$.

Below we will deal with linear operators in a complex Hilbert space which will be denoted by operator-valued integrals of the form (21) with

$$\varphi^{(k)} = \Phi^{(k)}_{\sharp_1,\ldots,\sharp_k}\big[h_1, \ldots, h_k, v^{(k)}\big] \in \mathbb{F}^{(k)}_{\sharp_1,\ldots,\sharp_k}. \tag{26}$$

We will also denote these operators by $I^{(k)}_{\sharp_1,\ldots,\sharp_k}\big[h_1, \ldots, h_k, v^{(k)}\big]$. Our next aim is to give a rigorous definition of the commutation relations (1)–(3) satisfied by these operators.

Let $k \geq 2$, $\sharp_1, \ldots, \sharp_k \in \{+, -\}$, and let $i \in \{1, \ldots, k-1\}$. Let us consider the operator-valued integral (21) with $\varphi^{(k)}$ given by (26). Assume that $\sharp_i = \sharp_{i+1}$. Then, at least informally, we calculate using either relation (1) or relation (2):

$$\int_{X^k} m^{\otimes k}(dx_1 \cdots dx_k)\, \varphi^{(k)}(x_1, \ldots, x_k) \partial_{x_1}^{\sharp_1} \cdots \partial_{x_k}^{\sharp_k}$$

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$$= \int_{X^k} m^{\otimes k}(dx_1 \cdots dx_k) \, h_1(x_1) \cdots h_k(x_k) v^{(k)}(x_1^1, \ldots, x_k^1) Q(x_{i+1}^1, x_i^1)$$
$$\times \partial_{x_1}^{\sharp_1} \cdots \partial_{x_{i-1}}^{\sharp_{i-1}} \partial_{x_{i+1}}^{\sharp_{i+1}} \partial_{x_i}^{\sharp_i} \partial_{x_{i+2}}^{\sharp_{i+2}} \cdots \partial_{x_k}^{\sharp_k}$$
$$= \int_{X^k} m^{\otimes k}(dx_1 \cdots dx_k) \, h_1(x_1) \cdots h_{i-1}(x_{i-1}) h_{i+1}(x_i) h_i(x_{i+1}) h_{i+2}(x_{i+2}) \cdots h_k(x_k)$$
$$\times (\Psi_i v^{(k)})(x_1^1, \ldots, x_k^1) \partial_{x_1}^{\sharp_1} \cdots \partial_{x_k}^{\sharp_k}. \tag{27}$$

Here

$$(\Psi_i v^{(k)})(s_1, \ldots, s_k)$$
$$:= v^{(k)}(s_1, \ldots, s_{i-1}, s_{i+1}, s_i, s_{i+2}, \ldots, s_k) Q(s_i, s_{i+1}) \in L^\infty(\mathbb{R}^k \to \mathbb{C}, \nu^{(k)}_{\sharp_1, \ldots, \sharp_k}). \tag{28}$$

Thus, inspired by (27) and (28), we give the following definition: Relation (1) (or relation (2), respectively) means that, for any $k \geq 2$, $i \in \{1, \ldots, k-1\}$ and $\sharp_1, \ldots, \sharp_k \in \{+, -\}$ such that $\sharp_i = \sharp_{i+1} = +$ (or $\sharp_i = \sharp_{i+1} = -$, respectively), we have

$$I^{(k)}_{\sharp_1, \ldots, \sharp_k}\big[h_1, \ldots, h_k, v^{(k)}\big] = I^{(k)}_{\sharp_1, \ldots, \sharp_k}\big[h_1, \ldots, h_{i-1}, h_{i+1}, h_i, h_{i+2}, \ldots, h_k, \Psi_i v^{(k)}\big]. \tag{29}$$

Analogously, relation (3) means that for any $k \geq 2$, $i \in \{1, \ldots, k-1\}$ and $\sharp_1, \ldots, \sharp_k \in \{+, -\}$ such that $\sharp_i = -$ and $\sharp_{i+1} = +$, we have

$$I^{(k)}_{\sharp_1, \ldots, \sharp_k}\big[h_1, \ldots, h_k, v^{(k)}\big]$$
$$= I^{(k-2)}_{\sharp_1, \ldots, \sharp_{i-1}, \sharp_{i+2}, \ldots, \sharp_k}\big[h_1, \ldots, h_{i-1}, h_{i+2}, \ldots, h_k, u^{(k-2)}\big]$$
$$+ I^{(k)}_{\sharp_1, \ldots, \sharp_{i-1}, \sharp_{i+1}, \sharp_i, \sharp_{i+2}, \ldots, \sharp_k}\big[h_1, \ldots, h_{i-1}, h_{i+1}, h_i, h_{i+2}, \ldots, h_k, \Psi_i' v^{(k)}\big], \tag{30}$$

where

$$u^{(k-2)}(s_1, \ldots, s_{k-2}) := \int_X h_i(x) h_{i+1}(x)$$
$$\times v^{(k)}(s_1, \ldots, s_{i-1}, x^1, x^1, s_i, \ldots, s_{k-2}) \, m(dx) \in L^\infty(\mathbb{R}^{k-2} \to \mathbb{C}, \nu^{(k-2)}_{\sharp_1, \ldots, \sharp_{i-1}, \sharp_{i+2}, \ldots, \sharp_k}) \tag{31}$$

and

$$(\Psi_i' v^{(k)})(s_1, \ldots, s_k) := v^{(k)}(s_1, \ldots, s_{i-1}, s_{i+1}, s_i, s_{i+2}, \ldots, s_k)$$
$$\times Q(s_{i+1}, s_i) \in L^\infty(\mathbb{R}^k \to \mathbb{C}, \nu^{(k)}_{\sharp_1, \ldots, \sharp_{i-1}, \sharp_{i+1}, \sharp_i, \sharp_{i+2}, \ldots, \sharp_k}). \tag{32}$$

Remark 2. In the case $k = 2$, the second addend on the right hand side of equality (30) is understood as the constant operator $u^{(0)}$, where

$$u^{(0)} := \int_X h_1(x) h_2(x) v^{(2)}(x^1, x^1) \, m(dx).$$

Remark 3. Note that the commutation relations (29), (30) do not depend on the representation of $\varphi^{(k)} \in \mathbb{F}^{(k)}_{\sharp_1,\ldots,\sharp_k}$ in the form (26).

Remark 4. Note that the commutation relations (29) do not depend on η. Indeed, for $\sharp_i = \sharp_{i+1}$, we have

$$\nu^{(k)}_{\sharp_1,\ldots,\sharp_k}\big(\{(s_1,\ldots,s_k) \mid s_i = s_{i+1}\}\big) = 0. \tag{33}$$

Hence, in (28), for $s_i = s_{i+1}$ the value $Q(s_i, s_{i+1}) = \eta$ plays no role. On the other hand, formula (33) is not true when $\sharp_i = -$ and $\sharp_{i+1} = +$. Therefore, for $s_i = s_{i+1}$ the value $Q(s_{i+1}, s_i) = \eta$ does matter for (32), hence also for the commutation relation (30).

2.2 Definition of the Q-CR algebra and the anyon exclusion principle

We are now in position to define the Q-CR algebra. Let $\mathscr{G}$ be a separable, complex Hilbert space. Let Θ be a dense linear subspace of $\mathscr{G}$. We assume that, for any $\sharp_1,\ldots,\sharp_k \in \{+,-\}$ and any $\varphi^{(k)} \in \mathbb{F}^{(k)}_{\sharp_1,\ldots,\sharp_k}$ we have a linear operator mapping Θ into Θ. This operator is denoted either as in (21) or by $I^{(k)}_{\sharp_1,\ldots,\sharp_k}\big(h_1,\ldots,h_k, v^{(k)}\big)$, given that $\varphi^{(k)}$ is as in (26). These operators will be called *operator-valued integrals*.

We will assume that the operator-valued integrals satisfy the following axioms.

(A1) *Consistency condition*: For any $g_1,\ldots,g_k, h_1,\ldots,h_k \in \mathscr{H}_{\mathbb{C}}$, and $v^{(k)}, w^{(k)} \in L^\infty(\mathbb{R}^k \to \mathbb{C}, \nu^{(k)}_{\sharp_1,\ldots,\sharp_k})$, if

$$\Phi^{(k)}_{\sharp_1,\ldots,\sharp_k}\big[g_1,\ldots,g_k, w^{(k)}\big] = \Phi^{(k)}_{\sharp_1,\ldots,\sharp_k}\big[h_1,\ldots,h_k, v^{(k)}\big],$$

then

$$I^{(k)}_{\sharp_1,\ldots,\sharp_k}(g_1,\ldots g_k, w^{(k)}) = I^{(k)}_{\sharp_1,\ldots,\sharp_k}(h_1,\ldots h_k, v^{(k)}).$$

(A2) *Linearity:* For any $\sharp_1,\ldots,\sharp_k \in \{+,-\}$, $I^{(k)}_{\sharp_1,\ldots,\sharp_k}(h_1,\ldots,h_k, v^{(k)})$ linearly depends on $h_i \in \mathscr{H}_{\mathbb{C}}$ $(i = 1,\ldots,k)$ and on $v^{(k)} \in L^\infty(\mathbb{R}^k \to \mathbb{C}, \nu^{(k)}_{\sharp_1,\ldots,\sharp_k})$.

(A3) *The adjoint operator:* The adjoint of any operator $I^{(k)}_{\sharp_1,\ldots,\sharp_k}(h_1,\ldots,h_k, v^{(k)})$ in the Hilbert space $\mathscr{G}$ contains Θ in its domain, and the restriction of this adjoint operator to Θ is equal to the operator $I^{(k)}_{\triangle_k,\ldots,\triangle_1}(\overline{h_k},\ldots,\overline{h_1}, v^{(k)*})$, where

$$\triangle_i := \begin{cases} + & \text{if } \sharp_i = -, \\ - & \text{if } \sharp_i = +, \end{cases} \qquad i = 1,\ldots,k,$$

$$v^{(k)*}(s_1,\ldots,s_k) = \overline{v^{(k)}(s_k,\ldots,s_1)} \in L^\infty(\mathbb{R}^k \to \mathbb{C}, \nu^{(k)}_{\triangle_k,\ldots,\triangle_1}).$$

(A4) *Q-commutation relations:* The operators ∂^+_x, ∂^-_x satisfy the Q-CR (1)–(3). A rigorous meaning of these relations is given by formulas (28)–(32).

(A5) *Multiplication of operator-valued integrals:* For any

$$h_1, \ldots, h_{k+n} \in \mathscr{H}_{\mathbb{C}}, \quad v^{(k)} \in L^\infty(\mathbb{R}^m, \nu^{(k)}_{\sharp_1, \ldots, \sharp_k}), \quad w^{(n)} \in L^\infty(\mathbb{R}^n, \nu^{(n)}_{\sharp_{k+1}, \ldots, \sharp_{k+n}}),$$

we have

$$I^{(k)}_{\sharp_1, \ldots, \sharp_k}(h_1, \ldots, h_k, v^{(k)}) I^{(n)}_{\sharp_{k+1}, \ldots, \sharp_{k+n}}(h_{k+1}, \ldots, h_{k+n}, w^{(n)})$$
$$= I^{(k+n)}_{\sharp_1, \ldots, \sharp_{k+n}}(h_1, \ldots, h_{k+n}, v^{(k)} \otimes w^{(n)}).$$

Here

$$(v^{(k)} \otimes w^{(n)})(s_1, \ldots, s_{k+n})$$
$$:= v^{(k)}(s_1, \ldots, s_k) w^{(n)}(s_{k+1}, \ldots, s_{k+n}) \in L^\infty(\mathbb{R}^{k+n}, \nu^{(k+n)}_{\sharp_1, \ldots, \sharp_{k+n}}).$$

Remark 5. We stress that the value η of the function Q on the diagonal does not matter for the relations (1), (2).

Let **A** denote the complex algebra generated by the operator-valued integrals satisfying axioms (A1)–(A5), with the usual multiplication of operators acting on Θ. We will call **A** the *algebra of Q-commutation relations*, or the *Q-CR algebra* for short. In the case where the function Q is given by (15), we will call **A** the *algebra of anyon commutation relations*, or the *ACR algebra* for short.

The following theorem shows that the anyon exclusion principle [19] (see also [10, Proposition 2.9]) holds in the ACR algebra with q being a root of 1.

Theorem 6. *Let $k \in \mathbb{N}$, $k \geq 2$. Let $q \in \mathbb{C}$ be such that $q \neq 1$ and $q^k = 1$. Then, in the ACR algebra, we have, for each $h \in \mathscr{H}_{\mathbb{C}}$,*

$$\left(\int_X m(dx) \, h(x) \partial_x^+ \right)^k = 0.$$

Proof. Note that $\nu^{(k)}_{+, \ldots, +} = \nu^{(k)}$, the Lebesgue measure on $\mathbb{R}^{(k)}$ ($\mathbb{R}^{(k)}$ consisting of all $(s_1, \ldots, s_k) \in \mathbb{R}^k$ with $s_i \neq s_j$ if $i \neq j$). By (29), we get, for any $i \in \{1, \ldots, k-1\}$ and $v^{(k)} \in L^\infty(\mathbb{R}^k, \nu^{(k)})$:

$$I^{(k)}_{+, \ldots, +}(h, \ldots, h, v^{(k)}) = I^{(k)}_{+, \ldots, +}(h, \ldots, h, \Psi_i v^{(k)}).$$

Here $\Psi_i v^{(k)}$ is given by formula (28) for $(s_1, \ldots, s_k) \in \mathbb{R}^{(k)}$. By the proof of Proposition 2.8 in [10], it follows from here that, for each permutation $\pi \in S_k$, we get

$$I^{(k)}_{+, \ldots, +}(h, \ldots, h, v^{(k)}) = I^{(k)}_{+, \ldots, +}(h, \ldots, h, \Psi_\pi v^{(k)}), \tag{34}$$

where

$$(\Psi_\pi v^{(k)})(s_1, \ldots, s_k) = Q_{\pi^{-1}}(s_1, \ldots, s_k) v^{(k)}(s_{\pi(1)}, \ldots, s_{\pi(k)}).$$

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Recall that the function Q_π was defined by (9), and we again used the obvious abuse of notation $Q_\pi(x_1, \ldots, x_k) = Q_\pi(x_1^1, \ldots, x_k^1)$ for $(x_1, \ldots, x_k) \in X^k$. By (10) and (34), we get

$$I_{+,\ldots,+}^{(k)}(h, \ldots, h, v^{(k)}) = I_{+,\ldots,+}^{(k)}(h, \ldots, h, P_k v^{(k)}), \tag{35}$$

and the function $P_k v^{(k)}$ is Q-symmetric on $\mathbb{R}^{(k)}$. It follows from the proof of Proposition 2.9 in [10] that if we choose $v^{(k)} \equiv 1$, we get

$$(P_k 1)(s_1, \ldots, s_k) = \frac{1 - q^k}{(1 - q)k!} = 0$$

for all $s_1 < s_2 < \cdots < s_k$. Hence, $P_k 1 = 0$ $\nu^{(k)}$-a.e. Now the statement of the theorem follows by the axioms (A5) and (A2). $\qquad\square$

2.3   Definition of a gauge-invariant quasi-free state

Let τ be a state on the Q-CR algebra $\mathbf{A}$. Because of the Q-CR, τ is completely determined by its n-point functions, which are defined by formula (22) with $\varphi^{(k+n)} \in \mathbb{F}_{\sharp_1, \ldots, \sharp_{k+n}}^{(k+n)}$, where $\sharp_1 = \cdots = \sharp_k = +$ and $\sharp_{k+1} = \cdots = \sharp_{k+n} = -$. We already discussed in subsection 1.3 that gauge invariance of τ means that $\mathbf{S}^{(k,n)} \equiv 0$ if $k \neq n$. So our aim now is to introduce a proper generalization of formulas (18), (19).

We denote the n-point functions by

$$\mathbf{S}^{(n,n)}(h_1, \ldots, h_{2n}, v^{(2n)}) := \tau\big(I_{\sharp_1, \ldots, \sharp_{2n}}^{(2n)}(h_1, \ldots, h_{2n}, v^{(2n)})\big), \tag{36}$$

where $\sharp_1 = \cdots = \sharp_n = +$ and $\sharp_{n+1} = \cdots = \sharp_{2n} = -$. By (A2), the right hand side of (36) identifies a linear functional of $v^{(2n)} \in L^\infty(\mathbb{R}^{2n}, \nu_{\sharp_1, \ldots, \sharp_{2n}}^{(2n)})$. If we assume that this functional continuously depends on $v^{(2n)}$, then, according to the general theory of linear continuous functionals on L^∞ spaces, this functional can be identified with a complex-valued, finite-additive measure on $\mathbb{R}^{2n}$ that is absolutely continuous with respect to the measure $\nu_{\sharp_1, \ldots, \sharp_{2n}}^{(2n)}$. We will actually assume the following stronger condition to be satisfied.

(M) For any $h_1, \ldots, h_{2n} \in \mathscr{H}_\mathbb{C}$, there exists a (unique) complex-valued measure $\mathbf{S}^{(n,n)}[h_1, \ldots, h_{2n}]$ on $\mathbb{R}^{2n}$ which is absolutely continuous with respect to $\nu_{\sharp_1, \ldots, \sharp_{2n}}^{(2n)}$, $\sharp_1 = \cdots = \sharp_n = +$ and $\sharp_{n+1} = \cdots = \sharp_{2n} = -$, and satisfies, for all $v^{(2n)} \in L^\infty(\mathbb{R}^{2n}, \nu_{\sharp_1, \ldots, \sharp_{2n}}^{(2n)})$,

$$\mathbf{S}^{(n,n)}(h_1, \ldots, h_{2n}, v^{(2n)}) = \int_{\mathbb{R}^{2n}} v^{(2n)}(s_1, \ldots, s_{2n})\, \mathbf{S}^{(n,n)}[h_1, \ldots, h_{2n}](ds_1 \cdots ds_{2n}). \tag{37}$$

We will denote by $\mathbf{S}^{(n,n)}[h_1, \ldots, h_{2n}](s_1, \ldots, s_{2n})$ the density of the measure $\mathbf{S}^{(n,n)}[h_1, \ldots, h_{2n}](ds_1 \cdots ds_{2n})$ with respect to the measure $\nu^{(2n)}_{\sharp_1, \ldots, \sharp_{2n}}$ with $\sharp_1 = \cdots = \sharp_n = +$ and $\sharp_{n+1} = \cdots = \sharp_{2n} = -$.

We say that τ is a *gauge-invariant quasi-free state on the Q-CR algebra* $\mathbf{A}$ if $\mathbf{S}^{(k,n)} \equiv 0$ if $k \neq n$, and for each $n \in \mathbb{N}$ and any $g_1, \ldots, g_n, h_1, \ldots, h_n \in \mathscr{H}_{\mathbb{C}}$, we have

$$\mathbf{S}^{(n,n)}[g_n, \ldots, g_1, h_1, \ldots, h_n](s_n, \ldots, s_1, s_{n+1}, \ldots, s_{2n})$$
$$= \sum_{\pi \in S_n} \left(\prod_{i=1}^n \mathbf{S}^{(1,1)}[g_i, h_{\pi(i)}](s_i, s_{n+\pi(i)}) \right) Q_\pi(s_1, \ldots, s_n). \tag{38}$$

Remark 7. Note the following slight difference in notations: in formulas (18), (19), the n-point function $\mathbf{S}^{(n,n)}(g_n, \ldots, g_1, h_1, \ldots, h_n)$ is linear in each g_i and antilinear in each h_i, while in our setting the n-point function in (38) depends linearly on both g_i and h_i.

Remark 8. Note that the measure $\nu^{(2n)}_{\sharp_1, \ldots, \sharp_{2n}}$ with $\sharp_1 = \cdots = \sharp_n = +$ and $\sharp_{n+1} = \cdots = \sharp_{2n} = -$ remains invariant under the transformation

$$\mathbb{R}^{2n} \ni (s_1, \ldots, s_{2n}) \mapsto (s_n, \ldots, s_1, s_{n+1}, \ldots, s_{2n}) \in \mathbb{R}^{2n}.$$

Hence, formulas (37), (38) mean that, for any $g_1, \ldots, g_n, h_1, \ldots, h_n \in \mathscr{H}_{\mathbb{C}}$ and $v^{(2n)} \in L^\infty(\mathbb{R}^{2n}, \nu^{(2n)}_{\sharp_1, \ldots, \sharp_{2n}})$, we have

$$\mathbf{S}^{(n,n)}(g_n, \ldots, g_1, h_1, \ldots, h_n, v^{(2n)}) = \int_{\mathbb{R}^{2n}} v^{(2n)}(s_n, \ldots, s_1, s_{n+1}, \ldots, s_{2n})$$
$$\times \sum_{\pi \in S_n} \left(\prod_{i=1}^n \mathbf{S}^{(1,1)}[g_i, h_{\pi(i)}](s_i, s_{n+\pi(i)}) \right) Q_\pi(s_1, \ldots, s_n) \, \nu^{(2n)}_{\sharp_1, \ldots, \sharp_{2n}}(ds_1 \cdots ds_{2n}). \tag{39}$$

Below, in Section 4, we will explicitly construct a class of gauge-invariant quasi-free states, but before doing this we wll now construct operator-valued integrals in the Q-Fock space.

3 Operator-valued integrals in the Q-symmetric Fock space

In this section, we will assume that X is a locally compact Polish space, $\mathscr{B}(X)$ is the Borel σ-algebra on X, and m is a reference measure on $(X, \mathscr{B}(X))$. We assume m to be a Radon measure (i.e., finite on any compact set in X) and non-atomic. Analogously to subsection 1.1, we assume that Δ is a measurable, symmetric subset of X^2 and satisfies $m^{\otimes 2}(\Delta) = 0$. We also assume that $D \subset \Delta$, where $D := \{(x, x) \mid x \in X\}$ is the diagonal in X^2. We denote $\widetilde{X}^2 := X^2 \setminus \Delta$. We fix $\eta \in \mathbb{R}$ and consider a function

$Q : X^2 \to \mathbb{C}$ such that $|Q(x, y)| = 1$ and $Q(x, y) = \overline{Q(y, x)}$ for all $(x, y) \in \widetilde{X}^2$, and $Q(x, y) = \eta$ for all $(x, y) \in \Delta$.

Analogously to $\widetilde{X}^2$, we define, for each $n \geq 3$

$$\widetilde{X}^n := \big\{ (x_1, \ldots, x_n) \in X^n \mid (x_i, x_j) \notin \Delta \text{ for all } 1 \leq i < j \leq n \big\}.$$

Let $n \geq 2$. A function $f^{(n)} : \widetilde{X}^n \to \mathbb{C}$ is called Q-*symmetric* if for any $i \in \{1, \ldots, n-1\}$ and $(x_1, \ldots, x_n) \in \widetilde{X}^n$, formula (8) holds. Since $m^{\otimes n}(X^n \setminus \widetilde{X}^n) = 0$, the function $f^{(n)}$ is defined $m^{\otimes n}$-a.e. on X^n. We define the function $Q_\pi : \widetilde{X}^n \to \mathbb{C}$ by formula (9), and the Q-symmetrization of a function $f^{(n)} : \widetilde{X}^n \to \mathbb{C}$ by (10). The definitions of $\mathscr{H}$, $\mathscr{H}_\mathbb{C}$, $\mathscr{H}_\mathbb{C}^{\otimes n}$, P_n, $\mathscr{F}^Q(\mathscr{H})$, $\mathscr{F}^Q_{\mathrm{fin}}(\mathscr{H})$, $a^+(h)$, and $a^-(h)$ are now similar to subsection 1.1.

Let $\sharp_1, \ldots, \sharp_k \in \{+, -\}$. Analogously to (24), (25), we define a linear mapping

$$\Xi^{(k)}_{\sharp_1, \ldots, \sharp_k} : \mathscr{H}^k_\mathbb{C} \times L^\infty(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1, \ldots, \sharp_k}) \to L^0(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1, \ldots, \sharp_k})$$

by

$$\Xi^{(k)}_{\sharp_1, \ldots, \sharp_k} \big[h_1, \ldots, h_k, \varkappa^{(k)} \big](x_1, \ldots, x_k) := h_1(x_1) \cdots h_k(x_k) \varkappa^{(k)}(x_1, \ldots, x_k), \qquad (40)$$

where $h_1, \ldots, h_k \in \mathscr{H}_\mathbb{C}$ and $\varkappa^{(k)} \in L^\infty(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1, \ldots, \sharp_k})$. We denote by $\mathbb{G}^{(k)}_{\sharp_1, \ldots, \sharp_k}$ the range of this mapping. Our aim now is to construct operator-valued integrals of the form (21) with $\varphi^{(k)} = \Xi^{(k)}_{\sharp_1, \ldots, \sharp_k} \big[h_1, \ldots, h_k, \varkappa^{(k)} \big]$.

The following proposition follows immediately from the definition of the creation operator, $a^+(h)$, and [10, Proposition 3.2].

Proposition 9. *Let $\mathscr{F}(\mathscr{H}) := \bigoplus_{n=0}^\infty \mathscr{H}_\mathbb{C}^{\otimes n}$ denote the full Fock space over $\mathscr{H}$, and let the space $\mathscr{F}_{\mathrm{fin}}(\mathscr{H})$ be defined analogously to $\mathscr{F}^Q_{\mathrm{fin}}(\mathscr{H})$. For $h \in \mathscr{H}_\mathbb{C}$, we define linear continuous operator $b^+(h)$ and $b^-(h)$ on $\mathscr{F}_{\mathrm{fin}}(\mathscr{H})$ by*

$$b^+(h) f^{(n)} := h \otimes f^{(n)},$$

$$(b^-(h) f^{(n)})(x_1, \ldots, x_{n-1}) := \sum_{i=1}^n \int_X h(y) Q(y, x_1) \cdots Q(y, x_{i-1})$$

$$\times f^{(n)}(x_1, \ldots, x_{i-1}, y, x_i, \ldots, x_{n-1}) \, m(dy), \quad f^{(n)} \in \mathscr{H}_\mathbb{C}^{\otimes n} \qquad (41)$$

Then, on $\mathscr{F}_{\mathrm{fin}}(\mathscr{H})$, we have

$$a^+(h) P = P b^+(h), \quad a^-(h) P = P b^-\big(\bar{h}\big). \qquad (42)$$

Here, for $f^{(n)} \in \mathscr{H}_\mathbb{C}^{\otimes n}$, we define $P f^{(n)} := P_n f^{(n)}$.

Using the notation (4), we conclude from here the following corollary.

Corollary 10. *For any $\sharp_1, \ldots, \sharp_k \in \{+, -\}$ and any $h_1, \ldots, h_k \in \mathscr{H}_{\mathbb{C}}$, we have on $\mathscr{F}_{\text{fin}}^Q(\mathscr{H})$:*

$$\int_{X^k} m^{\otimes k}(dx_1 \cdots dx_k) \, h_1(x_1) \cdots h_k(x_k) \partial_{x_1}^{\sharp_1} \cdots \partial_{x_k}^{\sharp_k} = P b^{\sharp_1}(h_1) \cdots b^{\sharp_k}(h_k). \tag{43}$$

Here we denoted

$$\int_{X^k} m^{\otimes k}(dx_1 \cdots dx_k) h_1(x_1) \cdots h_k(x_k) \partial_{x_1}^{\sharp_1} \ldots, \partial_{x_k}^{\sharp_k}$$
$$:= \left(\int_X m(dx_1) h(x_1) \partial_{x_1}^{\sharp_1} \right) \cdots \left(\int_X m(dx_k) h(x_k) \partial_{x_k}^{\sharp_k} \right).$$

Let $f^{(n)} \in \mathscr{H}_{\mathbb{C}}^{\otimes n}$. Using Corollary 10, we can write down the action of the operator in formula (43) on $f^{(n)}$ through the function

$$\varphi^{(k)}(x_1, \ldots, x_k) := h_1(x_1) \cdots h_k(x_k). \tag{44}$$

(Note that this function is defined $m_{\sharp_1, \ldots, \sharp_k}^{(k)}$-a.e.) For example,

$$\left(\int_{X^3} m^{\otimes 3}(dy_1 \, dy_2 \, dy_3) \varphi^{(3)}(y_1, y_2, y_3) \partial_{y_1}^- \partial_{y_2}^- \partial_{y_3}^+ f^{(n)} \right)(x_1, \ldots, x_{n-1})$$

$$= P_{n-1} \Bigg\{ \sum_{i=1}^n \int_{X^2} m^{\otimes 2}(dy_1 \, dy_2) \, \varphi^{(3)}(y_1, y_2, y_2)$$

$$\times \Big[Q(y_1, x_1) \cdots Q(y_1, x_{i-1}) f^{(n)}(x_1, \ldots, x_{i-1}, y_1, x_i, \ldots, x_{n-1})$$

$$+ Q(y_2, y_1) Q(y_2, x_1) \cdots Q(y_2, x_{i-1}) f^{(n)}(x_1, \ldots, x_{i-1}, y_2, x_i, \ldots, x_{n-1}) \Big]$$

$$+ \sum_{i=2}^{n+1} \int_{X^2} m^{\otimes 2}(dy_1 \, dy_2) \, \varphi^{(3)}(y_1, y_2, x_1) \Bigg[\sum_{j=2}^{i-1} Q(y_2, y_1) Q(y_1, x_1) \cdots Q(y_1, x_{j-1})$$

$$\times Q(y_2, x_1) \cdots Q(y_2, x_{i-2}) f^{(n)}(x_2, \ldots, x_{j-1}, y_1, x_j, \ldots, x_{i-2}, y_2, x_{i-1}, \ldots, x_{n-1})$$

$$+ \sum_{j=i}^n Q(y_1, x_1) \cdots Q(y_1, x_{j-1}) Q(y_2, x_1) \cdots Q(y_2, x_{i-1})$$

$$\times f^{(n)}(x_2, \ldots, x_{i-1}, y_2, x_i, \ldots, x_{j-1}, y_1, x_j, \ldots, x_{n-1}) \Bigg] \Bigg\}.$$

As easily seen, we can replace in the obtained formulas the function $\varphi^{(k)}$ of the form (44) with a function $\varphi^{(k)}$ being given by the right hand side of formula (40). As a result, for each $\varphi^{(k)} = \Xi_{\sharp_1, \ldots, \sharp_k}^{(k)}[h_1, \ldots, h_k, \varkappa^{(k)}] \in \mathbb{G}_{\sharp_1, \ldots, \sharp_k}^{(k)}$, we have constructed a continuous linear operator on $\mathscr{F}_{\text{fin}}^Q(\mathscr{H})$ which is denoted as in (21) or by $I_{\sharp_1, \ldots, \sharp_k}^{(k)}(h_1, \ldots, h_k, \varkappa^{(k)})$. Note that this operator indeed depends on the values of the function $\varphi^{(k)}(x_1, \ldots, x_k) = h_1(x_1) \cdots h_k(x_k) \varkappa^{(k)}(x_1, \ldots, x_k) \, m_{\sharp_1, \ldots, \sharp_k}^{(k)}$-a.e.

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The following proposition follows from the construction of the operator-valued integrals (compare with [10, Proposition 3.8] regarding the corresponding statement about the Q-CR).

Proposition 11. *The above constructed operator-valued integrals $I^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1,\dots,h_k,\varkappa^{(k)})$ satisfy the axioms* (A1)′–(A5)′ *that are obtained from the axioms* (A1)–(A5) *by replacing the space $L^\infty(\mathbb{R}^k \to \mathbb{C}, \nu^{(k)}_{\sharp_1,\dots,\sharp_k})$ by $L^\infty(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1,\dots,\sharp_k})$.*

Let us note that we initially had operator-valued integrals $I^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1,\dots,h_k,\varkappa^{(k)})$ for $\varkappa^{(k)} \equiv 1$ and then we extended them to an arbitrary function $\varkappa^{(k)} \in L^\infty(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1,\dots,\sharp_k})$. The following lemma shows that, under natural assumptions on the operator-valued integrals, this extension is, in fact, unique.

Lemma 12. *Assume that the operator-valued integrals $\tilde{I}^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1,\dots,h_k,\varkappa^{(k)})$ acting on $\mathscr{F}^Q_{\mathrm{fin}}(\mathscr{H})$ satisfy the axioms* (A1)′, (A2)′ *and the following assumption: for any $h_1,\dots,h_k \in \mathscr{H}_\mathbb{C}$ and any $F,G \in \mathscr{F}^Q_{\mathrm{fin}}(\mathscr{H})$ the linear functional*

$$L^\infty(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1,\dots,\sharp_k}) \ni \varkappa^{(k)} \to \big(\tilde{I}^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1,\dots,h_k,\varkappa^{(k)})F,G\big)_{\mathscr{F}^Q(\mathscr{H})} \in \mathbb{C}$$

is continuous and is given by a complex-valued measure on X^k. Then the equality

$$\tilde{I}^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1,\dots,h_k,1) = I^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1,\dots,h_k,1) \tag{45}$$

implies the equality

$$\tilde{I}^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1,\dots,h_k,\varkappa^{(k)}) = I^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1,\dots,h_k,\varkappa^{(k)})$$

for all $\varkappa^{(k)} \in L^\infty(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1,\dots,\sharp_k})$.

Proof. Denote by $\tilde{\mathfrak{m}}^{(k)}_{\sharp_1,\dots,\sharp_k}[F;G;h_1,\dots,h_k]$ the complex-valued measure on X^k that satisfies, for all $\varkappa^{(k)} \in L^\infty(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1,\dots,\sharp_k})$,

$$\int_{X^k} \varkappa^{(k)}(x_1,\dots,x_k)\,\tilde{\mathfrak{m}}^{(k)}_{\sharp_1,\dots,\sharp_k}[F;G;h_1,\dots,h_k](dx_1\cdots dx_k)$$
$$= \big(\tilde{I}^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1,\dots,h_k,\varkappa^{(k)})F,G\big)_{\mathscr{F}^Q(\mathscr{H})}. \tag{46}$$

As easily seen from the construction of the operator-valued integrals, there exists a complex-valued measure $\mathfrak{m}^{(k)}_{\sharp_1,\dots,\sharp_k}[F;G;h_1,\dots,h_k]$ on X^k that satisfies equality (46) in which $\tilde{\mathfrak{m}}^{(k)}_{\sharp_1,\dots,\sharp_k}$ and $\tilde{I}^{(k)}_{\sharp_1,\dots,\sharp_k}$ are replaced with $\mathfrak{m}^{(k)}_{\sharp_1,\dots,\sharp_k}$ and $I^{(k)}_{\sharp_1,\dots,\sharp_k}$, respectively. Hence, it suffices to prove that

$$\tilde{\mathfrak{m}}^{(k)}_{\sharp_1,\dots,\sharp_k}[F;G;h_1,\dots,h_k] = \mathfrak{m}^{(k)}_{\sharp_1,\dots,\sharp_k}[F;G;h_1,\dots,h_k]. \tag{47}$$

For $i \in \{1, \dots, k\}$, let $A_i \in \mathscr{B}(X)$ and denote by χ_{A_i} the indicator function of the set A_i. We have, by axiom (A1)$'$ and (45),

$$\tilde{\mathfrak{m}}^{(k)}_{\sharp_1,\dots,\sharp_k}[F; G; h_1, \dots, h_k](A_1 \times \cdots \times A_k) = \big(\tilde{I}^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1, \dots, h_k, \chi_{A_1} \cdots \chi_{A_k})F, G\big)_{\mathscr{F}^Q(\mathscr{H})}$$

$$= \big(\tilde{I}^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1\chi_{A_1}, \dots, h_k\chi_{A_k}, 1)F, G\big)_{\mathscr{F}^Q(\mathscr{H})} = \big(I^{(k)}_{\sharp_1,\dots,\sharp_k}(h_1\chi_{A_1}, \dots, h_k\chi_{A_k}, 1)F, G\big)_{\mathscr{F}^Q(\mathscr{H})}$$

$$= \mathfrak{m}^{(k)}_{\sharp_1,\dots,\sharp_k}[F; G; h_1, \dots, h_k](A_1 \times \cdots \times A_k),$$

which implies (47). $\qquad\qquad\square$

We note that, in this section, we have not yet used the value of the function Q on the diagonal.

Proposition 13. *The following relation between operators ∂_x^+ and ∂_y^- holds in the obtained representation of the Q-CR algebra in the Q-Fock space $\mathscr{F}^Q(\mathscr{H})$:*

$$\partial_x^+ \partial_y^- = Q(x, y)\partial_y^- \partial_x^+ - \eta\delta(x, y). \tag{48}$$

Here $\int_{X^2} f^{(2)}(x, y)\delta(x, y)\, m^{\otimes 2}(dx\, dy) := \int_X f^{(2)}(x, x)\, m(dx)$. A rigorous meaning of relation (48) *is given analogously to formulas* (30)–(32) *(see also formulas* (50), (52) *below).*

Proof. To simplify notation, let us consider the case of an operator-valued integral $I^{(2)}_{+,-}(h_1, h_2, \varkappa^{(2)})$. Since X^2 can be represented as the disjoint union of $\widetilde{X}^2$ and Δ, we have

$$I^{(2)}_{+,-}(h_1, h_2, \varkappa^{(2)}) = I^{(2)}_{+,-}(h_1, h_2, \varkappa^{(2)}\chi_{\widetilde{X}^2}) + I^{(2)}_{+,-}(h_1, h_2, \varkappa^{(2)}\chi_\Delta).$$

But $I^{(2)}_{+,-}(h_1, h_2, \varkappa^{(2)}\chi_\Delta) = I^{(2)}_{+,-}(h_1, h_2, 0) = 0$. Hence, in view of the relation

$$\partial_x^- \partial_y^+ = \delta(x, y) + Q(x, y)\partial_y^+ \partial_x^-,$$

we get

$$I^{(2)}_{+,-}(h_1, h_2, \varkappa^{(2)}) = I^{(2)}_{+,-}(h_1, h_2, \varkappa^{(2)}\chi_{\widetilde{X}^2}) = I^{(2)}_{-,+}(h_2, h_1, (\Psi_1'\varkappa^{(2)})\chi_{\widetilde{X}^2}), \tag{49}$$

where

$$\Psi_1'\varkappa^{(2)}(x_1, x_2) = \varkappa^{(2)}(x_2, x_1)Q(x_2, x_1). \tag{50}$$

On the other hand

$$I^{(2)}_{-,+}(h_2, h_1, \Psi_1'\varkappa^{(2)})$$

$$= I^{(2)}_{-,+}(h_2, h_1, (\Psi_1'\varkappa^{(2)})\chi_{\widetilde{X}^2}) + \int_X h_1(x)h_2(x)\varkappa^{(2)}(x, x)Q(x, x)\, m(dx)$$

$$= I^{(2)}_{-,+}(h_2, h_1, (\Psi_1'\varkappa^{(2)})\chi_{\widetilde{X}^2}) + \eta\int_X h_1(x)h_2(x)\varkappa^{(2)}(x, x)\, m(dx). \tag{51}$$

By (49) and (51),

$$I^{(2)}_{+,-}(h_1, h_2, \varkappa^{(2)}) = I^{(2)}_{-,+}(h_2, h_1, \Psi'_1 \varkappa^{(2)}) - \eta \int_X h_1(x) h_2(x) \varkappa^{(2)}(x, x) \, m(dx). \qquad (52)$$

$\square$

Remark 14. Assume that $X = \mathbb{R}^d$ with $d \geq 2$, m is the Lebesgue measure on X and $Q(x, y) = Q(x^1, y^1)$. Then, we have the inclusion $\mathbb{F}^{(k)}_{\sharp_1, \ldots, \sharp_k} \subset \mathbb{G}^{(k)}_{\sharp_1, \ldots, \sharp_k}$, where we identify each function $v^{(k)}(s_1, \ldots, s_k) \in L^\infty(\mathbb{R}^k \to \mathbb{C}, \nu^{(k)}_{\sharp_1, \ldots, \sharp_k})$ with the function

$$\varkappa^{(k)}(x_1, \ldots, x_k) := v^{(k)}(x^1_1, \ldots, x^1_k) \in L^\infty(X^k \to \mathbb{C}, m^{(k)}_{\sharp_1, \ldots, \sharp_k}).$$

Thus, the above constructed operator-valued integrals give a representation of the Q-CR algebra in the Fock space $\mathscr{F}^Q(\mathscr{H})$.

4 Construction of gauge-invariant quasi-free states

In this section we again assume $X = \mathbb{R}^d$ with $d \geq 2$, m is the Lebesgue measure, $Q(x, y) = Q(x^1, y^1)$ for $(x, y) \in \widetilde{X}^2$ and $Q(x, y) = \eta \in \mathbb{R}$ for $(x, y) \in \Delta$.

4.1 The operators K_1, K_2

We fix continuous linear operators K_1 and K_2 in $\mathscr{H}$. We assume that these operators satisfy the following condition.

(C) For a bounded measurable function $\psi : \mathbb{R} \to \mathbb{R}$, let M_ψ denote the continuous linear operator in $\mathscr{H}$ given by

$$(M_\psi f)(x^1, \ldots, x^d) := \psi(x^1) f(x^1, \ldots, x^d), \quad f \in \mathscr{H}.$$

Then, for any bounded measurable function $\psi : \mathbb{R} \to \mathbb{R}$, both operators K_1 and K_2 commute with M_ψ.

Remark 15. Condition (C) implies that, for any bounded measurable function $\psi : \mathbb{R} \to \mathbb{R}$, both operators K_1^* and K_2^* commute with M_ψ.

Remark 16. Condition (C) is satisfied if $K_i = \mathbf{1} \otimes \widetilde{K}_i$ $(i = 1, 2)$, where $\widetilde{K}_1$ and $\widetilde{K}_2$ are any continuous linear operators in $L^2(\mathbb{R}^{d-1}, dx^2 \cdots dx^d)$. In the general case, the operators K_i have the following structure:

$$(K_i f)(x^1, x^2, \ldots, x^d) = \big(\widetilde{K}_i(x^1) f(x^1, \cdot)\big)(x^2, \ldots, x^d), \quad i = 1, 2,$$

where, for each $x^1 \in \mathbb{R}$, $K_i(x^1)$ is a continuous linear operator in $L^2(\mathbb{R}^{d-1}, dx^2 \cdots dx^d)$ such that K_i is a continuous linear operator in $\mathscr{H}$.

Remark 17. The results of this section with a proper modification will also hold for $X = \mathbb{R}$. In this case, condition (C) just means that both K_1 and K_2 are multiplication operators. In fact, under the latter assumption, we could deal with an arbitrary locally compact Polish space X and a function Q as in Section 3.

We extend the operators K_1 and K_2 by linearity to $\mathscr{H}_{\mathbb{C}}$, the complexification of $\mathscr{H}$.

Remark 18. Note that, for each $h \in \mathscr{H}_{\mathbb{C}}$, we get $\overline{K_i h} = K_i \bar{h}$ and similarly for K_i^*, $i = 1, 2$.

Let $i_1, \ldots, i_k \in \{1, 2\}$ and let us consider the operator $K_{i_1} \otimes \cdots \otimes K_{i_k}$ in $\mathscr{H}_{\mathbb{C}}^{\otimes k}$. Let $h_1, \ldots, h_k \in \mathscr{H}_{\mathbb{C}}$ and let $v^{(k)} \in L^\infty(\mathbb{R}^k, \nu^{(k)})$. Then

$$\varphi^{(k)}(x_1, \ldots, x_k) := h_1(x_1) \cdots h_k(x_k) v^{(k)}(x_1^1, \ldots, x_k^1)$$

belongs to $\mathscr{H}_{\mathbb{C}}^{\otimes k}$. By using condition (C), we get the following equality in $\mathscr{H}_{\mathbb{C}}^{\otimes k}$, hence $m^{\otimes k}$-a.e.:

$$(K_{i_1} \otimes \cdots \otimes K_{i_k})\varphi^{(k)}(x_1, \ldots, x_k) = (K_{i_1} h_1)(x_1) \cdots (K_{i_k} h_k)(x^k) v^{(k)}(x_1^1, \ldots, x_k^1).$$

Hence, we may define a linear operator

$$K_{i_1} \otimes \cdots \otimes K_{i_k} : \mathbb{F}^{(k)}_{\sharp_1, \ldots, \sharp_k} \to \mathbb{F}^{(k)}_{\sharp_1, \ldots, \sharp_k}$$

by

$$(K_{i_1} \otimes \cdots \otimes K_{i_k})\Phi^{(k)}_{\sharp_1, \ldots, \sharp_k}\left[h_1, \ldots, h_k, v^{(k)}\right] := \Phi^{(k)}_{\sharp_1, \ldots, \sharp_k}\left[K_{i_1} h_1, \ldots, K_{i_k} h_k, v^{(k)}\right]. \quad (53)$$

Indeed, the action of $K_{i_1} \otimes \cdots \otimes K_{i_k}$ onto $\varphi^{(k)} \in \mathbb{F}^{(k)}_{\sharp_1, \ldots, \sharp_k}$ is independent of the representation (26).

4.2 The representation of the Q-CR algebra corresponding to the operators K_1, K_2

Given operators K_1, K_2 satisfying condition (C), we will now construct a corresponding representation of the Q-CR algebra. Our construction is reminiscent of construction of quasi-free states for the CCR and CAR cases using the representations of Araki, Woods [3] and Araki, Wyss [4], respectively.

Let X_1 and X_2 denote two copies of the space X. Let $Z := X_1 \sqcup X_2$ denote the disjoint union of X_1 and X_2. Thus, $Z = X \times \{1, 2\}$. We equip Z with the product topology of the space X and the trivial one on $\{1, 2\}$. In particular, Z is a locally compact Polish space. With an abuse of notation, we define a measure m on $(Z, \mathscr{B}(Z))$ so that the restriction of this measure to X_1 (or X_2, respectively) coincides with the measure m on $(X, \mathscr{B}(X))$. In particular, we get

$$L^2(Z \to \mathbb{C}, m) = L^2(X_1 \to \mathbb{C}, m) \oplus L^2(X_2 \to \mathbb{C}, m) = \mathscr{H}_{\mathbb{C}} \oplus \mathscr{H}_{\mathbb{C}}.$$

On some occasions, we will identify a point $(x, y) \in Z^2$ with the corresponding point $(x, y) \in X^2$, i.e., we forget which of the two copies of the space X the points x and y belong to. So, again with an abuse of notation, we define a subset Δ of Z^2 which consists of those points $(x, y) \in Z^2$ for which $(x, y) \in \Delta$, where the latter Δ is the above introduced subset of X^2. Note that $m^{\otimes 2}(\Delta) = 0$ and Δ contains the diagonal in Z^2. Similarly, if $\phi : X^2 \to \mathbb{C}$ is a function on X^2 and if $(x, y) \in Z^2$, we will denote by $\phi(x, y)$ the value of the function ϕ at the corresponding point $(x, y) \in X^2$.

Let a function $JQ : Z^2 \to \mathbb{C}$ be defined by

$$JQ(x, y) := \begin{cases} Q(x, y), & \text{if } x, y \in X_1 \text{ or } x, y \in X_2, \\ Q(y, x), & \text{if } x \in X_1, \ y \in X_2 \text{ or } x \in X_2, \ y \in X_1. \end{cases} \tag{54}$$

In particular, $JQ(x, y) = \eta$ for all $(x, y) \in \Delta$ and $|JQ(x, y)| = 1$, $JQ(y, x) = \overline{JQ(x, y)}$ for all $(x, y) \in \widetilde{Z}^2 := Z^2 \setminus \Delta$.

So, according to Section 3, we can define the JQ-Fock space over $L^2(Z, m)$, i.e., $\mathscr{F}^{JQ}(L^2(Z, m))$. For $x \in X$, we denote by $\partial_{x,i}^+$ and $\partial_{x,i}^-$ ($i \in \{1, 2\}$) the creation and annihilation operators at the point x being identified with the corresponding point of X_i. Thus, analogously to (4), we may write, for $h \in \mathscr{H}_\mathbb{C}$,

$$a^+(h, 0) = \int_X m(dx) \, h(x) \partial_{x,1}^+, \qquad a^-(h, 0) = \int_X m(dx) \, \overline{h(x)} \, \partial_{x,1}^-,$$

$$a^+(0, h) = \int_X m(dx) \, h(x) \partial_{x,2}^+, \qquad a^-(0, h) = \int_X m(dx) \, \overline{h(x)} \, \partial_{x,2}^-. \tag{55}$$

We now define (informal) operators D_x^+ and D_x^- ($x \in X$) which satisfy, for each $h \in \mathscr{H}_\mathbb{C}$:

$$\int_X m(dx) \, h(x) D_x^+ := \int_X m(dx) \, (K_1 h)(x) \partial_{x,1}^- + \int_X m(dx) \, (K_2 h)(x) \partial_{x,2}^+, \tag{56}$$

$$\int_X m(dx) \, h(x) D_x^- := \int_X m(dx) \, (K_1 h)(x) \partial_{x,1}^+ + \int_X m(dx) \, (K_2 h)(x) \partial_{x,2}^-. \tag{57}$$

We will now show that, under the assumption (64) below, the operators D_x^+, D_x^- satisfy the Q-CR and lead to a representation of the Q-CR algebra. The latter algebra will be generated by the operator-valued integrals

$$\mathfrak{I}_{\sharp_1, \ldots, \sharp_k}^{(k)}(h_1, \ldots, h_k, v^{(k)})$$
$$:= \int_{X^k} m^{\otimes k}(dx_1 \cdots dx_k) \, h_1(x_1) \cdots h_k(x_k) v^{(k)}(x_1^1, \ldots, x_k^1) D_{x_1}^{\sharp_1} \cdots D_{x_k}^{\sharp_k}, \tag{58}$$

where $\sharp_1, \ldots, \sharp_k \in \{+, -\}$, $h_1, \ldots, h_k \in \mathscr{H}_\mathbb{C}$, and $v^{(k)} \in L^\infty(\mathbb{R}^k, \nu_{\sharp_1, \ldots, \sharp_k}^{(k)})$. In view of (53)–(57), we can now easily formalize the definition (58).

We define operators $\mathscr{K}_1 : \mathscr{H}_{\mathbb{C}} \to \mathscr{H}_{\mathbb{C}} \oplus \mathscr{H}_{\mathbb{C}}$ by

$$\mathscr{K}_1 h := (K_1 h, 0), \quad \mathscr{K}_2 h := (0, K_2 h), \quad h \in \mathscr{H}_{\mathbb{C}}. \tag{59}$$

We also denote

$$s(1,+) := -, \quad s(2,+) := +, \quad s(1,-) := +, \quad s(2,-) := -. \tag{60}$$

Using these notations, we can rewrite formulas (56), (57) as follows:

$$\int_X m(dx)\, h(x) D_x^+ = \int_Z m(dx)\, (\mathscr{K}_1 h)(x) \partial_x^{s(1,+)} + \int_Z m(dx)\, (\mathscr{K}_2 h)(x) \partial_z^{s(2,+)},$$

$$\int_X m(dx)\, h(x) D_x^- = \int_Z m(dx)\, (\mathscr{K}_1 h)(x) \partial_z^{s(1,-)} + \int_Z m(dx)\, (\mathscr{K}_2 h)(x) \partial_x^{s(2,-)}.$$

For $g_1, \ldots, g_k \in \mathscr{H}_{\mathbb{C}} \oplus \mathscr{H}_{\mathbb{C}}$ and $\varkappa^{(k)} \in L^\infty(Z^k \to \mathbb{C}, m_{\sharp_1, \ldots, \sharp_k}^{(k)})$, we denote by

$$I_{\sharp_1, \ldots, \sharp_k}^{(k)}(g_1, \ldots, g_k, \varkappa^{(k)}) \tag{61}$$

the corresponding operator-valued integral in the JQ-Fock space $\mathscr{F}^{JQ}(L^2(Z, m))$ acting on $\mathscr{F}_{\text{fin}}^{JQ}(L^2(Z, m))$ as defined in Section 3.

We now give a rigorous formulation of the definition (58). For any $\sharp_1, \ldots, \sharp_k \in \{+, -\}$, $h_1, \ldots, h_k \in \mathscr{H}_{\mathbb{C}}$, and $v^{(k)} \in L^\infty(\mathbb{R}^k, \nu_{\sharp_1, \ldots, \sharp_k}^{(k)})$, we define

$$\mathfrak{I}_{\sharp_1, \ldots, \sharp_k}^{(k)}(h_1, \ldots, h_k, v^{(k)})$$

$$:= \sum_{(i_1, \ldots, i_k) \in \{1,2\}^k} I_{s(i_1, \sharp_1), \ldots, s(i_k, \sharp_k)}^{(k)}(\mathscr{K}_{i_1} h_1, \ldots, \mathscr{K}_{i_k} h_k, \mathscr{R}_{i_1, \ldots, i_k}^{(k)} v^{(k)}), \tag{62}$$

where the function $\mathscr{R}_{i_1, \ldots, i_k}^{(k)} v^{(k)} \in L^\infty(Z^k \to \mathbb{C}, m_{\sharp_1, \ldots, \sharp_k}^{(k)})$ is given by

$$\left(\mathscr{R}_{i_1, \ldots, i_k}^{(k)} v^{(k)}\right)(x_1, \ldots, x_k) := \begin{cases} v^{(k)}(x_1^1, \ldots, x_k^1), & \text{if } (x_1, \ldots, x_k) \in X_{i_1} \times \cdots \times X_{i_k}, \\ 0, & \text{otherwise.} \end{cases} \tag{63}$$

Theorem 19. *Let K_1 and K_2 be continuous linear operators in $\mathscr{H}$ which satisfy condition (C) and*

$$K_2^* K_2 = \mathbf{1} + \eta K_1^* K_1. \tag{64}$$

Let the function $JQ : Z \to \mathbb{C}$ be defined by (54). Let for any $\sharp_1, \ldots, \sharp_k \in \{+, -\}$, $h_1, \ldots, h_k \in \mathscr{H}_{\mathbb{C}}$, and $v^{(k)} \in L^\infty(\mathbb{R}^k, \nu_{\sharp_1, \ldots, \sharp_k}^{(k)})$, a continuous linear operator

$$\mathfrak{I}_{\sharp_1, \ldots, \sharp_k}^{(k)}(h_1, \ldots, h_k, v^{(k)})$$

on $\mathscr{F}_{\text{fin}}^{JQ}(L^2(Z, m))$ be defined by (62). These operators satisfy the axioms (A1)–(A5). Thus, the algebra $\mathbf{A}$ generated by these operators gives a representation of the Q-CR algebra.

Proof. Axiom (A1) follows from condition (C) and the considerations in the last paragraph of subsection 4.1. Axiom (A2) is trivially satisfied. Axiom (A3) follows from the corresponding property of the operator-valued integrals (61) and and Remark 18. Similarly, axiom (A5) is trivially satisfied. So we only have to check the axiom (A4), i.e., the Q-CR relations.

We will prove (A4) only in the case $k = 2$. The general case will follow from the case $k = 2$ by an easy generalization, which we leave to the interested reader.

So let $h_1, h_2 \in \mathcal{H}_{\mathbb{C}}$ and $v^{(2)} \in L^\infty(\mathbb{R}^2 \to \mathbb{C}, \nu^{(2)}_{+,+})$, we get from (54), (59), (60), (62), (63), Proposition 13, and the JQ-CR satisfied by the operator-valued integrals (61):

$$
\begin{aligned}
\mathfrak{I}^{(2)}_{+,+}(h_1, h_2, v^{(2)}) &= I^{(2)}_{-,-}(\mathcal{K}_1 h_1, \mathcal{K}_1 h_2, \mathcal{R}^{(2)}_{1,1} v^{(2)}) + I^{(2)}_{+,+}(\mathcal{K}_2 h_1, \mathcal{K}_2 h_2, \mathcal{R}^{(2)}_{2,2} v^{(2)}) \\
&\quad + I^{(2)}_{-,+}(\mathcal{K}_1 h_1, \mathcal{K}_2 h_2, \mathcal{R}^{(2)}_{1,2} v^{(2)}) + I^{(2)}_{+,-}(\mathcal{K}_2 h_1, \mathcal{K}_1 h_2, \mathcal{R}^{(2)}_{2,1} v^{(2)}) \\
&= I^{(2)}_{-,-}(\mathcal{K}_1 h_2, \mathcal{K}_1 h_1, \mathcal{R}^{(2)}_{1,1} \Psi_1 v^{(2)}) + I^{(2)}_{+,+}(\mathcal{K}_2 h_2, \mathcal{K}_2 h_1, \mathcal{R}^{(2)}_{2,2} \Psi_1 v^{(2)}) \\
&\quad + I^{(2)}_{+,-}(\mathcal{K}_2 h_2, \mathcal{K}_1 h_1, \mathcal{R}^{(2)}_{2,1} \Psi_1 v^{(2)}) + I^{(2)}_{-,+}(\mathcal{K}_1 h_2, \mathcal{K}_2 h_1, \mathcal{R}^{(2)}_{1,2} \Psi_1 v^{(2)}) \\
&= \mathfrak{I}^{(2)}_{+,+}(h_2, h_1, \Psi_1 v^{(2)}).
\end{aligned}
$$

By taking the adjoint operators, this also implies that

$$
\mathfrak{I}^{(2)}_{-,-}(h_1, h_2, v^{(2)}) = \mathfrak{I}^{(2)}_{+,+}(h_2, h_1, \Psi_1 v^{(2)}).
$$

Using additionally Remark 15 and (64), we get

$$
\begin{aligned}
\mathfrak{I}^{(2)}_{-,+}(h_1, h_2, v^{(2)}) &= I^{(2)}_{+,-}(\mathcal{K}_1 h_1, \mathcal{K}_1 h_2, \mathcal{R}^{(2)}_{1,1} v^{(2)}) + I^{(2)}_{-,-}(\mathcal{K}_2 h_1, \mathcal{K}_1 h_2, \mathcal{R}^{(2)}_{2,1} v^{(2)}) \\
&\quad + I^{(2)}_{+,+}(\mathcal{K}_1 h_1, \mathcal{K}_2 h_2, \mathcal{R}^{(2)}_{1,2} v^{(2)}) + I^{(2)}_{-,+}(\mathcal{K}_2 h_1, \mathcal{K}_2 h_2, \mathcal{R}^{(2)}_{2,2} v^{(2)}) \\
&= I^{(2)}_{-,+}(\mathcal{K}_1 h_2, \mathcal{K}_1 h_1, \mathcal{R}^{(2)}_{1,1} \Psi'_1 v^{(2)}) - \eta \int_X (K_1 h_1)(x)(K_1 h_2)(x) v^{(2)}(x^1, x^1) \, m(dx) \\
&\quad + I^{(2)}_{-,-}(\mathcal{K}_1 h_2, \mathcal{K}_2 h_1, \mathcal{R}^{(2)}_{1,2} \Psi'_1 v^{(2)}) + I^{(2)}_{+,+}(\mathcal{K}_2 h_2, \mathcal{K}_1 h_1, \mathcal{R}^{(2)}_{2,1} \Psi'_1 v^{(2)}) \\
&\quad + I^{(2)}_{+,-}(\mathcal{K}_2 h_2, \mathcal{K}_2 h_1, \mathcal{R}^{(2)}_{2,2} \Psi'_1 v^{(2)}) + \int_X (K_2 h_1)(x)(K_2 h_2)(x) v^{(2)}(x^1, x^1) \, m(dx) \\
&= \mathfrak{I}^{(2)}_{+,-}(h_2, h_1, \Psi'_1 v^{(2)}) + \int_X \left((-\eta K_1^* K_1 + K_2^* K_2) h_1\right)(x) h_2(x) v^{(2)}(x^1, x^1) \, m(dx) \\
&= \mathfrak{I}^{(2)}_{+,-}(h_2, h_1, \Psi'_1 v^{(2)}) + \int_X h_1(x) h_2(x) v^{(2)}(x^1, x^1) \, m(dx).
\end{aligned}
$$

$\square$

Corollary 20. *Let $k \in \mathbb{N}$, $k \geq 2$. Let $q \in \mathbb{C}$ be such that $q \neq 1$ and $q^k = 1$. Then, for each $h \in \mathcal{H}_{\mathbb{C}}$,*

$$
\left(\int_X m(dx) \, h(x) D_x^+ \right)^k = 0.
$$

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Proof. Immediate by Theorems 6 and 19. □

Remark 21. In fact, a more general statement can be shown on the space $\mathscr{F}_{\mathrm{fin}}^{JQ}(L^2(Z,m))$: *Let the conditions of Corollary 20 be satisfied. Then, for any $h_1, h_2 \in \mathscr{H}_{\mathbb{C}}$, we have*

$$\left(\int_X m(dx)h_1(x)\partial_{x,1}^- + \int_X m(dx)h_2(x)\partial_{x,2}^+ \right)^k = 0. \tag{65}$$

We leave the (nontrivial) proof of formula (65) to the interested reader.

4.3 The associated state

Let $\mathbf{A}$ be the complex algebra from Theorem 19. We define a state τ on $\mathbf{A}$ by

$$\tau(a) := (a\Omega, \Omega)_{\mathscr{F}^{JQ}(L^2(Z,m))}, \quad a \in \mathbf{A}. \tag{66}$$

Theorem 22. *The state τ defined by (66) is a gauge-invariant quasi-free state on the Q-CR algebra $\mathbf{A}$. The state τ is completely determined by the self-adjoint positive operator $T := K_1^* K_1$ in $\mathscr{H}_{\mathbb{C}}$ and satisfies, for any $g, h \in \mathscr{H}_{\mathbb{C}}$ and $v^{(2)} \in L^\infty(\mathbb{R}^2, \nu_{+,-}^{(2)})$,*

$$\mathbf{S}^{(1,1)}(g, h, v^{(2)}) = \int_X g(x)(Th)(x)v^{(2)}(x^1, x^1)\, m(dx), \tag{67}$$

or equivalently

$$\mathbf{S}^{(1,1)}[g, h](s_1, s_2) = \chi_D(s_1, s_2) \int_{\mathbb{R}^{d-1}} g(s_1, x^2, \dots, x^d)(Th)(s_1, x^2, \dots, x^d)\, dx^2 \cdots dx^d. \tag{68}$$

In formula (68), $D = \{(s_1, s_2) \in \mathbb{R}^2 \mid s_1 = s_2\}$.

Remark 23. Note that, by (64), in the case $\eta < 0$, the operator T additionally satisfies $T \le -1/\eta$.

Remark 24. Note that the representation of the Q-CR algebra from Theorem 19 depends on operators K_1 and K_2 satisfying (64), while the state τ from Theorem 22 depends only on $|K_1| := \sqrt{K_1^* K_1}$.

Proof of Theorem 22. For any $g_1, \dots, g_k, h_1, \dots, h_n \in \mathscr{H}_{\mathbb{C}}$ and $v^{(k+n)} \in L^\infty(\mathbb{R}^{k+n}, \nu_{\sharp_1, \dots, \sharp_{k+n}}^{(k+n)})$ with $\sharp_1 = \cdots = \sharp_k = +$, $\sharp_{k+1} = \cdots = \sharp_{k+n} = -$, we have by (59), (60), (62), (63), and (66),

$$\mathbf{S}^{(k,n)}(g_k, \dots, g_1, h_1, \dots, h_n, v^{(k+n)})$$
$$= \left(\mathfrak{I}_{\sharp_1, \dots, \sharp_{k+n}}^{(k+n)}(g_k, \dots, g_1, h_1, \dots, h_n, v^{(k+n)})\Omega, \Omega \right)_{\mathscr{F}^{JQ}(L^2(Z,m))}$$
$$= \left(I_{\sharp_{k+1}, \dots, \sharp_{k+n}, \sharp_1, \dots, \sharp_k}^{(k+n)}(\mathscr{K}_1 g_k, \dots, \mathscr{K}_1 g_1, \right.$$

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$$\Big(\mathscr{K}_1 h_1, \ldots, \mathscr{K}_1 h_n, \mathscr{R}_{1,\ldots,1}^{(k+n)} v^{(k+n)}\Big)\Omega, \Omega\Big)_{\mathscr{F}^{JQ}(L^2(Z,k))}. \tag{69}$$

Hence $\mathbf{S}^{(k,n)} \equiv 0$ if $k \neq n$, and for $k = n$ we get from (69):

$$\mathbf{S}^{(n,n)}(g_n, \ldots, g_1, h_1, \ldots, h_n, v^{(2n)})$$
$$= \Big(I_{\sharp_{n+1},\ldots,\sharp_{2n},\sharp_1,\ldots,\sharp_n}^{(2n)}\big(K_1 g_n, \ldots, K_1 g_1, K_1 h_1, \ldots, K_1 h_n, v^{(2n)}\big)\Omega, \Omega\Big)_{\mathscr{F}^Q(L^2(X,m))}. \tag{70}$$

Note that formulas (67), (68) trivially follow from (70) with $n = 1$.

For the general $n \in \mathbb{N}$, analogously to the proof of Lemma 12, it suffices to consider the case where

$$v^{(2n)}(s_1, \ldots, s_{2n}) = u_n(s_1) \cdots u_1(s_n) w_1(s_{n+1}) \cdots w_n(s_{2n}), \tag{71}$$

where $u_1, \ldots, u_n, w_1, \ldots, w_n$ are indicator functions of sets from $\mathscr{B}(\mathbb{R})$. Denote

$$g_i'(x) := g_i(x) u_i(x^1), \quad h_i'(x) := h_i(x) w_i(x^1), \quad x \in X, \; i = 1, \ldots, n.$$

We get from (A1), Remarks 15, 18, and formulas (10), (70), (71)

$$\mathbf{S}^{(n,n)}(g_n, \ldots, g_1, h_1, \ldots, h_n, v^{(2n)})$$
$$= \Big(I_{\sharp_{n+1},\ldots,\sharp_{2n},\sharp_1,\ldots,\sharp_n}^{(2n)}\big(K_1 g_n', \ldots, K_1 g_1', K_1 h_1', \ldots, K_1 h_n', 1\big)\Omega, \Omega\Big)_{\mathscr{F}^Q(L^2(X,m))}$$
$$= \Big(\int_X m(dx_n)\,(K_1 g_n')(x_n)\partial_{x_n}^-$$
$$\qquad \cdots \int_X m(dx_1)\,(K_1 g_1')(x_1)\partial_{x_1}^-\big((K_1 h_1') \circledast \cdots \circledast (K_1 h_n')\big)\Omega, \Omega\Big)_{\mathscr{F}^Q(L^2(X,m))}$$
$$= \big((K_1 h_1') \circledast \cdots \circledast (K_1 h_n'), (\overline{K_1 g_1'}) \circledast \cdots \circledast (\overline{K_1 g_n'})\big)_{\mathscr{F}^Q(L^2(X,m))}$$
$$= n!\,\big(P_n(K_1 h_1') \otimes \cdots \otimes (K_1 h_n'), (K_1 \overline{g_1'}) \otimes \cdots \otimes (K_1 \overline{g_n'})\big)_{\mathscr{H}_{\mathbb{C}}^{\otimes n}}$$
$$= \sum_{\pi \in S_n} \big(\big[(K_1 h_{\pi(1)}') \otimes \cdots \otimes (K_1 h_{\pi(n)}')\big]Q_\pi, (K_1 \overline{g_1'}) \otimes \cdots \otimes (K_1 \overline{g_n'})\big)_{\mathscr{H}_{\mathbb{C}}^{\otimes n}}$$
$$= \sum_{\pi \in S_n} \big((K_1^* \otimes \cdots \otimes K_1^*)Q_\pi(K_1 \otimes \cdots \otimes K_1)(h_{\pi(1)}' \otimes \cdots \otimes h_{\pi(n)}'), \overline{g_1'} \otimes \cdots \otimes \overline{g_n'}\big)_{\mathscr{H}_{\mathbb{C}}^{\otimes n}}$$
$$= \sum_{\pi \in S_n} \big(Q_\pi\big[(Th_{\pi(1)}') \otimes \cdots \otimes (Th_{\pi(n)}')\big], \overline{g_1'} \otimes \cdots \otimes \overline{g_n'}\big)_{\mathscr{H}_{\mathbb{C}}^{\otimes n}}$$
$$= \sum_{\pi \in S_n} \int_{X^n} Q_\pi(x_1^1, \ldots, x_n^1)(Th_{\pi(1)})(x_1)g_1(x_1) \cdots (Th_{\pi(n)})(x_n)g_n(x_n)$$
$$\qquad \times u_1(x_1^1) \cdots u_n(x_n^1) w_{\pi(1)}(x_1^1) \cdots w_{\pi(n)}(x_n^1)\,m^{\otimes n}(dx_1 \cdots dx_n)$$
$$= \sum_{\pi \in S_n} \int_{X^n} \Big(\prod_{i=1}^n g_i(x_i)(Th_{\pi(i)})(x_i)\Big)Q_\pi(x_1^1, \ldots, x_n^1)$$

$$\times\, v^{(2n)}(x_n^1, \ldots, x_1^1, x_{\pi^{-1}(1)}^1, \ldots, x_{\pi^{-1}(n)}^1)\, m^{\otimes n}(dx_1 \cdots dx_n)$$

$$= \sum_{\pi \in S_n} \int_{\mathbb{R}^n} \left(\prod_{i=1}^n \int_{\mathbb{R}^{d-1}} g_i(s_i, x^2, \ldots, x^d)(Th_{\pi(i)})(s_i, x^2, \ldots, x^d)\, dx^2 \cdots dx^d \right)$$

$$\times\, Q_\pi(s_1, \ldots, s_n) v^{(2n)}(s_n, \ldots, s_1, s_{\pi^{-1}(1)}, \ldots, s_{\pi^{-1}(n)})\, ds_1 \cdots ds_n$$

$$= \sum_{\pi \in S_n} \int_{\mathbb{R}^{2n}} \left(\prod_{i=1}^n \chi_D(s_i, s_{n+\pi(i)}) \int_{\mathbb{R}^{d-1}} g_i(s_i, x^2, \ldots, x^d)(Th_{\pi(i)})(s_i, x^2, \ldots, x^d)\, dx^2 \cdots dx^d \right)$$

$$\times\, Q_\pi(s_1, \ldots, s_n) v^{(2n)}(s_n, \ldots, s_1, s_{n+1}, \ldots, s_{2n})\, \nu^{(2n)}_{\sharp_1, \ldots, \sharp_{2n}}(ds_1 \cdots ds_{2n})$$

$$= \int_{\mathbb{R}^{2n}} \sum_{\pi \in S_n} \left(\prod_{i=1}^n \mathbf{S}^{(1,1)}[g_i, h_{\pi(i)}](s_i, s_{n+\pi(i)}) \right)$$

$$\times\, Q_\pi(s_1, \ldots, s_n) v^{(2n)}(s_n, \ldots, s_1, s_{n+1}, \ldots, s_{2n})\, \nu^{(2n)}_{\sharp_1, \ldots, \sharp_{2n}}(ds_1 \cdots ds_{2n}). \tag{72}$$

Hence, by (39), the theorem follows. $\qquad\square$

The following corollary immediately follows from (68) and (72). The reader is advised to compare it with formulas (18)–(20).

Corollary 25. *Let $v^{(2n)} : \mathbb{R}^{2n} \to \mathbb{C}$ be identically equal to 1. Then, for any $g_1, \ldots, g_n$, $h_1, \ldots, h_n \in \mathscr{H}_\mathbb{C}$,*

$$\mathbf{S}^{(n,n)}(g_n, \ldots, g_1, h_1, \ldots, h_n, 1)$$

$$= \sum_{\pi \in S_n} \int_{X^n} \left(\prod_{i=1}^n g_i(x_i)(Th_{\pi(i)})(x_i) \right) Q_\pi(x_1, \ldots, x_n)\, m^{\otimes n}(dx_1 \cdots dx_n).$$

Corollary 26. *Let $\eta \geq 0$. Let T be a continuous linear operator in $\mathscr{H}$ that is self-adjoint and positive. Assume that, for any bounded measurable function $\psi : \mathbb{R} \to \mathbb{R}$, the operator T commutes with M_ψ (see condition (C)). Extend T by linearity to $\mathscr{H}_\mathbb{C}$. Then there exists a gauge-invariant quasi-free state τ on the Q-CR algebra $\mathbf{A}$ that satisfies (67).*

If $\eta < 0$, the latter statement remains true if the operator T additionally satisfies $0 \leq T \leq -1/\eta$.

Proof. Choose $K_1 := \sqrt{T}$, $K_2 := \sqrt{1 + \eta T}$. Note that K_1 and K_2 satisfy condition (C), see Remark 16. Now the corollary follows from Theorem 22. $\qquad\square$

5 Particle density

Let operators ∂_x^+, ∂_x^- $(x \in X)$ satisfy the ACR. We heuristically define the particle density by

$$\rho(x) := \partial_x^+ \partial_x^-, \quad x \in X.$$

It follows from the Q-CR that these operators commute, cf. [19, 22]. Indeed, for any $x, y \in X$,

$$\begin{aligned}
\rho(x)\rho(y) &= \partial_x^+ \partial_x^- \partial_y^+ \partial_y^- = \delta(x-y)\partial_x^+ \partial_y^- + Q(x,y)\partial_x^+ \partial_y^+ \partial_x^- \partial_y^- \\
&= \delta(x-y)\partial_x^+ \partial_x^- + Q(y,x)\partial_y^+ \partial_x^+ \partial_y^- \partial_x^- \\
&= \delta(x-y)\partial_x^+ \partial_x^- + \partial_y^+ \partial_y^- \partial_x^+ \partial_x^- - \delta(x-y)\partial_x^+ \partial_x^- = \rho(y)\rho(x).
\end{aligned}$$

In this section, we will study the particle density corresponding to the gauge-invariant quasi-free state from Theorem 22 with $T = \kappa^2 \mathbf{1}$, where $\kappa > 0$ is a constant. In the case $\eta < 0$, we additionally assume that $\kappa^2 < 1/\eta$. Thus, we set $K_1 = \kappa \mathbf{1}$ and $K_2 = \sqrt{1 + \eta\kappa^2}\,\mathbf{1}$. We will see below that, in order to properly define $\int_X m(dx)\,\varphi(x)\rho(x)$ for a test function $\varphi : X \to \mathbb{R}$, we will need a certain renormalization.

We will also assume that $\eta = \Re q$. Note that, with this choice of η, we get

$$Q(x,x) = \frac{1}{2}\left(\lim_{x \to y,\ x^1 > y^1} Q(x,y) + \lim_{x \to y,\ x^1 < y^1} Q(x,y) \right), \quad x \in X.$$

We will use this value of $Q(x,x)$ as a 'limiting value' of $Q(x,y)$ $(x_1 \neq y_1)$ when performing renormalization.

5.1 Renormalization

We start with heuristic calculations. By (56) and (57) with the above choice of the operators K_1 and K_2, we get

$$\begin{aligned}
D_x^+ &= \kappa \partial_{x,1}^- + \sqrt{1 + \eta\kappa^2}\, \partial_{x,2}^+, \\
D_x^- &= \kappa \partial_{x,1}^+ + \sqrt{1 + \eta\kappa^2}\, \partial_{x,2}^-.
\end{aligned}$$

Hence, the corresponding particle density is given by

$$\rho(x) = D_x^+ D_x^- = \kappa\sqrt{1 + \eta\kappa^2}\, R(x), \tag{73}$$

where

$$R(x) := \partial_{x,2}^+ \partial_{x,1}^+ + \partial_{x,1}^- \partial_{x,2}^- + \beta^{-1}\partial_{x,1}^- \partial_{x,1}^+ + \beta \partial_{x,2}^+ \partial_{x,2}^- \tag{74}$$

with $\beta := \sqrt{\eta + \kappa^{-2}}$. We denote by $\mathscr{D}$ the space of all real-valued infinitely differentiable functions on X with compact support. For each $\varphi \in \mathscr{D}$, we denote $\rho(\varphi) := \int_X m(dx)\,\varphi(x)\rho(x)$. We denote by $\mathfrak{R}$ the real commutative algebra generated by $\rho(\varphi)$ $(\varphi \in \mathscr{D})$ and constants. The involution on this algebra is the indentity mapping. We would like to define a vacuum state τ on $\mathfrak{R}$ analogously to (66). However, we are not able to intepret $\rho(\varphi)$ as a linear operator in $\mathscr{F}^{JQ}(L^2(Z,m))$. So we need a proper renormalization. For this, as discussed in Introduction, we will use Ivanov's formula [25] in the form $\delta^2 = \delta$.

Below we will denote by $\odot$ symmetric tensor product. Let $\mathscr{D}^{\odot_{\mathrm{a}} n}$ denote the nth symmetric algebraic tensor power of $\mathscr{D}$, i.e., the vector space of finite sums of functions on X^n of the form $f_1 \odot \cdots \odot f_n$, where $f_1, \ldots, f_n \in \mathscr{D}$. For each $f^{(n)} \in \mathscr{D}^{\odot_{\mathrm{a}} n}$, we denote

$$W(f^{(n)}) := \int_{X^n} m^{\otimes n}(dx_1 \cdots dx_n) f^{(n)}(x_1, \ldots, x_n) \partial^+_{x_1,2} \partial^+_{x_1,1} \cdots \partial^+_{x_n,2} \partial^+_{x_n,1} \Omega.$$

We stress that $W(f^{(n)})$ is treated as a formal expression. Note that, for $f_1, \ldots, f_n \in \mathscr{D}$,

$$W(f_1 \odot \cdots \odot f_n) = \left(\int_X m(dx_1) f_1(x_1) \partial^+_{x_1,2} \partial^+_{x_1,1} \cdots \int_X m(dx_n) f_n(x_n) \partial^+_{x_n,2} \partial^+_{x_n,1} \right) \Omega,$$

where we used that $\partial^+_{x_i,2} \partial^+_{x_i,1}$ and $\partial^+_{x_j,2} \partial^+_{x_j,1}$ commute for $i \neq j$. We also set $\mathscr{D}^{\odot_{\mathrm{a}} 0} := \mathbb{R}$ and for $f^{(0)} \in \mathbb{R}$, we set $W(f^{(0)}) := f^{(0)} \Omega$. We denote by $\mathscr{F}_{\mathrm{fin}}(\mathscr{D})$ the real linear space of vectors of the form

$$F = (f^{(0)}, f^{(1)}, \ldots, f^{(n)}, 0, 0, \ldots),$$

where $f_i \in \mathscr{D}^{\odot_{\mathrm{a}} i}$. For such $F \in \mathscr{F}_{\mathrm{fin}}(\mathscr{D})$, we denote

$$W(F) := \sum_{i=0}^{n} W(f^{(i)}).$$

The following proposition will be central for our considerations.

Proposition 27. *For each $\varphi \in \mathscr{D}$, we define a linear operator*

$$\hat{R}(\varphi) : \mathscr{F}_{\mathrm{fin}}(\mathscr{D}) \to \mathscr{F}_{\mathrm{fin}}(\mathscr{D})$$

by

$$\hat{R}(\varphi) := a^+(\varphi) + (\beta + \beta^{-1}\eta)a^0(\varphi) + a_1^-(\varphi) + \eta a_2^-(\varphi) + \beta^{-1} \int_X \varphi(x) \, m(dx). \qquad (75)$$

Here $a^+(\varphi)$ is the symmetric creation operator: for $f^{(n)} \in \mathscr{D}^{\odot_{\mathrm{a}} n}$ we have $a^+(\varphi)f^{(n)} := \varphi \odot f^{(n)}$; $a^0(\varphi)$ is the neutral operator: for $f_1, \ldots, f_n \in \mathscr{D}$

$$a^0(\varphi)f_1 \odot \cdots \odot f_n := \sum_{i=1}^{n} f_1 \odot \cdots \odot (\varphi f_i) \odot \cdots \odot f_n;$$

$a_1^-(\varphi)$ is the annihilation operator:

$$a_1^-(\varphi)f_1 \odot \cdots \odot f_n := \sum_{i=1}^{n} \int_X \varphi(x) f_i(x) \, m(dx) \, f_1 \odot \cdots \odot \check{f}_i \odot \cdots \odot f_n,$$

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where $\check{f}_i$ denotes the absence of f_i; and $a_2^-(\varphi)$ is an annihilation operator which acts as follows:

$$a_2^-(\varphi)f_1 \odot \cdots \odot f_n := \sum_{i=1}^n \sum_{j \neq i} f_1 \odot \cdots \odot f_{\min\{i,j\}-1} \odot (\varphi f_i f_j) \odot \cdots \odot \check{f}_{\max\{i,j\}} \odot \cdots \odot f_n.$$

Let

$$R(\varphi) := \int_X m(dx)\varphi(x)R(x) = \left(\kappa\sqrt{1+\eta\kappa^2}\right)^{-1}\rho(\varphi).$$

Then, the JQ-commutation relations satisfied by $\partial_{x,i}^+$, $\partial_{x,i}^-$, the condition $\partial_{x,i}^-\Omega = 0$, and the renormalization formula $\delta^2 = \delta$ imply that, for each $F \in \mathscr{F}_{\text{fin}}(\mathscr{D})$,

$$R(\varphi)W(F) = W(\hat{R}(\varphi)F). \tag{76}$$

Proof. We trivially get

$$\int_X m(dy)\varphi(y)\partial_{y,2}^+\partial_{y,1}^+ W(F) = W(a^+(\varphi)F). \tag{77}$$

For each $f^{(n)} \in \mathscr{D}^{\odot_a n}$, we have

$$\int_X m(dy)\varphi(y)\partial_{y,1}^-\partial_{y,2}^- W(f^{(n)}) = \int_{X^{n+1}} m^{\otimes(n+1)}(dy\, dx_1 \cdots dx_n)\varphi(y)f^{(n)}(x_1,\ldots,x_n)$$
$$\times \partial_{y,1}^-\partial_{y,2}^-\partial_{x_1,2}^+\partial_{x_1,1}^+ \cdots \partial_{x_n,2}^+\partial_{x_n,1}^+\Omega. \tag{78}$$

Note that

$$\partial_{y,2}^-\partial_{x_i,2}^+\partial_{x_i,1}^+ = \delta(y-x_i)\delta_{x_i,1}^+ + Q(y,x_i)\partial_{x_i,2}^+\partial_{y,2}^-\partial_{x_i,1}^+$$
$$= \delta(y-x_i)\delta_{x_i,1}^+ + \partial_{x_i,2}^+\partial_{x_i,1}^+\partial_{y,2}^-,$$

and for $i \neq j$

$$\partial_{x_i,1}^+\partial_{x_j,2}^+\partial_{x_j,1}^+ = \partial_{x_j,2}^+\partial_{x_j,1}^+\partial_{x_i,1}^+. \tag{79}$$

Hence,

$$\partial_{y,1}^-\partial_{y,2}^-\partial_{x_1,2}^+\partial_{x_1,1}^+ \cdots \partial_{x_n,2}^+\partial_{x_n,1}^+\Omega$$
$$= \sum_{i=1}^n \delta(y-x_i)\partial_{y,1}^-\partial_{x_1,2}^+\partial_{x_1,1}^+ \cdots \left(\partial_{x_i,2}^+,\partial_{x_i,1}^+\right)^{\vee} \cdots \partial_{x_n,2}^+\partial_{x_n,1}^+\partial_{x_i,1}^+\Omega. \tag{80}$$

Here and below $(\cdots)^{\vee}$ denotes the absence of the corresponding term. Similarly, we have

$$\partial_{y,1}^-\partial_{x_j,2}^+\partial_{x_j,1}^+ = Q(x_j,y)\partial_{x_j,2}^+\partial_{y,1}^-\partial_{x_j,1}^+$$

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$$= Q(x_j, y)\partial^+_{x_j,2}\delta(y - x_j) + \partial^+_{x_j,2}\partial^+_{x_j,1}\partial^-_{y,1}$$
$$= \eta\partial^+_{x_j,2}\delta(y - x_j) + \partial^+_{x_j,2}\partial^+_{x_j,1}\partial^-_{y,1}, \tag{81}$$

Thus, using (79) and (81), we continue (80) as follows:

$$= \eta\sum_{i=1}^n\sum_{j\neq i}\delta(y - x_i)\delta(y - x_j)\partial^+_{x_1,2}\partial^+_{x_1,1}\cdots\left(\partial^+_{x_{\max\{i,j\}},2}\partial^+_{x_{\max\{i,j\}},1}\right)^{\smile}\cdots\partial^+_{x_n,2}\partial^+_{x_n,1}\Omega$$

$$+ \sum_{i=1}^n\delta(y - x_i)\partial^+_{x_1,2}\partial^+_{x_1,1}\cdots\left(\partial^+_{x_i,2},\partial^+_{x_i,1}\right)^{\smile}\cdots\partial^+_{x_n,2}\partial^+_{x_n,1}\partial^-_{y,1}\partial^+_{x_i,1}\Omega. \tag{82}$$

We also note that

$$\delta(y - x_i)\partial^-_{y,1}\partial^+_{x_i,1}\Omega = \delta^2(y - x_i)\Omega = \delta(y - x_i)\Omega. \tag{83}$$

Hence, by (80), (82), and (83),

$$\int_X m(dy)\varphi(y)\partial^-_{y,1}\partial^-_{y,2}W(f^{(n)}) = W\left(\left(a^-_1(\varphi) + \eta a^-_2(\varphi)\right)f^{(n)}\right). \tag{84}$$

Similarly,

$$\partial^+_{y,2}\partial^-_{y,2}\partial^+_{x_1,2}\partial^+_{x_1,1}\cdots\partial^+_{x_n,2}\partial^+_{x_n,1}\Omega$$

$$= \partial^+_{y,2}\sum_{i=1}^n\delta(y - x_i)\partial^+_{x_1,2}\partial^+_{x_1,1}\cdots\left(\partial^+_{x_i,2}\right)^{\smile}\partial^+_{x_i,1}\cdots\partial^+_{x_n,2}\partial^+_{x_n,1}\Omega$$

$$= \sum_{i=1}^n\delta(y - x_i)\partial^+_{x_1,2}\partial^+_{x_1,1}\cdots\left(\partial^+_{x_i,2}\right)^{\smile}\partial^+_{y,2}\partial^+_{x_i,1}\cdots\partial^+_{x_n,2}\partial^+_{x_n,1}\Omega$$

$$= \sum_{i=1}^n\delta(y - x_i)\partial^+_{x_1,2}\partial^+_{x_1,1}\cdots\partial^+_{x_i,2}\partial^+_{x_i,1}\cdots\partial^+_{x_n,2}\partial^+_{x_n,1}\Omega,$$

which implies

$$\int_X m(dy)\varphi(y)\partial^+_{y,2}\partial^-_{y,2}W(f^{(n)}) = W(a^0(\varphi)f^{(n)}). \tag{85}$$

Finally,

$$\partial^-_{y,1}\partial^+_{y,1}\partial^+_{x_1,2}\partial^+_{x_1,1}\cdots\partial^+_{x_n,2}\partial^+_{x_n,1}\Omega$$

$$= \partial^-_{y,1}\partial^+_{x_1,2}\partial^+_{x_1,1}\cdots\partial^+_{x_n,2}\partial^+_{x_n,1}\partial^+_{y,1}\Omega$$

$$= \eta\sum_{i=1}^n\delta(y - x_i)\partial^+_{x_1,2}\partial^+_{x_1,1}\cdots\partial^+_{x_i,2}\left(\partial^+_{x_i,1}\right)^{\smile}\cdots\partial^+_{x_n,2}\partial^+_{x_n,1}\partial^+_{y,1}\Omega$$

$$+ \partial^+_{x_1,2}\partial^+_{x_1,1}\cdots\partial^+_{x_n,2}\partial^+_{x_n,1}\partial^-_{y,1}\partial^+_{y,1}\Omega$$

$$= \eta \sum_{i=1}^{n} \delta(y - x_i) \partial^+_{x_1,2} \partial^+_{x_1,1} \cdots \partial^+_{x_i,2} \partial^+_{x_i,1} \cdots \partial^+_{x_n,2} \partial^+_{x_n,1} \Omega$$
$$+ \partial^+_{x_1,2} \partial^+_{x_1,1} \cdots \partial^+_{x_n,2} \partial^+_{x_n,1} \Omega.$$

Here we used that

$$\int_X m(dy) \varphi(y) \partial^-_{y,1} \partial^+_{y,1} \Omega = \int_{X^2} m^{\otimes 2}(dy\,du) \delta(y - u) \varphi(y) \partial^-_{y,1} \partial^+_{u,1} \Omega$$

$$= \int_{X^2} m^{\otimes 2}(dy\,du) \delta^2(y - u) \varphi(y) \Omega$$

$$= \int_{X^2} m^{\otimes 2}(dy\,du) \delta(y - u) \varphi(y) \Omega = \int_X \varphi(y)\, m(dy) \Omega.$$

Thus,

$$\int_X m(dy)\, \varphi(y) \partial^-_{y,1} \partial^+_{y,1} W(f^{(n)}) = W(\eta a^0(\varphi) f^{(n)}) + \int_X \varphi(y)\, m(dy) W(f^{(n)}). \tag{86}$$

Now, formula (76) follows from (74), (77), (84)–(86). $\qquad\square$

Corollary 28. *We have*

$$\{r\Omega \mid r \in \mathfrak{R}\} = \{W(F) \mid F \in \mathscr{F}_{\mathrm{fin}}(\mathscr{D})\}.$$

More precisely, for each $F \in \mathscr{F}_{\mathrm{fin}}(\mathscr{D})$, there exists a unique $r \in \mathfrak{R}$ such that $W(F) = r\Omega$. This correspondence is given through formula (76). Vice versa, for each $r \in \mathfrak{R}$, there exists a unique $F \in \mathscr{F}_{\mathrm{fin}}(\mathscr{D})$ such that $W(F) = r\Omega$ holds.

Proof. By Proposition 27, we have, for $\varphi \in \mathscr{D}$,

$$R(\varphi)\Omega = W(\varphi) + \beta^{-1}\Omega.$$

Hence
$$W(\varphi) = (R(\varphi) - \beta^{-1})\Omega. \tag{87}$$

By (75) and (76), for any $n \geq 2$ and $f_1, \ldots, f_n \in \mathscr{D}$,

$$W(f_1 \odot \cdots \odot f_n) = R(f_1)W(f_2 \odot \cdots \odot f_n)$$
$$- W\left(\left[(\beta + \beta^{-1}\eta)a^0(f_1) + a^-_1(f_1) + \eta a^-_2(f_1) + \beta^{-1}\int_X f_1(x)\, m(dx)\right] f_2 \odot \cdots \odot f_n\right). \tag{88}$$

Hence, for $F \in \mathscr{F}_{\mathrm{fin}}(\mathscr{D})$, a unique representation of $W(F)$ as $r\Omega$ ($r \in \mathfrak{R}$) follows by induction on n and formulas (87), (88). The converse statement follows immediately from Proposition 27 by induction. $\qquad\square$

Corollary 28 implies that, for each $r \in \mathfrak{R}$, there exists a unique vector

$$F = (f^{(0)}, f^{(1)}, \ldots, f^{(n)}, 0, 0, \ldots) \in \mathscr{F}_{\mathrm{fin}}(\mathscr{D}) \tag{89}$$

such that

$$r\Omega = f^{(0)}\Omega + \sum_{i=1}^{n} \int_{X^i} m^{\otimes i}(dx_1 \cdots dx_i) f^{(i)}(x_1, \ldots, x_i) \partial^+_{x_1,2} \partial^+_{x_1,1} \cdots \partial^+_{x_i,2} \partial^+_{x_i,1} \Omega.$$

Thus, we can rigorously define a linear mapping $\tau : \mathfrak{R} \to \mathbb{R}$ by $\tau(r) := f^{(0)}$. However, since we used the renormalization, from our construction of τ is not *a priori* clear whether τ is positive definite, i.e., whether $\tau(r^2) \geq 0$ for each $r \in \mathfrak{R}$. So, our next aim is to decide whether positive definiteness holds.

5.2 Measure-valued Lévy processes and positive definiteness of τ

We start with preliminaries on measure-valued Lévy process. For more detail, see e.g. [23, 26–28, 31].

Let $\mathbb{M}(X)$ denote the space of all Radon measures on X. We equip $\mathbb{M}(X)$ with the topology of vague convergence. Let $\mathscr{B}(\mathbb{M}(X))$ denote the corresponding Borel σ-algebra on $\mathbb{M}(X)$. Let $(\Omega, \mathscr{A}, P)$ be a provability space. A *random measure on X* is a measurable mapping $\gamma : \Omega \to \mathbb{M}(X)$. A *competely random measure* is a random measure γ such that, for any mutually disjoint sets $A_1, \ldots, A_n \in \mathscr{B}_0(X)$, the random variables $\gamma(A_1), \ldots, \gamma(A_n)$ are independent. Here $\mathscr{B}_0(X)$ denotes the subset of $\mathscr{B}(X)$ that consists of all bounded sets from $\mathscr{B}(X)$. A *measure-valued Lévy process* is a completely random measure γ such that, for any sets $A_1, A_2 \in \mathscr{B}_0(X)$ with $m(A_1) = m(A_2)$, the random variables $\gamma(A_1)$ and $\gamma(A_2)$ are identically distributed.

It follows from [27] that a random measure γ is a measure-valued Lévy process if and only if there exist a constant $c \geq 0$ and a measure ζ on $\mathbb{R}_+ := (0, \infty)$ satisfying

$$\int_{\mathbb{R}_+} \min\{s, 1\} \zeta(ds) < \infty$$

such that the Laplace transform of γ is given by

$$\mathbb{E}\left(e^{\langle f, \gamma \rangle}\right) = \exp\left[c \int_X f(x) \, m(dx) + \int_X \int_{\mathbb{R}_+} (e^{sf(x)} - 1) \, \zeta(ds) \, m(dx)\right], \quad f \in C_0(X), \ f \leq 0. \tag{90}$$

Here $C_0(X)$ denotes the space of real-valued continuous functions on X with compact support and $\langle f, \gamma \rangle := \int_X f(x) \, \gamma(dx)$. The measure ζ is called the *Lévy measure of the measure-valued Lévy process γ*.

Let $\mathbb{K}(X)$ denote the subset of $\mathbb{M}(X)$ consisting of all discrete Radon measures on X, i.e., Radon measures of the form $\sum_{i \in I} s_i \delta_{x_i}$, where the set I is either finite or countable, $s_i > 0$, and δ_{x_i} denotes here the Dirac measure with mass at x_i. We also assume that $x_i \neq x_j$ if $i \neq j$. A *discrete random measure* is a random measure which takes values in $\mathbb{K}(X)$ a.s. Each measure-valued Lévy process γ for which $c = 0$ in formula (90) is a discrete random measure.

The space $\ddot{\Gamma}(X)$ of multiple configurations in X is the subset of $\mathbb{K}(X)$ which consists of all Radon measures of the form $\sum_{i \in I} s_i \delta_{x_i}$ with $s_i \in \mathbb{N}$. A *point process on X* is a random measure γ which takes values in $\ddot{\Gamma}(X)$ a.s.

The configuration space $\Gamma(X)$ is defined as the subset of $\ddot{\Gamma}(X)$ which consists of all Radon measures of the form $\sum_{i \in I} \delta_{x_i}$. Each Radon measure $\sum_{i \in I} \delta_{x_i}$ can be identified with the locally finite set $\{x_i \mid i \in I\} \subset X$. A *simple point process on X* is a random measure γ which takes values in $\Gamma(X)$ a.s.

It should be noted that if $c = 0$ and $\zeta(\mathbb{R}_+) < \infty$, the measure-valued Lévy process with Fourier transform (90) has the property that a.s. $\gamma = \sum_{i \in I} s_i \delta_{x_i}$, where the set $\{x_i \mid i \in I\}$ is locally finite, i.e., a configuration in X. On the other hand, if $\zeta(\mathbb{R}_+) = \infty$, the set $\{x_i \mid i \in I\}$ is a.s. dense in X.

By choosing $c = 0$ and the measure ζ in (90) to be $z\delta_1$ with $z > 0$, one obtains the simple point process γ with Laplace transform

$$\mathbb{E}\left(e^{\langle f, \gamma \rangle}\right) = \exp\left[\int_X \left(e^{f(x)} - 1\right) zm(dx)\right].$$

This γ is called the *Poisson point process with intensity measure zm*, since for each $A \in \mathscr{B}_0(X)$, the random variable $\gamma(A)$ has Poisson distribution with parameter $zm(A)$.

Theorem 29. *The functional τ on the real algebra $\mathfrak{R}$ is positive definite (i.e., $\tau(r^2) \geq 0$ for each $r \in \mathfrak{R}$) if and only if $\eta \geq 0$.*

Furthermore, if $\eta = 0$, then

$$\tau(\rho(f_1) \cdots \rho(f_n)) = \mathbb{E}\left(\langle f_1, \gamma \rangle \cdots \langle f_1, \gamma \rangle\right), \quad f_1, \ldots, f_n \in \mathscr{D}, \tag{91}$$

where γ is the Poisson point process on X with intensity measure $\kappa^2 m$.

If $\eta > 0$, then (91) holds with γ being a negative binomial point process. More precisely, γ is the measure-valued Lévy process with Laplace transform (90) in which $c = 0$ and

$$\zeta = \frac{1}{\eta} \sum_{k=1}^{\infty} \left(\frac{\eta}{\eta + \kappa^{-2}}\right)^k \frac{1}{k} \delta_k. \tag{92}$$

For each $A \in \mathscr{B}_0(X)$, the distribution of the random variable $\gamma(A)$ is the negative binomial distribution

$$(1 + \kappa^2 \eta)^{-m(A)/\eta} \sum_{n=0}^{\infty} \left(\frac{\eta}{\eta + \kappa^{-2}}\right)^n \frac{\left(\frac{m(A)}{\eta}\right)^{(n)}}{n!} \delta_n. \tag{93}$$

Here, we used the standard symbol $a^{(n)} := a(a+1)(a+2)\cdots(a+n-1)$, the so-called rising factorial.

The proof of Theorem 29 is based on the property of orthogonal polynomials of a Lévy white noise proved in [43, Theorem 2.1 and Corollaries 2.1, 2.3] and [5], see also [9, Theorem 1.2]. We will now briefly explain this result in the special case of a measure-valued Lévy process.

Let γ be a measure-valued Lévy process such that $c = 0$ in (90) (so that γ is a discrete random measure). We denote $\zeta'(ds) := s^2\zeta(ds)$, the so-called *Kolmogorov measure of the measure-valued Lévy process* γ. We assume that ζ' is a probability measure on $\mathbb{R}_+$, and furthermore,

$$\int_{\mathbb{R}_+} e^{\varepsilon s}\, \zeta'(ds) < \infty \quad \text{for some } \varepsilon > 0. \tag{94}$$

The latter assumption implies that the set of polynomials is dense in $L^2(\mathbb{R}_+, \zeta')$. If the support of the measure ζ' has infinitely many points, we will denote by $(p_k)_{k=0}^\infty$ the system of monic polynomials that are orthogonal with respect to ζ'. These polynomials satisfy the recurrence relation

$$sp_k(s) = p_{k+1}(s) + b_k p_k(s) + a_k p_{k-1}(s), \quad k = 0, 1, 2, \ldots \tag{95}$$

with $p_{-1}(s) := 0$, $a_k > 0$, and $b_k \in \mathbb{R}$.

Let assume that the σ-algebra $\mathscr{A}$ from the probability space $(\Omega, \mathscr{A}, P)$ is the minimal σ-algebra with respect to which $\gamma(A)$ is measurable for each $A \in \mathscr{B}_0(X)$. We denote by $\mathscr{CP}$ the *set of continuous polynomials of* γ, i.e., the set of random variables of the form

$$f^{(0)} + \sum_{i=1}^n \langle f^{(i)}, \gamma^{\otimes i}\rangle, \tag{96}$$

where $f^{(i)} \in \mathscr{D}^{\odot_a i}$ and $n \in \mathbb{N}$. If $f^{(n)} \neq 0$, we call the random variable in (96) a continuous polynomial of γ of degree n. Condition (94) implies that $\mathscr{CP}$ is a dense subset of $L^2(\Omega, P)$.

Let us denote $\mathscr{CP}_n$ the subset of $\mathscr{CP}$ which consists of all polynomials of γ of degree $\leq n$. Let $\mathscr{MP}_n$ denote the closure of $\mathscr{CP}_n$ in $L^2(\Omega, P)$ (*measurable polynomials of degree* $\leq n$). Let $\mathscr{OP}_n := \mathscr{MP}_n \ominus \mathscr{MP}_{n-1}$, where $\ominus$ means the orthogonal difference in $L^2(\Omega, P)$ (*orthogonal polynomials of degree n*). As a result, we get the orthogonal decomposition $L^2(\Omega, P) = \bigoplus_{n=0}^\infty \mathscr{OP}_n$. For $f^{(n)} \in \mathscr{D}^{\odot_a n}$, we denote by $P(f^{(n)})$ the orthogonal projection of the monomial $\langle f^{(n)}, \gamma^{\otimes n}\rangle$ onto $\mathscr{OP}_n$. Note that $P(f^{(n)})$ does not need to belong to $\mathscr{CP}$. For $F \in \mathscr{F}_{\text{fin}}(\mathscr{D})$ as in (89), we denote $P(F) := \sum_{i=0}^n P(f^{(i)})$. The set of all such random variables we denote by $\mathscr{OCP}$ (*orthogonalized continuous polynomials*).

Theorem 30. *Let γ be a measure-valued Lévy process that satisfies the above assumptions. We have $\mathscr{OCP} \subset \mathscr{CP}$ (and, in fact, $\mathscr{OCP} = \mathscr{CP}$) if and only is there exist constants $\eta \geq 0$ and $\lambda > 0$ with $\lambda \geq 2\sqrt{\eta}$ such that: if $\eta = 0$ then $\zeta = \lambda^{-2}\delta_\lambda$; and if $\eta > 0$ then the measure ζ' has infinitely many points in its support and the system $(p_k)_{k=0}^\infty$ of monic polynomials that are orthogonal with respect to ζ' satisfies the recurrence relation (95) with $a_k = \eta k(k+1)$ and $b_k = \lambda(k+1)$. Furthermore, for any η and λ as above, we have, for $\varphi \in \mathscr{D}$ and $F \in \mathscr{F}_{\mathrm{fin}}(\mathscr{D})$,*

$$\langle \varphi, \gamma \rangle P(F) = P\left(\left[a^+(\varphi) + \lambda a^0(\varphi) + a_1^-(\varphi) + \eta a_2^-(\varphi) + 2\big(\lambda + \sqrt{\lambda^2 - 4\eta}\big)^{-1} \right] F \right),$$

where the operators $a^+(\varphi)$, $a^0(\varphi)$, and $a_1^-(\varphi)$, and $a_2^-(\varphi)$ were defined in Proposition 27.

Remark 31. If $\eta > 0$ and $\lambda > 2\sqrt{\eta}$, we get from Theorem 30 and e.g. [12, Chapter VI, Section 3] that the Lévy measure of the corresponding measure-valued Lévy process γ is

$$\zeta = \frac{1}{\eta} \sum_{k=1}^\infty \left(\frac{\eta}{\upsilon^2} \right)^k \frac{1}{k} \, \delta_{(\upsilon - \frac{\eta}{\upsilon})k} \,,$$

with

$$\upsilon := (\lambda + \sqrt{\lambda^2 - 4\eta})/2,$$

and for $A \in \mathscr{B}_0(X)$, the distribution of the random variable $\gamma(A)$ is the negative binomial distribution

$$\left(\frac{\upsilon^2 - \eta}{\upsilon^2} \right)^{m(A)/\eta} \sum_{k=0}^\infty \left(\frac{\eta}{\upsilon^2} \right)^k \frac{1}{k!} \left(\frac{m(A)}{\eta} \right)^{(k)} \delta_{(\upsilon - \frac{\eta}{\upsilon})k} \,.$$

Thus, γ is a *negative binomial random measure*.

In the case where $\eta > 0$ and $\lambda = 2\sqrt{\eta}$, the Lévy measure of γ is

$$\zeta(ds) = \frac{1}{s\eta} \, e^{-\frac{s}{\sqrt{\eta}}} \, ds,$$

and for $A \in \mathscr{B}_0(X)$, the distribution of the random variable $\gamma(A)$ is the gamma distribution

$$\left[\eta^{\frac{m(A)}{2\eta}} \Gamma\left(\frac{m(A)}{\eta} \right) \right]^{-1} u^{\frac{m(\Delta)}{\eta} - 1} e^{-\frac{u}{\sqrt{\eta}}} \chi_{\mathbb{R}_+}(u) \, du.$$

Thus, γ in this case is the *gamma random measure*, see e.g. [29, 30, 47]. The Laplace transform of γ can also be written in the form

$$\mathbb{E}\left(e^{\langle f, \gamma \rangle} \right) = \exp\left[-\frac{1}{\eta} \int_X \log\big(1 - \sqrt{\eta} \, f(x) \big) \, m(dx) \right], \quad f \in C_0(X), \ f \leq 0. \tag{97}$$

Proof of Theorem 29. Let $\eta > 0$. Set $\lambda = \beta + \beta^{-1}\eta$. We have

$$2\big(\lambda + \sqrt{\lambda^2 - 4\eta}\,\big)^{-1} = \beta^{-1}.$$

As easily seen, $\beta > \sqrt{\eta}$. Hence,

$$\lambda = \beta + \frac{\eta}{\beta} > 2\sqrt{\eta}.$$

Let $\gamma_{\lambda,\eta}$ be the measure-valued Lévy process on X from Theorem 30 that corresponds to the parameters λ, η. By Proposition 27 and Theorem 30, we get

$$\tau\big(R(f_1)\cdots R(f_n)\big) = \mathbb{E}\big(\langle f_1, \gamma_{\lambda,\eta}\rangle \cdots \langle f_n, \gamma_{\lambda,\eta}\rangle\big), \quad f_1, \ldots, f_n \in \mathscr{D}. \tag{98}$$

We define the measure-valued Lévy process $\gamma := \kappa\sqrt{1 + \eta\kappa^2}\,\gamma_{\lambda,\eta}$. Let ζ and $\zeta_{\lambda,\eta}$ denote the Lévy measure of γ and $\gamma_{\lambda,\eta}$, respectively. Then ζ is the pushforward of the measure $\zeta_{\lambda,\eta}$ under the mapping $\mathbb{R}_+ \ni s \mapsto \kappa\sqrt{1 + \eta\kappa^2}\,s \in \mathbb{R}_+$. By (73) and (98), we get (91). Formulas (92), (93) follow by direct calculations from Remark 31. Note that, since the Lévy measure ζ is concentrated on $\mathbb{N}$, γ is a point process. The proof for $\eta = 0$ is analogous.

Let us now prove that τ is not positive definite for $\eta < 0$. Assume the contrary. Let $f \in \mathscr{D}$. We easily calculate

$$\tau\big(W(f \otimes f)^2\big) = 2\bigg(\int_X f^2(x)\,m(dx)\bigg)^2 + 2\eta \int_X f^4(x)\,m(dx) \ge 0.$$

From here we conclude by approximation that, for any cube A in X,

$$m(A)^2 + \eta m(A) \ge 0,$$

which obviously fails for A small enough. $\qquad\square$

Remark 32. In view of Theorem 29, we can rigorously understand $\rho(\varphi)$ $(\varphi \in \mathscr{D})$ as the operator of multiplication by $\langle \varphi, \gamma\rangle$ in the Hilbert space $L^2(\Omega, P)$, which maps $\mathscr{C}\mathscr{P}$ into itself. It follows from [43, Lemma 2.1 and Theorem 2.1] that each $\rho(\varphi)$ is essentially self-adjoint on $\mathscr{C}\mathscr{P}$.

Corollary 33. *Let $\eta > 0$. Denote by τ_κ the state τ on $\mathfrak{R}$ which corresponds to the operator $T = \kappa^2\mathbf{1}$. Let γ be the gamma random measure with Laplace transform* (97). *Then, for any $f_1, \ldots, f_n \in \mathscr{D}$, we have*

$$\lim_{\kappa \to \infty} \tau_\kappa\big(R(f_1)\cdots R(f_n)\big) = \lim_{\kappa \to \infty} \big(\kappa\sqrt{1 + \eta\kappa^2}\big)^{-n} \tau_\kappa\big(\rho(f_1)\cdots \rho(f_n)\big)$$
$$= \mathbb{E}\big(\langle f_1, \gamma\rangle \cdots \langle f_n, \gamma\rangle\big).$$

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Proof. The statement follows immediately from Proposition 27, Theorem 30, and Remark 31 if we note that

$$\lim_{\kappa\to\infty}\left((\eta+\kappa^{-2})^{1/2}+(\eta+\kappa^{-2})^{-1/2}\eta\right)=2\sqrt{\eta},\quad \lim_{\kappa\to\infty}(\eta+\kappa^{-2})^{-1/2}=\eta^{-1/2}.$$

$\square$

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