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Stochastic averaging principle for differential equations with non-Lipschitz coefficients driven by fractional Brownian motion

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In this paper, we are concerned with the stochastic averaging principle for stochastic differential equations (SDEs) with non-Lipschitz coefficients driven by fractional Brownian motion (fBm) of the Hurst parameter $H \in (\frac{1}{2}, 1)$. We define the stochastic integrals with respect to the fBm in the integral formulation of the SDEs as pathwise integrals and we adopt the non-Lipschitz condition proposed by Taniguchi (1992) which is a much weaker condition with wider range of applications. The averaged SDEs are established. We then use their corresponding solutions to approximate the solutions of the original SDEs both in the sense of mean square and of probability. One can find that the similar asymptotic results are suitable for those non-Lipschitz SDEs with fBm under different types of stochastic integrals.

Keywords: Stochastic differential equations; non-Lipschitz coefficients; fractional Brownian motion; stochastic averaging; pathwise integrals.

AMS Subject Classification: 34F05, 37H10, 60H10, 93E03

1. Introduction

We are concerned with the following SDEs with non-Lipschitz coefficients driven by fractional Brownian motion (fBm) on R

$$X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dB^H(s), \quad t \in [0, T], \tag{1.1}$$

where the initial data $X(0) = X_0$ is a random variable and $E|X_0|^2 < \infty$, $0 < T < \infty$, the process $B^H(t)$ represents the fBm with Hurst index $H \in (\frac{1}{2}, 1)$ defined in a complete probability space $(\Omega, \mathcal{F}, P)$ and $b(t, X(t)) : [0, T] \times R \to R$ is a measurable function, $\sigma(t, X(t)) : [0, T] \times R \to R \times R$ is a measurable function and $\int_0^t \cdot \, dB^H(s)$ stands for the stochastic integral with respect to fBm.

The above mentioned fBm, which is a family of Gaussian processes, was introduced by Kolmogorov [11], Hurst [8] and Mandelbrot and Van Ness [16]. Due to the long-range dependence of the fBm (expecially for $\frac{1}{2} < H < 1$), the SDEs driven by fBm have been used as models of a number of practical problems [2, 3, 19]. However, the powerful tools for the classical SDEs theory are not applicable, because the fBm is neither a semi-martingale nor a Markov process. This motivates us to investigate other techniques to study such SDEs with fBm.

Now, we would like to mention that the theory of the stochastic averaging for SDEs has been studied intensively. In Khasminskii [10], the author initiated to study a stochastic averaging method for SDEs with Gaussian random fluctuations. Stoyanov and Kolomiets [12, 20] investigated the stochastic averaging method for SDEs driven by Poisson noises. Xu [24, 26–28] proved the stochastic averaging for SDEs driven by Lévy noise or by fBm, where an averaged system is presented to replace the original one both in the sense of convergence in mean square and in probability.

In all the above works, one notices that the coefficients of SDEs are usually assumed to satisfy the Lipschitz condition. However, many practical models of SDEs do not satisfy the Lipschitz condition. For example, the one-dimensional semi-linear SDEs with Markov switching

$$dy(t) = a(r(t))y(t)dt + \tilde{b}(r(t))\tilde{\sigma}(|y(t)|)dW(t), t \in [0, T],$$

where the process $W(t)$ represents Brownian motion (Bm), $r(t)$ is a continuous-time Markov chain and $\tilde{\sigma} : R^+ \to R^+$ is defined in the following manner

$$\tilde{\sigma}(u) = \begin{cases} u\sqrt{-\ln(u)}, & 0 \leq u \leq e^{-1}, \\ e^{-1} + 0.5(u - e^{-1}), & u > e^{-1}. \end{cases}$$

Such models appear widely in many branches of science, engineering, industry and finance [4, 6, 13]. In view of the pressing need, the importance, and the impact on many diverse applications, it is necessary and also significant to consider some weaker conditions than the Lipschitz one. Fortunately, Yamada [29] and Taniguchi [21] have given much weaker conditions which are regarded as the so-called non-Lipschitz conditions.

Up to now, there are only a few results on the stochastic averaging of non-Lipschitz SDEs [25] and most of the papers concentrated on the case of the Lipschitz condition. Given the widespread applications of the long-range dependence of the fBm and non-Lipschitz SDEs, in this paper, we will make the first attempt to study the stochastic averaging for SDEs driven by fBm with non-Lipschitz coefficients proposed. Let us also point out that, the SDEs driven by fBm could not be treated by the method for SDEs driven by Bm. For example, due to the fact that the stochastic integral with respect to fBm is no longer a martingale, we definitely lost good inequalities such as Burkholder–Davis–Gundy inequality which is crucial for SDEs driven by Bm. This point motivates us to carry out the present study. As expected, our obtained convergence results is in general weaker than those results for SDEs driven by Bm.

We close this section by mentioning the paper Lan and Wu [14] for the most recent account of SDEs with (much weaker) non-Lipschitizian coefficients driven by Brownian motion. It would be interesting to investigate the stochastic averaging for SDEs with weaker non-Lipschitz coefficients, for instance, of Lan and Wu type, driven by fBm. This topic will be addressed in different work.

2. Preliminaries

In this section we present some notations, conceptions on the pathwise integrals with respect to fBm and we also introduce non-Lipschitz condition proposed by Taniguchi [21].

2.1. *Fractional Brownian motion*

Let $\varphi : R^+ \times R^+ \to R^+$ be defined by

$$\varphi(t,s) = H(2H-1)|t-s|^{2H-2}, \quad t,s \in R^+,$$

where H is a constant with $\frac{1}{2} < H < 1$. Let $g : R^+ \to R$ be Borel measurable. Define

$$L^2_\varphi(R^+) = \left\{ g : \|g\|^2_\varphi = \int_{R^+}\int_{R^+} g(t)g(s)\varphi(t,s)dsdt < \infty \right\}.$$

If we equip $L^2_\varphi(R^+)$ with the inner product

$$\langle g_1, g_2\rangle_\varphi = \int_{R^+}\int_{R^+} g_1(t)g_2(s)\varphi(t,s)dsdt, g_1, g_2 \in L^2_\varphi(R^+),$$

then $L^2_\varphi(R^+)$ becomes a separable Hilbert space.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F(\omega) = f\left(\int_0^T \psi_1(t)dB^H_t, \ldots, \int_0^T \psi_n(t)dB^H_t\right),$$

where $n \geq 1, f \in \mathcal{C}^\infty_b(R^n)$ (i.e. f and all its partial derivatives are bounded), and $\psi_i \in \mathcal{H}, i = 1,2,\ldots,n$. $\mathcal{H}$ is the completion of the measurable functions such that $\|\psi\|^2_\varphi < \infty$ and $\{\psi_n\}$ is a sequence in $\mathcal{H}$ such that $\langle\psi_i,\psi_j\rangle_\varphi = \delta_{ij}$. The elements of $\mathcal{H}$ may not be functions but distributions of negative order. Thanks to this reason, it is convenient to introduce the space $|\mathcal{H}|$ of measurable function h on $[0,T]$ satisfying

$$\|h\|^2_{|\mathcal{H}|} = \int_0^T\int_0^T |h(t)||h(s)|\varphi(t,s)dsdt < \infty.$$

And it is not difficult to show that $|\mathcal{H}|$ is a Banach space with the norm $\|\cdot\|^2_{|\mathcal{H}|}$.

The Malliavin derivative D_t^H of a smooth and cylindrical random variable $F \in \mathcal{S}$ is defined as the $\mathcal{H}$-valued random variable:

$$D_t^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(\int_0^T \psi_1(t)dB^H(t), \dots, \int_0^T \psi_n(t)dB^H(t)\right)\psi_i(t).$$

Then, for any $p \geq 1$, the derivative operator D_t^H is a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$. In addition, we denote $D_t^{H,k}$ as the iteration of the derivative operator for any integer $k \geq 1$. And the Sobolev space $\mathbf{D}^{k,p}$ is the closure of $\mathcal{S}$ with respect to the norm, for any $p \geq 1$ ($\bigotimes$ denotes the tensor product)

$$\|F\|_{k,p}^p = E|F|^p + E\sum_{j=1}^k \|D_t^{H,j} F\|_{\mathcal{H}^{\otimes j}}^p.$$

Similarly, for a Hilbert space U, we denote by $\mathbf{D}^{k,p}(U)$ the corresponding Sobolev space of U-valued random variables. For any $p > 0$ we denote by $\mathbf{D}^{1,p}(|\mathcal{H}|)$ the subspace of $\mathbf{D}^{1,p}(\mathcal{H})$ formed by the elements h such that $h \in |\mathcal{H}|$.

Now, we introduce the φ-derivative of F:

$$D_t^\varphi F = \int_{R^+} \varphi(t,v) D_v^H F dv.$$

Biagini [2], Alos [1], Hu [7] and Borkowska [9] have given more details about the fBm.

2.2. *The pathwise integrals with respect to fBm*

There are several ways to define integrals with respect to fBm. Here we follow [2] (see Definition 6.1.15 on p. 154 there) to introduce the forward and backward integrals, and the reader is referred to [2] for more detailed explanations.

Definition 1. ([18]) Let $u(t)$ be a stochastic process with integrable trajectories.

(D1) The symmetric integral of $u(t)$ with respect to $B^H(t)$ is defined as

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^T u(s)[B^H(s+\varepsilon) - B^H(s-\varepsilon)]ds,$$

provided that the limit exists in probability, and is denoted by $\int_0^T u(s)d^\circ B^H(s)$.

(D2) The forward integral of $u(t)$ with respect to $B^H(t)$ is defined as

$$\lim_{\varepsilon \to 0} \int_0^T u(s)\left[\frac{B^H(s+\varepsilon) - B^H(s)}{\varepsilon}\right] ds,$$

provided that the limit exists in probability, and is denoted by $\int_0^T u(s)d^- B^H(s)$.

(D3) The backward integral of $u(t)$ with respect to $B^H(t)$ is defined as

$$\lim_{\varepsilon\to 0}\int_0^T u(s)\left[\frac{B^H(s-\varepsilon)-B^H(s)}{\varepsilon}\right]ds,$$

provided that the limit exists in probability, and is denoted by $\int_0^T u(s)d^+B^H(s)$.

Remark 1. ([2], Proposition 6.2.3, p. 159) Let $u(t)$ be a stochastic process in the space $\mathbf{D}^{1,2}(|\mathcal{H}|)$, and satisfies

$$\int_0^T\int_0^T |D_s^H u(t)||t-s|^{2H-2}dsdt<\infty,$$

then the symmetric integral coincides with the forward and the backward integrals. In terms of the result of Proposition 6.2.3 in [20], we obtain Remark 1.

Definition 2. The space $\mathcal{L}_\varphi[0,T]$ of integrands is defined as the family of stochastic processes $u(t)$ on $[0,T]$, such that $E\|u(t)\|_\varphi^2<\infty$, $u(t)$ is φ-differentiable, the trace of $D_s^\varphi u(t)$ exists, $0\le s\le T$, $0\le t\le T$, and

$$E\int_0^T\int_0^T [D_t^\varphi u(s)]^2 dsdt<\infty,$$

and for each sequence of partitions $(\pi_n, n\in\mathbb{N})$ such that $|\pi_n|\to 0$ as $n\to\infty$,

$$\sum_{i=0}^{n-1}E\left[\int_{t_i^{(n)}}^{t_{i+1}^{(n)}}\int_{t_j^{(n)}}^{t_{j+1}^{(n)}}|D_s^\varphi u^\pi(t_i^{(n)})D_t^\varphi u^\pi(t_j^{(n)})-D_s^\varphi u(t)D_t^\varphi u(s)|dsdt\right]$$

and

$$E[\|u^\pi-u\|_\varphi^2],$$

tend to 0 as $n\to\infty$,where $\pi_n=t_0^{(n)}<t_1^{(n)}<\cdots<t_{n-1}^{(n)}<t_n^{(n)}=T$.

2.3. *Non-Lipschitz condition*

Hypothesis 1. There exists a function $G(t,x):[0,+\infty)\times R^+\to R^+$ such that

(a) for any fixed $x\ge 0$, $t\in[0,+\infty)\mapsto G(t,x)\in R^+$ is locally integrable, and for any fixed $t\ge 0$, $x\in R^+\mapsto G(t,x)\in R^+$ is continuous, non-decreasing, concave, and fulfills $G(t,0)=0$ and for any fixed t, $\int_{0+}\frac{1}{G(t,x)}dx=\infty$,

(b) for any fixed $t\in[0,T], b(t,\cdot),\sigma(t,\cdot)\in\mathbf{D}^{1,2}(|\mathcal{H}|)\cap\mathcal{L}_\varphi[0,T]$, we have

$$\begin{aligned}&|b(t,X)-b(t,Y)|^2+|\sigma(t,X)-\sigma(t,Y)|^2+|D_t^\varphi(\sigma(t,X)-\sigma(t,Y))|^2\\&\quad\le G(t,|X-Y|^2),\end{aligned}$$

(c) for any constant $K>0$, if a non-negative function $Z(t)$ satisfies that

$$Z(t)\le K\int_0^t G(s,Z(s))ds,$$

for all $t\in R^+$, then $Z(t)\equiv 0$.

Hypothesis 2. There exists a function $H(t,x) : [0,+\infty) \times R^+ \to R^+$ such that

(a) $H(t,x)$ is locally integral in $t \geq 0$ for any fixed $x \geq 0$ and is continuous monotone non-decreasing concave in x for any fixed $t \geq 0$,

(b) for any fixed $t \in [0,T], b(t,\cdot), \sigma(t,\cdot) \in \mathbf{D}^{1,2}(|\mathcal{H}|) \cap \mathcal{L}_\varphi[0,T]$, we have

$$|b(t,X)|^2 + |\sigma(t,X)|^2 + |D_t^\varphi \sigma(t,X)|^2 \leq H(t,|X|^2),$$

(c) for any constant $K > 0$, the differential equation

$$\frac{dx}{dt} = KH(t,x),$$

has a global solution for any initial value x_0.

To obtain more detailed descriptions about the above non-Lipschitz condition, see e.g., [15, 17, 21–23].

Remark 2. Let $G(t,x) = \lambda(t)\Gamma(x), t \geq 0, x \in R^+$, where $\lambda(t) : [0,\infty) \to R^+$ is a locally integrable function, $\Gamma(x)$ is non-decreasing, continuous and concave function from R^+ to R^+ such that $\Gamma(0) = 0$ and

$$\int_{0+} \frac{1}{\Gamma(x)} dx = \infty.$$

Then the function $G(t,x)$ satisfies Hypothesis 1.

Now, we give some concrete examples of the function Γ. Let $K > 0$ and let $\mu \in]0,1[$ be sufficiently small.

Define

$$\Gamma_1(x) = Kx, x \geq 0.$$

$$\Gamma_2(x) = \begin{cases} x\log(x^{-1}), & 0 \leq x \leq \mu, \\ \mu\log(\mu^{-1}) + \Gamma_2'(\mu-)(x-\mu), & x > \mu, \end{cases}$$

$$\Gamma_3(x) = \begin{cases} x\log(x^{-1})\log\log(x^{-1}), & 0 \leq x \leq \mu, \\ \mu\log(\mu^{-1})\log\log(\mu^{-1}) + \Gamma_3'(\mu-)(x-\mu), & x > \mu, \end{cases}$$

where Γ' denotes the derivative of function Γ. They are all concave and non-decreasing functions satisfying

$$\int_{0+} \frac{1}{\Gamma_i(x)} dx = \infty, \quad i = 1,2,3.$$

In particular, we see clearly that if let $\Gamma(x) = x, \lambda(t) = K$ then the non-Lipschitz Hypotheses 1–2 reduce to Lipschitz condition. In other words, non-Lipschitz condition is weaker than the Lipschitz condition.

Hypothesis 3. Suppose that there exist functions $\bar{b}(X), \bar{\sigma}(X) \in \mathbf{D}^{1,2}(|\mathcal{H}|) \cap \mathcal{L}_\varphi[0,T]$, we have

$$\frac{1}{T_1}\int_0^{T_1} |b(s,X) - \bar{b}(X)|ds \le \varphi_1(T_1)\rho(|X|), \tag{2.1}$$

$$\frac{1}{T_1}\int_0^{T_1} |\sigma(s,X) - \bar{\sigma}(X)|^2 ds \le \varphi_2(T_1)\rho(|X|^2), \tag{2.2}$$

where $T_1 \in [0,T]$ and $\varphi_i(T_1)$ are positive bounded functions with $\lim_{T_1\to\infty}\varphi_i(T_1) = 0, i = 1,2$, $\rho(\cdot)$ is a non-decreasing, continuous and concave function from R^+ to R^+ and $\bar{\sigma} : R \to R$, $\bar{b} : R \to R$ are all measurable functions.

3. The Stochastic Averaging Principle

In this paper, we are concerned with SDEs involving forward stochastic integral with respect to fBm. One can follow [2, 21] to verify the existence and uniqueness of the solutions of the SDEs under Hypotheses 1–2 introduced in the previous section. We present the existence and uniqueness results and their proofs in the Appendix.

Now, we consider the approximate solutions for non-Lipschitz SDEs (the forward integral case) with fBm as follows:

$$X_\varepsilon(t) = X(0) + \varepsilon^{2H}\int_0^t b(s, X_\varepsilon(s))ds + \varepsilon^H \int_0^t \sigma(s, X_\varepsilon(s))d^- B^H(s), \tag{3.1}$$

where $X(0) = X_0$ is a given random variable as the initial condition, $t \in [0,T]$ and the coefficients satisfy Hypotheses 1–2, and $\varepsilon \in (0, \varepsilon_0]$ is a positive parameter with ε_0 a fixed number.

Let us present some auxiliary results.

Lemma 1. *If $u(t) \in \boldsymbol{D}^{1,2}(|\mathcal{H}|) \cap \mathcal{L}_\varphi[0,T]$, then the symmetric integral is well defined and the following relations hold*

$$\int_0^T u(s)d^\circ B^H(s) = \int_0^T u(s)\diamondsuit dB^H(s) + \int_0^T D_s^\varphi u(s)ds, \tag{3.2}$$

where $\diamondsuit$ denotes the Wick product (see Duncan [5], *p.* 588). *Moreover, the forward and backward integrals are also well defined, and by Remark* 1, *they both coincide with the symmetric integral under the condition of Lemma* 1.

Lemma 1 can be found in the work of Biagini and Hu *et al.* ([2], Theorem 6.2.5). By Remark 1, the symmetric integral coincides with the forward and backward integrals under the condition of Lemma 1, so three types of pathwise integrals have same relations as Eq. (3.2).

Lemma 2. *Let $B^H(t)$ be a fBm with $\frac{1}{2} < H < 1$, and $u(t)$ be a stochastic process in $\boldsymbol{D}^{1,2}(|\mathcal{H}|) \cap \mathcal{L}_\varphi[0,T]$, then for every $T < \infty$, there exists a constant C such that*

$$E\left[\int_0^T u(s)d^- B^H(s)\right]^2 \leq 2HT^{2H-1}E\left[\int_0^T |u(s)|^2 ds\right] + 4CT^2.$$

Proof. We have

$$\begin{aligned}
E\left[\int_0^T u(s)d^- B^H(s)\right]^2 &= E\left[\int_0^T u(s)\diamondsuit dB^H(s) + \int_0^T D_s^\varphi u(s)ds\right]^2 \\
&\leq 2E\left[\int_0^T u(s)\diamondsuit dB^H(s)\right]^2 + 2E\left[\int_0^T D_s^\varphi u(s)ds\right]^2 \\
&\leq 2HT^{2H-1}E\left[\int_0^T |u(s)|^2 ds\right] + 4E\left[\int_0^T D_s^\varphi u(s)ds\right]^2 \\
&\leq 2HT^{2H-1}E\left[\int_0^T |u(s)|^2 ds\right] + 4TE\int_0^T [D_s^\varphi u(s)]^2 ds. \\
&\leq 2HT^{2H-1}E\left[\int_0^T |u(s)|^2 ds\right] + 4CT^2.
\end{aligned}$$

This completes the proof. □

The detailed proof of Lemma 2 can be found in the authors' work [26].

Remark 3. In the same conditions with Lemmas 1–2, and under Remark 1, the symmetric, forward and backward integral cases have same conclusions.

In the rest of the paper, K_l are all constants, $l = 1, 2, 3, 4, 5, 6, 7$. Our main result is the following.

Theorem. *If Hypotheses 1–3 are satisfied and $Z_\varepsilon(t)$ denotes the solution process to the SDEs*

$$Z_\varepsilon(t) = X(0) + \varepsilon^{2H}\int_0^t \bar{b}(Z_\varepsilon(s))ds + \varepsilon^H \int_0^t \bar{\sigma}(Z_\varepsilon(s))d^- B^H(s), \tag{3.3}$$

then for a given arbitrarily small number $\delta_1 > 0$, there exist $\varepsilon_1 \in (0, \varepsilon_0]$, such that for any $\varepsilon \in (0, \varepsilon_1], t \in [0, T]$,

$$E(|X_\varepsilon(t) - Z_\varepsilon(t)|^2) \leq \delta_1.$$

Proof. Through Eqs. (3.1) and (3.3), taking expectation and employing the following inequality for $m \in \mathbb{N}_+$ and $x_1, x_2, \ldots, x_m \in R$:

$$|x_1 + x_2 + \cdots + x_m|^2 \leq m(|x_1|^2 + |x_2|^2 + \cdots + |x_m|^2), \tag{3.4}$$

we obtain that

$$\begin{aligned}E|X_\varepsilon(t)-Z_\varepsilon(t)|^2 &\le 2\varepsilon^{4H}E\left|\int_0^t[b(s,X_\varepsilon(s))-\bar{b}(Z_\varepsilon(s))]ds\right|^2\\&\quad+2\varepsilon^{2H}E\left|\int_0^t[\sigma(s,X_\varepsilon(s))-\bar{\sigma}(Z_\varepsilon(s))]d^-B^H(s)\right|^2\\&=I_1^2+I_2^2,\end{aligned}$$

where $[0,t]\subseteq[0,\tilde{u}]\subseteq[0,T], I_i, i=1,2$ denote the above terms respectively.

Firstly, we apply inequality (3.4) for I_1^2 to yield

$$\begin{aligned}I_1^2&=2\varepsilon^{4H}E\left|\int_0^t[b(s,X_\varepsilon(s))-\bar{b}(Z_\varepsilon(s))]ds\right|^2\\&\le 4\varepsilon^{4H}E\left|\int_0^t[b(s,X_\varepsilon(s))-b(s,Z_\varepsilon(s))]ds\right|^2\\&\quad+4\varepsilon^{4H}E\left|\int_0^t[b(s,Z_\varepsilon(s))-\bar{b}(Z_\varepsilon(s))]ds\right|^2\\&=I_{11}^2+I_{12}^2.\end{aligned}$$

By Cauchy–Schwarz inequality for I_{11}^2 and Hypothesis 1(b), we arrive at

$$\begin{aligned}I_{11}^2&\le 4\varepsilon^{4H}E\left(t\int_0^t|b(s,X_\varepsilon(s))-b(s,Z_\varepsilon(s))|^2ds\right)\\&\le 4\varepsilon^{4H}\tilde{u}E\int_0^t G(s,|X_\varepsilon(s)-Z_\varepsilon(s)|^2)ds.\end{aligned}$$

Then, for I_{12}^2, using Eq. (2.1) and $\varphi_1(T_1)$ a positive bounded function to yield:

$$\begin{aligned}I_{12}^2&\le 4\varepsilon^{4H}t^2E\left[\frac{1}{t}\int_0^t|b(s,Z_\varepsilon(s))-\bar{b}(Z_\varepsilon(s))|ds\right]^2\\&\le 4\varepsilon^{4H}\tilde{u}^2\left(\sup_{0\le t\le\tilde{u}}\varphi_1^2(t)\right)E\left[\rho\left(\sup_{0\le t\le\tilde{u}}|Z_\varepsilon(t)|\right)\right]^2.\end{aligned}$$

Now, we employ inequality (3.4) for I_2^2 to obtain

$$\begin{aligned}I_2^2&\le 4\varepsilon^{2H}E\left|\int_0^t[\sigma(s,X_\varepsilon(s))-\sigma(s,Z_\varepsilon(s))]d^-B^H(s)\right|^2\\&\quad+4\varepsilon^{2H}E\left|\int_0^t[\sigma(s,Z_\varepsilon(s))-\bar{\sigma}(Z_\varepsilon(s))]d^-B^H(s)\right|^2\\&=I_{21}^2+I_{22}^2.\end{aligned}$$

By, Lemma 2 and Hypothesis 1(b), we obtain

$$\begin{aligned} I_{21}^2 &= 4\varepsilon^{2H} E\left|\int_0^t [\sigma(s, X_\varepsilon(s)) - \sigma(s, Z_\varepsilon(s))] d^- B^H(s)\right|^2 \\ &\le 4\varepsilon^{2H}\left\{2Ht^{2H-1} E\int_0^t [\sigma(s, X_\varepsilon(s)) - \sigma(s, Z_\varepsilon(s))]^2 ds + 4C\tilde{u}^2\right\} \\ &\le 8\varepsilon^{2H}\tilde{u}^{2H-1} K_1 \left(E\int_0^t G(s, |X_\varepsilon(s) - Z_\varepsilon(s)|^2) ds\right) + 16\varepsilon^{2H} K_2 \tilde{u}^2. \end{aligned}$$

Due to Eq. (2.2), we have

$$\begin{aligned} I_{22}^2 &= 4\varepsilon^{2H} E\left|\int_0^t [\sigma(s, Z_\varepsilon(s)) - \bar{\sigma}(Z_\varepsilon(s))] d^- B^H(s)\right|^2 \\ &\le 4\varepsilon^{2H}\left\{2H\tilde{u}^{2H-1} E\left[\int_0^t |\sigma(s, Z_\varepsilon(s)) - \bar{\sigma}(Z_\varepsilon(s))|^2 ds\right] + 4C\tilde{u}^2\right\} \\ &\le 8\varepsilon^{2H} H\tilde{u}^{2H-1} t E\left\{\frac{1}{t}\int_0^t |\sigma(s, Z_\varepsilon(s)) - \bar{\sigma}(Z_\varepsilon(s))|^2 ds\right\} + 16\varepsilon^{2H}\tilde{u}^2 K_3 \\ &\le 8\varepsilon^{2H} H\tilde{u}^{2H}\left(\sup_{0\le t\le\tilde{u}} \varphi_2(t)\right) E\left[\rho\left(\sup_{0\le t\le\tilde{u}} |Z_\varepsilon(t)|^2\right)\right] + 16\varepsilon^{2H}\tilde{u}^2 K_3. \end{aligned}$$

Therefore, from the above discussions, and $G(t,x)$ is concave in x for any fixed $t \ge 0$, we get

$$\begin{aligned} E|X_\varepsilon(t) - Z_\varepsilon(t)|^2 \le{}& (4\varepsilon^{4H}\tilde{u} + 8\varepsilon^{2H}\tilde{u}^{2H-1}K_1)\int_0^t G(s, E|X_\varepsilon(s) - Z_\varepsilon(s)|^2) ds \\ &+ 4\varepsilon^{4H}\tilde{u}^2\left(\sup_{0\le t\le\tilde{u}} \varphi_1^2(t)\right) E\left[\rho\left(\sup_{0\le t\le\tilde{u}} |Z_\varepsilon(t)|\right)\right]^2 \\ &+ 8\varepsilon^{2H} H\tilde{u}^{2H}\left(\sup_{0\le t\le\tilde{u}} \varphi_2(t)\right) E\left[\rho\left(\sup_{0\le t\le\tilde{u}} |Z_\varepsilon(t)|^2\right)\right] \\ &+ 16\varepsilon^{2H}\tilde{u}^2 K_2 + 16\varepsilon^{2H}\tilde{u}^2 K_3. \end{aligned}$$

Note that there exist $a(t) \ge 0, b(t) \ge 0$, such that

$$G(t,x) \le a(t) + b(t)x, \quad \int_0^T a(t)dt < \infty, \quad \int_0^T b(t)dt < \infty,$$

we have

$$\begin{aligned} &E|X_\varepsilon(t) - Z_\varepsilon(t)|^2 \\ &\quad\le (4\varepsilon^{4H}\tilde{u} + 8\varepsilon^{2H}\tilde{u}^{2H-1}K_1)\left(\sup_{0\le t\le\tilde{u}} a(t)\right) \\ &\qquad + \left(4\varepsilon^{4H}\tilde{u} + 8\varepsilon^{2H}\tilde{u}^{2H-1}K_1\right)\left(\sup_{0\le t\le\tilde{u}} b(t)\right)\int_0^t E|X_\varepsilon(s) - Z_\varepsilon(s)|^2 ds \\ &\qquad + 4\varepsilon^{4H}\tilde{u}^2 K_4 + 8\varepsilon^{2H}\tilde{u}^{2H} K_5 + 16\varepsilon^{2H}\tilde{u}^2 K_2 + 16\varepsilon^{2H}\tilde{u}^2 K_3 \end{aligned}$$

$$\begin{aligned}
&\leq (4\varepsilon^{4H}\tilde{u}K_6 + 8\varepsilon^{2H}\tilde{u}^{2H-1}K_1)\int_0^t E|X_\varepsilon(s) - Z_\varepsilon(s)|^2 ds + 4\varepsilon^{4H}K_7\tilde{u}\\
&\quad + 8\varepsilon^{2H}\tilde{u}^{2H-1}K_1 + 4\varepsilon^{4H}\tilde{u}^2K_4 + 8\varepsilon^{2H}\tilde{u}^{2H}K_5\\
&\quad + 16\varepsilon^{2H}\tilde{u}^2K_2 + 16\varepsilon^{2H}\tilde{u}^2K_3\\
&= (4\varepsilon^{4H}\tilde{u}K_6 + 8\varepsilon^{2H}\tilde{u}^{2H-1}K_1)\int_0^t E|X_\varepsilon(s) - Z_\varepsilon(s)|^2 ds\\
&\quad + 4\varepsilon^{2H}(K_7\varepsilon^{2H}\tilde{u} + K_1\tilde{u}^{2H-1} + K_4\varepsilon^{2H}\tilde{u}^2 + K_5\tilde{u}^{2H} + K_2\tilde{u}^2 + K_3\tilde{u}^2).
\end{aligned}$$

Now by the Gronwall–Bellman inequality, we obtain

$$\begin{aligned}
E|X_\varepsilon(t) - Z_\varepsilon(t)|^2 &\leq 4\varepsilon^{2H}[K_7\varepsilon^{2H}\tilde{u} + K_2\tilde{u}^2 + \tilde{u}^{2H-1}(K_1 + K_5\tilde{u}) + (K_4\varepsilon^{2H} + K_3)\tilde{u}^2]\\
&\quad \times \exp(4\varepsilon^{4H}\tilde{u}^2K_6 + 8\varepsilon^{2H}\tilde{u}^{2H}K_1).
\end{aligned}$$

Select $\beta \in (0,1)$, $L > 0$, such that for all $t \in [0, L\varepsilon^{-H\beta}] \subseteq [0,T]$, we have

$$E|X_\varepsilon(t) - Z_\varepsilon(t)|^2 \leq Q\varepsilon^{1-H\beta},$$

where

$$\begin{aligned}
Q &= 4\varepsilon^{2H+H\beta-1}\exp(4\varepsilon^{4H-2H\beta}L^2K_6 + 8L^{2H}\varepsilon^{2H-2H^2\beta}K_1)[K_7\varepsilon^{2H}L\varepsilon^{-H\beta}\\
&\quad + K_2L^2\varepsilon^{-2H\beta} + L^{2H-1}\varepsilon^{-H\beta(2H-1)}(K_1 + K_5L\varepsilon^{-H\beta}) + L^2(K_4\varepsilon^{2H} + K_3)\varepsilon^{-2H\beta}]
\end{aligned}$$

is a constant.

Consequently, given any number $\delta_1 > 0$, we can select $\varepsilon_1 \in (0, \varepsilon_0]$, such that for every $\varepsilon \in (0, \varepsilon_1]$, and for $t \in [0, L\varepsilon^{-H\beta}] \subseteq [0,T]$,

$$E|X_\varepsilon(t) - Z_\varepsilon(t)|^2 \leq \delta_1.$$

This completes the proof. □

Corollary. *Assume that the original SDEs* (3.1) *and the averaged SDEs* (3.3) *both satisfy the Hypotheses* 1–3. *Then for any number* $\delta_2 > 0$, *there exist* $L > 0$ *and* $\beta \in (0,1)$, *such that*

$$\lim_{\varepsilon\to 0} P(|X_\varepsilon(t) - Z_\varepsilon(t)| > \delta_2) = 0.$$

Proof. On the basis of Theorem and Chebyshev–Markov inequality, for any given number $\delta_2 > 0$, one can have

$$P(|X_\varepsilon(t) - Z_\varepsilon(t)| > \delta_2) \leq \frac{1}{\delta_2^2}E|X_\varepsilon(t) - Z_\varepsilon(t)|^2 \leq \frac{Q\varepsilon^{1-H\beta}}{\delta_2^2}.$$

Let $\varepsilon \to 0$ and the required result follows.

This completes the proof. □

Remark 4. Similarly, we get the same results for three types of pathwise integrals of SDEs.

Remark 5. We would like to compare our main result with the corresponding result for SDEs with standard Brownian motion ($H = \frac{1}{2}$) in which the stochastic integral is just the usual Itô integral. In fact, in the Brownian motion case, we have

$$E\left(\sup_{t\in[0,L\varepsilon^{-1}]}|X_\varepsilon(t) - Z_\varepsilon(t)|^2\right) \le \delta_1.$$

However, due to the fact that stochastic integral with respect to fBm is no longer a martingale (Itô integral is a martingale but it is not well defined with respect to fBm), we definitely lose good inequalities such as Burkholder–Davis–Gundy inequality which is crucial for SDEs driven by Bm. So we cannot obtain the uniform convergence result. As expected, our obtained convergence result is in general weaker than those results for SDEs driven by Bm as follows:

$$E|X_\varepsilon(t) - Z_\varepsilon(t)|^2 \le \delta_1, \quad t \in [0, L\varepsilon^{-H\beta}].$$

The obtained convergence result in this paper is in general weaker than that of Bm case, but this is an interesting theoretical result to study the solution of complex equations with fBm through the solution of simplified equation.

Remark 6. In this paper, Hypotheses 3 are to ensure that the coefficients between original Eq. (3.1) and averaged Eq. (3.3) can be controlled. If not, we cannot obtain the convergence for the solution. Here, we mainly emphasize the convergence of the solutions. In fact, the rate of convergence in this result is about $-H\beta$ compared with -1 in Brownian motion case (namely, $H = 1/2$).

Finally, let us give an example of SDEs driven by fBm to illustrate the computation of $\bar{\sigma}$ and $\bar{b}$. We also derive the associated averaging process.

Example. Consider the following SDEs driven by fBm:

$$dX_\epsilon = -\epsilon^{2H}\lambda X_\epsilon \sin^2(t)dt + \epsilon^H d^- B^H(t), \tag{3.5}$$

where $X_\epsilon(0) = X_0$ and $E|X_0|^2 < \infty, t \in [0, T]$, $b(t, X_\epsilon) = -\lambda X_\epsilon \sin^2(t), \sigma(t, X_\epsilon) = 1$, and λ is a positive constant, $B^H(t)$ is a fBm. Then

$$\bar{b}(X_\epsilon) = \frac{1}{\pi}\int_0^\pi b(t, X_\epsilon)dt = -\frac{1}{2}\lambda X_\epsilon, \quad \bar{\sigma}(X_\epsilon) = 1,$$

and define a new averaged SDEs

$$dZ_\epsilon = \epsilon^{2H}\bar{b}(Z_\epsilon)dt + \epsilon^H\bar{\sigma}(Z_\epsilon)d^- B^H(t),$$

namely,

$$dZ_\epsilon = -\frac{1}{2}\epsilon^{2H}\lambda Z_\epsilon dt + \epsilon^H d^- B^H(t). \tag{3.6}$$

Obviously, $Z_\epsilon(t)$ is the well-known Ornstein–Uhlenbeck process, and the solution process can be obtained as:

$$Z_\epsilon(t) = \exp\left(-\frac{1}{2}\epsilon\lambda t\right) X_0 + \sqrt{\epsilon}\int_0^t \exp\left(-\frac{1}{2}\epsilon\lambda(t-s)\right) d^- B^H(t).$$

As all the Hypotheses 1–3 are satisfied for functions $b, \sigma, \bar{b}, \bar{\sigma}$ in SDEs (3.5)–(3.6), thus the Theorem and Corollary hold. That is,

$$\mathbb{E}(|X_\epsilon(t) - Z_\epsilon(t)|^2) \leq \delta_1,$$

and consequently, as $\epsilon \to 0$,

$$|X_\epsilon(t) - Z_\epsilon(t)| \to 0 \quad \text{in probability}.$$

Appendix: Existence and Uniqueness of Solutions of SDEs with Forward Stochastic Integrals

We are concerned with the following SDEs

$$X(t) = X_0 + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))d^- B^H(s), \quad t \in [0, T], \tag{A.1}$$

with the initial data X_0 being a random variable. Following [2, 21], we have the following.

Theorem A. *Under the assumption Hypotheses* 1–2, *there is a unique solution of* (A.1).

Proof. By using an iteration of the Picard type, we construct an approximate sequence of stochastic process $\{X_k(t), k \in \mathbb{N}_+\}$ as follows:

$$X_{k+1}(t) = X_0 + \int_0^t b(s, X_k(s))ds + \int_0^t \sigma(s, X_k(s))d^- B^H(s), \quad t \in [0, T], \tag{A.2}$$

where $X_1(t) = X_0$ is the initial condition with $E|X_0|^2 < \infty$.

By Lemmas 1–2 and Hypothesis 2(b), we have

$$\begin{aligned} E|X_{k+1}(t)|^2 &\leq 3E|X_0|^2 + 3E\left|\int_0^t b(s, X_k(s))ds\right|^2 + 3E\left|\int_0^t \sigma(s, X_k(s))d^- B^H(s)\right|^2 \\ &\leq 3E|X_0|^2 + 3TE\int_0^t |b(s, X_k(s))|^2 ds + 6HT^{2H-1}E\int_0^t |\sigma(s, X_k(s))|^2 dt \\ &\quad + 12TE\int_0^t |D_s^\varphi \sigma(s, X_k(s))|^2 dt \\ &\leq 3E|X_0|^2 + C_{T,H}\int_0^t H(s, E|X_k(s)|^2)ds, \end{aligned} \tag{A.3}$$

for all $t \in [0, T]$, where the constant $C_{T,H}$ only depends on T, H.

Next, by Hypothesis 2(c) and inequality (A.3), there is a $u(t), t \in [0, T]$ satisfying

$$u(t) = 3E|X_0|^2 + C_{T,H}\int_0^t H(s, u(s))ds.$$

By induction, we obtain for any $k \in \mathbb{N}_+$,

$$E|X_{k+1}(t)|^2 \leq u(t) \leq u(T) < \infty.$$

This proves the uniform boundedness of $\{X_k(t), 0 \leq t \leq T, k \in \mathbb{N}_+\}$.

On the other hand, by the same way as above, for $t \in [0,T]$, we have

$$\begin{aligned} E|X_{k+1}(t) - X_{m+1}(t)|^2 &\le 2E\left|\int_0^t [b(s,X_k(s))ds - b(s,X_m(s))]ds\right|^2 \\ &\quad + 2E\left|\int_0^t [\sigma(s,X_k(s)) - \sigma(s,X_m(s))]d^-B^H(s)\right|^2 \\ &\le 2TE\int_0^t |b(s,X_k(s))ds - b(s,X_m(s))|^2 ds \\ &\quad + 4HT^{2H-1}E\int_0^t |\sigma(s,X_k(s)) - \sigma(s,X_m(s))|^2 dt \\ &\quad + 8TE\int_0^t |D_s^\varphi[\sigma(s,X_k(s)) - \sigma(s,X_m(s))]|^2 dt \\ &\le C_{T,H}\int_0^t G(s, E|X_k(s) - X_m(s)|^2)ds, \end{aligned}$$

where $C_{T,H}$ is also a constant.

To proceed, it is routine to derive

$$\begin{aligned} &\sup_{0\le s\le t} E|X_{k+1}(s) - X_{m+1}(s)|^2 \\ &\quad \le C_{T,H}\int_0^t G\left(s, \sup_{0\le s_1\le s} E|X_k(s_1) - X_m(s_1)|^2\right) ds. \end{aligned} \tag{A.4}$$

Let

$$Z(t) = \limsup_{k,m\to\infty} \sup_{0\le s\le t} E|X_{k+1}(s) - X_{m+1}(s)|^2, \tag{A.5}$$

then Eqs. (A.4)–(A.5) together with Fatous lemma yield

$$Z(t) \le C_{T,H}\int_0^t G(s, Z(s))ds. \tag{A.6}$$

At last, through the above inequality (A.6) and Hypothesis 1(c), we immediately get $Z(t) \equiv 0$, i.e.

$$Z(t) = \lim_{k,m\to\infty} \sup_{0\le s\le t} E|X_{k+1}(s) - X_{m+1}(s)|^2 = 0,$$

indicating that $\{X_k(t), 0 \le t \le T, k \in \mathbb{N}_+\}$ is a Cauchy sequence. The limit is denoted by $X(t)$. Let $k \to \infty$ in (A.2) yields that $X(t)$ satisfies (A.1) for all $t \in [0,T]$. In other words, we have shown the existence of solutions of (A.1). Then, by the same way, we obtain the uniqueness of solution of (A.1). □

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