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Paper:

Lv, G., Duan, J., Gao, H. & Wu, J. (2016). On a stochastic nonlocal conservation law in a bounded domain. *Bulletin des Sciences Mathématiques*

<http://dx.doi.org/10.1016/j.bulsci.2016.03.003>

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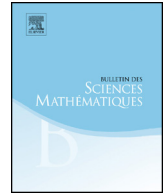
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Bulletin des Sciences Mathématiques

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On a stochastic nonlocal conservation law in a bounded domain



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ARTICLE INFO

Article history:

Received 6 December 2015

Available online 10 March 2016

MSC:

35K20

60H15

60H40

Keywords:

Anomalous diffusion

Itô's formula

Stochastic Burgers equation

Conservation laws

ABSTRACT

In this paper, we are interested in the Dirichlet boundary value problem for a multi-dimensional nonlocal conservation law with a multiplicative stochastic perturbation in a bounded domain. Using the concept of measure-valued solutions and Kruzhkov's semi-entropy formulations, a result of existence and uniqueness of entropy solution is proved.

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1. Introduction

Partial differential equations with a nonlocal (i.e., fractional) Laplacian operator have attracted a lot of attention recently. The usual Laplacian operator Δ may be thought as macroscopic manifestation of Brownian motion, as known from the Fokker–Planck equation for a stochastic differential equation driven by a Brownian motion (a Gaussian process), whereas the nonlocal Laplacian operator $(-\Delta)^{\frac{\gamma}{2}}$ is associated with a γ -stable Lévy motion (a non-Gaussian process) L_t^γ , $\gamma \in (0, 2)$. See [2,17] for a discussion about this microscopic–macroscopic relation.

Nonlocal Laplacian operator also appears in mathematical models for viscoelastic materials (e.g., Kelvin–Voigt model), certain heat transfer processes in fractal and disordered media, and fluid flows and acoustic propagation in porous media, see e.g. [9,26, 27], just mention a few. Interestingly, a nonlocal diffusion equation also arises in pricing derivative securities in financial markets [9].

In this paper we aim to solve the following problem

$$\begin{cases} du + ((-\Delta)^{\gamma/2}u - \operatorname{div}(f(u)))dt = h(u)dw, & t > 0, \quad x \in D, \\ u(0, x) = u_0(x), & x \in D, \\ u|_{D^c} = 0, \end{cases} \tag{1.1}$$

where $0 < \gamma \leq 1$, $D \subset \mathbb{R}^n$ is a bounded domain, with a Lipschitz boundary if $d \geq 2$, $Q = (0, T) \times D$, T is positive number and $w = \{w_t, \mathcal{F}_t : 0 \leq t \leq T\}$ denotes a standard adapted one-dimensional continuous Brownian motion with $w_0 = 0$, defined on the classical Wiener space $C_0([0, T])$.

It is well-known that equation (1.1) can be interpreted as Fokker–Planck equation with noise perturbation associated to some stochastic differential equation in the sense of McKean, see the paper [20,21]. In paper [22], the authors provided a numerical probabilistic scheme for the fractional scalar conservation law (1.1).

It is well-known that the specific value of $\gamma \in (0, 2)$ plays a key role:

- $1 < \gamma < 2$: In this case, $(-\Delta)^{\gamma/2}$ is the dominant term, so the equation (1.1) is a stochastic equation of parabolic type. Thus the existence of the solution to (1.1) can be obtained by a fixed point or contraction mapping argument [12,30].
- $\gamma = 1$: In this case, the two terms $(-\Delta)^{\gamma/2}$ and $\nabla \cdot f(u)$ have the same order in equations (1.1). Caffarelli–Figalli [10] considered the equations with square root operator $(-\Delta)^{\frac{1}{2}}$.
- $0 < \gamma < 1$: In this case, $\nabla \cdot f(u)$ is the leading term, and we do not expect to have a regularity theory for (1.1). So it is natural to think that (1.1) with $0 < \gamma < 1$ could behave as the following hyperbolic equation

$$du - \operatorname{div}(f(u))dt = h(u)dw \tag{1.2}$$

in the bounded domain. That is to say, we must introduce the notion of *entropy solution*.

In the absence of noise ($h = 0$), equation (1.1) reduces to a *deterministic* partial differential equation known as the nonlocal conservation law

$$\partial_t u(t, x) + (-\Delta)^\alpha u(t, x) + \nabla \cdot f(u(t, x)) = 0, \quad x \in \mathbb{R}^d, \quad t > 0, \quad (1.3)$$

which has been extensively studied [1,13,14]. When $\alpha \in (\frac{1}{2}, 1)$, equation (1.3) has been studied by [5–8,28]. When $\alpha \in (0, \frac{1}{2})$, Alibaud [1] defined an entropy solution to (1.3), and showed the existence and uniqueness of a solution to (1.3) in L^∞ . Moreover, Silvestre [28] studied the regularity of the solution of (1.3)

Let us recall some results about the stochastic conservation laws. H. Holden and N.H. Risebro [19] proved the existence of a weak solution to the Cauchy problem with multiplicative noise by using an operator splitting method. J.V. Kim [23] proposed a method of compensated compactness to prove, via vanishing viscosity approximation, the existence of a stochastic weak entropy solution to the Cauchy problem with additive noise. Vallet and Wittbold [29] extended the results of Kim to the multi-dimensional Dirichlet problem with additive noise. Recently, Bauzet et al. [3] studied the problem (1.2) in the whole space. And in another paper [4], they obtained the well-posedness of (1.2) in a bounded domain. Lv et al. [25] consider the problem (1.1) in the whole space by using the method of [18,11].

A cautious remark The problems (1.1) and (1.2) in a bounded domain are more difficult than those in the whole space. The reasons are the followings. Firstly, the definition of entropy solution in a bounded domain, which makes the proof of uniqueness more difficult, is different from that in the whole space (see Section 2 for details). Secondly, there is the effect of boundary, which implies that we must find special test function to prove the uniqueness. We must compare any weak entropy solution to a solution coming from the artificial viscosity. What is the most difficult is that, unlike in the whole space, the definition of entropy solution in a bounded domain destroys the symmetry of test function, but the operator $(-\Delta)^{\frac{\gamma}{2}}$ is a symmetric operator. And thus we need more calculations in order to obtain Kato's inequalities. In paper [4], the authors defined a special test function and used the following Kato's inequality

$$\Delta u j'(u) \leq \Delta j(u) \quad \text{in } \mathcal{D}' \quad \text{for } j(u) = (k^+ - u)^+. \quad (1.4)$$

In the present paper, we shall give a different test function which turns out to be easier to calculate. And the Kato's inequality (1.4) will not appear, see Remark 3.1 for more details. Another difficulty in this paper is the effect of nonlocal operator $(-\Delta)^{\gamma/2}$. Because it is defined in the whole space, and so it will bring more trouble for the bounded domain. The biggest difference is the working space, which is different from that in paper [4].

Another remark is that in paper [24], the authors found that the classical Sobolev space $H^s(D)$ is not suitable to describe the operator $(-\Delta)^{\gamma/2}$ and they introduced the weighted Sobolev space $W_\rho^{s,p}(D)$, where $\rho(x) = \text{dist}(x, \partial D)$. In fact, $W_\rho^{s,p}(D)$ is equivalent to

$$\hat{H}^s(\mathbb{R}^d) = \{u \in H^s(\mathbb{R}^d), u \equiv 0 \text{ in } \mathbb{R}^d \setminus D\}.$$

See [15,16] for more details about the nonlocal operator.

There are two highlights in this paper. First, the problem (1.1) is entirely new, and there is no result about the nonlocal operator in a bounded domain. The nonlocal operator will bring a lot of trouble. Second, we define a different test function and use a different method to prove the uniqueness. Even for equation (1.2), our method is easier than that in [4].

The rest of this paper is organized as follows. In section 2, we introduce the notion of stochastic entropy solution for equation (1.1) and propose a result of existence of a measure-valued entropy solution for (1.1) via a vanishing viscosity approximation. Section 3 is concerned with the proof of the main result on the uniqueness of entropy solution. As a by-product, we deduce the existence and uniqueness of the entropy solution of the Dirichlet problem for (1.1).

2. Entropy solution and existence of a measure-valued solution

In this section, we first present the definition of an entropy solution. To present our formulation for (1.1), we recall the following results on the operator $(-\Delta)^{\gamma/2}$.

Lemma 2.1. (See [14].) For $\gamma \in (0, 2)$, $\forall \phi \in \mathcal{S}(\mathbb{R}^d)$ and $\forall r > 0$

$$\begin{aligned} (-\Delta)^{\gamma/2}\phi(x) &= -C_d(\gamma) \int_{|z|\geq r} \frac{\phi(x+z) - \phi(x)}{|z|^{d+\gamma}} dz \\ &\quad - C_d(\gamma) \int_{|z|<r} \frac{\phi(x+z) - \phi(x) - \nabla\phi(x) \cdot z}{|z|^{d+\gamma}} dz \end{aligned} \tag{2.1}$$

where the constant $C_d(\gamma) := \frac{\gamma\Gamma(\frac{d+\gamma}{2})}{2\pi^{\frac{d}{2}+\gamma}\Gamma(1-\frac{\gamma}{2})} > 0$ (only depends on d and α). Moreover, in the case that $\gamma \in (0, 1)$, one can take $r = 0$ such that

$$(-\Delta)^{\gamma/2}\phi(x) = -C_d(\gamma) \int_{\mathbb{R}^d} \frac{\phi(x+z) - \phi(x)}{|z|^{d+\gamma}} dz$$

and in the case that $\gamma \in (1, 2)$, one can take $r = +\infty$ such that

$$(-\Delta)^{\gamma/2}\phi(x) = -C_d(\gamma) \int_{\mathbb{R}^d} \frac{\phi(x+z) - \phi(x) - \nabla\phi(x) \cdot z}{|z|^{d+\gamma}} dz.$$

In this paper, we mainly focus on the case that $\gamma \in (0, 1)$ and so we let $r = 0$. For simplicity, we would like to drop the constant $C_d(\gamma)$ in the integral representation by letting $C_d(\gamma) = 1$. Thus, we simply take the following formula for the nonlocal Laplacian $(-\Delta)^{\gamma/2}$

$$(-\Delta)^{\gamma/2}\phi(x) = - \int_{\mathbb{R}^d} \frac{\phi(x+z) - \phi(x)}{|z|^{d+\gamma}} dz \tag{2.2}$$

and this integral representation will be taken in force throughout the rest of the paper.

For a given separable Banach space X , we denote by $N_{\omega}^2(0, T, X)$ the space of the predictable X -valued processes. This space is the space $L^2((0, T) \times \Omega, X)$ for the product measure $dt \otimes dP$ on \mathcal{P}_T , the predictable σ -field (i.e. the σ -field generated by the sets $\{0\} \times \mathcal{F}_0$ and the rectangles $(s, t] \times A$ for any $A \in \mathcal{F}_s$).

Denote \mathcal{E}^+ as the set of non-negative convex functions η in $C^{2,1}(\mathbb{R})$, approximating the semi-Kruzhkov entropies $x \rightarrow x^+$ such that $\eta(x) = 0$ if $x \leq 0$ and that there exists $\delta > 0$ such that $\eta'(x) = 1$ if $x > \delta$. Then η'' has a compact support and η and η' are Lipschitz-continuous functions. \mathcal{E}^- denotes the set $\{\check{\eta} := \eta(-\cdot), \eta \in \mathcal{E}^+\}$; and for the definition of the entropy inequality, one denotes

$$\begin{aligned} \mathbb{A}^+ &= \{(k, \phi, \eta) \in \mathbb{R} \times \mathcal{D}^+(\mathbb{R}^{d+1}) \times \mathcal{E}^+, k < 0 \Rightarrow \phi \in \mathcal{D}^+([0, T] \times D)\}, \\ \mathbb{A}^- &= \{(k, \phi, \eta), (-k, \phi, \check{\eta}) \in \mathbb{A}^+\} \quad \text{and } \mathbb{A} = \mathbb{A}^+ \cup \mathbb{A}^-. \end{aligned}$$

Then, for convenience, denote

$$\begin{aligned} \text{sgn}_0^+(x) &= 1 \text{ if } x > 0 \text{ and } 0 \text{ else; } \text{sgn}_0^-(x) = -\text{sgn}_0^+(-x), \quad \text{sgn}_0 = \text{sgn}_0^+ + \text{sgn}_0^-, \\ F(a, b) &= \text{sgn}_0(a - b)[f(a) - f(b)]; \quad F^{+(-)}(a, b) = \text{sgn}_0^{+(-)}(a - b)[f(a) - f(b)], \end{aligned}$$

$$\text{and for any } \eta \in \mathcal{E}^+ \cup \mathcal{E}^-, \quad F^\eta(a, b) = \int_b^a \eta'(\sigma - b) f'(\sigma) d\sigma.$$

For any function u of $N_{\omega}^2(0, T, L^2(D)) \cap \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)$, any real k and any regular function η , denote dP -a.s. in Ω by $\mu_{\eta, k}$, the distribution in \mathbb{R}^{d+1} , defined by

$$\begin{aligned} \phi \rightarrow \mu_{\eta, k}(\phi) &= \int_Q \eta(u - k) \partial_t \phi - F^\eta(u, k) \nabla \phi dx dt + \int_Q \eta'(u - k) h(u) \phi dx dw(t) \\ &+ \int_Q \int_{\mathbb{R}^d \setminus \{0\}} \eta'(u - k) \frac{u(t, x + z) - u(t, x)}{|z|^{d+\gamma}} dz \phi dx dt \\ &+ \frac{1}{2} \int_Q h^2(u) \eta''(u - k) \phi dx dt + \int_D \eta(u_0 - k) \phi(0) dx. \end{aligned}$$

Remark 2.1. It is easy to see that the integration

$$\int_Q \int_{\mathbb{R}^d \setminus \{0\}} \eta'(u - k) \frac{u(t, x + z) - u(t, x)}{|z|^{d+\gamma}} dz \phi dx dt \tag{2.3}$$

makes sense because $u \in \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)$. In fact, due to $\langle u, (-\Delta)^{\frac{\gamma}{2}} u \rangle = \|u\|_{\hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)}$, the Holder inequality yields the above integration makes sense. On the other hand, the definition of nonlocal operator used here is different from that in [1].

Now, let us define the notion of entropy solution.

Definition 2.1. A function $u \in N^2_\omega(0, T, L^2(D)) \cap \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)$ is called an entropy solution of the stochastic nonlocal conservation law (1.1) with the initial condition $u_0 \in L^2(D)$ if

$$u \in L^\infty(0, T, L^2(\Omega, L^2(D) \cap \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)))$$

and

$$\forall(k, \phi, \eta) \in \mathbb{A}, \quad 0 \leq \mu_{\eta, k}(\phi) \quad dP\text{-a.s.}$$

Following the idea of [4], we need the following generalized notion of entropy solution. By the result of uniqueness, we are able to deduce the existence of an entropy solution in the sense of Definition 2.1.

Definition 2.2. A function

$$u \in N^2_\omega(0, T, L^2((0, 1); L^2(D) \cap \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)) \cap L^\infty((0, T), L^2(\Omega \times (0, 1); L^2(D) \cap \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)))$$

is called a (Young) measure-valued entropy solution of the stochastic nonlocal conservation law (1.1) with the initial condition $u_0 \in L^2(D) \cap \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)$ if

$$u \in L^\infty(0, T, L^2(\Omega, L^2(D) \cap \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)))$$

and

$$\forall(k, \phi, \eta) \in \mathbb{A}, \quad 0 \leq \int_0^1 \mu_{\eta, k}(\phi) d\alpha \quad dP\text{-a.s.}$$

Throughout this paper, we assume that

- H_1 : $f = (f_1, \dots, f_d) : \mathbb{R} \rightarrow \mathbb{R}^d$ is a Lipschitz-continuous function and $f(0) = 0$;
- H_2 : $h : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous function and $h(0) = 0$;
- H_3 : $u_0 \in L^2(D) \cap \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)$.

Now we are ready to state out our main results.

Theorem 2.1. *Under the assumptions H_1 – H_3 , there exists a unique measure-valued entropy solution in the sense of Definition 2.2 and this solution is obtained by viscous approximation.*

It is the unique entropy solution in the sense of Definition 2.1.

If u_1, u_2 are entropy solutions of (1.1) corresponding to initial data $u_{10}, u_{20} \in L^2(D) \cap \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)$, respectively, then for any $t > 0$

$$\mathbb{E} \int_D (u_1 - u_2)^+ dx \leq \int_D (u_{10} - u_{20})^+ dx.$$

The technique to prove the result of existence is based on the notion of narrow convergence of Young measures. Since the operator $(-\Delta)^{\gamma/2}$ is a divergence operator, one can easily prove the existence of Young measure-valued solution for (1.1) by using the method in [4]. Thus we leave the details to readers. In fact, for any $\varepsilon > 0$, there exists a unique weak solution u_ε of the stochastic viscous parabolic equation:

$$\partial_t \left[u - \int_0^t h(u) dw(s) \right] - \varepsilon \Delta u + (-\Delta)^{\gamma/2} u - \operatorname{div}(f(u)) = 0 \tag{2.4}$$

associated with a regular initial condition u_0^ε .

Lemma 2.2. *Under the assumptions H_1 – H_3 , there exists a unique solution u_ε of (2.4), satisfying*

- $u_\varepsilon \in N_\omega^2(0, T, H^1(D) \cap \hat{H}^{\gamma/2}(\mathbb{R}^d)) \cap C([0, T], L^2(\Omega \times D))$;
- If u_0^ε is bounded in $C^2(D)$ and $\|u_0\|_{C^2} \leq \|u_0\|_{L^2}$, then there exists a positive constant C , which does not depend on ε , such that

$$\|u_\varepsilon\|_{L^\infty(0, T; L^2(\Omega \times D))}^2 + \varepsilon \|u_\varepsilon\|_{L^2((0, T) \times \Omega); H^1(D)} + \|u_\varepsilon\|_{L^2((0, T) \times \Omega); \hat{H}^{\gamma/2}(D)} \leq C;$$

- $\forall (k, \phi, \eta) \in \mathbb{A}, 0 \leq \mu_{\eta, k}(\phi) - \varepsilon \int_Q \eta'(u_\varepsilon - k) \nabla \phi dx dt \, dP$ -a.s.

It follows from Lemma 2.1 that u_ε is a bounded sequence in $N_\omega^2(0, T, L^2(D) \cap \hat{H}^{\gamma/2}(\mathbb{R}^d))$, and so the associated sequence u_ε converges to (up to a subsequence still indexed in the same way) to a Young measure denoted by u . Thanks to the a priori estimates and the compatibility of the Itô integration with respect to the weak convergence in $N_\omega^2(0, T, L^2(D) \cap \hat{H}^{\gamma/2}(\mathbb{R}^d))$, one gets that this Young measure is a measure-valued entropy solution.

3. Uniqueness

In this section, we will prove the uniqueness of the stochastic entropy solution of (1.1). The proof is divided into two steps. The first step is to establish the local Kato inequality and the second step is to get the global Kato inequality.

3.1. Local Kato inequality

Lemma 3.1. *Let u, \hat{u} be Young measure-valued entropy solutions to (1.1) with initial data $u_0, \hat{u}_0 \in L^2(D)$, respectively, and assume that at least one of them is obtained by viscous approximation. Then, for any $\mathcal{D}^+([0, T] \times D)$ -function ϕ , one has that*

$$\begin{aligned} 0 \leq & \mathbb{E} \int_{Q \times (0,1)^2} (\hat{u}(t, x, \beta) - u(t, x, \alpha))^+ \partial_t \phi dx dt d\alpha d\beta \\ & - \mathbb{E} \int_{Q \times (0,1)^2} (u(t, x) - v(t, x))^+ (-\Delta)^{\frac{\gamma}{2}} \phi(t, x) dx dt d\alpha d\beta \\ & - \mathbb{E} \int_{Q \times (0,1)^2} F^+(\hat{u}(t, x, \beta), u(t, x, \alpha)) \cdot \nabla \phi dx dt d\alpha d\beta + \int_D (\hat{u}_0 - u_0)^+ dx. \end{aligned}$$

The proof of Lemma 3.1 is similar to that in [25] (see section 3 for details in [25]). The reason why there is no difference between the whole space and bounded domain for the local Kato inequality is the definition of stochastic entropy solution. Looking at the Definitions 2.1 and 2.2, we find if $\phi \in \mathcal{D}^+([0, T] \times D)$, then we need not assume $k \geq 0$. And thus we can use the same test functions as in [25]. But for the following global Kato inequality, there will be different.

3.2. Global Kato inequality

Lemma 3.2. *Let u, \hat{u} be Young measure-valued entropy solutions to (1.1) with initial data $u_0, \hat{u}_0 \in L^2(D)$, respectively, and assume that at least one of them is obtained by viscous approximation. Then, for any $\mathcal{D}^+([0, T] \times \mathbb{R}^d)$ -function ϕ , one has that*

$$\begin{aligned} 0 \leq & \mathbb{E} \int_{Q \times (0,1)^2} (\hat{u}(t, x, \alpha) - u(t, x, \beta))^+ \partial_t \phi dx dt d\alpha d\beta \\ & - \mathbb{E} \int_{Q \times (0,1)^2} (\hat{u}(t, x, \alpha) - u(t, x, \beta))^+ (-\Delta)^{\frac{\gamma}{2}} \phi(t, x) dx dt d\alpha d\beta \\ & - \mathbb{E} \int_{Q \times (0,1)^2} F^+(\hat{u}(t, x, \alpha), u(t, x, \beta)) \cdot \nabla \phi dx dt d\alpha d\beta + \int_D (\hat{u}_0 - u_0)^+ \phi(0) dx. \quad (3.1) \end{aligned}$$

It follows from the [Definitions 2.1 and 2.2](#) that k must be positive. Hence we can not obtain [\(3.1\)](#) directly. We will use the following fact that

$$(a - b)^+ = (a - b^+)^+ + (-b - a^-)^+, \quad \forall a, b \in \mathbb{R}. \tag{3.2}$$

In the sequel, without restriction, we assume that u is obtained by viscous approximation and choose a partition of unity subordinate to a covering of \bar{D} by balls \mathcal{B}_i , $i = 1, \dots, k$ satisfying $\mathcal{B}_0 \cap \partial D = \emptyset$ and, for $i = 1, \dots, k$, $\mathcal{B}_i \subset \mathcal{B}'_i$ with $\mathcal{B}'_i \cap \partial D$ part of a Lipschitz graph. We let

- $\phi \in \mathcal{D}^+([0, T] \times \mathbb{R}^d)$ with $\text{supp}\phi(t, \cdot) \subset B := \mathcal{B}_i$ for some $i \in \{1, \dots, k\}$;
- ρ_n is a sequence of mollifiers in \mathbb{R} with $\text{supp}\rho_n \subset [-2/n, 0]$;
- ρ_m is a shifted sequence of mollifiers in \mathbb{R}^d such that $y \rightarrow \rho_m(x - y) \in \mathcal{D}(D)$ for all $x \in B \cap D$.

We point out that ρ_m is chosen as in [\[4\]](#) and it follows that for m big enough, $y \rightarrow \phi(s, y)\rho_m(x - y) \in \mathcal{D}(D)$. Denote

$$\theta_m(y) := \int_D \rho_m(x - y)dx \quad \text{and} \quad \sigma_n(s) := \int_0^T \rho_n(t - s)dt,$$

which are non-negative, non-decreasing sequences bounded by 1.

To simplify matters, denote $p := (t, x, \alpha)$, $q := (s, y, \beta)$ and $\tilde{\mathcal{B}}^l_k := \rho_l(\eta_{\bar{\delta}}(u_\varepsilon(s, y)) - k)$, where the definition of ρ_l is the same as that of ρ_n . We divide the proof of [Lemma 3.2](#) into three steps.

Step 1: Estimate the first part of [\(3.2\)](#) on the right-side hand, that is, estimate $(a - b^+)^+$.

Since $\hat{u}(p)$ is a Young measure-valued entropy solutions to [\(1.1\)](#), we have

$$\begin{aligned} 0 \leq & \mathbb{E} \int_Q \int_{\mathbb{R}} \int_D \eta_{\bar{\delta}}(\hat{u}_0(x) - k) \phi(s, y) \rho_n(-s) \rho_m(x - y) dx \tilde{\mathcal{B}}^l_k dk dy ds \\ & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta'_{\bar{\delta}}(\hat{u}(p) - k) \phi(s, y) \partial_t \rho_n(t - s) \rho_m(x - y) dp \tilde{\mathcal{B}}^l_k dk dy ds \\ & - \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 F^{\eta_{\bar{\delta}}}(\hat{u}(p), k) \nabla_x \rho_m(x - y) \rho_n(t - s) \phi(s, y) dp \tilde{\mathcal{B}}^l_k dk dy ds \\ & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta'_{\bar{\delta}}(\hat{u}(p) - k) \int_{|z| \geq r} \frac{\hat{u}(t, x + z) - \hat{u}(t, x)}{|z|^{d+\gamma}} dz \phi \rho_n \rho_m dp \tilde{\mathcal{B}}^l_k dk dy ds \end{aligned}$$

$$\begin{aligned}
 & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta'_\delta(\hat{u}(p) - k) \int_{|z| < r} \frac{\hat{u}(t, x + z) - \hat{u}(t, x)}{|z|^{d+\gamma}} dz \phi \rho_n \rho_m dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 & + \frac{1}{2} \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 h^2(\hat{u}(p)) \eta''(\hat{u}(p) - k) \rho_m(x - y) \rho_n(t - s) \phi(s, y) dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta'_\delta(\hat{u}(p) - k) h(\hat{u}(p)) d\alpha \rho_m(x - y) \rho_n(t - s) \phi(s, y) dx dw(t) \tilde{\mathcal{B}}_k^l dk dy ds \\
 & =: I_1 + I_2 + \dots + I_6.
 \end{aligned}$$

On the other hand, if one denotes $\hat{\mathcal{A}}_k^l := \rho_l(k - \hat{u}(p))$, since u_ε is a viscous solution, the Itô formula gives

$$\begin{aligned}
 0 & \leq \mathbb{E} \int_Q \int_{\mathbb{R}} \int_D \eta_\delta(k - \eta_\delta(u_0^\varepsilon(y))) \phi(0, y) \rho_n(t) \rho_m(x - y) dy \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta_\delta(k - \eta_\delta(u_\varepsilon(s, y))) \partial_s \phi(s, y) \rho_n(t - s) \rho_m(x - y) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta_\delta(k - \eta_\delta(u_\varepsilon(s, y))) \phi(s, y) \partial_s \rho_n(t - s) \rho_m(x - y) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & - \varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta'_\delta(k - \eta_\delta(u_\varepsilon(s, y))) \eta'_\delta(u_\varepsilon(s, y)) \Delta_y u_\varepsilon(s, y) \phi(s, y) \\
 & \quad \times \rho_n(t - s) \rho_m(x - y) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & - \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta'_\delta(k - \eta_\delta(u_\varepsilon(s, y))) \eta'_\delta(u_\varepsilon(s, y)) \int_{\mathbb{R}^d \setminus \{0\}} \frac{u_\varepsilon(s, y + z) - u_\varepsilon(s, y)}{|z|^{d+\gamma}} dz \\
 & \quad \times \phi(s, y) \rho_n(t - s) \rho_m(x - y) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & - \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q F^{\eta_\delta(k - \eta_\delta(\cdot + k))}(u_\varepsilon(s, y), k) \nabla_y \phi(s, y) \rho_n(t - s) \rho_m(x - y) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & - \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q F^{\eta_\delta(k - \eta_\delta(\cdot + k))}(u_\varepsilon(s, y), k) \phi(s, y) \rho_n(t - s) \nabla_y \rho_m(x - y) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2} \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \{ \eta''_{\delta}(k - \eta_{\bar{\delta}}(u_{\varepsilon}(s, y))) (\eta'_{\bar{\delta}}(u_{\varepsilon}(s, y)))^2 \\
 & - \eta'_{\delta}(k - \eta_{\bar{\delta}}(u_{\varepsilon}(s, y))) \eta''_{\bar{\delta}}(u_{\varepsilon}(s, y)) \} \\
 & \times h^2(u_{\varepsilon}(s, y)) \phi(s, y) \rho_n(t - s) \rho_m(x - y) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & - \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta'_{\delta}(k - \eta_{\bar{\delta}}(u_{\varepsilon}(s, y))) \eta'_{\bar{\delta}}(u_{\varepsilon}(s, y)) h(u_{\varepsilon}(s, y)) \phi(s, y) \rho_n(t - s) dy dw(s) \\
 & \times \rho_m(x - y) \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & =: J_1 + J_2 + \dots + J_9.
 \end{aligned}$$

First, note that $\text{supp} \rho_n \subset [-2/n, 0]$, we have $\rho_n(t) = 0, t \in [0, T]$ and thus $J_1 = 0$. In paper [4], the authors obtained the followings:

$$\begin{aligned}
 I_1 & \rightarrow_{l, \delta, \bar{\delta}, \varepsilon, m, n} \mathbb{E} \int_D (\hat{u}_0 - u_0^+)^+ \phi(0, y) dy; \\
 J_2 & \rightarrow_{l, \delta, \bar{\delta}, \varepsilon, m, n} \mathbb{E} \int_Q \int_0^1 \int_0^1 (\hat{u}(t, x, \alpha) - u^+(t, x, \beta))^+ \partial_t \phi(t, x) d\alpha d\beta dp.
 \end{aligned}$$

By changing variable method, we have $I_2 + J_3 = 0$, see p. 2517 of [4] for details. Now, we consider the J_4 :

$$\begin{aligned}
 J_4 & = -\varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta'_{\delta}(k - \eta_{\bar{\delta}}(u_{\varepsilon}(s, y))) \eta'_{\bar{\delta}}(u_{\varepsilon}(s, y)) \Delta_y u_{\varepsilon}(s, y) \\
 & \times \phi(s, y) \rho_n(t - s) \rho_m(x - y) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & = \varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q [\Delta_y \eta_{\delta}(k - \eta_{\bar{\delta}}(u_{\varepsilon}(s, y))) - \eta''_{\delta}(k - \eta_{\bar{\delta}}(u_{\varepsilon}(s, y))) (\eta'_{\bar{\delta}}(u_{\varepsilon}(s, y)))^2] |\nabla u_{\varepsilon}|^2 \\
 & + \eta'_{\delta}(k - \eta_{\bar{\delta}}(u_{\varepsilon}(s, y))) \eta''_{\bar{\delta}}(u_{\varepsilon}(s, y)) |\nabla u_{\varepsilon}|^2 \phi(s, y) \rho_n(t - s) \rho_m(x - y) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & \leq \varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \Delta_y \eta_{\delta}(k - \eta_{\bar{\delta}}(u_{\varepsilon}(s, y))) \phi(s, y) \rho_n(t - s) \rho_m(x - y) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp
 \end{aligned}$$

$$\begin{aligned}
 & + \varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta'_\delta(k - \eta_{\bar{\delta}}(u_\varepsilon(s, y))) \eta''_\delta(u_\varepsilon(s, y)) |\nabla u_\varepsilon|^2 \phi(s, y) \rho_n(t - s) \rho_m(x - y) dy ds \\
 & \times \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 =: & J_{41} + J_{42}.
 \end{aligned}$$

We first look at J_{42} . We remark that $y \rightarrow \phi(s, y) \rho_m(x - y)$ have compact support and that $y \rightarrow u_\varepsilon(s, y) \in H^2_{loc}(D)$. Thus we have $\{u_\varepsilon = 0, \text{a.e.}\} \subset \{\nabla u_\varepsilon = 0, \text{a.e.}\}$. By using the facts $\lim_{\delta \rightarrow 0} \eta''_\delta(x) = \delta_0(x)$ and η'_δ is bounded, we get $\lim_{\bar{\delta} \rightarrow 0} J_{42} = 0$.

Now, we consider J_{41} . By Green formula, we have

$$\begin{aligned}
 J_{41} & = -\varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \nabla \eta_\delta(k - \eta_{\bar{\delta}}(u_\varepsilon(s, y))) \cdot [\nabla \phi(s, y) \rho_m(x - y) + \phi(s, y) \nabla_y \rho_m(x - y)] \\
 & \times \rho_n(t - s) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & = \varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta_\delta(k - \eta_{\bar{\delta}}(u_\varepsilon(s, y))) [\Delta \phi(s, y) \rho_m(x - y) + 2 \nabla \phi(s, y) \nabla_y \rho_m(x - y) \\
 & + \phi(s, y) \Delta \rho_m(x - y)] \rho_n(t - s) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp.
 \end{aligned}$$

For simplicity, denote

$$\mathcal{J} := \Delta \phi(s, y) \rho_m(x - y) + 2 \nabla \phi(s, y) \nabla_y \rho_m(x - y) + \phi(s, y) \Delta \rho_m(x - y).$$

Note that $\eta''_\delta \geq 0$, $\eta_\delta(x) = 0$ if $x \leq 0$ and $\eta'_\delta(x) = 1$ if $x > \delta$, we get

$$\eta_\delta(x + \eta_{\bar{\delta}}(y)) \leq |x + \eta_{\bar{\delta}}(y)| \leq |x| + |y|, \quad \forall x, y \in \mathbb{R}.$$

Then, by Hölder inequality, we have

$$\begin{aligned}
 \lim_{l \rightarrow \infty} J_{41} & = \varepsilon \mathbb{E} \int_Q \int_Q \int_0^1 \eta_\delta(\hat{u}(p) - \eta_{\bar{\delta}}(u_\varepsilon)) \mathcal{J} \rho_n(t - s) dy ds dp \\
 & \leq \varepsilon \mathbb{E} \int_Q \int_Q \int_0^1 |\hat{u}(p)| |\mathcal{J}| \rho_n(t - s) dy ds dp + \varepsilon \mathbb{E} \int_Q \int_Q \int_0^1 |u_\varepsilon| |\mathcal{J}| \rho_n(t - s) dy ds dp
 \end{aligned}$$

$$\begin{aligned} &\leq \varepsilon \mathbb{E} \int_Q \int_Q \int_0^1 |\hat{u}(p)| |\mathcal{J}| \rho_n(t-s) dy ds dp \\ &\quad + \varepsilon C(T, Q) \mathbb{E} \left(\int_Q |u_\varepsilon|^2 dy ds \right)^{1/2} \int_Q \int_0^1 \left(\int_Q |\mathcal{J}|^2 dy ds \right)^{1/2} \rho_n(t-s) dp, \end{aligned} \tag{3.3}$$

where $C(T, Q)$ is a positive constant, which depends on T and Q .

It follows from [Lemma 2.2](#) that

$$\|u_\varepsilon\|_{L^\infty(0, T; L^2(\Omega \times D))} \leq C,$$

where C does not depend on ε . Thus, letting $\varepsilon \rightarrow 0$ in [\(3.3\)](#) and using the assumptions on the functions ϕ , ρ_n and ρ_m , we have

$$\lim_{\varepsilon} \lim_l J_{41} \leq 0.$$

Combining the above results, we have $\lim_{\delta} \lim_{\varepsilon} \lim_l J_4 \leq 0$.

Remark 3.1. It is remarked that we do not use the local Kato inequality [\(1.4\)](#). In paper [\[4\]](#), the authors take the test function as

$$\psi_{\delta, \tilde{\delta}}^k(x) = \eta_\delta(k - \eta_{\tilde{\delta}}(x)) + \eta_\delta(-x),$$

which is different from that in this paper. It is easy to see that our proof is easier.

In paper [\[4\]](#), the authors introduce the term $\eta_\delta(-x)$ in order to estimate J_4 . They first took the limit of n , l , δ , $\tilde{\delta}$, and then by using [\(1.4\)](#) got that

$$\text{sgn}_0^+(\hat{u}^+(p) - u_\varepsilon) \Delta[\hat{u}^+(p) - u_\varepsilon] \leq \Delta[\hat{u}^+(p) - u_\varepsilon]^+.$$

Finally, they used integration by part to obtain the desired result. Meanwhile, one can find our discussions in this paper is easier to read.

Noting that $\eta'' \geq 0$, we have

$$\begin{aligned} \eta(a) - \eta(b) &= \eta'(b)(a - b) + \frac{1}{2} \eta''(\xi)(a - b)^2 \\ &\geq \eta'(b)(a - b), \quad \xi \in (\min\{a, b\}, \max\{a, b\}). \end{aligned}$$

By using the above inequality, we get

$$I_4 = \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta'_\delta(\hat{u}(p) - k) \int_{\mathbb{R}^d \setminus \{0\}} \frac{\hat{u}(t, x + z, \alpha) - \hat{u}(t, x, \alpha)}{|z|^{d+\gamma}} dz \phi \rho_n \rho_m dp \tilde{\mathcal{B}}_k^l dk dy ds$$

$$\begin{aligned}
 &\leq \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \int_{\mathbb{R}^d \setminus \{0\}} \frac{\eta_\delta(\hat{u}(t, x + z, \alpha) - k) - \eta_\delta(\hat{u}(t, x, \alpha) - k)}{|z|^{d+\gamma}} dz \phi \rho_n \rho_m dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 &= \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\hat{u}(p) - k) \int_{\mathbb{R}^d \setminus \{0\}} \frac{\rho_m(x + z - y) - \rho_m(x - y)}{|z|^{d+\gamma}} dz \phi \rho_n dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 &= \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\hat{u}(p) - k) \int_{\mathbb{R}^d \setminus \{0\}} \frac{\rho_m(x - y + z)\phi - \rho_m\phi(s, y + z)}{|z|^{d+\gamma}} dz \rho_n dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 &\quad + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta_\delta(\hat{u}(p) - k) \rho_m \rho_n (-\Delta)^{\gamma/2} \phi(s, y) dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 &=: I_{41} + I_{42},
 \end{aligned}$$

where in the above derivation $\rho_m := \rho_m(x - y)$, $\phi := \phi(s, y)$ and we have used the facts $\hat{u}|_{D^c} = 0$ and

$$\int_{\mathbb{R}^d} v(-\Delta)^{\gamma/2} u dx = \int_{\mathbb{R}^d} u(-\Delta)^{\gamma/2} v dx.$$

Clearly,

$$\lim_{l, m, n, \delta, \tilde{\delta}, \varepsilon} I_{42} \leq \int_Q \int_0^1 \int_0^1 (\hat{u}(p) - u^+(t, x, \beta))^+ (-\Delta)^{\gamma/2} \phi(t, x) dt dx d\alpha d\beta.$$

We first note that by the assumptions on ρ_m and ϕ , I_{41} makes sense.

$$\begin{aligned}
 \lim_{l, m, n, \varepsilon} I_{41} &= \mathbb{E} \int_Q \int_D \int_0^1 \int_0^1 \eta_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(u(t, y, \beta))) \\
 &\quad \times \int_{\mathbb{R}^d \setminus \{0\}} \frac{\delta_0(x - y + z)\phi - \delta_0(x - y)\phi(t, y + z)}{|z|^{d+\gamma}} dz d\beta dp dy \\
 &\leq \mathbb{E} \int_Q \int_0^1 \int_0^1 \int_{\mathbb{R}^d \setminus \{0\}} \frac{\eta_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(u(t, x + z, \beta))) - \eta_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(u(t, x, \beta)))}{|z|^{d+\gamma}} \\
 &\quad \times \phi(t, x + z) dz d\beta dp.
 \end{aligned}$$

Thanks to the properties of $\eta_\delta, \eta_{\tilde{\delta}}$, we know that the above integration is well-posed. By the assumptions of ϕ , $\text{supp}\phi(t, \cdot) \subset B := \mathcal{B}_i$ for some $i \in \{1, \dots, k\}$, there exists a

constant $c_1 > 0$ which does not depend on ϕ such that $|\phi(x + z) - \phi(x)| \leq c_1|z|$. When $|z| > 1$, it is easy to prove the above integration makes sense. We only consider the case $0 < |z| \leq 1$. Obviously,

$$\begin{aligned} & \int_{0 < |z| \leq 1} \frac{\eta_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(u(t, x + z, \beta))) - \eta_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(u(t, x, \beta)))}{|z|^{d+\gamma}} \phi(t, x + z) dz \\ & \leq C(|\hat{u}(p)| + |u(t, x, \beta)|) \int_{0 < |z| \leq 1} \frac{|u(t, x + z, \beta)[\phi(x + z) - \phi(x)]|}{|z|^{d+\gamma}} dz \\ & \quad + \phi(t, x) \int_{0 < |z| \leq 1} \frac{\eta_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(u(t, x + z, \beta))) - \eta_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(u(t, x, \beta)))}{|z|^{d+\gamma}} dz. \end{aligned}$$

The first integration is well-posedness because $\hat{u}, u \in L^p, \forall p \geq 2$. Moreover, the first integration is uniform bounded for $\delta, \tilde{\delta} > 0$. By Taylor expansion, we have

$$\begin{aligned} & \int_{0 < |z| \leq 1} \frac{\eta_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(u(t, x + z, \beta))) - \eta_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(u(t, x, \beta)))}{|z|^{d+\gamma}} dz \\ & = \eta'_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(u(t, x, \beta)))\eta'_{\tilde{\delta}}(u(t, x, \beta))(-\Delta)^{\frac{\gamma}{2}} u(t, x, \beta) \\ & \quad + \int_{0 < |z| \leq 1} [\eta''_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(\xi))\eta'_{\tilde{\delta}}(\xi) - \eta'_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(\xi))\eta''_{\tilde{\delta}}(\xi)] \\ & \quad \times \frac{|u(t, x + z, \beta) - u(t, x, \beta)|^2}{|z|^{d+\gamma}} dz \\ & \leq \eta'_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(u(t, x, \beta)))\eta'_{\tilde{\delta}}(u(t, x, \beta))|(-\Delta)^{\frac{\gamma}{2}} u(t, x, \beta)| \\ & \quad + \left| \int_{0 < |z| \leq 1} |\eta''_\delta(\hat{u}(p) - \eta_{\tilde{\delta}}(\xi))\eta'_{\tilde{\delta}}(\xi)| \frac{|u(t, x + z, \beta) - u(t, x, \beta)|^2}{|z|^{d+\gamma}} dz, \right. \end{aligned}$$

where $\xi = \theta u(t, x + z, \beta) + (1 - \theta)u(t, x, \beta), 0 \leq \theta \leq 1$. Since $u \in \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)$, the above integration is uniform bounded for $\delta, \tilde{\delta} > 0$. That is to say, the following integration makes sense

$$\mathbb{E} \int_Q \int_0^1 \int_0^1 \int_{\mathbb{R}^d \setminus \{0\}} \frac{(\hat{u}(p) - u^+(t, x + z, \beta))^+ - (\hat{u}(p) - u^+(t, x, \beta))^+}{|z|^{d+\gamma}} \phi(t, x + z) dz d\beta dp.$$

For J_5 , we have

$$J_5 = -\mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta'_\delta(k - \eta_{\tilde{\delta}}(u_\varepsilon(s, y)))\eta'_{\tilde{\delta}}(u_\varepsilon(s, y)) \int_{\mathbb{R}^d \setminus \{0\}} \frac{u_\varepsilon(s, y + z) - u_\varepsilon(s, y)}{|z|^{d+\gamma}} dz$$

$$\begin{aligned} & \times \phi(s, y)\rho_n(t - s)\rho_m(x - y)dyds \int_0^1 \hat{A}_k^l dkdp \\ \rightarrow_{l,m,n,\delta,\bar{\delta},\varepsilon} & -\mathbb{E} \int_Q \int_0^1 \int_0^1 \text{sgn}_0^+(\hat{u}(p) - u^+(t, x, \beta)) \text{sgn}_0^+(u(t, x, \beta))\phi(-\Delta)^{\frac{\gamma}{2}} ud\beta dp. \end{aligned}$$

Due to $u \in \hat{H}^{\frac{\gamma}{2}}(\mathbb{R}^d)$, the above integration makes sense.

Following [4], we have $I_3 + J_7 = 0$ and

$$J_6 \rightarrow_{l,m,n,\delta,\bar{\delta},\varepsilon} -\mathbb{E} \int_Q \int_0^1 \int_0^1 F^+(\hat{u}(p), u^+(t, x, \beta))\nabla_x \phi(t, x)d\alpha d\beta dp.$$

$$\begin{aligned} I_5 + J_8 &= \frac{1}{2}\mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 h^2(\hat{u}(p))\eta''(\hat{u}(p) - k)\rho_m(x - y)\rho_n(t - s)\phi(s, y)dp\tilde{B}_k^l dkdyds \\ &+ \frac{1}{2}\mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \{ \eta''_{\delta}(k - \eta_{\bar{\delta}}(u_{\varepsilon}(s, y))) (\eta'_{\bar{\delta}}(u_{\varepsilon}(s, y)))^2 \\ &- \eta'_{\delta}(k - \eta_{\bar{\delta}}(u_{\varepsilon}(s, y)))\eta''_{\bar{\delta}}(u_{\varepsilon}(s, y)) \} h^2(u_{\varepsilon}(s, y))\phi(s, y)\rho_n(t - s) \\ &\times \rho_m(x - y)dyds \int_0^1 \hat{A}_k^l dkdp \\ \rightarrow_{l,n} & \frac{1}{2}\mathbb{E} \int_Q \int_D \int_0^1 h^2(\hat{u}(p))\eta''(\hat{u}(p) - \eta_{\bar{\delta}}(u_{\varepsilon}(t, y)))\rho_m(x - y)\phi(t, y)dydp \\ &+ \frac{1}{2}\mathbb{E} \int_Q \int_D \int_0^1 \{ \eta''_{\delta}(\hat{u}(p) - \eta_{\bar{\delta}}(u_{\varepsilon}(t, y))) (\eta'_{\bar{\delta}}(u_{\varepsilon}(t, y)))^2 \\ &- \eta'_{\delta}(\hat{u}(p) - \eta_{\bar{\delta}}(u_{\varepsilon}(t, y)))\eta''_{\bar{\delta}}(u_{\varepsilon}(t, y)) \} h^2(u_{\varepsilon}(t, y))\phi(t, y)\rho_m(x - y)dydp. \end{aligned}$$

Since $\alpha(t) = \int_0^1 \rho_l(k - \hat{u}(t, x, \tau))d\tau$ is predictable and if one denotes

$$\beta(s) \int_D \eta'_{\delta}(k - \eta_{\bar{\delta}}(u_{\varepsilon}))\eta'_{\bar{\delta}}(u_{\varepsilon})h(u_{\varepsilon})\phi(s, y)\rho_n(t - s)dy,$$

we have

$$\mathbb{E} \left[\alpha(t) \int_t^T \beta(s)dw(s) \right] = \mathbb{E} \left[\alpha(t) \int_0^T \beta(s)dw(s) \right] - \mathbb{E} \left[\alpha(t) \int_0^t \beta(s)dw(s) \right] = 0,$$

where we used the fact that

$$\mathbb{E} \left[\alpha(t) \int_0^T \beta(s)dw(s) \right] = \mathbb{E} \left[\alpha(t) \mathbb{E} \int_0^T \beta(s)dw(s) | \mathcal{F}_t \right] = \mathbb{E} \left[\alpha(t) \int_0^t \beta(s)dw(s) \right].$$

Then, by the same type of arguments with $\rho_l(\eta_{\bar{\delta}}(u_\varepsilon(s - \frac{2}{n}, y)) - k)$, we deduce

$$\begin{aligned} I_6 + J_9 &= \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta'_{\bar{\delta}}(\hat{u}(p) - k)h(\hat{u}(p))d\alpha\rho_m(x - y)\rho_n(t - s)\phi(s, y)dx dw(t) \\ &\quad \times \hat{\mathcal{B}}_k^l dk dy ds \\ &\quad - \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta'_{\bar{\delta}}(k - \eta_{\bar{\delta}}(u_\varepsilon(s, y)))\eta'_{\bar{\delta}}(u_\varepsilon(s, y))h(u_\varepsilon(s, y))\phi(s, y)\rho_n(t - s)dy dw(s) \\ &\quad \times \rho_m(x - y) \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\ &= \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta'_{\bar{\delta}}(\hat{u}(p) - k)h(\hat{u}(p))d\alpha\rho_m(x - y)\rho_n(t - s)\phi(s, y)dx dw(t) \\ &\quad \times \left[\rho_l(\eta_{\bar{\delta}}(u_\varepsilon(s, y)) - k) - \rho_l(\eta_{\bar{\delta}}(u_\varepsilon(s - \frac{2}{n}, y)) - k) \right] dk dy ds. \end{aligned}$$

As $du_\varepsilon = [\varepsilon\Delta u_\varepsilon - (-\Delta)^{\gamma/2}u_\varepsilon + \operatorname{div} f(u_\varepsilon)]dt + h(u_\varepsilon)dw = A_\varepsilon dt + h(u_\varepsilon)dw$, we get

$$\begin{aligned} &\rho_l(\eta_{\bar{\delta}}(u_\varepsilon(s, y)) - k) - \rho_l(\eta_{\bar{\delta}}(u_\varepsilon(s - \frac{2}{n}, y)) - k) \\ &= \int_{(s-\frac{2}{n})^+}^s \rho'_l(\eta_{\bar{\delta}}(u_\varepsilon(\sigma, y)) - k)\eta'_{\bar{\delta}}(u_\varepsilon(\sigma, y))A_\varepsilon(\sigma, y)d\sigma \\ &\quad + \int_{(s-\frac{2}{n})^+}^s \rho'_l(\eta_{\bar{\delta}}(u_\varepsilon(\sigma, y)) - k)\eta'_{\bar{\delta}}(u_\varepsilon(\sigma, y))h(u_\varepsilon(\sigma, y))dw(\sigma) \\ &\quad + \frac{1}{2} \int_{(s-\frac{2}{n})^+}^s \{ \rho''_l(\eta_{\bar{\delta}}(u_\varepsilon(\sigma, y)) - k) (\eta'_{\bar{\delta}}(u_\varepsilon(\sigma, y)))^2 \\ &\quad + \rho'_l(\eta_{\bar{\delta}}(u_\varepsilon(\sigma, y)) - k)\eta''_{\bar{\delta}}(u_\varepsilon(\sigma, y)) \} h^2(u_\varepsilon(\sigma, y))d\sigma \\ &= -\frac{\partial}{\partial k} \left\{ \int_{(s-\frac{2}{n})^+}^s \rho_l(\eta_{\bar{\delta}}(u_\varepsilon(\sigma, y)) - k)\eta'_{\bar{\delta}}(u_\varepsilon(\sigma, y))A_\varepsilon(\sigma, y)d\sigma \right. \end{aligned}$$

$$\begin{aligned}
 &+ \int_{(s-\frac{2}{n})^+}^s \rho_l(\eta_{\bar{\delta}}(u_\varepsilon(\sigma, y)) - k)\eta'_{\bar{\delta}}(u_\varepsilon(\sigma, y))h(u_\varepsilon(\sigma, y))d\omega(\sigma) \\
 &+ \frac{1}{2} \int_{(s-\frac{2}{n})^+}^s \left\{ \rho'_l(\eta_{\bar{\delta}}(u_\varepsilon(\sigma, y)) - k) (\eta'_{\bar{\delta}}(u_\varepsilon(\sigma, y)))^2 + \rho_l(\eta_{\bar{\delta}}(u_\varepsilon(\sigma, y)) - k)\eta''_{\bar{\delta}}(u_\varepsilon(\sigma, y)) \right\} \\
 &\times h^2(u_\varepsilon(\sigma, y))d\sigma \Big\}.
 \end{aligned}$$

Noting that $\int_Q A_\varepsilon^2(\sigma, y)d\sigma dy < \infty$, similar to that in [4], we have

$$\limsup_{\delta, \bar{\delta}} \lim_{l, n} I_5 + J_8 + I_6 + J_9 \leq 0.$$

Combining all the estimates then yields

$$\begin{aligned}
 0 &\leq \mathbb{E} \int_{Q \times (0,1)^2} (\hat{u}(p) - u^+(t, x, \beta))^+ \partial_t \phi dp d\beta \\
 &- \mathbb{E} \int_{Q \times (0,1)^2} (\hat{u}(p) - u^+(t, x, \beta))^+ (-\Delta)^{\gamma/2} \phi(t, x) dp d\beta \\
 &- \mathbb{E} \int_{Q \times (0,1)^2} \text{sgn}_0^+(\hat{u}(p) - u^+(t, x, \alpha)) [f(\hat{u}(p)) - f(u^+(t, x, \beta))] \cdot \nabla \phi dp d\beta \\
 &+ \int_D (\hat{u}_0 - u_0^+)^+ \phi(0) dx + \hat{\mathcal{L}}(\phi),
 \end{aligned}$$

where

$$\begin{aligned}
 \hat{\mathcal{L}}(\phi) &= \mathbb{E} \int_Q \int_0^1 \int_0^1 \int_{\mathbb{R}^d \setminus \{0\}} \frac{(\hat{u}(p) - u^+(t, x + z, \beta))^+ - (\hat{u}(p) - u^+(t, x, \beta))^+}{|z|^{d+\gamma}} \\
 &\times \phi(t, x + z) dz d\beta dp \\
 &- \mathbb{E} \int_Q \int_0^1 \int_0^1 \text{sgn}_0^+(\hat{u}(p) - u^+(t, x, \beta)) \text{sgn}_0^+(u(t, x, \beta)) \phi(-\Delta)^{\frac{\gamma}{2}} u d\beta dp.
 \end{aligned}$$

Step 2: Estimate the second part of (3.2) on the right-side hand, that is, estimate $(-b - a^-)^+$.

In this step, the proof is similar to that in [4]. Note that the test function $\phi \rho_m \rho_n$ vanishes on the boundary. By denoting again $\hat{\mathcal{B}}_k^l := \rho_l(\eta_{\bar{\delta}}(u_\varepsilon(s, y)) - k)$, one has

$$\begin{aligned}
 0 \leq & \mathbb{E} \int_Q \int_{\mathbb{R}} \int_D \check{\eta}_\delta(\hat{u}_0(x) - k) \phi(0, x) \rho_n(-s) \rho_m(y - x) dx \tilde{\mathcal{B}}_k^l dk dy ds \\
 & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \check{\eta}_\delta(\hat{u}(p) - k) \partial_t \phi(t, x) \rho_n(t - s) \rho_m(y - x) dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \check{\eta}_\delta(\hat{u}(p) - k) \phi(t, x) \partial_t \rho_n(t - s) \rho_m(y - x) dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 & - \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 F^{\check{\eta}_\delta}(\hat{u}(p), k) \nabla_x \phi(t, x) \rho_m(y - x) \rho_n(t - s) dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 & - \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 F^{\check{\eta}_\delta}(\hat{u}(p), k) \nabla_x \rho_m(y - x) \rho_n(t - s) \phi(t, x) dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \check{\eta}'_\delta(\hat{u}(p) - k) \int_{\mathbb{R}^d \setminus \{0\}} \frac{\hat{u}(t, x + z) - \hat{u}(t, x)}{|z|^{d+\gamma}} dz \phi \rho_n \rho_m dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 & + \frac{1}{2} \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 h^2(\hat{u}(p)) \check{\eta}''(\hat{u}(p) - k) \rho_m(y - x) \rho_n(t - s) \phi(t, x) dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \check{\eta}'_\delta(\hat{u}(p) - k) h(\hat{u}(p)) d\alpha \rho_m(y - x) \rho_n(t - s) \phi(t, x) dx dw(t) \tilde{\mathcal{B}}_k^l dk dy ds \\
 =: & I_1 + I_2 + \dots + I_8.
 \end{aligned}$$

Moreover, the entropy formulation, with $k = 0$ and any regular non-negative ϕ , yields

$$\begin{aligned}
 0 \leq & \int_D \eta_\delta(-\hat{u}_0) \phi(0) dx + \mathbb{E} \int_Q \int_0^1 \left[\eta_\delta(-\hat{u}) \partial_t \phi - F^{\check{\eta}_\delta}(\hat{u}, 0) \nabla \phi + \frac{1}{2} h^2(\hat{u}) \eta''_\delta(-u) \phi \right] dp \\
 & + \mathbb{E} \int_Q \int_0^1 \eta_\delta(-\hat{u}) (-\Delta)^{\gamma/2} \phi dp,
 \end{aligned}$$

where we used the convex of the function η .

Since $\lim_{\delta \rightarrow 0} \eta_\delta(x) = x^+$, $\lim_{\delta \rightarrow 0} \eta''_\delta(x) = \delta_0(x)$ and $h(0) = 0$, we have

$$0 \leq \int_D \hat{u}_0^- \phi(0) dx + \mathbb{E} \int_Q \int_0^1 \left[\hat{u}^- \partial_t \phi - F^{(\cdot)^-}(\hat{u}, 0) \nabla \phi + \hat{u}^- (-\Delta)^{\gamma/2} \phi \right] dp.$$

Denote

$$\mathcal{L}(\phi) := \int_D \hat{u}_0^- \phi(0) dx + \mathbb{E} \int_Q \int_0^1 [\hat{u}^- \partial_t \phi - F^{(\cdot)-}(\hat{u}, 0) \nabla \phi + \hat{u}^- (-\Delta)^{\gamma/2} \phi] dp.$$

Clearly, \mathcal{L} is linear and non-negative over $\mathcal{D}([0, T] \times \mathbb{R}^d)$. Since $0 \leq \phi\theta_m \leq \phi\theta_{m+1} \leq \phi$, we conclude that $\mathcal{L}(\phi\theta_m)$ has a limit in $[0, \infty)$ when $m \rightarrow \infty$. Thus

$$\begin{aligned} \lim_{m \rightarrow \infty} \mathcal{L}(\phi\theta_m) &= \int_D \hat{u}_0^- \phi(0) dx + \mathbb{E} \int_Q \int_0^1 \hat{u}^- \partial_t \phi dp - \mathbb{E} \int_Q \int_0^1 F^{(\cdot)-}(\hat{u}, 0) \nabla \phi dp \\ &\quad - \lim_{m \rightarrow \infty} \mathbb{E} \int_Q \int_0^1 \phi F^{(\cdot)-}(\hat{u}, 0) \nabla \theta_m dp + \mathbb{E} \int_Q \int_0^1 \hat{u}^- (-\Delta)^{\gamma/2} \phi dp \\ &\quad + \lim_{m \rightarrow \infty} \mathbb{E} \int_Q \int_0^1 \hat{u}^- \int_{\mathbb{R}^d \setminus \{0\}} \frac{\theta_m(x+z) - \theta_m(x)}{|z|^{d+\gamma}} \phi(t, x+z) dz dp \\ &=: \tilde{I}_1 + \dots + \tilde{I}_6. \end{aligned}$$

On the other hand, denoting again $\hat{\mathcal{A}} := \rho_l(k - \hat{u})$, since u_ε is a viscous solution, the Itô formula applied to $\int_D \eta_\delta(\eta_\delta(u_\varepsilon) - k) \rho_n(t-s) \rho_m(y-x) \phi(t, x) dy ds$ yields

$$\begin{aligned} 0 \leq & \mathbb{E} \int_Q \int_{\mathbb{R}} \int_D \eta_\delta(\eta_\delta(u_\varepsilon^\xi(y)) - k) \phi(t, x) \rho_n(t) \rho_m(y-x) dy \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\ & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta_\delta(\eta_\delta(u_\varepsilon(s, y)) - k) \phi(t, x) \partial_s \rho_n(t-s) \rho_m(y-x) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\ & - \varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \nabla_y u_\varepsilon(s, y) [\nabla_y \phi(t, x) \rho_m(y-x) + \nabla_y \rho_m(y-x) \phi(t, x)] \rho_n(t-s) \\ & \times \eta'_\delta(\eta_\delta(u_\varepsilon(s, y)) - k) \eta'_\delta(u_\varepsilon(s, y)) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\ & - \varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q [\eta''_\delta(\eta_\delta(u_\varepsilon(s, y)) - k) (\eta'_\delta(u_\varepsilon(s, y)))^2 + \eta'_\delta(\eta_\delta(u_\varepsilon(s, y)) - k) \eta''_\delta(u_\varepsilon(s, y))] \\ & \times |\nabla_y u_\varepsilon(s, y)|^2 \phi(t, x) \rho_m(y-x) \rho_n(t-s) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \end{aligned}$$

$$\begin{aligned}
 & - \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q F^{\eta_{\delta}(\eta_{\delta}(\cdot) - k)}(u_{\varepsilon}(s, y), k) \nabla_y \rho_m(y - x) \rho_n(t - s) \phi(t, x) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta'_{\delta}(\eta_{\delta}(u_{\varepsilon}(s, y)) - k) \eta'_{\delta}(u_{\varepsilon}(s, y)) \int_{\mathbb{R}^d \setminus \{0\}} \frac{u_{\varepsilon}(s, y + z) - u_{\varepsilon}(s, y)}{|z|^{d+\gamma}} dz \\
 & \times \phi(t, x) \rho_n(t - s) \rho_m(y - x) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & + \frac{1}{2} \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \{ \eta''_{\delta}(\eta_{\delta}(u_{\varepsilon}(s, y)) - k) (\eta'_{\delta}(u_{\varepsilon}(s, y)))^2 \\
 & + \eta'_{\delta}(k - \eta_{\delta}(u_{\varepsilon}(s, y))) \eta''_{\delta}(u_{\varepsilon}(s, y)) \} \\
 & \times h^2(u_{\varepsilon}(s, y)) \phi(t, x) \rho_n(t - s) \rho_m(y - x) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 & + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta'_{\delta}(\eta_{\delta}(u_{\varepsilon}(s, y)) - k) \eta'_{\delta}(u_{\varepsilon}(s, y)) h(u_{\varepsilon}(s, y)) \phi(t, x) \rho_n(t - s) dy dw(s) \\
 & \times \rho_m(y - x) \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 =: & J_1 + J_2 + \dots + J_8.
 \end{aligned}$$

Due to that the support $\text{supp} \rho_n \subset [-\frac{2}{n}, 0]$, $\rho_n(t) = 0$ for $t \in [0, T]$, and thus $J_1 = 0$. By the known results of [4], we have

$$\begin{aligned}
 I_1 - \tilde{I}_1 & \rightarrow_{l, \delta, \bar{\delta}, \varepsilon, m, n} \mathbb{E} \int_D [u_0^+ - \hat{u}_0]^+ \phi(0, \cdot) dy - \int_D \hat{u}_0^- \phi(0) dy; \\
 I_2 - \tilde{I}_2 & \rightarrow_{l, \delta, \bar{\delta}, \varepsilon, m, n} \mathbb{E} \int_Q \int_0^1 \int_0^1 [u^+(t, x, \beta) - \hat{u}(p)]^+ \partial_t \phi dp d\beta - \mathbb{E} \int_Q \int_0^1 \hat{u}^- \partial_t \phi dp; \\
 I_4 - \tilde{I}_3 & \rightarrow_{l, \delta, \bar{\delta}, \varepsilon, m, n} - \mathbb{E} \int_Q \int_0^1 \int_0^1 \text{sgn}_0^+ [u^+(t, x, \beta) - \hat{u}(p)] [f(u^+(t, x, \beta)) \\
 & - f(\hat{u}(p))] \nabla_x \phi(t, x) dp d\beta \\
 & - \mathbb{E} \int_Q \int_0^1 \int_0^1 f(-\hat{u}^-) \nabla_x \phi(t, x) dp;
 \end{aligned}$$

$$I_5 + J_5 - \tilde{I}_4 = 0.$$

By changing variable, we get $I_3 + J_2 = 0$ (see [4] for details). Noting that $\nabla_y \phi(t, x) = 0$, we get

$$\begin{aligned}
 J_3 &= -\varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta'_\delta(\eta_{\bar{\delta}}(u_\varepsilon(s, y)) - k) \eta'_\delta(u_\varepsilon(s, y)) \nabla_y u_\varepsilon(s, y) \nabla_y \rho_m(y - x) \\
 &\quad \times \phi(t, x) \rho_n(t - s) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 &= -\varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \nabla_y \eta_\delta(\eta_{\bar{\delta}}(u_\varepsilon(s, y)) - k) \nabla_y \rho_m(y - x) \phi(t, x) \rho_n(t - s) dy ds \\
 &\quad \times \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 &= \varepsilon \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta_\delta(\eta_{\bar{\delta}}(u_\varepsilon(s, y)) - k) \Delta_y \rho_m(y - x) \phi(t, x) \rho_n(t - s) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\
 &\rightarrow_{l, n, \delta, \bar{\delta}} \varepsilon \mathbb{E} \int_Q \int_D \int_0^1 (u_\varepsilon^+(t, x) - \hat{u}(p))^+ \Delta_y \rho_m(y - x) \phi(t, x) dy dp \\
 &\leq \varepsilon \mathbb{E} \int_Q \int_D \int_0^1 |\hat{u}(p)| \Delta_y \rho_m(y - x) \phi(t, x) dy dp \\
 &\quad + \varepsilon C \mathbb{E} \int_0^T \left(\int_D u_\varepsilon^2(t, x) dx \right)^{\frac{1}{2}} \left(\int_D \left(\int_D \Delta_y \rho_m(y - x) dy \right)^2 \right)^{\frac{1}{2}} dt \\
 &\rightarrow_\varepsilon 0.
 \end{aligned}$$

Noting that $\eta', \eta'' \geq 0$, we have $J_4 \leq 0$. By Lemma 2.2, we know that $\int_Q A_\varepsilon^2(\sigma, y) d\sigma dy < \infty$, similar to that in [4], one can prove

$$I_7 + J_7 + I_8 + J_8 \rightarrow_{l, \delta, \bar{\delta}, \varepsilon, m, n} 0.$$

Now, we consider the terms I_6, J_6, \tilde{I}_5 and \tilde{I}_6 .

$$\begin{aligned}
 I_6 &= -\mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \eta'_\delta(k - \hat{u}(p)) \int_{\mathbb{R}^d \setminus \{0\}} \frac{\hat{u}(t, x + z) - \hat{u}(t, x)}{|z|^{d+\gamma}} dz \phi \rho_n \rho_m dp \tilde{\mathcal{B}}_k^l dk dy ds \\
 &\leq \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \int_0^1 \int_{\mathbb{R}^d \setminus \{0\}} \frac{\eta_\delta(k - \hat{u}(t, x + z, \alpha)) - \eta_\delta(k - \hat{u}(p))}{|z|^{d+\gamma}} dz \phi \rho_n \rho_m dp \tilde{\mathcal{B}}_k^l dk dy ds
 \end{aligned}$$

$$\begin{aligned}
 &= \mathbb{E} \int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta_\delta(k - \hat{u}(p)) \int_{\mathbb{R}^d \setminus \{0\}} \frac{\rho_m(y - x - z)\phi(t, x + z) - \rho_m\phi(t, x)}{|z|^{d+\gamma}} dz \\
 &\quad \times \rho_n(t - s) d\alpha dx dt \tilde{\mathcal{B}}_k^l dk dy ds \\
 &= \mathbb{E} \int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta_\delta(k - \hat{u}(p)) \int_{\mathbb{R}^d \setminus \{0\}} \frac{\rho_m(y - x - z) - \rho_m}{|z|^{d+\gamma}} \phi(t, x + z) dz \\
 &\quad \times \rho_n(t - s) d\alpha dx dt \tilde{\mathcal{B}}_k^l dk dy ds \\
 &\quad + \mathbb{E} \int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \int_0^1 \eta_\delta(k - \hat{u}(p)) \rho_n(t - s) \rho_m(y - x) (-\Delta)^{\frac{\gamma}{2}} \phi d\alpha dx dt \tilde{\mathcal{B}}_k^l dk dy ds \\
 &:= I_{61} + I_{62}.
 \end{aligned}$$

Note that $y \rightarrow \rho_m(x - y) \in \mathcal{D}(D)$, we have

$$I_{62} \rightarrow_{l, \delta, \tilde{\delta}, \varepsilon, m, n} \mathbb{E} \int_Q \int_0^1 \int_0^1 [u^+(t, x, \beta) - \hat{u}(p)]^+ (-\Delta)^{\gamma/2} \phi(t, x) dp d\beta,$$

which implies

$$I_{62} - \tilde{I}_5 \rightarrow_{l, \delta, \tilde{\delta}, \varepsilon, m, n} \mathbb{E} \int_Q \int_0^1 \int_0^1 [(u^+(t, x, \beta) - \hat{u}(p))^+ - u^-] (-\Delta)^{\gamma/2} \phi(t, x) dp d\beta.$$

Similar to the discussion about the I_{41} and J_5 in first step, the following limits exist and the resulting integrations make sense

$$\begin{aligned}
 I_{61} &= \mathbb{E} \int_Q \int_{\mathbb{R}} \int_0^T \int_{\mathbb{R}^d} \int_0^1 \int_{\mathbb{R}^d \setminus \{0\}} \eta_\delta(k - \hat{u}(t, x - z, \alpha)) \frac{\rho_m - \rho_m(y - x + z)}{|z|^{d+\gamma}} dz \\
 &\quad \times \phi(t, x) \rho_n(t - s) d\alpha dx dt \tilde{\mathcal{B}}_k^l dk dy ds \\
 &\rightarrow_{l, n, m, \delta, \tilde{\delta}, \varepsilon} \mathbb{E} \int_Q \int_0^1 \int_0^1 \int_{\mathbb{R}^d \setminus \{0\}} \phi(t, x) [(u^+(t, x, \beta) - \hat{u}(t, x + z, \alpha))^+ \\
 &\quad - (u^+(t, x + z, \beta) - \hat{u}(t, x + z, \alpha))^+] / |z|^{d+\gamma} dz d\beta dp;
 \end{aligned}$$

$$J_6 = \mathbb{E} \int_Q \int_{\mathbb{R}} \int_Q \eta'_\delta(\eta_{\tilde{\delta}}(u_\varepsilon(s, y)) - k) \eta'_\delta(u_\varepsilon(s, y)) \int_{\mathbb{R}^d \setminus \{0\}} \frac{u_\varepsilon(s, y + z) - u_\varepsilon(s, y)}{|z|^{d+\gamma}} dz$$

$$\begin{aligned} & \times \phi(t, x) \rho_n(t - s) \rho_m(y - x) dy ds \int_0^1 \hat{\mathcal{A}}_k^l dk dp \\ \rightarrow_{l, m, n, \delta, \tilde{\delta}, \varepsilon} & \mathbb{E} \int_Q \int_0^1 \int_0^1 \operatorname{sgn}_0^+(u^+(t, x, \beta) - \hat{u}(p)) \operatorname{sgn}_0^+(u(t, x, \beta)) \\ & \times \phi(t, x) (-\Delta)^{\frac{\gamma}{2}} u(t, x, \beta) d\beta dp. \end{aligned}$$

Denote

$$\mathcal{I} := \lim_{l, n, m, \delta, \tilde{\delta}, \varepsilon} (I_{61} + J_6).$$

Then, combining all the above estimates, we get

$$\begin{aligned} 0 \leq & \mathbb{E} \int_D [(u_0^+ - \hat{u}_0)^+ - \hat{u}_0^-] \phi(0, x) dx \\ & + \mathbb{E} \int_Q \int_0^1 \int_0^1 [(u^+(t, x, \beta) - \hat{u}(p))^+ - \hat{u}^-] (-\Delta)^{\gamma/2} \phi(t, x) d\alpha d\beta dt dx \\ & + \mathbb{E} \int_Q \int_0^1 \int_0^1 [(u^+(t, x, \beta) - \hat{u}(p))^+ - \hat{u}^-] \partial_t \phi dp d\beta \\ & - \mathbb{E} \int_Q \int_0^1 \int_0^1 \operatorname{sgn}_0^+[u^+(t, x, \beta) - \hat{u}(p)] [f(u^+(t, x, \beta)) - f(\hat{u}(p))] \nabla_x \phi(t, x) dp d\beta \\ & - \mathbb{E} \int_Q \int_0^1 \int_0^1 f(-\hat{u}^-) \nabla_x \phi(t, x) dp + \lim_m (\mathcal{L}(\phi\theta_m) - \tilde{I}_6 + \mathcal{I}). \end{aligned}$$

Denote $\tilde{\mathcal{L}}(\phi\theta_m) := (\mathcal{L}(\phi\theta_m) - \tilde{I}_6)$, we have

$$\begin{aligned} \lim_{m \rightarrow \infty} \tilde{\mathcal{L}}(\phi\theta_m) &= \int_D \hat{u}_0^- \phi(0) dx + \mathbb{E} \int_Q \int_0^1 \hat{u}^- \partial_t \phi dp - \mathbb{E} \int_Q \int_0^1 F^{(\cdot)-}(\hat{u}, 0) \nabla \phi dp \\ &\quad - \lim_{m \rightarrow \infty} \mathbb{E} \int_Q \int_0^1 \phi F^{(\cdot)-}(\hat{u}, 0) \nabla \theta_m dp + \mathbb{E} \int_Q \int_0^1 \hat{u}^- (-\Delta)^{\gamma/2} \phi dp. \end{aligned}$$

By using the fact that $(a^+ - b)^+ - b^- = (a - b^+)^+$, we get

$$\begin{aligned}
 0 \leq & \int_D (u_0 - \hat{u}_0^+)^+ \phi(0, x) dx + \lim_m [\tilde{\mathcal{L}}(\phi\theta_m) + \tilde{\mathcal{I}}] \\
 & + \mathbb{E} \int_{Q \times (0,1)^2} (\hat{u}(t, x, \alpha) - u(t, x, \beta))^+ \partial_t \phi dx dt d\alpha d\beta \\
 & - \mathbb{E} \int_{Q \times (0,1)^2} (\hat{u}(t, x, \alpha) - u(t, x, \beta))^+ (-\Delta)^\alpha \phi(t, x) dx dt d\alpha d\beta \\
 & - \mathbb{E} \int_{Q \times (0,1)^2} \text{sgn}_0^+ [\hat{u}(p) - u(t, x, \beta)] [f(\hat{u}(p)) - f(u(t, x, \beta))] \nabla_x \phi(t, x) dp d\beta.
 \end{aligned}$$

Note that $-\hat{u}$ and $-u$ are measure-valued entropy solution of $dv = (\text{div } \tilde{f}(v) - (-\Delta)^{\gamma/2} v) dt + \tilde{h}(v) dw$ with $\tilde{f}(x) = -f(-x)$, $\tilde{h}(x) = -h(-x)$ and initial data $-\hat{u}_0$ and $-u_0$, respectively, where u is obtained by the viscous approximation u_ε . Consequently, replacing \hat{u} by $-\hat{u}$ and u by $-u$ in above inequality, we get the following estimate

$$\begin{aligned}
 0 \leq & \int_D (-u_0 - \hat{u}_0^-)^+ \phi(0, x) dx + \lim_m [\tilde{\mathcal{L}}(\phi\theta_m) + \tilde{\mathcal{I}}] \\
 & + \mathbb{E} \int_Q \int_0^1 \int_0^1 (-u(t, x, \beta) - \hat{u}^-(p))^+ (-\Delta)^{\gamma/2} \phi(t, x) d\alpha d\beta dt dx \\
 & + \mathbb{E} \int_Q \int_0^1 \int_0^1 (-u(t, x, \beta) - \hat{u}^-(p))^+ \partial_t \phi dp d\beta \\
 & - \mathbb{E} \int_Q \int_0^1 \int_0^1 \text{sgn}_0^+ [-u(t, x, \beta) - \hat{u}^-(p)] [f(u(t, x, \beta)) - f(-\hat{u}^-(p))] \nabla_x \phi(t, x) dp d\beta,
 \end{aligned}$$

where

$$\begin{aligned}
 \tilde{\mathcal{I}}(\phi) := & \mathbb{E} \int_Q \int_0^1 \int_0^1 \int_{\mathbb{R}^d \setminus \{0\}} \phi(t, x) [(u^-(t, x, \beta) + \hat{u}(t, x + z, \alpha))^+ \\
 & - (u^-(t, x + z, \beta) + \hat{u}(t, x + z, \alpha))^+] / |z|^{d+\gamma} dz d\beta dp \\
 & - \mathbb{E} \int_Q \int_0^1 \int_0^1 \text{sgn}_0^+ (u^-(t, x, \beta) + \hat{u}(p)) \text{sgn}_0^+ (u(t, x, \beta)) \\
 & \times \phi(t, x) (-\Delta)^{\frac{\gamma}{2}} u(t, x, \beta) d\beta dp.
 \end{aligned}$$

Step 3: by using (3.2) and the identity

$$-\operatorname{sgn}_0^+(b-a)[f(b)-f(a)] = -\operatorname{sgn}_0^+(b-a^+)[f(b)-f(a^+)] + \operatorname{sgn}_0^+(-a-b^-)[f(a)-f(-b^-)]$$

we obtain that

$$\begin{aligned} 0 \leq & \int_D (\hat{u}_0 - u_0)^+ \phi(0) dx + \mathbb{E} \int_{Q \times (0,1)^2} (\hat{u}(t, x, \alpha) - u(t, x, \beta))^+ \partial_t \phi dx dt d\alpha d\beta \\ & - \mathbb{E} \int_{Q \times (0,1)^2} (\hat{u}(t, x, \alpha) - u(t, x, \beta))^+ (-\Delta)^\alpha \phi(t, x) dx dt d\alpha d\beta \\ & - \mathbb{E} \int_{Q \times (0,1)^2} \operatorname{sgn}_0^+[\hat{u}(p) - u(t, x, \beta)][f(\hat{u}(p)) - f(u(t, x, \beta))]\nabla_x \phi(t, x) dp d\beta \\ & + \lim_m [\tilde{\mathcal{L}}(\phi\theta_m) + \tilde{\mathcal{I}}(\phi)] + \hat{\mathcal{L}}(\phi). \end{aligned}$$

Now, let $\phi \in \mathcal{D}^+([0, T] \times B)$, then $\phi = \theta_n \phi + (1 - \theta_n)\phi$ and $\theta_n \phi \in \mathcal{D}^+([0, T] \times D)$ for n sufficiently large. Then applying the local Kato inequality with $\theta_n \phi$ and the global one with $(1 - \theta_n)\phi$, yields

$$\begin{aligned} 0 \leq & \int_D (\hat{u}_0 - u_0)^+ \phi(0) dx + \mathbb{E} \int_{Q \times (0,1)^2} (\hat{u}(t, x, \alpha) - u(t, x, \beta))^+ \partial_t \phi dx dt d\alpha d\beta \\ & - \mathbb{E} \int_{Q \times (0,1)^2} (\hat{u}(t, x, \alpha) - u(t, x, \beta))^+ (-\Delta)^\alpha \phi(t, x) dx dt d\alpha d\beta \\ & - \mathbb{E} \int_{Q \times (0,1)^2} \operatorname{sgn}_0^+[\hat{u}(p) - u(t, x, \beta)][f(\hat{u}(p)) - f(u(t, x, \beta))]\nabla_x \phi(t, x) dp d\beta \\ & + \lim_m \tilde{\mathcal{L}}(\phi(1 - \theta_n)\theta_m) + [\tilde{\mathcal{I}} + \tilde{\mathcal{L}}]((1 - \theta_n)\phi). \end{aligned}$$

As $\tilde{\mathcal{L}}, \hat{\mathcal{L}}$ and $\tilde{\mathcal{I}}$ are linear operators and $\theta_n \theta_m = \theta_n$ if m is large, one gets that

$$\lim_m \tilde{\mathcal{L}}(\phi(1 - \theta_n)\theta_m) = \lim_m \tilde{\mathcal{L}}(\phi\theta_m) - \lim_m \tilde{\mathcal{L}}(\phi\theta_n)$$

and $\lim_n \lim_m \tilde{\mathcal{L}}(\phi(1 - \theta_n)\theta_m) = 0 = \lim_n [\tilde{\mathcal{I}} + \tilde{\mathcal{L}}]((1 - \theta_n)\phi)$. Thus the global Kato inequality holds for any $\phi \in \mathcal{D}^+([0, T] \times B)$, and by using a partition of unity, it holds for any $\phi \in \mathcal{D}^+([0, T] \times \mathbb{R}^d)$. This completes the proof. \square

Proof of uniqueness. Similar to Lemma 3.2, one can get that

$$0 \leq \int_D (u_0 - \hat{u}_0)^+ \phi(0) dx + \mathbb{E} \int_{Q \times (0,1)^2} (u(t, x, \beta) - \hat{u}(t, x, \alpha))^+ \partial_t \phi dx dt d\alpha d\beta$$

$$\begin{aligned}
 & - \mathbb{E} \int_{Q \times (0,1)^2} (u(t, x, \beta) - \hat{u}(t, x, \alpha))^+ (-\Delta)^\alpha \phi(t, x) dx dt d\alpha d\beta \\
 & - \mathbb{E} \int_{Q \times (0,1)^2} \text{sgn}_0^+[u(t, x, \beta) - \hat{u}(p)][f(u(t, x, \beta)) - f(\hat{u}(p))]\nabla_x \phi(t, x) dp d\beta.
 \end{aligned}$$

Combining Lemma 3.2, we have

$$\begin{aligned}
 0 & \leq \int_D |\hat{u}_0 - u_0| \phi(0) dx + \mathbb{E} \int_{Q \times (0,1)^2} |\hat{u}(t, x, \alpha) - u(t, x, \beta)| \partial_t \phi dx dt d\alpha d\beta \\
 & - \mathbb{E} \int_{Q \times (0,1)^2} |\hat{u}(t, x, \alpha) - u(t, x, \beta)| (-\Delta)^\alpha \phi(t, x) dx dt d\alpha d\beta \\
 & - \mathbb{E} \int_{Q \times (0,1)^2} \text{sgn}[\hat{u}(p) - u(t, x, \beta)][f(\hat{u}(p)) - f(u(t, x, \beta))]\nabla_x \phi(t, x) dp d\beta. \tag{3.4}
 \end{aligned}$$

For each $n \in N$, define

$$\phi_n(x) = \begin{cases} 1 & \text{if } |x| \leq n, \\ 2(1 - \frac{|x|}{2n}) & \text{if } n < |x| \leq 2n, \\ 0 & \text{if } |x| > 2n. \end{cases}$$

For each $h > 0$ and $t \geq 0$, define

$$\psi_h(s) = \begin{cases} 1 & \text{if } s \leq t, \\ 1 - \frac{s-t}{h} & \text{if } t < s \leq t + h, \\ 0 & \text{if } s > t + h. \end{cases}$$

Then, by standard approximation, truncation and mollification argument, (3.4) holds with

$$\phi(t, x) = \psi_h(s) K(t, \cdot) * \phi_n(\cdot)(x),$$

where K stands for the Green function on D with Dirichlet boundary condition. Similar to the proof of theorem 3.1 in [25], one can get the desired results. This completes the proof. \square

Conflict of interest statement

No conflict of interest.

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China grants 11301146, 11171064, 11371367, and 11271290. The authors would like to thank Professor David Nualart and the referees for their careful reading of the manuscript and for the clarification comments which lead to an improvement of the presentation of the paper.

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