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Paper:

Giansiracusa, J. & Giansiracusa, N. (in press). Equations of tropical varieties. *Duke Mathematical Journal*

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EQUATIONS OF TROPICAL VARIETIES

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ABSTRACT. We introduce a scheme-theoretic enrichment of the principal objects of tropical geometry. Using a category of semiring schemes, we construct tropical hypersurfaces as schemes over idempotent semirings such as $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \max, +)$ by realizing them as solution sets to explicit systems of tropical equations that are uniquely determined by idempotent module theory. We then define a tropicalization functor that sends closed subschemes of a toric variety over a ring R with non-archimedean valuation to closed subschemes of the corresponding tropical toric variety. Upon passing to the set of $\mathbb{T}$-points this reduces to Kajiwara-Payne's extended tropicalization, and in the case of a projective hypersurface we show that the scheme structure determines the multiplicities attached to the top-dimensional cells. By varying the valuation, these tropicalizations form algebraic families of $\mathbb{T}$-schemes parameterized by a moduli space of valuations on R that we construct. For projective subschemes, the Hilbert polynomial is preserved by tropicalization, regardless of the valuation. We conclude with some examples and a discussion of tropical bases in the scheme-theoretic setting.

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Date: 5 January, 2016.
1991 *Mathematics Subject Classification.* 14T05 (primary), 14A20 (secondary).

Dedicated to Max and Add(ie)

1. Introduction

Tropical geometry is an emerging tool in algebraic geometry that can transform certain questions into combinatorial problems by replacing a variety with a polyhedral object called a tropical variety. It has had striking applications to a range of subjects, such as enumerative geometry [Mik05, FM10, GM08, AB13], classical geometry [CDPR12, Bak08], intersection theory [Kat09, GM12, OP13], moduli spaces and compactifications [Tev07, HKT09, ACP15, RSS14], mirror symmetry [Gro10, GPS10, Gro11], abelian varieties [Gub07, CV10], representation theory [FZ02, GL13], algebraic statistics and mathematical biology [PS04, Man11] (and many more papers by many more authors). Since its inception, it has been tempting to look for algebraic foundations of tropical geometry, e.g., to view tropical varieties as varieties in a more literal sense and to understand tropicalization as a degeneration taking place in one common algebro-geometric world. However, tropical geometry is based on the idempotent semiring $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \max, +)$, which is an object outside the traditional scope of algebraic geometry.

Motivated by the desire to do algebraic geometry over the *field with one element*, $\mathbb{F}_1$, various authors have constructed extensions of Grothendieck's scheme theory to accommodate geometric

objects whose functions form algebraic objects outside the category of rings, such as semirings and monoids—the context of $\mathbb{F}_1$-geometry. The three theories developed in [Dur07, TV09, Lor12] essentially coincide over semirings, where the resulting schemes can be described in familiar terms either as spaces equipped with a sheaf of semirings, or as functors of points extended from rings to the larger category of semirings. Although these three theories produce distinct categories of $\mathbb{F}_1$-schemes, there is a smaller category of naive $\mathbb{F}_1$-schemes (containing the category of split toric varieties with torus-equivariant morphisms) that embeds as a full subcategory of each and is appropriate for the purposes of this paper; moreover, there are base-change functors from (each version of) $\mathbb{F}_1$-schemes to schemes over any ring or semiring. The above-cited authors have each speculated that the category of schemes over the semiring $\mathbb{T}$ might provide a useful foundation for tropical geometry. However, tropicalization, as it currently exists in the literature, produces tropical varieties as sets rather than as solutions to systems of tropical equations. Since the set of geometric points of a scheme is very far from determining the scheme, the challenge is to lift tropicalization to schemes in an appropriate way.

In traditional tropical geometry (e.g., [MS15]) one considers subvarieties of a torus defined over a non-archimedean valued field k; usually one assumes the valuation is nontrivial and the field is algebraically closed and complete with respect to the valuation. Tropicalization sends a subvariety Z of the torus $(k^\times)^n$ to the polyhedral subset $trop(Z)$ of the tropical torus $(\mathbb{T}^\times)^n = \mathbb{R}^n$ constructed as the Euclidean closure of the image of coordinate-wise valuation. Kajiwara and Payne extended this set-theoretic tropicalization to subvarieties of a toric variety by using the stratification by torus orbits [Kaj08, Pay09]. A fan determines a toric scheme X over $\mathbb{F}_1$ and base-change to k yields a familiar toric variety X_k, while base-change to $\mathbb{T}$ yields a tropical toric scheme $X_\mathbb{T}$. The $\mathbb{T}$-points of $X_\mathbb{T}$ (or equivalently, of X) form the partial compactification of $N_\mathbb{R}$ dual to the fan, and the Kajiwara-Payne tropicalization map $trop$ sends subvarieties of X_k to subsets of $X(\mathbb{T})$. We give a scheme-theoretic enhancement, $\mathcal{Trop}$, of this process.

Theorem A. *Let R be a ring equipped with a non-archimedean valuation (see Definition 2.5.1) $v : R \to S$, where S is an idempotent semiring (such as $\mathbb{T}$), and let X be a toric scheme over $\mathbb{F}_1$. There is a tropicalization map*

$$\mathcal{Trop}_X^v \colon \{closed\ subschemes\ of\ X_R\} \to \{closed\ subschemes\ of\ X_S\}$$

such that

(1) *it is functorial in X with respect to torus-equivariant morphisms, and*
(2) *when $S = \mathbb{T}$ the composition with $\mathrm{Hom}_{\mathrm{Sch}/\mathbb{T}}(\mathrm{Spec}\,\mathbb{T}, -)$ recovers the set-theoretic tropicalization of Kajiwara-Payne.*

If $Z \subset X_R$ is irreducible of dimension d and not contained in the toric boundary, then the restriction of the set-theoretic tropicalization $trop(Z)$ to the tropical torus admits the structure of a polyhedral complex of pure dimension d and there are natural number multiplicities associated to the facets such that the well-known balancing condition is satisfied (see, e.g., [DFS07, §2]). This balanced polyhedral complex is often thought of as the tropical analogue of the algebraic cycle $[Z]$. We show in Corollary 7.2.2 that, when Z is a hypersurface in projective space, the scheme $\mathcal{Trop}_X^v(Z)$ determines these facet multiplicities.[1]

Theorem B. *Let $v : k \to S$ be a valued field with S a totally ordered idempotent semifield. Given a closed subscheme $Z \subset \mathbb{P}_k^n$, the tropicalization $\mathcal{Trop}_{\mathbb{P}^n}^v(Z) \subset \mathbb{P}_S^n$ has a well-defined Hilbert polynomial, and it coincides with that of Z.*

This suggests that the process of sending a variety to its tropicalization behaves like a flat degeneration.

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[1]In an earlier preprint version of this paper we conjectured this to be true in higher codimension, and Maclagan and Rincón subsequently proved this [MR14].

We briefly explain the idea behind the construction of this scheme-theoretic tropicalization. Due to the nature of $(\max, +)$-algebra, the graph of a tropical polynomial f is piecewise linear; the regions of linearity are where a single monomial in f strictly dominates and the "bend locus," where the function is nonlinear, is the set of points where the maximum is attained by at least two monomials simultaneously. The bend locus (often called a tropical hypersurface or locus of tropical vanishing) is the tropical analogue of the zero locus of a polynomial over a ring. We enrich the bend locus of f with a scheme structure by realizing it as the solution set to a natural system of tropical algebraic equations, the *bend relations* of f (§5.1). These equations are given by equating f with each polynomial obtained from f by deleting a single monomial. By the Fundamental Theorem of Tropical Geometry [MS15, Theorem 3.2.4] (Kapranov's Theorem in the case of a hypersurface), set-theoretic tropicalization can be recast as intersecting the bend loci of the coefficient-wise valuations of all polynomials in the ideal defining an affine variety. Our scheme-theoretic tropicalization is defined by replacing this set-theoretic intersection with the intersection of bend loci as schemes. This yields a solution to the implicitization problem for the coordinate-wise valuation map.

An alternative description of scheme-theoretic tropicalization is as follows. An ideal I defining an affine subscheme is, in particular, a linear subspace of the coordinate ring $k[x_1, \ldots, x_n]$ of the ambient space. The set-theoretic tropicalization of I is then an (infinite-dimensional) tropical linear space $trop(I)$ inside $\mathbb{T}[x_1, \ldots, x_n]$. The $\mathbb{T}$-module quotient of $\mathbb{T}[x_1, \ldots, x_n]$ by the bend relations of each $f \in trop(I)$ turns out to coincide with the $\mathbb{T}$-algebra quotient (Lemma 5.1.4) and thus yields a closed subscheme of tropical affine space. For a homogeneous ideal defining a projective subscheme, the bend relations are compatible with the grading, and thus the tropicalization can be computed degree by degree in terms of $\mathbb{T}$-module theory and valuated matroids. From this and the fact that set-theoretic tropicalization preserves dimension, the Hilbert polynomial result follows.

The essential data in writing down the bend relations, and thus the scheme-theoretic tropicalization, is a basis of monomials in the coordinate algebra of the ambient space. Toric varieties are a natural class of varieties where there is a well-behaved class of monomials in each coordinate patch and this allows for a global extension of these affine constructions. We use the language of schemes over $\mathbb{F}_1$ (in the naive sense that is common to the theories of Durov, Lorscheid, and Toën-Vaquié) as a convenient way to keep track of monomials and to provide a slight generalization of the ambient toric varieties in which tropicalization usually takes place. These $\mathbb{F}_1$ schemes are locally modelled on monoids; we no longer require the monoids to be toric, though integrality (§2.1) is still necessary.

One can ask how the tropicalization of $Z \subset X_R$ depends on the valuation $v : R \to \mathbb{T}$. We show that the valuations on R form an algebraic moduli space and the corresponding tropicalizations form an algebraic family over this space.

Theorem C. *Let R be a ring, X a locally integral scheme over $\mathbb{F}_1$, and $Z \subset X_R$ a closed subscheme.*

(1) *The moduli space $\mathcal{Val}(R)$ of valuations on R is represented in affine idempotent semiring schemes, and there is a universal valuation $v_{\mathrm{univ}} : R \to \Gamma(\mathcal{Val}(R), \mathcal{O}_{\mathcal{Val}(R)})$ through which all others factor uniquely. In particular, $\mathcal{Val}(R)(\mathbb{T})$ is the set of non-archimedean valuations on R.*

(2) *The fiber of the algebraic family $\mathcal{Trop}_X^{v_{\mathrm{univ}}}(Z) \to \mathcal{Val}(R)$ over each $\mathbb{T}$-point $v : R \to \mathbb{T}$ is the tropicalization $\mathcal{Trop}_X^v(Z) \subset X_{\mathbb{T}}$. If $X = \mathbb{P}^n$ and R is a field then the Hilbert polynomials of the fibers exist and are all equal.*

Remark 1.0.1. If k is a non-archimedean field and R is a k-algebra, then the set of valuations on R extending the valuation on k is the Berkovich analytification of $\mathrm{Spec}\, R$ relative to k [Ber90]. The $\mathbb{T}$-points of our moduli space $\mathcal{Val}(R)$ can thus be viewed as an absolute version of the analytification of $\mathrm{Spec}\, R$. This is explored further in [GG14].

1.1. **Organization of the paper.** We begin in §2 by recalling some standard material on monoids and semirings and then giving our slightly generalized definition of valuation. In §3 we discuss the construction of $\mathbb{F}_1$-schemes and semiring schemes, and in §4 we review some constructions in toric schemes within this setting. The core of the paper is §5, where we define bend loci as schemes, and §6, where we use this to define and study scheme-theoretic tropicalization. In §7 we study the tropical Hilbert function and the multiplicities on the facets of a tropical hypersurface and in §8 we investigate tropical bases.

2. ALGEBRAIC PRELIMINARIES: MONOIDS, SEMIRINGS, AND VALUATIONS

Throughout this paper all monoids, semirings, and rings will be assumed commutative and unital. All monoids are viewed multiplicatively unless otherwise specified.

2.1. **Monoids and $\mathbb{F}_1$ algebra.** In this paper we shall work with a naive version of algebra over the so-called "field with one element", $\mathbb{F}_1$, which is entirely described in terms of monoids. More sophisticated notions of $\mathbb{F}_1$ algebra exist, such as Durov's commutative algebraic monads [Dur07], but the naive version recalled here is the one that appears most appropriate for tropical geometry and it provides a convenient language for working with monoids and (semi)rings in parallel. This naive $\mathbb{F}_1$ theory (or a slight variation on it) and its algebraic geometry have been studied by many authors, including [CC10, Dei08, TV09, FW14].

Rather than defining an object $\mathbb{F}_1$, one starts by defining the category of modules, $\mathbb{F}_1$-Mod, to be the category of pointed sets. The basepoint of an $\mathbb{F}_1$-module M is denoted 0_M and is called the zero element of M. This category has a closed symmetric monoidal tensor product $\otimes$ given by the smash product of pointed sets (take the cartesian product and then collapse the subset $M \times \{0_N\} \cup \{0_M\} \times N$ to the basepoint). The two-point set $\{0, 1\}$ is a unit for this tensor product.

An $\mathbb{F}_1$-*algebra* is an $\mathbb{F}_1$-module A equipped with a commutative and unital product map $A \otimes A \to A$ (i.e., it is a commutative monoid in $\mathbb{F}_1$-Mod). Concretely, an $\mathbb{F}_1$-algebra is a commutative and unital monoid with a (necessarily unique) element 0_A such that $0_A \cdot x = 0_A$ for all x; thus $\mathbb{F}_1$-algebras, as defined here, are sometimes called *monoids-with-zero*. The two-point set $\{0, 1\}$ admits a multiplication making it an $\mathbb{F}_1$-algebra, and it is clearly an initial object, so we can denote it by $\mathbb{F}_1$ and speak of $\mathbb{F}_1$-algebras without ambiguity.

Example 2.1.1. The $\mathbb{F}_1$ polynomial algebra $\mathbb{F}_1[x_1, \ldots, x_n]$ is the free abelian monoid-with-zero on n generators. The Laurent polynomial algebra $\mathbb{F}_1[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the free abelian group on n generators, together with a disjoint basepoint. Written additively, these are $\mathbb{N}^n \cup \{-\infty\}$ and $\mathbb{Z}^n \cup \{-\infty\}$, respectively.

Definition 2.1.2. An $\mathbb{F}_1$-algebra A is *integral* if the subset $A \smallsetminus \{0_A\}$ is multiplicatively closed (no zero divisors) and the natural map from $A \smallsetminus \{0_A\}$ to its group completion is injective.

An A-*module* M is an $\mathbb{F}_1$-module equipped with an associative and unital action of A given by a map $A \otimes M \to M$. Concretely, this is a pointed set with an action of the monoid A such that 0_A sends everything to 0_M. An A-*algebra* is an $\mathbb{F}_1$-algebra morphism $A \to B$

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[2]MacPherson has been developing related ideas in his thesis [Mac13] and has independently discovered the equations for scheme-theoretic tropicalization that we propose here.

[3]These discussions also led Maclagan and Rincón in [MR14] to further develop some of the ideas put forward in this paper.

2.2. Semirings.

Commutative monoids (viewed additively) admit a tensor product $\otimes$ generalizing that of abelian groups. If M and N are commutative monoids then $M \otimes N$ is the quotient of the free commutative monoid generated by $M \times N$ by the relations

(1) $(x_1 + x_2, y) \sim (x_1, y) + (x_2, y)$,
(2) $(x, y_1 + y_2) \sim (x, y_1) + (x, y_2)$,
(3) $(0, y) \sim 0$ and $(x, 0) \sim 0$,

for all $x, x_1, x_2 \in M$ and $y, y_1, y_2 \in N$. It satisfies the analogue of the usual universal property: additive maps $M \otimes N \to L$ are in bijection with bilinear maps $M \times N \to L$.

Just as a ring can be defined as a monoid in the category of abelian groups, a *semiring* is a monoid in the monoidal category of commutative monoids—that is, an object satisfying all the axioms of a ring except for the existence of additive inverses. For a semiring S, an *S-module* is a commutative monoid M equipped with an associative action $S \otimes M \to M$. An *S-algebra* is a morphism of semirings $S \to T$. Polynomial algebras $S[x_1, \ldots, x_n]$, and Laurent polynomial algebras, are defined as they are for rings. The category of semirings has an initial object, $\mathbb{N}$, so the category of semirings is equivalent to the category of $\mathbb{N}$-algebras. A semiring is a *semifield* if every nonzero element admits a multiplicative inverse.

A semiring S is *idempotent* if $a + a = a$ for all $a \in S$. In this case (and more generally, for an idempotent commutative monoid) there is a canonical partial order defined by

$$a \leq b \text{ if } a + b = b.$$

The least upper bound of any finite set $\{a_i\}$ of elements exists and is given by the sum $\sum a_i$. If the partial order is actually a total order then $\sum a_i$ is equal to the maximum of the a_i.

From the perspective of tropical geometry, the central example of an idempotent semiring is the semifield of *tropical numbers*, $\mathbb{T}$. As a set,

$$\mathbb{T} := \mathbb{R} \cup \{-\infty\}.$$

The addition operation is defined by the maximum: $a + b = \max\{a, b\}$ if both a and b are finite. Multiplication $a \cdot b$ in $\mathbb{T}$ is defined as the usual addition of real numbers $a + b$ if both are finite. The additive and multiplicative units are $0_{\mathbb{T}} = -\infty$ and $1_{\mathbb{T}} = 0$, respectively, and this defines the extension of addition and multiplication to $-\infty$.

This is a special case of a general construction: given a commutative monoid $(\Gamma, +)$ equipped with a translation-invariant total order, the set $\Gamma \cup \{-\infty\}$ equipped with the operations $(\max, +)$ forms an idempotent semiring, and if Γ is a group then this yields a semifield. These semifields are the targets of Krull valuations. The tropical numbers $\mathbb{T}$ are the rank 1 case, which results when Γ is $(\mathbb{R}, +)$ with its canonical total order. Taking $\Gamma = \mathbb{R}^n$ equipped with the lexicographic total order yields the idempotent semifield $\mathbb{R}^n_{lex} \cup \{-\infty\}$, which is the target of some higher rank Krull valuations and whose tropical geometry has been studied in [Ban15].

Remark 2.2.1. Idempotent totally ordered semifields appear to play much of the role in idempotent algebra and geometry of fields in classical algebra and geometry.

The *boolean semiring* is the subsemiring

$$\mathbb{B} := \{-\infty, 0\} \subset \mathbb{T}.$$

The boolean semiring is initial in the category of idempotent semrings and every $\mathbb{B}$-algebra is idempotent, so $\mathbb{B}$-algebras are the same as idempotent semirings.

2.3. **Scalar extension and restriction.** Given a (semi)ring S, there is an adjoint pair of functors

$$\mathbb{F}_1\text{-Mod} \leftrightarrows S\text{-Mod};$$

the right adjoint sends an S-module to its underlying set with the additive unit as the basepoint, and the left adjoint, denoted $- \otimes S$, sends a pointed set M to the free S-module generated by the non-basepoint elements of M. If M is an $\mathbb{F}_1$-algebra then $M \otimes S$ has an induced S-algebra structure. Note that $- \otimes S$ sends polynomial algebras over $\mathbb{F}_1$ to polynomial algebras over S.

In this paper, S-modules equipped with an $\mathbb{F}_1$-descent datum (i.e., modules of the form $M \otimes S$ for M a specified $\mathbb{F}_1$-module) play a particularly important role. For $f \in M \otimes S$, the *support* of f, denoted $\mathrm{supp}(f)$, is the subset of M corresponding to the terms appearing in f.

Given a semiring homomorphism $\varphi : S \to T$ one obtains an adjoint pair

$$S\text{-Mod} \leftrightarrows T\text{-Mod}$$

in the standard way. As usual, the left adjoint is denoted $- \otimes_S T$, and it sends S-algebras to T-algebras and coincides with the pushout of S-algebras along φ.

2.4. **Ideals, congruences and quotients.** Let A be either an $\mathbb{F}_1$-algebra or a semiring. We can regard A as an A-module and define an *ideal* in A to be a submodule of A. When A is a ring this agrees with the usual definition of an ideal.

Quotients of semirings generally cannot be described by ideals, since a quotient might identify elements f and g without the existence of an element $f - g$ to identify with zero. The same issue arises when constructing quotients of modules over semirings. For this reason, one must instead work with congruences to describe quotients. Here we we present some basic results illustrating the behavior and use of congruences. Further details on properties of congruences can be found in the books [Gol99] and [HW98].

Definition 2.4.1. Let S and M be a semiring and S-module respectively. A *semiring congruence* on S is an equivalence relation $J \subset S \times S$ that is a sub-semiring, and a *module congruence* on M is an S-submodule $J \subset M \times M$ that is an equivalence relation. If the type is clear from context, we refer to such an equivalence relation simply as a *congruence*.

Given a semiring congruence J on S, we write S/J for the set of equivalence classes. Note that, just as for rings, finite sums and products in the category of semirings agree, and moreover, the pushout of the two projection maps $S \leftarrow J \to S$ has S/J as its underlying set, and

$$
\begin{array}{ccc}
J & \longrightarrow & S \\
\downarrow & & \downarrow \\
S & \longrightarrow & S/J
\end{array}
$$

is both a pushout square and a pullback square in semirings; the analogous statement for modules also holds. From this one sees the following (see also [HW98, Theorem 7.3]):

Proposition 2.4.2. *Let J be an equivalence relation on a semiring S (or module M over a semiring). The semiring (or module) structure descends to the set of equivalence classes S/J (or M/J) if and only if J is a semiring (or module) congruence.*

Definition 2.4.3. Given a morphism of semirings $\varphi : S \to R$, we define the *kernel* congruence

$$\ker \varphi := S \times_R S = \{(f, g) \in S \times S \mid \varphi(f) = \varphi(g)\}.$$

Using congruences in place of ideals, the usual isomorphism theorems extend to semirings:

Proposition 2.4.4. (1) *Let $\varphi : S \to R$ be a homomorphism of semirings. The image is a semiring, the kernel is a congruence, and $S/\ker \varphi \cong \mathrm{im}\, \varphi$.*

(2) *Let R be a semiring, $S \subset R$ a sub-semiring, I a congruence on R, and let $S + I$ denote the I-saturation of S (the union of all I-equivalence classes that contain an element of S). Then $S + I$ is a sub-semiring of R, I restricts to a congruence I' on $S + I$ and a congruence I'' on S, and there is an isomorphism $(S + I)/I' \cong S/I''$*

(3) *For $J \subset I$ congruences on S, we have a congruence I/J (the image of I in $S/J \times S/J$) on S/J with $(S/J)/(I/J) \cong S/I$. This yields a bijection between congruences on S/J and congruences on S containing J.*

Proof. Part (1) is Theorem 7.5 in [HW98], and part (3) follows directly from Theorem 7.12 there. For part (2), first observe that two elements $s_1, s_2 \in S$ are equivalent modulo I'' if and only if they are equivalent modulo I as elements in R. From this it follows that S/I'' maps isomorphically onto its image in R/I. By construction, $(S + I)/I'$ has the same image in R/I as S/I'', and by the same reasoning it also maps isomorphically onto its image. $\square$

Since the intersection of congruences is a congruence, for a collection $\{f_\alpha, g_\alpha \in S\}_{\alpha \in A}$ there is a unique smallest (or finest) congruence identifying f_α with g_α for each α; this is the congruence generated by the pairs (f_α, g_α). In the case of a semiring congruence, we denote this by $\langle f_\alpha \sim g_\alpha \rangle_{\alpha \in A}$. More generally, for any subset $J \subset S \times S$, we denote by $\langle J \rangle$ the semiring congruence it generates. If $\varphi : S \to R$ is a semiring (or module) homomorphism and J is a congruence on S, then $\varphi(J)$ need not be a congruence on R because transitivity and reflexivity can fail; we denote by $\varphi_* J$ the congruence generated by $\varphi(J)$.

Lemma 2.4.5. *The semiring congruence $\langle f_\alpha \sim g_\alpha \rangle_{\alpha \in A}$ consists of the transitive closure of the sub-semiring of $S \times S$ generated by the elements (f_α, g_α), (g_α, f_α), and the diagonal $S \subset S \times S$. The analogous statement for module congruences also holds.*

Proof. The sub-semiring generated clearly gives a binary relation that is symmetric and reflexive, so it suffices to check that if $R \subset S \times S$ is a sub-semiring, then the transitive closure R' is also a sub-semiring. Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_k$ be sequences of elements in S such that each consecutive pair (x_i, x_{i+1}) and (y_i, y_{i+1}) is in R. Thus (x_1, x_n) and (y_1, y_k) are in R', and we must show that the product and sum of these are each in R'. We may assume $k \leq n$, and by padding with $n - k$ copies of (y_k, y_k), which is in R' since R' contains the diagonal, we can assume that $k = n$. By adding or multiplying the two sequences term by term we obtain the result. $\square$

An ideal I in a semiring S induces a congruence $\langle f \sim 0_S \rangle_{f \in I}$ on S.

Proposition 2.4.6. *If S is a ring then the above map from ideals to congruences is a bijection with inverse given by sending a congruence $\{f_i \sim g_i\}_{i \in \Lambda}$ to the ideal $\{f_i - g_i\}_{i \in \Lambda}$.*

Proof. Congruences on S are in bijection with quotients of S as a semiring, and ideals in S are of course in bijection with quotients of S as a ring, so it suffices to show that every semiring quotient of S is in fact a ring. This is immediate since a homomorphism of semirings sends additively invertible elements to additively invertible elements. $\square$

However, if S is a semiring that is not a ring then there can me multiple distinct ideals inducing the same congruence. For example, in the semiring $\mathbb{N}$ the maximal ideal $\mathbb{N} \smallsetminus \{1\}$ and the unit ideal $\mathbb{N}$ both induce the same congruence. Indeed, the former congruence contains $(1,1) + (0,2) = (1,3)$ and $(3,0)$ so by transitivity it also contains $(1,0)$.

More significantly, many congruences on semirings are not induced by an ideal as above. For instance, the tropical numbers $\mathbb{T}$ form a semifield and hence the only proper ideal is the trivial one; however, there is a non-trivial congruence on $\mathbb{T}$ with quotient the Boolean semiring $\mathbb{B}$. Another example is provide by the diagonal $\mathbb{A}^1_{\mathbb{T}}$ inside $\mathbb{A}^2_{\mathbb{T}} = \operatorname{Spec} \mathbb{T}[x, y]$; this congruence is generated by the relation $x \sim y$.

2.5. **Valuations.** The term "non-archimedean valuation" on a ring R usually means a homomorphism of multiplicative monoids $v : R \to \mathbb{T}$ satisfying $v(0_R) = -\infty$ and the subadditivity condition $v(a+b) \leq v(a) + v(b)$ for all $a, b \in R$. (Many authors use the opposite sign convention, and some would call this a "semi-valuation" unless the non-degeneracy condition $v^{-1}(-\infty) = 0$ holds.) The subadditivity condition appears semi-algebraic but, as observed in [Man11], it can be reformulated as an algebraic condition:

$$v(a+b) + v(a) + v(b) = v(a) + v(b).$$

We use this observation in §6.5 when constructing the moduli space of valuations on a ring.

It is useful—for example, when studying families of tropical varieties—to allow a more general codomain, so throughout this paper the term "valuation" shall refer to the following generalization. Note that, when passing from $\mathbb{T}$ to an arbitrary idempotent semiring, the total order is replaced by a partial order (cf., §2.2).

Definition 2.5.1. A *valuation* on a ring R is an idempotent semiring S (called the *semiring of values*), and a map $v : R \to S$ satisfying,

(1) (unit) $v(0_R) = 0_S$, and $v(1_R) = 1_S$,
(2) (sign) $v(-1_R) = 1_S$,
(3) (multiplicativity) $v(ab) = v(a)v(b)$,
(4) (subadditivity) $v(a+b) + v(a) + v(b) = v(a) + v(b)$.

for any $a, b \in R$. A valuation v is said to be *non-degenerate* if $v(a) = 0_S$ implies $a = 0_R$.

For $S = \mathbb{T}$ this coincides with the usual notion of a non-archimedean valuation described above. When S is totally ordered the resulting valuations coincide with Krull valuations and considering these leads to Huber's "adic spaces" approach to non-archimedean analytic geometry [Hub96]. Note that any valuation on a field is automatically non-degenerate.

Remark 2.5.2. If S is totally ordered then the sign condition $v(-1_R) = 1_S$ is implied by the multiplicativity and the unit condition $v(1_R) = 1_S$ in Definition 2.5.1. Indeed, $v(-1_R)^2 = v(1_R) = 1_S$ and the total ordering implies that square roots, when they exist, are unique. For semirings that are not totally ordered this need not be the case. We believe that the sign condition is fundamental to our generalization of valuations, as it is necessary for the functoriality of tropicalization (Proposition 6.4.1) — see Remark 6.4.2. Note that the sign condition is also satisfied by the valuations considered in [Mac13].

Lemma 2.5.3. *Let* $v : R \to S$ *be a valuation and* $a, b \in R$.

(1) $v(a+b) + v(a) = v(a+b) + v(a) + v(b)$.

Assume now that the partial order on S is a total order.

(2) *If* $v(a) \neq v(b)$ *then* $v(a+b) = v(a) + v(b)$.
(3) *The image of* v *is a subsemiring of S, and* $R \twoheadrightarrow \operatorname{im} v$ *is a valuation.*

Proof. For part (1), let $x = a+b$ and $y = -a$. The subadditivity equation, applied to x and y, becomes

$$v(b) + v(a+b) + v(-a) = v(a+b) + v(-a),$$

and using the sign condition $v(-1_R) = 1_S$, this becomes the desired equation.

For (2), assume without loss of generality that $v(a) < v(b)$; we then need to show that $v(a+b) = v(b)$. As above, we apply the subadditivity property with $x = a+b$ and $y = -a$, which here yields

$$v(b) + v(a+b) = v(a) + v(a+b).$$

Adding $v(a)$ to both sides then using idempotency, subadditivity, and the condition $v(a) + v(b) = v(b)$, we deduce that $v(b) = v(a) + v(a+b)$. The result then follows from the total order hypothesis.

Part (3) follows immediately from part (2). $\square$

A *valued ring* is a triple $(R, S, v : R \to S)$ where R is a ring and v is a valuation. Valued rings form a category in which a morphism $\varphi : (R, S, v) \to (R', S', v')$ consists of a ring homomorphism $\varphi_1 : R \to R'$ and a semiring homomorphism $\varphi_2 : S \to S'$ such that $v' \circ \varphi_1 = \varphi_2 \circ v$. Note that the composition of a valuation $v : R \to S$ with a semiring homomorphism $S \to S'$ is again a valuation.

As an illustration of the utility of considering the general class of valuations defined above, we show that, for a fixed ring R, there exists a *universal valuation* $v_{univ}^R : R \to S_{univ}^R$ on R from which any other valuation can be obtained by composition with a unique semiring homomorphism. This will be used to show that, as one varies the valuation on R, the set of all tropicalizations of a fixed subscheme forms an algebraic family over $\operatorname{Spec} S_{univ}^R$ (Theorem C part (1)). Recall that $\mathbb{B} = \{0, -\infty\} \subset \mathbb{T}$ is the boolean semiring and consider the polynomial $\mathbb{B}$-algebra $\mathbb{B}[x_a \mid a \in R]$ with one generator x_a for each element $a \in R$. The universal semiring of values S_{univ}^R is the quotient of $\mathbb{B}[x_a \mid a \in R]$ by the congruence generated by the relations

(1) $x_0 \sim 0_S$ and $x_1 \sim x_{-1} \sim 1_S$,
(2) $x_a x_b \sim x_{ab}$ for any $a, b \in R$,
(3) $x_{a+b} + x_a + x_b \sim x_a + x_b$ for any $a, b \in R$.

The universal valuation v_{univ}^R sends a to x_a.

Proposition 2.5.4. *Given a valuation $v : R \to T$, there exists a unique homomorphism $\phi : S_{univ}^R \to T$ such that $\phi \circ v_{univ}^R = v$. Hence valuations with semiring of values T are in bijection with homomorphisms $S_{univ}^R \to T$*

Proof. The homomorphism ϕ is defined by sending each generator x_a to $v(a)$. Since the relations in S_{univ}^R correspond exactly to the relations satisfied by a valuation, ϕ is well-defined. Uniqueness is immediate. $\square$

3. $\mathbb{F}_1$-SCHEMES AND SEMIRING SCHEMES

3.1. **Construction of $\mathbb{F}_1$-schemes and semiring schemes.** The papers [TV09], [Lor12], and [Dur07] each construct categories of schemes over semirings and some notion of $\mathbb{F}_1$. For the purposes of the present paper we do not require the full generality of their constructions, so we present below a streamlined construction that follows the classical construction of schemes and yields a category that admits a full embedding into each of their categories.

Remark 3.1.1. Over a semiring, the category of schemes described here is equivalent to that of Toën-Vaquié, and it is a full subcategory of Durov's generalized schemes. There is a functor from it to Lorscheid's category of blue schemes, and this functor is faithful but not full. See [LPL11] for a comparison of these three threories over each of their notions of $\mathbb{F}_1$.

The construction of schemes modelled on $\mathbb{F}_1$-algebras or semirings proceeds exactly as in the classical setting of rings. Let A be a Q-algebra, where Q is either a semiring or an $\mathbb{F}_1$-algebra. A proper ideal in A is *prime* if its complement is closed under multiplication. Given a prime ideal $\mathfrak{p} \subset A$, one can form the localization $A_\mathfrak{p}$ via equivalence classes of fractions in the usual way. As a space, the prime spectrum $|\operatorname{Spec} A|$ is the set of prime ideals in A equipped with the Zariski topology in which the open sets are the collections of primes not containing a given ideal (a basis is given by sets of the form $D(f) = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}$ for $f \in A$). Any A-module (or algebra) M determines a sheaf $\widetilde{M}$ of Q-modules (or algebras) that sends a principal open set $D(f)$ to the localization $M_f = A_f \otimes M$

in which f is inverted. In particular, A itself gives a sheaf of Q-algebras, and this is the structure sheaf $\mathscr{O}_A$.

An *affine scheme* (over Q) is a pair $(X, \mathscr{O})$ consisting of a topological space X and a sheaf of Q-algebras that is isomorphic to a pair of the form $(|\operatorname{Spec} A|, \mathscr{O}_A)$. A general Q-scheme is a pair that is locally affine. A morphism of schemes is a morphism of pairs that is given in suitable affine patches by a homomorphism of Q-algebras. As explained in [Dur07, 6.5.2], for rings this coincides with the usual construction in terms of locally ringed spaces. The category of affine Q-schemes is equivalent to the opposite of the category of Q-algebras.

Proposition 3.1.2. *Given a Q-algebra A, the category of A-schemes is canonically equivalent to the category of Q-schemes over* $\operatorname{Spec} A$.

A condition that will be fundamental later in our tropicalization construction is the following. An $\mathbb{F}_1$-scheme is *locally integral* if it admits an open affine cover by the spectra of integral $\mathbb{F}_1$-algebras (recall Definition 2.1.2).

Proposition 3.1.3. *Any locally integral $\mathbb{F}_1$-scheme has a topological basis given by the spectra of integral $\mathbb{F}_1$-algebras.*

Proof. This follows from the fact that any localization of an integral $\mathbb{F}_1$-algebra is integral. $\square$

Remark 3.1.4. A related construction is that of a Kato fan, which first appeared in Kato's seminal work on log geometry [Kat89] and was later explored in relation to tropicalization by Ulirsch [Uli14]. An affine Kato fan consists of the prime spectrum of a monoid M, defined as above, but equipped with the structure sheaf associating to each basic open subset $D(f)$ the quotient of M_f by its subgroup of units. Thus in the case of a monoid-with-zero, an affine Kato fan is homeomorphic to an affine $\mathbb{F}_1$-scheme. An arbitrary Kato fan is obtained by gluing affine Kato fans. The advantage of these modified structure sheaves in Kato's construction is that they allow more flexible gluing. For instance, associated to a toroidal embedding is a Kato fan homeomorphic to the set of generic points of the toroidal strata together with their specialization relations [Uli14, Remark 4.16(ii)], whereas $\mathbb{F}_1$-schemes more closely resemble toric varieties (cf., §4) since inverting all non-zero elements of an integral monoid yields the coordinate algebra of a (possibly infinite-dimensional) algebraic torus. However, in Kato's framework a torus has no non-trivial regular functions; since the tropical geometry we generalize in this paper is based on subschemes of tori, this appears to be an insurmountable obstruction to using Kato fans for our purposes. A further variant of these constructions is that of Artin fans, developed in [ACMW14, AW13] and extending ideas in [Ols03]; these are algebraic stacks étale locally isomorphic to the stack quotient of a toric variety by its big torus. Thus an Artin fan has the flexibility of Kato fans yet retains in its stack structure some of the crucial information that Kato fans discard. Many of the constructions developed in this paper should extend from the context of $\mathbb{F}_1$-schemes and the Zariski topology to the context of Artin fans and an appropriate étale site.

3.2. **Base change functors.** The scalar extension and restriction functors of §2.3 admit globalizations that we briefly describe here.

Using the fact that $\mathbb{F}_1$-Mod and S-Mod (for S a semiring) are cocomplete, all fiber products exist in the categories of $\mathbb{F}_1$-schemes and S-schemes and they are constructed in the usual way. In particular, if T is an S-algebra and X is an S-scheme, then $X_T := \operatorname{Spec} T \times_{\operatorname{Spec} S} X$ exists and by Proposition 3.1.2 it can be regarded as a T-scheme. Thus $\operatorname{Spec} T \times_{\operatorname{Spec} S} -$ defines a base change functor from S-schemes to T-schemes, and this is the right adjoint of the forgetful functor (defined using Proposition 3.1.2) that regards a T-scheme as an S-scheme.

For R a ring or semiring, the scalar extension functor $- \otimes R$ clearly sends localizations of $\mathbb{F}_1$-algebras to localizations of R-algebras, so it globalizes to give a base change functor from $\mathbb{F}_1$-schemes to R-schemes. Given an $\mathbb{F}_1$-scheme X, we write X_R for the base change of X to R-schemes.

This base change functor is right adjoint to the forgetful functor from R-schemes to $\mathbb{F}_1$-schemes that globalizes the corresponding forgetful functor from R-Mod to $\mathbb{F}_1$-Mod. Given an $\mathbb{F}_1$-scheme X, the set of R-points of X and of X_R coincide and we denote this set by $X(R)$.

3.3. Closed subschemes.
At a formal level, the classical theory of schemes and the extended theory of semiring schemes are nearly identical when considering open subschemes and gluing. However, novel features appear when considering closed subschemes; this is essentially because the bijection between ideals and congruences (Proposition 2.4.6) fails in general for semirings.

Over a ring, a closed immersion is a morphism $\Phi : Y \to X$ such that $\Phi(Y)$ is topologically a closed subspace of X, the induced map $Y \to \Phi(Y)$ is a homeomorphism, and the sheaf map $\Phi^\sharp : \mathscr{O}_X \to \Phi_* \mathscr{O}_Y$ is surjective. These conditions on Φ are equivalent to requiring that Φ be an affine morphism with $\Phi^\sharp$ surjective. A closed subscheme is then an equivalence class of closed immersions, where $\Phi : Y \to X$ and $\Phi' : Y' \to X$ are equivalent if there is an isomorphism $Y \cong Y'$ commuting with these morphisms. There is a bijection between closed subschemes of X and quasi-coherent ideal sheaves on X.

Over a semiring, the equivalence between the above two characterizations of a closed immersion breaks down; see Remark 3.3.2 below. The prevailing attitude (e.g., in [Dur07]), and the choice that we follow here, is to adopt the second perspective: for a scheme X over a semiring, a *closed immersion* is an affine morphism $\Phi : Y \to X$ such that $\Phi^\sharp : \mathscr{O}_X \to \Phi_* \mathscr{O}_Y$ is surjective. As before, a *closed subscheme* of X is an equivalence class of closed immersions into X. Closed subschemes of $\mathbb{F}_1$-schemes are defined in the same way.

A *congruence sheaf* $\mathscr{J}$ on X is a subsheaf of $\mathscr{O}_X \times \mathscr{O}_X$ such that $\mathscr{J}(U)$ is a congruence on $\mathscr{O}_X(U)$ for each open $U \subset X$. A congruence sheaf is *quasi-coherent* if it is quasi-coherent when regarded as a sub-$\mathscr{O}_X$-module of $\mathscr{O}_X \times \mathscr{O}_X$.

Proposition 3.3.1. *Let S be a semiring.*

(1) *Let $X = \operatorname{Spec} A$ be an affine S-scheme. Taking global sections induces a bijection between quasi-coherent congruence sheaves on $\operatorname{Spec} A$ and congruences on A.*
(2) *For X an arbitrary S-scheme, there is a bijection between closed subschemes of X and quasi-coherent congruence sheaves on X.*

Proof. Part (1) follows directly from [Har77, Corollary 5.5], whose proof is unaffected by the generalization from rings to semirings.

For (2), given a closed immersion $\Phi : Y \to X$, the congruence kernel $\ker \Phi^\sharp$ is a quasi-coherent congruence sheaf. Conversely, a quasi-coherent congruence sheaf on X determines determines a closed subscheme of each affine open subscheme U, and the quasi-coherence condition together with part (1) ensure that these glue together to form a well-defined closed subscheme of X.   □

Remark 3.3.2. Curiously, morphisms that are scheme-theoretic closed immersions defined in this way are often not closed embeddings at the level of topological spaces. For instance, a point $\Phi : \operatorname{Spec} \mathbb{T} \to \mathbb{A}^n_{\mathbb{T}}$ corresponding to a $\mathbb{T}$-algebra morphism $\varphi : \mathbb{T}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{T}$ sending each x_i to some finite value $\varphi(x_i) \in \mathbb{R}$ is a closed immersion, but the image of this map is not Zariski closed—in fact, it is a dense point! Indeed, $\varphi^{-1}(-\infty) = \{-\infty\}$, which is contained in all primes, so every point of $|\operatorname{Spec} \mathbb{T}[x_1, \ldots, x_n]|$ is in the closure of the image of the point $|\operatorname{Spec} \mathbb{T}|$.

The congruence sheaf generated by a family of quasi-coherent congruence sheaves is again a quasi-coherent congruence sheaf, and this defines the intersection of the corresponding closed subschemes.

Remark 3.3.3. One can view the prime spectrum and its Zariski topology as a technical scaffolding whose purpose is to define the functor of points, which is then regarded as the fundamental geometric object as in [TV09]. For instance, as we see in the following example, the $\mathbb{T}$-points of a tropical variety more closely reflect familiar geometry than its prime spectrum. Moreover, there is a natural topology on the $\mathbb{T}$-points of a $\mathbb{T}$-scheme such that closed subschemes, as defined above, induce closed subsets of the $\mathbb{T}$-points; see [GG14, §3.4] where this notion is introduced and used to show that the Berkovich analytification of a scheme is homeomorphic to the $\mathbb{T}$-points of a certain tropicalization of the scheme.

3.4. **Example: the affine tropical line.** The set of $\mathbb{T}$-points of the affine line $\mathbb{A}^1_{\mathbb{T}} = \operatorname{Spec} \mathbb{T}[x]$ is clearly $\mathbb{T}$ itself, but the ideal-theoretic kernels of the corresponding homomorphisms $\mathbb{T}[x] \to \mathbb{T}$ are all trivial except for the point $x \mapsto -\infty$ for which the ideal is maximal. On the other hand, one can of course distinguish all these points using the congruence-theoretic kernel, by the First Isomorphism Theorem.

The semiring $\mathbb{T}[x]$ has a rather intricate structure; however, it admits a quotient with the same set of $\mathbb{T}$-points that behaves more like univariate polynomials over an algebraically closed field:

$$\overline{\mathbb{T}[x]} := \mathbb{T}[x]/\sim, \text{ where } f \sim g \text{ if } f(t) = g(t) \text{ for all } t \in \mathbb{T}.$$

Polynomials in this quotient split uniquely into linear factors. More specifically, if

$$b_t := 0 + t^{-1}x \in \overline{\mathbb{T}[x]} \text{ for } t \in \mathbb{T}^\times = \mathbb{R} \text{ and } b_{-\infty} := x \in \overline{\mathbb{T}[x]},$$

then any element of $\overline{\mathbb{T}[x]}$ can be written uniquely as $c \prod b_{t_i}^{d_i}$ for $c, t_i \in \mathbb{T}$. Nonetheless, the prime spectrum of $\overline{\mathbb{T}[x]}$ is larger than one might guess based on analogy with the case of algebraically closed fields. For any subset $K \subset \mathbb{T}$ we define the ideal $I_K := (\{b_t \mid t \in K\}) \subset \overline{\mathbb{T}[x]}$.

Proposition 3.4.1. *If $K \subset \mathbb{T}$ is an interval (not necessarily closed or open) then $I_K \setminus \{-\infty\}$ is the set of functions that have a bend in K. As a set, $|\operatorname{Spec} \overline{\mathbb{T}[x]}| = \{I_K \mid K \subset \mathbb{T} \text{ is an interval }\}$. The finitely generated primes correspond to closed intervals and the principal primes to points of $\mathbb{T}$.*

Proof. If $f \in \overline{\mathbb{T}[x]}$ has a bend at $t \in K \subset \mathbb{T}$ then $f \in I_{\{t\}} \subset I_K$. Conversely, if $f \in I_K$ then $f = \sum_{i=1}^n g_i b_{t_i}$ for some $t_i \in K$ and $g_i \in \overline{\mathbb{T}[x]}$. Each summand $g_i b_{t_i}$ has a bend at t_i, and the tropical sum of a function with a bend at t_i and a function with a bend at t_j must have a bend in the closed interval $[t_i, t_j]$. Thus when K is convex (i.e., an interval) we indeed have that the non-constant functions of I_K are precisely the functions with a bend in K.

From this it follows that if K is an interval then I_K is prime: if $f, g \in \overline{\mathbb{T}[x]} \setminus I_K$ then neither f nor g has a bend in K so the same is true of fg, hence $fg \in \overline{\mathbb{T}[x]} \setminus I_K$. Conversely, if $\mathfrak{p} \subset \overline{\mathbb{T}[x]}$ is prime then by the factorization property of $\overline{\mathbb{T}[x]}$, any element of $\mathfrak{p}$ must be divisible by b_t for some $t \in \mathbb{T}$. The identity

$$t_1 r^{-1} b_{t_1} + b_{t_2} = b_r \text{ for any } r \in [t_1, t_2] \subset \mathbb{T}$$

then shows that $\mathfrak{p} = I_K$ where K is the convex hull of all such t. The statement about finitely generated primes and principal primes immediately follows. $\qquad\square$

4. Toric varieties and their tropical models

4.1. **Toric schemes over $\mathbb{F}_1$ and $\mathbb{T}$.** Let $N \cong \mathbb{Z}^n$ be a lattice with dual lattice M. The datum of a rational polyhedral fan Δ in $N_\mathbb{R}$ determines an $\mathbb{F}_1$-scheme as in the usual construction of toric varieties. For each cone $\sigma \in \Delta$, there is a corresponding monoid $M_\sigma = M \cap \sigma^\vee$. If $\tau \subset \sigma$ is a face then M_τ is a localization of M_σ. Hence adjoining zeros to these monoids, after writing them multiplicatively, and taking Spec results in a collection of affine $\mathbb{F}_1$-schemes that glue together according to the incidence relations of the fan Δ to give an $\mathbb{F}_1$-scheme X^Δ. Base change to a ring R yields the usual toric variety over R associated with the fan Δ. The full subcategory of

$\mathbb{F}_1$-schemes spanned by the objects of the form X^Δ is equivalent to the category of toric varieties and torus-equivariant morphisms.

Kajiwara [Kaj08] and Payne [Pay09] have each studied toric varieties over $\mathbb{T}$. The $\mathbb{T}$-points of the open torus stratum are canonically identified with the points of $N_\mathbb{R} \cong \mathbb{R}^n$, and $X^\Delta(\mathbb{T})$ is then the polyhedral partial compactification of $N_\mathbb{R}$ dual to the fan Δ, with a codimension i stratum at infinity for each i-dimensional cone. For example, $\mathbb{P}^n(\mathbb{T})$ is homeomorphic, and combinatorially equivalent, to an n-simplex.

Remark 4.1.1. Given a toric variety X_k, where k is a valued field, some authors refer to the corresponding tropical scheme $X_\mathbb{T}$ as the tropicalization of X_k.

Observe that the toric $\mathbb{F}_1$-schemes X^Δ described above are locally integral. However, the class of locally integral $\mathbb{F}_1$-schemes is larger; it allows objects that are disconnected, non-normal, and/or not of finite type. In the scheme-theoretic tropical geometry that we develop in this paper, the class of ambient spaces in which tropicalization makes sense can naturally be enlarged from toric varieties to locally integral $\mathbb{F}_1$-schemes.

4.2. Cox's quotient construction.

We now explain how Cox's construction of toric varieties as quotients of affine space can be defined over $\mathbb{F}_1$. Let $X = X^\Delta$ be as above and suppose the rays $\Delta(1)$ span $N_\mathbb{R}$, i.e., X has no torus factors. We define the *Cox algebra* as the free $\mathbb{F}_1$-algebra on the set of rays:

$$\mathrm{Cox}(X) := \mathbb{F}_1[x_\rho \mid \rho \in \Delta(1)].$$

For any field k the toric variety X_k is split and the divisor class group is independent of the field k, so we can formally define $\mathrm{Cl}(X) := \mathbb{Z}^{\Delta(1)}/M$, where

$$M \hookrightarrow \mathbb{Z}^{\Delta(1)}, \ m \mapsto (m \cdot u_\rho)_{\rho \in \Delta(1)},$$

and u_ρ denotes the first lattice point on the ray $\rho \subset N_\mathbb{R}$.

The Cox algebra has a grading by the divisor class group, via the composition

$$\mathrm{Cox}(X) \setminus \{0\} \cong \mathbb{N}^{\Delta(1)} \hookrightarrow \mathbb{Z}^{\Delta(1)} \twoheadrightarrow \mathrm{Cl}(X),$$

where the above isomorphism is from a multiplicative to an additive monoid. The graded pieces are the eigenspaces for the action of the dual group

$$G := \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{Z}) \subset \mathrm{Hom}(\mathbb{Z}^{\Delta(1)}, \mathbb{Z})$$

on $\mathrm{Spec}\,\mathrm{Cox}(X) \cong \mathbb{A}_{\mathbb{F}_1}^{\Delta(1)}$.

Each class $[D] \in \mathrm{Cl}(X)$ is represented by a torus-invariant Weil divisor D and determines a coherent sheaf $\mathcal{O}_X(D)$ on X, the global sections of which are naturally isomorphic to the $\mathbb{F}_1$-module of homogeneous elements in $\mathrm{Cox}(X)$ of multi-degree $[D]$. If X is complete then each graded piece is finite and the sections of this $\mathbb{F}_1$-sheaf are naturally the lattice-points in a polytope.

The irrelevant ideal in $\mathrm{Cox}(X)$ is generated by the elements $x_\sigma := \prod_{\rho \notin \sigma(1)} x_\rho$ for all cones $\sigma \in \Delta$. This determines an open subscheme

$$U := \bigcup_{\sigma \in \Delta} \mathrm{Spec}\,\mathrm{Cox}(X)[x_\sigma^{-1}] \subset \mathbb{A}_{\mathbb{F}_1}^{\Delta(1)}.$$

Proposition 4.2.1. *With notation as above, X is the categorical quotient U/G in $\mathbb{F}_1$-schemes.*

Proof. This is an immediate translation of [Cox95, Theorem 2.1] and its proof to the setting of monoids. We cover U by $\mathbb{F}_1$-open affine G-invariant charts $U_\sigma := \mathrm{Spec}\,\mathrm{Cox}(X)[x_\sigma^{-1}]$ and observe that Cox's argument carries over to show that

$$\mathrm{Cox}(X)[x_\sigma^{-1}]^G = \mathrm{Cox}[x_\sigma^{-1}]_0 \cong \sigma^\vee \cap M.$$

This implies that for this chart we have the categorical quotient

$$U_\sigma/G = \operatorname{Spec} \sigma^\vee \cap M,$$

and following Cox's argument again we see that the way these affine quotients glue together to yield the categorical quotient U/G is identical to the way the affine charts corresponding to the cones in the fan Δ glue together to produce the toric variety X. □

5. Bend loci

In this section we define the *bend locus scheme* of a tropical regular function; locally, a bend locus is the tropical analogue of the zero locus of a regular function. These bend loci will be the basic building blocks of scheme-theoretic tropicalization.

Recall that over a ring R, a polynomial $f \in R[x_1, \dots, x_n]$ determines a zero locus $V(f) \subset \mathbb{A}_R^n$ as the set of points where f vanishes, but $V(f)$ has the additional structure of a scheme over R given by $\operatorname{Spec} R[x_1, \dots, x_n]/(f)$. There are various heuristic arguments (e.g., [RGST05, §3], [Mik06, §3.1]) that the correct analogue of zero locus in the tropical setting is the locus of points where the piecewise linear graph of a tropical polynomial f is nonlinear—i.e., the locus where the graph "bends". This set appears in the literature under various names: the tropical hypersurface, tropical vanishing locus, corner locus, or bend locus of f. We will refer to it as the *set-theoretic bend locus*, and denote it $bend(f)$.

In embedding tropicalization into semiring scheme theory, the relevant question is then how to realize this set $bend(f)$ as the $\mathbb{T}$-points of a naturally associated semiring scheme, and to do so in a way that generalizes from affine space to a larger class of $\mathbb{F}_1$-schemes and allows for coefficients in an arbitrary idempotent semiring rather than just $\mathbb{T}$. Associating a closed subscheme structure to the set-theoretic bend locus means realizing it as the set of solutions to a system of polynomial equations over $\mathbb{T}$—more precisely, we must construct a congruence on the coordinate algebra of the ambient affine scheme (and a quasi-coherent congruence sheaf in the non-affine case) such that the $\mathbb{T}$-points of the quotient form the set-theoretic bend locus. While the $\mathbb{T}$-points alone are not enough to uniquely determine this congruence, the particular congruence we propose here appears quite natural and allows for a robust theory of scheme-theoretic tropicalization to be developed.

Given an idempotent semiring S, an $\mathbb{F}_1$-algebra M (which is the set of monomials) and $f \in M \otimes S$, we construct a congruence $\langle \mathcal{B}(f) \rangle$ which defines the bend locus $\mathcal{B}end(f)$ of f as a closed subscheme of $\operatorname{Spec} M \otimes S$. (Note the capitalization: we use $\mathcal{B}end$ for scheme-theoretic and $bend$ for set-theoretic bend loci, consistent with our use of $\mathcal{T}rop$ for scheme-theoretic and $trop$ for set-theoretic tropicalization.) The generators of this congruence are called the *bend relations* of f. When the ambient space is a torus and $S = \mathbb{T}$, the set of $\mathbb{T}$-points of $\mathcal{B}end(f)$ equals $bend(f)$. The scheme $\mathcal{B}end(f)$ contains strictly more information than the set $bend(f)$. It determines the multiplicities (see §7.2), and while the set of S-points of $\mathcal{B}end(f)$ does not in general determine f up to a scalar, the scheme structure does in some cases, such as when S is a semifield and f is homogeneous (see Lemma 5.1.8 below).

Remark 5.0.2. A word of caution: in the literature, the set-theoretic bend locus of a tropical polynomial is often called a "tropical hypersurface," and the set-theoretic tropicalization of a hypersurface is an example of one. Indeed, the set-theoretic tropicalization of the hypersurface $V(f)$ coincides with the set-theoretic bend locus of the tropical polynomial $\nu(f)$ obtained by coefficient-wise valuation: $trop(V(f)) = bend(\nu(f))$. However, this equality breaks at the level of schemes. We shall define scheme-theoretic tropicalization, $\mathcal{T}rop$, below in §6 by taking the bend relations of the coefficient-wise valuations of *all* elements in an ideal. When enriched with this scheme structure, the tropicalization of a hypersurface is usually cut out by more relations than just the bend relations of a single tropical polynomial: $\mathcal{T}rop(V(f))$ is always a closed subscheme of $\mathcal{B}end(\nu(f))$, but this containment can be strict. See §8.1 for examples where additional relations are needed and hence that $\mathcal{T}rop(V(f)) \subsetneq \mathcal{B}end(\nu(f))$.

5.1. **The bend relations.** Let S be an idempotent semiring and M an $\mathbb{F}_1$-module (or algebra). Given $f \in M \otimes S$ and $m \in \operatorname{supp}(f)$, we write $f_{\widehat{m}}$ for the result of deleting the m-term from f.

Definition 5.1.1. The *bend relations* of $f \in M \otimes S$ are the relations
$$\{f \sim f_{\widehat{m}}\}_{m \in \operatorname{supp}(f)}.$$
We write $\mathcal{B}(f)$ for the S-module congruence on $M \otimes S$ generated by the bend relations of f, and if $J \subset M \otimes S$ is an S-submodule then we write $\mathcal{B}(J)$ for the S-module congruence generated by the bend relations of each $f \in J$; these congruences are called the *bend congruences* of f and J respectively. The set of S-module homomorphisms $M \otimes S/\mathcal{B}(f) \to S$, or S-algebra homomorphisms $M \otimes S/\langle \mathcal{B}(f) \rangle \to S$ when M is an $\mathbb{F}_1$-algebra, is called the *set-theoretic bend locus of f, denoted* $\mathfrak{bend}(f)$.

Example 5.1.2. If $f = a_1 x_1 + a_2 x_2 + a_3 x_3 \in S[x_1, x_2, x_3]$ then the bend relations of f are
$$a_1 x_1 + a_2 x_2 + a_3 x_3 \sim a_2 x_2 + a_3 x_3$$
$$\sim a_1 x_1 + a_3 x_3$$
$$\sim a_1 x_1 + a_2 x_2.$$

Lemma 5.1.3. *Let $f \in M \otimes S$.*

(1) *If $\lambda \in S$ is a unit then $\mathcal{B}(\lambda f) = \mathcal{B}(f)$.*
(2) *Suppose $\{f_i\}$ are elements in $M \otimes S$ and N is the S-submodule they generate. The module congruence $\mathcal{B}(N)$ is equal to the module congruence generated by $\{\mathcal{B}(f_i)\}$.*

Proof. The arguments are straightforward. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The first item above is analogous to the fact that, classically, all nonzero scalar multiples of a polynomial define the same hypersurface. The second item is used multiple times throughout this paper; it allows one to work explicitly with a set of generators for the relations defining the closed subschemes we shall be studying in the idempotent world.

The following result sheds light on the behavior described in Remark 5.0.2, namely, the presence of relations in a scheme-theoretically tropicalized hypersurface beyond those coming from the defining polynomial. Indeed, the second part of Lemma 5.1.4 below implies that these extra relations are due to the interplay between the valuation and the ring structure prior to tropicalizing, as opposed to something taking place entirely on the idempotent side of the story. The first part of the lemma is also crucial to our theory and will be relied up heavily in §6 where tropicalization is studied; note in particular that, via the bend relations, ideals in an idempotent algebra play an important role when forming quotient algebras.

Lemma 5.1.4. *Suppose M is an $\mathbb{F}_1$-algebra, $f \in M \otimes S$, and $J \subset M \otimes S$ is an ideal.*

(1) *The module congruence $\mathcal{B}(J)$ is in fact a semiring congruence.*
(2) *If M is integral (recall Definition 2.1.2) and $J = (f)$ is a principal ideal then the semiring congruence $\langle \mathcal{B}(f) \rangle$ generated by $\mathcal{B}(f)$ is equal to $\mathcal{B}(J)$.*

Proof. By Lemma 2.4.5, it suffices to show that $\mathcal{B}(J)$ is closed under multiplication by bend relations $g \sim g_{\widehat{m}}$ for $g \in J$.

We first show that $\mathcal{B}(J)$ is not just an S-module congruence but actually an $M \otimes S$-module congruence. Multiplying a generating relation $g \sim g_{\widehat{m}}$ by a monomial $x \in M$ yields a relation of the form $xg \sim xg_{\widehat{m}}$, and we must show that this is in $\mathcal{B}(J)$. If multiplication by x is injective then this is one of the generating relations of $\mathcal{B}(J)$; if the map is not injective then let $b_1, \ldots, b_n$ be those monomials in $\operatorname{supp}(g) \smallsetminus \{m\}$ that are identified with m after multiplication by x and let $\lambda_i \in S$ be the coefficient of b_i in g. Then $xg \in J$, so $xg \sim (xg)_{\widehat{xm}}$ is a relation in $\mathcal{B}(J)$, and adding $\sum_i \lambda_i b_i$ to both sides (using the idempotency of addition) yields the desired relation $xg \sim xg_{\widehat{m}}$.

Now suppose $h \sim h'$ is an arbitrary relation in $\mathcal{B}(J)$ and $g \sim g_{\widehat{m}}$ is a generating bend relation. Then since $\mathcal{B}(J)$ is an $M \otimes S$-submodule, the two relations

$$hg \sim h'g \text{ and } h'g \sim h'g_{\widehat{m}}$$

are both in $\mathcal{B}(J)$, and hence, by transitivity, the relation $hg \sim h'g_{\widehat{m}}$ is as well, and this proves part (1).

We now turn to part (2). By part (1) we have $\langle \mathcal{B}(f) \rangle \subset \mathcal{B}(J)$, so it suffices to prove the opposite containment. By Lemma 5.1.3 part (2), $\mathcal{B}(J)$ is generated as a module congruence by the module congruences $\mathcal{B}(xf)$ for $x \in M$. By the integrality hypothesis on M, multiplication by any monomial x yields an injective map $M \to M$, and so $\mathcal{B}(xf) = x\mathcal{B}(f) \subset \langle \mathcal{B}(f) \rangle$. □

Remark 5.1.5. If M is not an integral $\mathbb{F}_1$-algebra then the bend congruence of a principal ideal $J = (f)$ can be strictly larger than the semiring congruence $\langle \mathcal{B}(f) \rangle$. For example, if $M = \mathbb{F}_1[x,y]/\langle x^2 \sim xy \rangle$ and $f = x + y \in M \otimes S$ then $\mathcal{B}(xf)$ contains the relation $x^2 \sim 0_S$, while the semiring congruence $\langle \mathcal{B}(f) \rangle$ does not contain this relation.

This next proposition states that the set-theoretic bend locus $bend(f)$ of a tropical polynomial (i.e., the solution set to its bend relations $\mathcal{B}(f)$, recall Definition 5.1.1) is the usual set-theoretic tropical hypersurface defined by f. However, since these set-theoretic bend loci are defined slightly more generally and in order to emphasize the distinction that exists at the scheme-theoretic level between bend loci and tropicalizations of hypersurfaces (see Remark 5.0.2), we will adhere to the 'bend locus' terminology.

Proposition 5.1.6. *Let S be a totally ordered idempotent semiring, M be an $\mathbb{F}_1$-module and $f \in M \otimes S$.*

(1) *An S-module homomorphism $p : M \otimes S \to S$ descends to the quotient by $\mathcal{B}(f)$, hence yields a point of $bend(f)$, if and only if either the maximum of the terms of $p(f)$ is attained at least twice or $p(f) = 0_S$.*

(2) *If M is an $\mathbb{F}_1$-algebra then the S-algebra homomorphisms $M \otimes S \to S$ descending to the quotient by $\langle \mathcal{B}(f) \rangle$ are in bijection with the S-module homomorphisms $M \otimes S \to S$ descending to the quotient by $\mathcal{B}(f)$; consequently, these two notions of $bend(f)$ coincide.*

(3) *If $X = \operatorname{Spec} \mathbb{F}_1[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a torus and $f \in \mathbb{T}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, then $X(\mathbb{T}) = (\mathbb{T}^{\times})^n = \mathbb{R}^n$ and $bend(f)$ is the subset of points at which the function $X(\mathbb{T}) \to \mathbb{T}$ defined by f is nonlinear.*

Proof. For part (1), first note if $p : M \otimes S \to S$ is an S-module homomorphism then $p(f)$ is a sum of terms given by evaluating p on each monomial term of f. Thus p factors through the quotient by $\langle \mathcal{B}(f) \rangle$ if and only if $p(f) = p(f_{\widehat{i}})$ for each i. This happens if and only if either $|\operatorname{supp}(f)| \geq 2$ and no single summand in $p(f)$ is strictly larger than all others, or if all summands are 0_S.

Part (2) is clear, so we now prove part (3). A homomorphism $p : \mathbb{T}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \to \mathbb{T}$ is determined by the n-tuple of tropical numbers $p(x_1), \ldots, p(x_n) \in \mathbb{T}^{\times} = \mathbb{R}$, so we identify p with a point in $\mathbb{R}^n$. This Euclidean space is divided into convex polyhedral chambers as follows. For each term of f there is a (possibly empty) chamber consisting of all p for which that term dominates, with the interior consisting of points where this term strictly dominates. Since f is the tropical sum (Euclidean maximum) of its terms, the chamber interiors are where the graph of f is linear and the walls are where the maximum is attained at least twice and hence the graph is nonlinear. □

Remark 5.1.7. If M is a finitely generated $\mathbb{F}_1$-module and $f \in M \otimes \mathbb{T}$, or if M is a (Laurent) polynomial algebra over $\mathbb{F}_1$ and $f \in M \otimes \mathbb{T}$ is homogeneous of degree one, then by the preceding proposition $bend(f)$ is a tropical hyperplane. Tropical hyperplanes were first defined in [SS04] as the set-theoretic tropicalization of classical hyperplanes and later recast in terms of *tropical Plücker vectors*, or equivalently, *valuated matroids*, in [Spe08]. A more algebraic exposition of tropical

hyperplanes, and tropical linear spaces more generally, closer to the spirit of this paper is Frenk's thesis [Fre13, Chapter 4].

In general one cannot recover a tropical polynomial from its set-theoretic bend locus (consider, e.g., $x^2 + ax + 0 \in \mathbb{T}[x]$ as $a \in \mathbb{T}$ varies). In the case of homogeneous polynomials this is manifest as the statement that the tropicalization of the Hilbert scheme of projective hypersurfaces is not a parameter space for set-theoretically tropicalized hypersurfaces (see [AN13, §6.1]). The following result says in particular that when enriched with its scheme structure, one can indeed recover, up to a scalar, a homogeneous tropical polynomial from its bend locus. This result is used in §7.2 to show that the scheme structure of a tropicalized hypersurface determines the multiplicities associated to the facets of the corresponding balanced polyhedral complex.

Lemma 5.1.8. *Suppose S is a semifield and $f \in M \otimes S$.*

(1) *The module congruence $\mathcal{B}(f)$ determines f uniquely up to a scalar.[4]*
(2) *If M is an $\mathbb{F}_1$-algebra that admits a grading by an abelian group such that $M_0 = 0_M$ and f is homogeneous, then $\langle \mathcal{B}(f) \rangle$ determines f up to a scalar.*

Remark 5.1.9. The hypotheses for (2) are satisfied by the Cox algebra of a toric scheme X over $\mathbb{F}_1$ whose base change to a ring is proper. We show below in §5.5 that a homogeneous polynomial in $\mathrm{Cox}(X_{\mathbb{T}})$ defines a closed subscheme of $X_{\mathbb{T}}$, generalizing the case of a homogeneous polynomial (in the usual sense) defining a closed subscheme of tropical projective space.

Proof. For (1), write $f = \sum_{i=1}^{n} a_i m_i$ with $a_i \in S, m_i \in M$. If $n = 1$ then the result is obvious, otherwise consider the elements ϕ of the dual module $\mathrm{Hom}(M \otimes S, S)$ of the form $m_i \mapsto 0_S$ for all i except two indices, say j_1 and j_2. Such a homomorphism descends to the quotient by $\mathcal{B}(f)$ if and only if $a_{j_1}\phi(m_{j_1}) = a_{j_2}\phi(m_{j_2})$. In this way we recover the ratio of each pair of coefficients a_{j_1}, a_{j_2}, and hence the vector of all coefficients $(a_1, \ldots, a_n)$ up to a scalar. Item (2) follows from (1) since the hypotheses guarantee that $\langle \mathcal{B}(f) \rangle_{\deg(f)} = \mathcal{B}(f)$, where the latter is viewed as a congruence on the module $M_{\deg(f)} \otimes S$. $\square$

5.2. Unicity of the bend relations.
Let S be a totally ordered idempotent semifield, M a finitely generated $\mathbb{F}_1$-module (i.e., a finite pointed set), and $f \in M \otimes S$. In this section we show that the congruence $\mathcal{B}(f)$ of bend relations of f is determined in a canonical way, via idempotent linear algebra, by the set-theoretic tropical hyperplane of f, i.e., its set-theoretic bend locus $bend(f)$. To explain this result we must first develop some idempotent linear algebra.

Given an S-module V, we will write

$$V^\vee = \mathrm{Hom}_S(V, S)$$

for the S-linear dual. As usual, $V^\vee$ inherits an S-module structure via $(s \cdot \alpha)(v) = \alpha(s \cdot v)$ for $s \in S$, $v \in V$ and $\alpha \in V^\vee$.

Proposition 5.2.1. *There is a canonical isomorphism of S-modules $(M \otimes S)^\vee \cong M \otimes S$.*

Proof. Modules of the form $M \otimes S$ are free with a canonical finite basis given by the non-basepoint elements of the $\mathbb{F}_1$-module M. $\square$

---

[4] Maclagan and Rincón [MR14] have subsequently observed that Proposition 5.1.6 combined with the duality theory of tropical linear spaces easily imples the following more general statement. Suppose M is a finitely generated $\mathbb{F}_1$-module and $L \subset M \otimes S$ is a tropical linear space. Let N denote the set-theoretic bend locus of L (i.e., the intersection of all bend loci of elements $f \in L$); it is given by $N = \mathrm{Hom}_S(M \otimes S/\mathcal{B}(L), S)$. Then N is a tropical linear space, and L and N are dual tropical linear spaces (in the sense of, e.g., [Spe08]): $L^\perp = N$ and $N^\perp = L$. This allows one to recover L from $\mathcal{B}(L)$.

The dual of a quotient of $M \otimes S$ is canonically a submodule of $(M \otimes S)^\vee$. Going in the other direction is not quite so straightforward: the dual of an arbitrary submodule $W \subset M \otimes S$ is canonically a quotient of $(M \otimes S)^\vee$ if and only if W has the property that every linear map $W \to S$ admits an extension to all of $M \otimes S$; conveniently, by [Fre13, Lemma 3.2.1], the assumption that S is a totally ordered semiring implies that this condition holds for any W.

Recall from Definition 5.1.1 that the set-theoretic bend locus $bend(f)$ is defined as the S-linear dual of the quotient $(M \otimes S)/\mathcal{B}(f)$. (As explained in Remark 5.1.7, the set-theoretic bend locus is essentially the tropical hyperplane defined by f.) In particular, in the setting here,

$$bend(f) \subset (M \otimes S)^\vee$$

is a submodule, and its S-linear dual, $bend(f)^\vee$, is thus a quotient of $M \otimes S^{\vee\vee} \cong M \otimes S$. The main result of this section identifies this quotient precisely as the quotient by the bend relations of f. Since one can easily recover a congruence J on a module V from its associated quotient V/J as the pullback of the diagram $V \to V/J \leftarrow V$, this allows one to recover the congruence $\mathcal{B}(f)$ from the set $bend(f)$ together with its S-module structure.

Theorem 5.2.2. *Let S be a totally ordered idempotent semifield, M a finitely generated $\mathbb{F}_1$-module, and $f \in M \otimes S$. The canonical map from $M \otimes S/\mathcal{B}(f)$ to its double dual is an isomorphism; in particular, there is a canonical isomorphism $bend(f)^\vee \cong M \otimes S/\mathcal{B}(f)$.*

Proof. Consider the following commutative diagram of S-modules:

$$
\begin{array}{ccc}
M \otimes S & \xrightarrow{\;\cong\;} & (M \otimes S)^{\vee\vee} \\
\downarrow & & \downarrow \\
M \otimes S/\mathcal{B}(f) & \longrightarrow & M \otimes S/\mathcal{B}(f)^{\vee\vee}.
\end{array}
$$

Since $(M \otimes S)/\mathcal{B}(f)^\vee$ is a submodule of $(M \otimes S)^\vee$ and S is totally ordered, the right vertical arrow is surjective, and hence the bottom horizontal arrow is also surjective.

To show that the bottom arrow is also injective, we will show that if $g, g' \in M \otimes S$ are equal at each point p in $bend(f)$, then they are equal in the quotient by $\mathcal{B}(f)$. We will examine their coefficients one at a time.

If $m \in M$ is a monomial in the complement of $\mathrm{supp}(f)$ then $f(m) = 0_S$ and so $m \in bend(f)$. Since g and g' agree on $bend(f)$, the coefficients of m in g and g' are identical.

We next consider the coefficients of monomials in $\mathrm{supp}(f)$.

First suppose $\mathrm{supp}(f) = \{a\}$ is a single monomial. In this case $\mathcal{B}(f)$ is spanned by the single relation $a \sim 0_S$. Since the coefficients of g and g' are identical for all monomials $b \neq a$, g and g' are identified in the quotient by $\mathcal{B}(f)$.

Now suppose $|\mathrm{supp}(f)| \geq 2$. For any $a \in M$, let $\chi_a \in (M \otimes S)^\vee$ denote the map sending a to 1_S and all other basis elements to 0_S. For any pair of distinct elements $a, b \in \mathrm{supp}(f)$, consider the element $p_{ab} \in (M \otimes S)^\vee$ given by the formula

$$p_{ab} = \left(\frac{1_S}{\chi_a(f)}\right) \chi_a + \left(\frac{1_S}{\chi_b(f)}\right) \chi_b.$$

Idempotency of addition implies that p_{ab} factors through the quotient by $\mathcal{B}(f)$, i.e., $p_{ab} \in bend(f)$, since for any $c \in \mathrm{supp}(f)$ we have

$$p_{ab}(f) = 1_S + 1_S = 1_S = p_{ab}(f_{\hat{c}}).$$

For each $a \in \mathrm{supp}(f)$, consider the element $g_a := \chi_a(g)/\chi_a(f) \in S$ and likewise for g', and let

$$m = \min_{a \in M} g_a, \quad \text{and} \quad m' = \min_{a \in M} g'_a.$$

By hypothesis, $p_{ab}(g) = p_{ab}(g')$ for all $a, b \in \mathrm{supp}(f)$, which yields the set of equations

$$(5.2.1) \qquad (\mathscr{R}_{ab}) : g_a + g_b = g'_a + g'_b.$$

Modulo the congruence $\mathcal{B}(f)$, we may assume the minima m and m' are each attained at least twice by the following argument. If m is attained only once by some g_a, and g_b achieves the minimum of the remaining non-minimal terms, then

$$g = g + g_b f_{\widehat{a}}$$

since for any $c \in M$, $\chi_c(g + g_b f_{\widehat{a}}) = \chi_c(g) + g_b \chi_c(f_{\widehat{a}})$, and if $c \notin \mathrm{supp}(f_{\widehat{a}}) = \mathrm{supp}(f) \smallsetminus a$ then this reduces to $\chi_c(g)$, while when $c \in \mathrm{supp}(f_{\widehat{a}})$ we have

$$g_b \chi_c(f_{\widehat{a}}) \leq g_c \chi_c(f_{\widehat{a}}) = \chi_c(g),$$

and so in all cases $\chi_c(g) = \chi_c(g + g_b f_{\widehat{a}})$. Then

$$g = g + g_b f_{\widehat{a}} \sim g + g_b f,$$

and in the final expression the minimum is equal to g_b and is attained at least twice (at the monomials a and b), so we replace g with this element, and likewise for g'. Now, for a and b such that $g_a = g_b = m$, the equation $(\mathscr{R}_{ab})$ implies that $m \geq m'$, and choosing a and b such that $g'_a = g'_b = m'$ we likewise see that $m \leq m'$. Hence $m = m'$.

Now let $a_1, \ldots, a_n$ be the elements of $\mathrm{supp}(f)$ ordered so that $g_{a_1} = g_{a_2} \leq \cdots \leq g_{a_n}$. Since $m = m'$, the equation $(\mathscr{R}_{a_1 a_2})$ implies that $g_{a_1} = g_{a_2} = g'_{a_1} = g'_{a_2}$. For any $k > 2$, g_{a_k} and g'_{a_k} are both greater than or equal to m, and so the equation $(\mathscr{R}_{a_1 a_k})$ implies that $g_{a_k} = g'_{a_k}$. Thus we have shown that g and g' are equal in $M \otimes S / \mathcal{B}(f)$. $\qquad\square$

5.3. Functoriality of the bend relations.

Here we give two lemmas expressing how the bend relations behave with respect to maps induced by morphisms of $\mathbb{F}_1$-modules and $\mathbb{F}_1$-algebras. These results are fundamental to the development that follows. Recall that for an $\mathbb{F}_1$-module M and a semiring S, we view the elements of M as the monomials of the S-module $M \otimes S$. An $\mathbb{F}_1$-module morphism $\varphi : M \to N$ is simply a map of pointed sets. This induces an S-module homomorphism $\varphi \otimes \mathrm{id}_S : M \otimes S \to N \otimes S$ that we denote simply by φ_S. If φ is a morphism of $\mathbb{F}_1$-algebras then φ_S is an S-algebra homomorphism. In either case, φ_S sends monomials to monomials, and in the latter case it is multiplicative on monomials.

Any S-algebra homomorphism $\psi : A \to B$ induces a map ψ_* that sends ideals in A to ideals in B by sending J to the ideal generated by the image of J, and likewise for semiring congruences.

Lemma 5.3.1. *Suppose $\varphi : M \to N$ is an $\mathbb{F}_1$-module morphism and $f \in M \otimes S$.*

(1) $(\varphi_S)_* \mathcal{B}(f) \subset \mathcal{B}(\varphi_S(f))$.
(2) *If M and N are $\mathbb{F}_1$-algebras and φ is an $\mathbb{F}_1$-algebra morphism then $(\varphi_S)_*\langle \mathcal{B}(f) \rangle \subset \langle \mathcal{B}(\varphi_S(f)) \rangle$, and if $J \subset M \otimes S$ is an ideal then $(\varphi_S)_* \mathcal{B}(J) \subset \mathcal{B}((\varphi_S)_* J)$.*
(3) *The above inclusions are equalities if φ is injective.*

Proof. Since $(\varphi_S)_* \mathcal{B}(f)$ is generated as an S-module congruence by the image of the generators of $\mathcal{B}(f)$, it suffices to show that any relation of the form $\varphi_S(f) \sim \varphi_S(f_{\widehat{m}})$ for $m \in M$ is implied by the relation $\varphi_S(f) \sim \varphi_S(f)_{\widehat{\varphi(m)}}$ in $\mathcal{B}(\varphi_S(f))$. Let $g_0, \cdots, g_k$ be the terms of f corresponding to the monomials in $\varphi^{-1}(\varphi(m)) \subset M$, with g_0 being the term of f whose support is m. The relation $\varphi_S(f)_{\widehat{\varphi(m)}} \sim \varphi_S(f)$ implies

$$\varphi_S(f_{\widehat{m}}) = \varphi_S(f)_{\widehat{\varphi(m)}} + \varphi_S(g_1 + \cdots + g_k) \sim \varphi_S(f) + \varphi_S(g_1 + \cdots + g_k)$$

$$= \varphi_S(f + g_1 + \cdots + g_k) = \varphi_S(f),$$

where the last equality follows from the idempotency of addition in S. This proves (1), from which part (2) follows immediately. When φ is injective it is clear that $\varphi_S(f_{\widehat{m}}) = \varphi_S(f)_{\widehat{\varphi_S(m)}}$, and so (3) follows from (1) and (2). $\qquad\square$

The next lemma asserts that formation of the bend congruences commutes with monomial localization.

Lemma 5.3.2. *Let M be an $\mathbb{F}_1$-algebra, $\varphi : M \to M[x^{-1}]$ a localization map, and $J \subset M \otimes S$ an ideal. The induced localization $\varphi_S : M \otimes S \to M[x^{-1}] \otimes S \cong (M \otimes S)[x^{-1}]$ satisfies $(\varphi_S)_* \mathcal{B}(J) = \mathcal{B}((\varphi_S)_* J)$.*

Proof. By Lemma 5.3.1, there is a containment $(\varphi_S)_* \mathcal{B}(J) \subset \mathcal{B}((\varphi_S)_* J)$ which is an equality if φ is injective (for example, if M is integral). So suppose that φ is not injective. The congruence $\mathcal{B}((\varphi_S)_* J)$ is generated by relations of the form

$$\varphi_S(f) \sim \varphi_S(f)_{\widehat{m}}$$

for $f \in J$ and $m \in \mathrm{supp}(\varphi_S(f)) \subset M[x^{-1}]$. Note that $\varphi(m) = \varphi(n)$ if and only if $x^k m = x^k n$ for some sufficiently large natural number k. Since $f \in J$ implies $x^k f \in J$, it now suffices to observe that $\varphi_S(f) = x^{-k} \varphi_S(x^k f)$ and $\varphi_S(f)_{\widehat{m}} = x^{-k} \varphi_S((x^k f)_{\widehat{x^k m}})$. $\qquad \square$

5.4. Construction of bend loci.
Let S be an idempotent semiring and X an $\mathbb{F}_1$-scheme. Let X_S be the base change of X from $\mathbb{F}_1$ to S. We now construct bend loci as closed subschemes of X_S.

Lemma 5.4.1. *Given a quasi-coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_{X_S}$, the association*

$$\mathrm{Spec}\, M \otimes S \mapsto \mathcal{B}(\mathscr{I}(\mathrm{Spec}\, M \otimes S))$$

defined for affine open subschemes $\mathrm{Spec}\, M \subset X$ determines a quasi-coherent congruence sheaf $\mathcal{B}(\mathscr{I})$ on X_S.

Proof. For $\mathrm{Spec}\, M \subset X$, the ideal $\mathscr{I}(\mathrm{Spec}\, M \otimes S) \subset M \otimes S$ determines a semiring congruence $\mathcal{B}(\mathscr{I}(\mathrm{Spec}\, M \otimes S))$ on $M \otimes S$ by Lemma 5.1.4(1). By Lemma 5.3.2, formation of this congruence commutes with restriction to a smaller open affine subscheme. $\qquad \square$

Definition 5.4.2. The *bend locus* of a quasi-coherent congruence sheaf $\mathscr{I}$ on X_S is the closed subscheme $\mathcal{B}end(\mathscr{I}) \subset X_S$ defined by the quasi-coherent congruence sheaf $\mathcal{B}(\mathscr{I})$. If $X = \mathrm{Spec}\, M \otimes S$ is affine and $I \subset M \otimes S$ is an ideal, we denote the corresponding closed subscheme simply by $\mathcal{B}end(I)$.

If $\mathscr{L}$ is a line bundle on X (i.e., a locally free sheaf of rank one) and $f \in \Gamma(X_S, \mathscr{L}_S)$ is a global section of the base change to S, then f determines a quasi-coherent ideal sheaf: for a local trivialization $\mathscr{L}|_U \cong \mathscr{O}_X|_U$ this is given by the principal ideal generated by the image of $f|_{U_S}$ in $\mathscr{O}_{X_S}(U_S)$. We denote the corresponding closed subscheme by $\mathcal{B}end(f) \subset X_S$. Note that, by Lemma 5.1.4(2) and Proposition 3.1.3, if X is locally integral then the bend relations locally defined by f and by the principal ideal generated by f coincide. Since we will be primarily concerned with situations in which X is locally integral, we can and will use the notation $\mathcal{B}end(f)$ without ambiguity.

Proposition 5.4.3. *Let $\varphi : X \to Y$ be a morphism of $\mathbb{F}_1$-schemes, $\mathscr{L}$ a line bundle on Y, and $f \in \Gamma(Y_S, \mathscr{L}_S)$. Then $\varphi_S : X_S \to Y_S$ maps $\mathcal{B}end(\varphi_S^* f)$ into $\mathcal{B}end(f)$.*

Proof. It suffices to check on affine patches, where the result follows from Lemma 5.3.1. $\qquad \square$

5.5. Tropical Proj and Cox.
If S is a semiring and A is an $\mathbb{N}$-graded S-algebra, then we can form an S-scheme $\mathrm{Proj}\, A$ in the usual way, either by topologizing the set of homogeneous prime ideals in A and constructing a structure sheaf from them, or by gluing the affine patches obtained by dehomogenizing. We say that a congruence J on A is *homogeneous* if the grading on A descends to the quotient by J. In this case, $\mathrm{Proj}\, A/J$ is naturally a closed subscheme of $\mathrm{Proj}\, A$.

Suppose M is an $\mathbb{N}$-graded monoid-with-zero. Then $M \otimes S$ is an $\mathbb{N}$-graded S-algebra, and if $I \subset M \otimes S$ is a homogeneous ideal then the congruence $\mathcal{B}(I)$ is homogeneous. Indeed, the bend

relations of a homogeneous polynomial are manifestly homogeneous, and $\mathcal{B}(I)$ is generated by the bend relations of the homogeneous elements of I. We denote the resulting closed subscheme $\mathrm{Proj}\, M \otimes S/\mathcal{B}(I)$ by $\mathcal{B}end\,(I)$. As usual, the context should make it clear whether $\mathcal{B}end\,(I)$ refers to this projective subscheme or to the affine scheme $\mathrm{Spec}\, M \otimes S/\mathcal{B}(I)$. Implicit here is the claim that using the bend relations of homogeneous elements in I to define a scheme via the Proj construction is equivalent to gluing the affine subschemes obtained by taking the bend relations after dehomogenizing the ideal. This is a special case of Proposition 5.5.1 below.

More generally, let $X = X^{\Delta}$ be a toric variety over S without torus factors and consider its $\mathrm{Cl}(X)$-graded algebra $\mathrm{Cox}(X) = S[x_{\rho} \mid \rho \in \Delta(1)]$. Identical to the case of toric schemes over $\mathbb{F}_1$ discussed in §4.2, each divisor class $[D] \in \mathrm{Cl}(X)$ is represented by a torus-invariant Weil divisor D and the homogeneous polynomials in $\mathrm{Cox}(X)$ of multi-degree $[D]$ are in bijection with global sections of a coherent sheaf $\mathscr{O}_X(D)$. Just as in the usual setting of toric varieties, these coherent sheaves need not be line bundles, but there is an inclusion $\mathrm{Pic}(X) \subset \mathrm{Cl}(X)$ of line bundle isomorphism classes into divisor classes and via this inclusion we can identify the global sections of any line bundle on X with the homogeneous elements of $\mathrm{Cox}(X)$ of the corresponding multi-degree.

Recall (Proposition 4.2.1) that $X = U/G$, where U is the complement of the vanishing of the irrelevant ideal and $G = \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{Z})$. Suppose $I \subset \mathrm{Cox}(X)$ is a homogeneous ideal generated by global sections of line bundles on X. Then every homogeneous $f \in I$ is the global section of a line bundle and we can consider the intersection of the bend loci $\mathcal{B}end\,(f) \subset X$ defined by these global sections, or we can consider the affine subscheme $\mathcal{B}end\,(I) \subset \mathrm{Spec}\,\mathrm{Cox}(X) \cong \mathbb{A}_S^{\Delta(1)}$ and attempt to take its quotient by the group G to get a closed subscheme of X. The following result says that these two constructions coincide.

Proposition 5.5.1. *Let $I \subset \mathrm{Cox}(X)$ be as above. Then*

$$\bigcap_{f \in I\ homogeneous} \mathcal{B}end\,(f) = (\mathcal{B}end\,(I) \cap U)/G$$

as closed subschemes of X, where the latter is the categorical quotient in the category of S-schemes.

Proof. Since I is generated as an S-module by its homogeneous elements, Lemmas 5.1.3(2) and 5.1.4(2) reduce us to proving the result in the case of a single homogeneous polynomial: $I = (f)$.

For each cone $\sigma \in \Delta$, let $x_{\widehat{\sigma}} := \prod_{\rho \notin \sigma(1)} x_{\rho}$. The restriction of $\mathcal{B}end\,(f)$ to the affine open $\mathrm{Spec}\,\mathrm{Cox}(X)[x_{\widehat{\sigma}}^{-1}] \subset U$ is defined by $\iota_*\langle \mathcal{B}(f)\rangle$, where $\iota : \mathrm{Cox}(X) \to \mathrm{Cox}(X)[x_{\widehat{\sigma}}^{-1}]$ is the localization map. The subalgebra of G-invariants on this chart is the degree zero piece

$$(\mathrm{Cox}(X)[x_{\widehat{\sigma}}^{-1}]/\iota_*\langle \mathcal{B}(f)\rangle)_0 = \mathrm{Cox}(X)[x_{\widehat{\sigma}}^{-1}]_0/\iota_*\langle \mathcal{B}(f)\rangle_0,$$

so this defines the restriction of $(\mathcal{B}end\,(f) \cap U)/G \subset U/G$ to the affine open

$$X_{\sigma} := \mathrm{Spec}\,(\mathrm{Cox}(X)[x_{\widehat{\sigma}}^{-1}]_0) \subset X = U/G.$$

On the other hand, a trivialization on X_{σ} of a line bundle $\mathscr{L}$ on X for which f is a section corresponds to a choice of unit $g \in \mathrm{Cox}(X)[x_{\widehat{\sigma}}^{-1}]$ with $\deg(g) = \deg(f)$. Then the bend locus of $f \in \Gamma(X, \mathscr{L})$ is defined on this affine patch by $\langle \mathcal{B}(\frac{f}{g})\rangle = \iota_*\langle \mathcal{B}(f)\rangle_0$, exactly as above. $\square$

6. Scheme-theoretic tropicalization

Let $v \colon k \to \mathbb{T}$ be a nontrivially valued field. The usual set-theoretic tropicalization is a map,

$$trop \colon \{\text{subvarieties of } k^n\} \to \{\text{subsets of } \mathbb{T}^n\}.$$

Given $Z \subset k^n$, the set $trop(Z)$ can be described either as the Euclidean closure of the image of Z under coordinate-wise valuation, or as the intersection of the set-theoretic bend loci of the coefficient-wise valuations of the polynomials in the defining ideal I_Z of Z:

$$(6.0.1) \qquad trop(Z) = \overline{\nu(Z)} = \bigcap_{f \in I_Z} bend(\nu(f)).$$

If Z is a linear subvariety, then $trop(Z) \subset \mathbb{T}^n$ is a $\mathbb{T}$-submodule.

In this section we construct and study a scheme-theoretic refinement/generalization $Trop$ of the set-theoretic tropicalization map $trop$. The setup is as follows. Let X be a locally integral $\mathbb{F}_1$-scheme and $\nu : R \to S$ a valued ring. Our scheme-theoretic tropicalization is then a map of posets (with respect to inclusion)

$$Trop_X^\nu : \{\text{closed subschemes of } X_R\} \to \{\text{closed subschemes of } X_S\}$$

(we will often drop the superscript and subscript if they are clear from the context).

Locally on an affine patch $\mathrm{Spec}\, A \subset X$ it sends $Z \subset \mathrm{Spec}\, A \otimes R$ to the scheme-theoretic intersection of the bend loci of the coefficient-wise valuations of all functions in the defining ideal $I_Z \subset A \otimes R$:

$$Trop(Z) = \bigcap_{f \in I_Z} Bend(\nu(f)).$$

This is functorial in X, compatible with the Cox construction, and when $S = \mathbb{T}$ the composition with $\mathrm{Hom}_{\mathrm{Sch}/\mathbb{T}}(\mathrm{Spec}\,\mathbb{T}, -)$ recovers the extended tropicalization functor of Kajiwara-Payne (which generalizes the above set-theoretic tropicalization from affine space to toric varieties). Moreover, these scheme-theoretic tropicalizations form an algebraic family as the valuation varies.

6.1. Construction of the tropicalization functor.
We first construct $Trop_X^\nu$ in the case when $X = \mathrm{Spec}\, A$ is an affine $\mathbb{F}_1$-scheme and then show that these affine pieces glue together to define it in general. In the affine case the scheme-theoretic tropicalization sends ideals in $A \otimes R$ to semiring congruences on $A \otimes S$, and it then globalizes to send quasi-coherent ideal sheaves to quasi-coherent congruence sheaves.

Let M be an $\mathbb{F}_1$-module and consider the following two operations.

(1) Given an R-submodule $N \subset M \otimes R$, let $trop(N) \subset M \otimes S$ denote the S-submodule generated by the image of N under the coefficient-wise valuation map $\nu : M \otimes R \to M \otimes S$. (Note that when R is a field and $S = \mathbb{T}$ then this is just the usual set-theoretic tropicalization map applied to linear subspaces.)
(2) The bend congruence construction $\mathcal{B}(-)$ sending S-submodules of $M \otimes S$ to S-module congruences on $M \otimes S$.

The composition of these operations, $N \mapsto \mathcal{B}trop(N)$, sends R-submodules to S-module congruences. Now suppose A is an $\mathbb{F}_1$-algebra and $I \subset A \otimes R$ is an ideal. Regarding I simply as a module, we can apply the above two operations to form the S-module congruence $\mathcal{B}trop(I)$ on $A \otimes S$.

Proposition 6.1.1. *If A is integral then the S-submodule $trop(I) \subset A \otimes S$ is an ideal, and hence $\mathcal{B}trop(I)$ is a semiring congruence.*

Proof. It suffices to check that $trop(I)$ is closed under multiplication by monomials. An arbitrary element of $trop(I)$ is a linear combination of elements of the form $\nu(f)$ for $f \in I$. If x is a monomial and $f \in I$ then $x\nu(f) = \nu(xf) \in trop(I)$ since multiplication by x is injective on monomials. The second claim then follows from Lemma 5.1.4 part (1). $\square$

Remark 6.1.2. The intergality hypothesis is necessary here, as the following example shows. Let $A = \mathbb{F}_1[x, y]/\langle x^2 \sim xy \sim y^2 \rangle$ and let $f = x - y \in A \otimes R$. Then $trop$ sends the principal ideal (f) to the S-submodule spanned by the single element $x + y$, and this is not an ideal.

Definition 6.1.3 (Affine tropicalization). If A is an integral $\mathbb{F}_1$-algebra and $Z \subset \operatorname{Spec} A \otimes R$ is the closed subscheme corresponding to an ideal I, then we define $\mathit{Trop}(Z) \subset \operatorname{Spec} A \otimes S$ to be the closed subscheme determined by the semiring congruence $\mathcal{B}\mathit{trop}(I)$, i.e.,

$$\mathit{Trop}(Z) := \mathcal{B}\mathit{end}\left(\mathit{trop}(I)\right).$$

Proposition 6.1.4. *The subscheme $\mathit{Trop}(Z)$ is the scheme-theoretic intersection $\bigcap_{f \in I} \mathcal{B}\mathit{end}\left(\nu(f)\right)$.*

Proof. The set $\{\nu(f)\}_{f \in I}$ generates $\mathit{trop}(I)$ as an S-module by definition, and so by Lemma 5.1.3 part (2), the congruence $\mathcal{B}\mathit{trop}(I)$ is generated as an S-module congruence by $\{\mathcal{B}(\nu(f))\}_{f \in I}$. $\square$

We now globalize the above picture. Let X be a locally integral $\mathbb{F}_1$-scheme and $\mathscr{I}$ a quasi-coherent ideal sheaf on X_R. If $U = \operatorname{Spec} A$ is an affine patch then over U_R the sheaf $\mathscr{I}$ is given by an ideal $I \subset A \otimes R$. By taking U to be sufficiently small we can, by Proposition 3.1.3, assume that A is an integral $\mathbb{F}_1$-algebra. Then $\mathit{trop}(I)$ is an ideal in $A \otimes S$, by Proposition 6.1.1. Hence we have a quasi-coherent ideal sheaf $\mathit{trop}(\mathscr{I}|_{U_R})$ over the affine patch U_S. The following lemma shows that these locally defined sheaves assemble to give a well-defined quasi-coherent ideal sheaf $\mathit{trop}(\mathscr{I})$ over X_S.

Lemma 6.1.5. *If A is integral then the construction $I \mapsto \mathit{trop}(I)$ commutes with monomial localizations. I.e., given a localization map $\varphi : A \to A[x^{-1}]$, one has $\mathit{trop}((\varphi_R)_* I) = (\varphi_S)_* \mathit{trop}(I)$.*

Proof. The ideal $\mathit{trop}((\varphi_R)_* I)$ is spanned as an S-module by the elements of the form $\nu(x^{-n}f)$ for $f \in I$. Since A is integral, multiplication by x commutes with valuation and therefore $\nu(x^{-n}f) = x^{-n}\nu(f)$, and elements of this form span $(\varphi_S)_* \mathit{trop}(I)$. $\square$

By Lemma 5.3.2, applying $\mathcal{B}(-)$ locally to the quasi-coherent ideal sheaf $\mathit{trop}(\mathscr{I})$ yields a quasi-coherent congruence sheaf $\mathcal{B}\mathit{trop}(\mathscr{I})$.

Definition 6.1.6. Let X be a locally integral $\mathbb{F}_1$-scheme and $Z \subset X_R$ a closed subscheme defined by an ideal sheaf $\mathscr{I}_Z$. The scheme-theoretic tropicalization of Z is the subscheme $\mathit{Trop}_X^\nu(Z) \subset X_S$ defined by the congruence sheaf $\mathcal{B}\mathit{trop}(\mathscr{I}_Z)$, i.e.,

$$\mathit{Trop}_X^\nu(Z) := \mathcal{B}\mathit{end}\left(\mathit{trop}(\mathscr{I}_Z)\right).$$

6.2. **Basic properties of Trop_X^ν.** In this section we let X be a locally integral $\mathbb{F}_1$-scheme and $\nu : R \to S$ a valued ring. We will present some general properties of tropicalization, but first let us consider the projective case.

Suppose that $X = \mathbb{P}^n$ and $Z \subset \mathbb{P}_R^n$ is a closed subscheme over R given by a homogeneous ideal $I \subset R[x_0, \ldots, x_n]$. Tropicalization is compatible with the Proj construction (cf. Theorem 6.6.1 below) in the following sense: $\mathit{trop}(I)$ is a homogeneous ideal in $S[x_0, \ldots, x_n]$, and $\mathcal{B}\mathit{trop}(I)$ is then a homogeneous congruence, which is to say that the grading descends to the quotient, and one has

$$\mathit{Trop}(\operatorname{Proj} R[x_0, \ldots, x_n]/I) = \operatorname{Proj}\left(S[x_0, \ldots, x_n]/\mathcal{B}\mathit{trop}(I)\right).$$

A key observation (used in defining the Hilbert polynomial and in Theorem 7.1.6) is that the S-linear dual,

$$\left(S[x_0, \ldots, x_n]/\mathcal{B}\mathit{trop}(I)\right)_d^\vee,$$

of the degree d graded piece of the homogeneous coordinate semiring is a tropical linear space in $S[x_0, \ldots, x_n]_d^\vee$; namely, it is the tropicalization of the linear subspace $(R[x_0, \ldots, x_n]/I)_d^\vee \subset R[x_0, \ldots, x_n]_d^\vee$ (as observed in [MR14], this tropical linear space is dual to the tropical linear space $\mathit{trop}(I_d)$). This is a special property, and arbitrary projective $\mathbb{T}$-schemes do not satisfy it in general, as illustrated in the following example.

Example 6.2.1. Consider the family

$$Z := \operatorname{Proj} \mathbb{T}[x, y, t] / \langle x^2 \sim x^2 + txy, y^2 \sim -\infty \rangle \subset \mathbb{P}^1_{\mathbb{T}} \times \mathbb{A}^1_{\mathbb{T}} \to \mathbb{A}^1_{\mathbb{T}},$$

where x and y are in degree 1, and t is in degree 0 and is the parameter on $\mathbb{A}^1_{\mathbb{T}}$. The fiber Z_t over $t = -\infty$ is cut out by the single equation $y^2 = -\infty$, and it is thus the tropicalization of the projective scheme cut out by $y^2 = 0$. On the other hand, when $t \neq -\infty$, the degree 2 piece of the homogeneous coordinate semiring of the fiber over t dualizes to the submodule $\{(a, b, -\infty) \mid a \leq b\} \subset \mathbb{T}^3$, which is not a tropical linear space, and so these fibers cannot be tropicalizations.

This property of projective $\mathbb{T}$-subschemes having homogeneous coordinate semiring whose graded pieces all dualize to tropical linear spaces is an important one. There is recent and upcoming work investigating this further and developing a theory of such schemes [MR14, MR15, FGG16].

Now let X be an arbitrary locally integral $\mathbb{F}_1$-scheme and $v : R \to S$ a valued ring.

Proposition 6.2.2. $\mathcal{T}rop_X(X_R) = X_S$.

Proof. Consider an affine patch $\operatorname{Spec} A \subset X$. If $I = (0)$ then $trop(I) = (0)$, and so $\mathcal{B} trop(I)$ is the trivial congruence. $\square$

One can therefore view the tropical model X_S of X as a canonical tropicalization of X_R. This next lemma is a slight extension of the statement that, when X is a toric variety, tropicalization commutes with restriction to toric strata and with restriction to open toric subvarieties.

Lemma 6.2.3. *Let $W \subset X$ be a locally closed locally integral subscheme such that W is locally defined by equations of the form $x \sim 0$. Then $\mathcal{T}rop_X(Z) \cap W_S = \mathcal{T}rop_W(Z \cap W_R)$. In particular, $\mathcal{T}rop_X(W_R) = W_S$.*

Proof. It suffices to show this in the affine case. By Lemmas 5.3.2 and 6.1.5, tropicalization commutes with restriction to an open subscheme coming from the $\mathbb{F}_1$ level, so we are reduced to the case when W is a closed subscheme, and then the result follows from Lemma 8.1.4 below. The equality $\mathcal{T}rop_X(W_R) = W_S$ then follows from Proposition 6.2.2. $\square$

For X an $\mathbb{F}_1$-scheme and R a (semi)ring, a morphism $\operatorname{Spec} R \to X_R$ is given locally by a multiplicative map from a monoid to R. Thus, a valuation $v : R \to S$ determines a map $\widetilde{v} : X(R) \to X(S)$. In particular, if $X = \mathbb{A}^n_{\mathbb{F}_1}$ then $\widetilde{v} : R^n \to S^n$ is coordinate-wise valuation.

Proposition 6.2.4. *If S is totally ordered, then the tropicalization of a point $p \in X(R)$ is the image of the point under $\widetilde{v}$; more precisely, if $Z \subset X_R$ is the closed subscheme corresponding to p, then $\mathcal{T}rop^v_X(Z)$ is the closed subscheme corresponding to the point $\widetilde{v}(p) \in X(S)$.*

Proof. Locally, $X = \operatorname{Spec} A$ with A an integral $\mathbb{F}_1$-algebra; let $\{x_i\}_{i \in \Lambda}$ be a set of generators for the monoid A. The point p is a multiplicative map $A \to R$ and is thus determined by the collection $\{p(x_i)\}_{i \in \Lambda}$ of elements of R; the corresponding subscheme Z is defined by the ideal $I := (x_i - p(x_i))_{i \in \Lambda}$. On the other hand, the point $\widetilde{v}(p) \in X(S)$ is determined by the family of elements $\widetilde{v}(p)(x_i) = v(p(x_i)) \in S$ and corresponds to the congruence $\langle x_i \sim v(p(x_i)) \rangle_{i \in \Lambda}$.

The ideal $trop(I)$ (recall Proposition 6.1.1) contains the elements $x_i - v(p(x_i)) \in A \otimes S$, and hence $\mathcal{B} trop(I)$ contains the relations $x_i \sim v(p(x_i))$ that define the point $\widetilde{v}(p)$ as a subscheme. It only remains to show that there are no additional relations in $\mathcal{B} trop(I)$, i.e., if $f \in I$ then the bend relations of $v(f)$ are implied by the relations $x_i \sim v(p(x_i))$. Since $A \otimes S / \langle x_i \sim v(p(x_i)) \rangle_{i \in \Lambda} \cong S$, it is equivalent to showing that the map $x_i \mapsto v(p(x_i))$ defines an S-point of $A \otimes S / \mathcal{B} trop(I)$. Since

$$\operatorname{Hom}(A \otimes S / \mathcal{B} trop(I), S) = \bigcap_{f \in I} \operatorname{Hom}(A \otimes S / \mathcal{B}(v(f)), S),$$

it suffices by Proposition 5.1.6 to show that each $\nu(f)$ tropically vanishes at $\widetilde{\nu}(p)$, in the sense of that proposition. Since $\nu(f(p)) = \nu(0_R) = 0_S$ but $\nu(f)(\widetilde{\nu}(p)) \neq 0_S$, the valuation does not commute with taking the sum of the terms in f, so the result follows from Lemma 2.5.3(2). $\square$

6.3. Relation to the Kajiwara-Payne extended tropicalization functor.
We now show that the above scheme-theoretic tropicalization recovers the Kajiwara-Payne extended tropicalization functor [Kaj08, Pay09] upon composition with $\mathrm{Hom}_{\mathrm{Sch}/\mathbb{T}}(\mathrm{Spec}\,\mathbb{T}, -)$.

Let X be a toric variety over $\mathbb{F}_1$, and let k be an algebraically closed field equipped with a non-trivial valuation $\nu : k \to \mathbb{T}$. The Kajiwara-Payne extended tropicalization is a map trop_X that sends subvarieties of X_k to subsets of $X(\mathbb{T})$ by sending $Z \subset X_k$ to the Euclidean closure of the image of $Z(k)$ under the map $\widetilde{\nu} : X(k) \to X(\mathbb{T})$. This extends the usual set-theoretic tropicalization map from tori or affine spaces to toric varieties.

Theorem 6.3.1. *The set of $\mathbb{T}$-points of $\mathcal{T}\!rop_X(Z)$ coincides with $\mathrm{trop}_X(Z)$ as a subset of $X(\mathbb{T})$.*

Proof. By [Pay09, Prop. 3.4], the set-theoretic tropicalization can be computed stratum by stratum. I.e., if W is a torus orbit in X then $\mathrm{trop}_X(Z) \cap W(\mathbb{T}) = \mathrm{trop}_W(Z \cap W_k)$. By the Fundamental Theorem of tropical geometry [MS15, Theorem 3.2.4] (a.k.a. Kapranov's Theorem in the case of a hypersurface), $\mathrm{trop}_W(Z \cap W_k)$ is the subset of points in $W(\mathbb{T}) \cong \mathbb{R}^n$ where the graph of each nonzero function in the ideal defining $Z \cap W_k$ is nonlinear. By Proposition 5.1.6 and Lemma 6.2.3, this is equal to the set of $\mathbb{T}$-points of $\mathcal{T}\!rop_X(Z) \cap W_\mathbb{T}$. $\square$

6.4. Functoriality of tropicalization.
We now examine the functoriality properties of the scheme-theoretic tropicalization map $\mathcal{T}\!rop_X^\nu$. We show that it is functorial in X in the sense described below, and under certain additional hypotheses it is functorial in the valuation ν.

For a (semi)ring R, let $\mathcal{P}(R)$ denote the category of pairs

$$(X \text{ a locally integral } \mathbb{F}_1\text{-scheme}, Z \subset X_R \text{ a closed subscheme}),$$

where a morphism $(X, Z) \to (X', Z')$ is an $\mathbb{F}_1$-morphism $\Phi : X \to X'$ such that $\Phi_R(Z) \subset Z'$.

Proposition 6.4.1. *The tropicalization maps $\{\mathcal{T}\!rop_X^\nu\}$ determine a functor $\mathcal{T}\!rop^\nu : \mathcal{P}(R) \to \mathcal{P}(S)$ sending (X, Z) to $(X, \mathcal{T}\!rop_X^\nu(Z))$.*

Proof. Given an arrow $(X, Z) \to (X', Z')$ in $\mathcal{P}(R)$, we must show that $\Phi_S(\mathcal{T}\!rop_X(Z)) \subset \mathcal{T}\!rop_{X'}(Z')$. It suffices to show this in the affine case: $X = \mathrm{Spec}\,A$, $X' = \mathrm{Spec}\,A'$, the map Φ is given by a monoid homomorphism $\varphi : A' \to A$, and Z and Z' are given by ideals $I \subset A \otimes R$ and $I' \subset A' \otimes R$ with $\varphi_R(I') \subset I$. The claim is now that $(\varphi_S)_* \mathcal{B}\,\mathrm{trop}(I') \subset \mathcal{B}\,\mathrm{trop}(I)$, and for this it suffices to show that $(\varphi_S)_* \mathcal{B}(\nu(f)) \subset \mathcal{B}(\nu(\varphi_R(f)))$ for any $f \in A' \otimes R$. In fact, we will show that each generating relation

$$(6.4.1) \qquad\qquad \varphi_S(\nu(f)) \sim \varphi_S(\nu(f)_{\widehat{i}})$$

of $(\varphi_S)_* \mathcal{B}(\nu(f))$ is implied by the corresponding relation

$$(6.4.2) \qquad\qquad \nu(\varphi_R(f)) \sim \nu(\varphi_R(f))_{\widehat{\varphi(i)}}$$

in $\mathcal{B}(\nu(\varphi_R(f)))$ by adding the RHS of (6.4.1) to both sides. We show this by comparing coefficients term-by-term. For $\ell \in \mathrm{supp}(f)$, let $a_\ell \in R$ denote the coefficient of ℓ. For each $m \in \mathrm{supp}(\varphi_R(f))$ with $m \neq \varphi(i)$, the coefficients of m on boths sides of (6.4.1) are equal to

$$(6.4.3) \qquad\qquad \sum_{\ell \in \varphi^{-1}(m)} \nu(a_\ell).$$

The coefficients of m on either side in (6.4.2) are both equal to

$$(6.4.4) \qquad\qquad \nu\left(\sum_{\ell \in \varphi^{-1}(m)} a_\ell \right).$$

By the subadditivity property of the valuation, adding (6.4.4) to (6.4.3) yields (6.4.3).

We now examine the coefficients of $\varphi(i)$ in (6.4.1) and (6.4.2); they are, respectively,

$$(6.4.5) \qquad \sum_{\ell \in \varphi^{-1}(\varphi(i))} v(a_\ell), \qquad\qquad \sum_{\ell \in \varphi^{-1}(\varphi(i)) \smallsetminus \{i\}} v(a_\ell)$$
$$\textit{(LHS)} \qquad\qquad\qquad\qquad \textit{(RHS)}$$

and

$$(6.4.6) \qquad v\left(\sum_{\ell \in \varphi^{-1}(\varphi(i))} a_\ell \right) \qquad\qquad 0_S.$$
$$\textit{(LHS)} \qquad\qquad\qquad \textit{(RHS)}$$

By Lemma 2.5.3 part (1), adding the RHS of (6.4.5) to both sides of (6.4.6) yields (6.4.5). $\qquad \square$

Remark 6.4.2. The sign condition in Definition 2.5.1 is necessary for the above proposition to hold. For example, suppose $f = \ell_1 - \ell_2$, φ maps the two monomials ℓ_1 and ℓ_2 in the support of f to the same monomial. When $i = \ell_2$ Then (6.4.5) becomes $v(1_R) + v(-1_R) \sim v(1_R)$, and (6.4.6) is $v(0_R) \sim 0_s$. On the other hand, taking $i = \ell_1$, (6.4.5) now gives $v(1_R) + v(-1_R) \sim v(-1_R)$. Thus it must be the case that $v(-1_R) = v(1_R)$.

We now turn to the dependence on v.

Proposition 6.4.3. *Let $v : R \to S$ be a valuation and $\varphi : S \to T$ a map of semirings. Then*

$$\mathcal{T}rop_X^{\varphi \circ v}(Z) = \mathcal{T}rop_X^v(Z) \times_{\operatorname{Spec} S} \operatorname{Spec} T$$

as subschemes of X_T.

Proof. It suffices to prove this in the case X is affine, so assume $X = \operatorname{Spec} A$ for some integral $\mathbb{F}_1$-algebra A, and let $I \subset A \otimes R$ be the ideal defining $Z \subset X$. Given a module congruence K on $A \otimes S$, the canonical isomorphism $(A \otimes S) \otimes_S T \cong A \otimes T$ descends to an isomorphism

$$\left(A \otimes S / K \right) \otimes_S T \cong A \otimes T / \varphi_* K.$$

The claim follows from this by taking $K = \mathcal{B}trop^v(I)$ and observing that $\varphi_* \mathcal{B}trop^v(I) = \mathcal{B}trop^{\varphi \circ v}(I)$. $\qquad \square$

6.5. Moduli of valuations and families of tropicalizations.
Let $\mathcal{V}al(R) := \operatorname{Spec} S_{univ}^R$ be the affine $\mathbb{B}$-scheme corresponding to the semiring of values associated with the universal valuation on R defined in §2.5. By Proposition 2.5.4, $\mathcal{V}al(R)$ represents the functor on affine $\mathbb{B}$-schemes,

$$\operatorname{Spec} S \mapsto \{\text{valuations } R \to S\}.$$

Thus $\mathcal{V}al(R)$ is the moduli scheme of valuations on R. This is a refinement of the observation of Manon [Man11] that the set of all valuations with semiring of values $\mathbb{T}$ forms a fan. In particular, the $\mathbb{T}$-points of $\mathcal{V}al(R)$ are the usual non-archimedean valuations on R.

As a special case of Proposition 6.4.3 we have the following (Theorem C part (1) from the introduction).

Theorem 6.5.1. *Given a locally integral $\mathbb{F}_1$-scheme X, a ring R, and a subscheme $Z \subset X_R$, the tropicalization of Z with respect to the universal valuation, $\mathcal{T}rop_X^{v_{univ}^R}(Z)$, forms an algebraic family of $\mathbb{B}$-schemes over $\mathcal{V}al(R)$ such that the fiber over each valuation v is $\mathcal{T}rop_X^v(Z)$.*

6.6. **Compatibility with Cox's quotient construction.** Let $X = X^\Delta$ be a toric scheme over $\mathbb{F}_1$ and recall (§4.2) that $X = U/G$, where $U \subset \mathbb{A}^{\Delta(1)}$ is the complement of the vanishing of the irrelevant ideal and $G = \mathrm{Hom}(\mathrm{Cl}(X),\mathbb{Z})$. A homogeneous ideal $I \subset \mathrm{Cox}(X_R) = R[x_\rho \mid \rho \in \Delta(1)]$ determines a closed subscheme $Z \subset X_R$, and if Δ is simplicial then every closed subscheme arises in this way [Cox95, Theorem 3.7]. The scheme Z is the categorical quotient of the G-invariant locally closed subscheme $\widetilde{Z} \cap U_R \subset \mathbb{A}_R^{\Delta(1)}$, where $\widetilde{Z} := V(I)$. In other words, we have

$$Z = (\widetilde{Z} \cap U_R)/G \subset U_R/G = X_R.$$

Theorem 6.6.1. *Tropicalization commutes with the Cox quotient:*

$$\mathit{Trop}_X(Z) = (\mathit{Trop}_{\mathbb{A}^{\Delta(1)}}(\widetilde{Z}) \cap U_S)/G \subset U_S/G = X_S.$$

Proof. In the notation of §5.5, we can cover X by open affines X_σ for $\sigma \in \Delta$. The subscheme $Z \subset X_R$ is defined in each such chart as $Z_\sigma := \mathrm{Spec}\,(\mathrm{Cox}(X_R)[x_{\widehat\sigma}^{-1}]_0/I_0')$, where I' denotes the image of I in this localization and I_0' its degree zero part. The tropicalization $\mathit{Trop}_X(Z)$ is then obtained by gluing the affine tropicalizations

$$\mathit{Trop}_{X_\sigma}(Z_\sigma) = \mathrm{Spec}\,\mathrm{Cox}(X_S)[x_{\widehat\sigma}^{-1}]_0/\mathcal{B}\mathit{trop}(I_0').$$

Since the valuation preserves degree and taking quotients commutes with taking degree zero part, this is the spectrum of $\big(\mathrm{Cox}(X_S)[x_{\widehat\sigma}^{-1}]/\mathcal{B}\mathit{trop}(I')\big)_0$. As in §4.2, taking degree zero here coincides with taking the subalgebra of G-invariants, and by Lemma 6.2.3 tropicalization commutes with $\mathbb{F}_1$-localization, so this is the categorical quotient of $\mathit{Trop}_{\mathbb{A}^{\Delta(1)}}(\widetilde{Z}) \setminus V(x_{\widehat\sigma})$. In the usual way, these categorical quotients patch together to yield the categorical quotient of $\mathit{Trop}_{\mathbb{A}^{\Delta(1)}}(\widetilde{Z}) \cap U_S$. $\qquad\square$

7. NUMERICAL INVARIANTS

Here we show that there is a natural way to define Hilbert polynomials for the class of projective subschemes over idempotent semirings that arise as tropicalizations, and that tropicalization preserves the Hilbert polynomial. We also show that for a projective hypersurface, the multiplicities (sometimes called weights) decorating the facets of its tropicalization, which are frequently used in tropical intersection theory, are encoded in the tropical scheme structure.

7.1. **The Hilbert polynomial.** First recall the classical setup. Let k be a field, $A := \mathbb{F}_1[x_0,\dots,x_n]$ the homogeneous coordinate algebra of $\mathbb{P}_{\mathbb{F}_1}^n$, and $Z \subset \mathbb{P}_k^n$ a subscheme defined by a homogeneous ideal $I \subset A \otimes k$. The Hilbert function of $A \otimes k/I$ is usually defined to be the map $d \mapsto \dim_k(A \otimes k/I)_d$; however, one could equally well replace $\dim_k(A \otimes k/I)_d$ with $\dim_k(A \otimes k/I)_d^\vee$, where $(-)^\vee$ is the k-linear dual. This may seem like a trivial observation since a finite dimensional vector space and its dual are noncanonically isomorphic, but the choice between the two becomes contentful when we come to define the Hilbert function over idempotent semirings (cf. Theorem 5.2.2). All homogeneous ideals defining Z have the same saturation, so the corresponding Hilbert functions coincide for $d \gg 0$ and this determines the Hilbert polynomial of $Z \subset \mathbb{P}_k^n$.

To define the Hilbert function for a homogeneous congruence K on $A \otimes S$, where S is an idempotent semiring, one first needs an appropriate definition of the dimension of an S-module. We assume here S is a totally ordered semifield. The following definition is from [MZ08], in the case $S = \mathbb{T}$.

Definition 7.1.1. Let S be a totally ordered semifield and L an S-module.

   (1) A collection $v_1,\dots,v_k \in L$ is *linearly dependent* if any linear combination of the v_i can be written as a linear combination of a proper subset of the v_i; otherwise it is *linearly independent*.
   (2) The *dimension* of L, denoted $\dim_S L$, is the largest number d such that there exists a set of d linearly independent elements in L.

We now show that the above notion of dimension is preserved under base change.

Lemma 7.1.2. *Let $\varphi : S \to T$ be a homomorphism of totally ordered idempotent semifields. If L is a submodule of a finitely generated free S-module then $\dim_S L = \dim_T L \otimes_S T$.*

Proof. It suffices to prove the result when $T = \mathbb{B}$ and φ is the unique homomorphism to $\mathbb{B}$, defined by sending all nonzero elements to $1_\mathbb{B}$. Moreover, since L is a submodule of a finitely generated free module, it suffices to show that a set $v_1, \ldots, v_d \in S^n$ is linearly independent (in the sense of the above definition) if and only if the set $\varphi(v_1), \ldots, \varphi(v_d)$ is linearly independent. Clearly if the v_i are S-linearly dependent then their images under φ are $\mathbb{B}$-linearly dependent. Conversely, suppose $\varphi(v_1), \ldots, \varphi(v_d)$ are $\mathbb{B}$-linearly dependent, so that (without loss of generality)

$$\sum_{i=1}^{d} \varphi(v_i) = \sum_{i=1}^{d-1} \varphi(v_i).$$

This condition says that, for each k, if the kth component of v_i vanishes for $i < d$ then it does so for v_d as well. Since S is a totally ordered semifield, given any notrivial elements $a, b \in S$, there exists $c \in S$ such that $ca \geq b$. Hence for each $i < d$, we may choose an $a_i \in S$ large enough so that each component of $a_i v_i$ is greater than or equal to the corresponding component of each v_d. By construction we then have

$$\sum_{i=1}^{d} a_i v_i = \sum_{i=1}^{d-1} a_i v_i,$$

which shows that the v_i are S-linearly dependent. $\qquad\qquad\square$

Lemma 7.1.3. *If $L \subset S^n$ is a tropical linear space of rank r (in the sense of [Fre13] or [SS04, Spe08]) then $\dim_S L = r$.*

Proof. Let $\psi : S \to \mathbb{T}$ be any homomorphism (for example, one can take the unique homomorphism to $\mathbb{B}$ followed by the unique homomorphism $\mathbb{B} \hookrightarrow \mathbb{T}$). The base change $L \otimes_S \mathbb{T}$ is a tropical linear space of rank d in $\mathbb{T}^n$ (this can easily be seen in terms of the corresponding valuated matroids). By Lemma 7.1.2, $\dim_S L = \dim_\mathbb{T} L \otimes_S \mathbb{T}$ and by [MZ08, Proposition 2.5], $\dim_\mathbb{T} L \otimes_S \mathbb{T}$ is equal to the maximum of the local topological dimensions of the polyhedral set underlying $L \otimes_S \mathbb{T}$. The statement now follows from the fact that a tropical linear space in $\mathbb{T}^n$ is a polyhedral complex of pure dimension equal to its rank. $\qquad\qquad\square$

Definition 7.1.4. Given a homogenous congruence K on $A \otimes S = S[x_0, \ldots, x_n]$, the *Hilbert function* of K is the map $d \mapsto \dim_S (A \otimes S/K)_d^\vee$.

Two homogeneous congruences (cf. §5.5) define the same projective subscheme if and only if they coincide in all sufficiently large degrees. Given a homogeneous congruence K, the *saturation* K^{sat} is the maximal homogeneous congruence that agrees with K in sufficiently high degrees (this exists since the sum of any two congruences that coincide with K in high degrees will itself coincide with K in high degrees), and we say that K is *saturated* if $K = K^{sat}$.

Definition 7.1.5. The Hilbert function of a projective subscheme $Z \subset \mathbb{P}^n_S$ is the Hilbert function of the unique saturated homogeneous congruence on $S[x_0, \ldots, x_n]$ defining Z.

Since modules over a semiring do not form an abelian category, it does not appear automatic that the Hilbert function of an arbitrary projective subscheme over S is eventually polynomial, but remarkably, this is the case for schemes in the image of the tropicalization functor.

Theorem 7.1.6. *Let $v : k \to S$ be a valued field. If $I \subset A \otimes k$ is a homogenous ideal then the Hilbert function of I coincides with the Hilbert function of $\mathcal{B}\mathrm{trop}(I)$. Consequently, for any subscheme $Z \subset \mathbb{P}^n_k$, the tropicalization $\mathrm{Trop}^\vee_{\mathbb{P}^n}(Z) \subset \mathbb{P}^n_S$ has a well-defined Hilbert polynomial and it coincides with that of Z.*

Proof. The operation $\mathcal{B}trop(-)$ commutes with restriction to the degree d graded piece, so

$$\big(A\otimes S/\mathcal{B}trop(I)\big)_d = A_d\otimes S/\mathcal{B}trop(I_d).$$

By Propositions 5.1.6 and 6.1.4, the dual, $\big(A_d\otimes S/\mathcal{B}trop(I_d)\big)^\vee$, is the tropical linear space in $(A_d\otimes S)^\vee$ that is the tropicalization of the linear subspace $(A_d\otimes k/I_d)^\vee \subset (A_d\otimes k)^\vee$. Since the tropicalization of a subspace of dimension r is a rank r tropical linear space, the statement that $\mathcal{B}trop(-)$ preserves the Hilbert function of I now follows from Lemma 7.1.3. The statement about the Hilbert polynomials then follows since, by Theorem 6.6.1, $\mathcal{T}rop_{\mathbb{P}^n}^\vee(Z)$ is defined by the homogeneous congruence $\mathcal{B}trop(I)$. $\qquad\square$

Recall that, classically, a family of subschemes in projective space is flat if and only if the Hilbert polynomials of the fibers are all equal. The above result therefore suggests that if one views tropicalization as some kind of degeneration of a variety, then the numerical behavior is that of a flat degeneration. Moreover, this next result (Theorem C part (2)) shows that the family of all tropicalizations of a projective subscheme Z has the numerical behaviour of a flat family.

Corollary 7.1.7. *For S a totally ordered idempotent semifield, the Hilbert polynomial of the fiber of the family $\mathcal{T}rop^{\nu_{univ}^k}(Z) \to \mathcal{V}al(k)$ over any S-point is equal to the Hilbert polynomial of Z.*

Proof. This follows directly from Theorems 6.5.1 and 7.1.6 since the Hilbert polynomials of the fibers are all equal to the Hilbert polynomial of Z. $\qquad\square$

7.2. **Recovering the multiplicities and the defining polynomial of a tropical hypersurface.**

Proposition 7.2.1. *For any valued ring $\nu : R \to S$ such that S is a semifield, and any projective hypersurface $Z = V(f) \subset \mathbb{P}_R^n$, the tropicalized scheme $\mathcal{T}rop(Z) \subset \mathbb{P}_S^n$ determines the defining homogeneous polynomial $\nu(f) \in (A\otimes S)_d$ uniquely up to scalar.*

Proof. Since $\mathcal{T}rop(Z) = \operatorname{Proj}A\otimes S/\langle\mathcal{B}(\nu(g))\rangle_{g\in(f)}$, and this homogeneous congruence in degree d coincides with the congruence $\mathcal{B}(\nu(f))$, the result follows from Lemma 5.1.8. $\qquad\square$

Corollary 7.2.2. *For an algebraically closed valued field $\nu : k \to \mathbb{T}$, and an irreducible projective hypersurface $Z \subset \mathbb{P}_k^n$ that is not contained in any coordinate hyperplane, the scheme $\mathcal{T}rop(Z) \subset \mathbb{P}_{\mathbb{T}}^n$ determines the multiplicities on the facets of the restriction of its $\mathbb{T}$-points to the tropical torus $\mathbb{R}^n$.[5]*

Proof. This follows immediately from Proposition 7.2.1, since the multiplicities for a tropical hypersurface are lattice lengths in the Newton polytope of f [DFS07, §2]. $\qquad\square$

8. HYPERSURFACES, BEND LOCI, AND TROPICAL BASES

Let $\nu : R \to S$ be a valued ring and $f \in R[x_1,\ldots,x_n]$. This data gives rise to two subschemes of $\mathbb{A}_S^n$ that are distinct in general. The first is the bend locus $\mathcal{B}end(\nu(f))$ of the coefficient-wise valuation of f — it is cut out by the bend relations of the single polynomial $\nu(f)$. The second is the tropicalization $\mathcal{T}rop(V(f))$ of the classical hypersurface determined by f — it is cut out by the bend relations of the coefficient-wise valuations of all elements in the principal ideal (f).

In this section we compare $\mathcal{B}end(\nu(f))$ and $\mathcal{T}rop(V(f))$ as schemes. The S-points of each coincide (at least when $S = \mathbb{T}$ — see [MS15, Example 2.5.5]), so the set-theoretic bend locus of $\nu(f)$ is equal to the set-theoretic tropicalization of $V(f)$; in the literature this set is usually referred to as the tropical hypersurface of $\nu(f)$. However, as schemes they are generally different, though they do sometimes agree, such as when f is a monomial or binomial (see Proposition 8.1.3).

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[5]Maclagan and Rincón have now extended this result from hypersurfaces to arbitrary irreducible projective varieties [MR14].

If R is a field then the discrepancy between these two schemes can be understood in terms of Theorem 7.1.6: the tropicalization of a projective hypersurface must have enough relations in its homogeneous coordinate algebra to yield the Hilbert polynomial of a codimension one subscheme, but the bend relations of a single tropical polynomial do not typically suffice for this numerical constraint.

This discussion leads naturally to the notion of a scheme-theoretic tropical basis, a term we introduce as a replacement for the usual set-theoretic notion considered in the tropical literature (e.g., [MS15, §2.5]).

8.1. **Bend loci versus tropical hypersurfaces.** Let A be an integral $\mathbb{F}_1$-algebra, $v : R \to S$ a valued ring, and $f \in A \otimes R$. With a slight abuse of notation, we will write $trop(f)$ for $trop(-)$ applied to the principal ideal (f). We are concerned with comparing the congruences $\langle \mathcal{B}(v(f)) \rangle$ and $\mathcal{B}\,trop(f)$. The former defines the bend locus $\mathcal{B}end(v(f))$, while the latter defines the tropicalization of the hypersurface $V(f)$. To illustrate that $\mathcal{T}rop(V(f))$ can be strictly smaller than $\mathcal{B}end(v(f))$ consider the following example.

Example 8.1.1. Let R be a ring equipped with the trivial valuation $v : R \to \mathbb{B}$, and let $f = x^2 + xy + y^2 \in R[x,y]$. One can see as follows that the congruence $\mathcal{B}\,trop(f)$ is strictly larger than the semiring congruence $\langle \mathcal{B}(v(f)) \rangle$. This latter congruence is generated by the degree 2 relations $x^2 + y^2 \sim x^2 + xy \sim xy + y^2$. The degree 3 part of $\langle \mathcal{B}(v(f)) \rangle$ is generated (as a module congruence) by the relations $\mathcal{B}(x^3 + x^2y + xy^2)$ and $\mathcal{B}(x^2y + xy^2 + y^3)$. If g,h are polynomials in $\mathbb{B}[x,y]$, then gh and $g + h$ each have at least as many monomial terms as g, and from this observation it follows that any nontrivial degree 3 relation in $\langle \mathcal{B}(v(f)) \rangle$ involves only polynomials with at least 2 terms. However, $(x-y)f = x^3 - y^3$, and this gives the degree 3 monomial relation $x^3 \sim y^3$ in $\mathcal{B}\,trop(f)$. This behavior, where $\mathcal{B}(v(f))$ does not generate all the relations $\mathcal{B}\,trop(f)$, appears to be generic. Suppose now that $f = x^2 + xy + ty^2$ for some $t \neq 0, 1$. The degree 3 part of $\langle \mathcal{B}(v(f)) \rangle$ is generated as a module congruence by the bend relations of $v(xf) = x^3 + x^2y + v(t)xy^2$ and $v(yf) = x^2y + xy^2 + v(t)y^3$. However, in $\mathcal{B}\,trop(f)$ one also has the bend relations of $v((x - ty)f) = x^3 + v(1-t)x^2y + v(t)^2y^3$; among these is the relation

$$x^3 + v(t)^2y^3 \sim x^3 + v(1-t)x^2y$$

which cannot be obtained from $\mathcal{B}(v(xf))$ and $\mathcal{B}(v(yf))$. In fact, one can check that these relations now generate all relations in the degree 3 part of $\mathcal{B}\,trop(f)$.

In general, when passing from $\langle \mathcal{B}(v(f)) \rangle$ to $\mathcal{B}\,trop(f)$, the additional relations appearing in $\mathcal{B}\,trop(f)$ are not uniquely determined by the single tropical polynomial $v(f)$, so the tropicalization of a hypersurface is not uniquely determined by the bend locus of the valuation of a defining polynomial. The following is a simple example illustrating this: two polynomials with the same valuation but whose associated hypersurfaces have distinct tropicalizations as schemes.

Example 8.1.2. Let $k = \mathbb{C}$ with the trivial valuation $v : k \to \mathbb{B}$, and consider the polynomials in $\mathbb{C}[x,y]$,

$$f = a_1x^2 + a_2xy + a_3y^2, \text{ and } g = x^2 + xy + y^2,$$

where the coefficients in f do not satisfy the quadratic relation $a_2^2 = a_1a_3$. Clearly $v(f) = v(g)$, and as seen in Example 8.1.1, $\mathcal{B}\,trop(g)$ contains the relation $x^3 \sim y^3$. However, for any nonzero linear form $h = b_1x + b_2y \in \mathbb{C}[x,y]$ the polynomial fh has at least three terms, so $\mathcal{B}\,trop(f)$ cannot contain the relation $x^3 \sim y^3$.

There are, however, certain nice situations where the tropicalization of an ideal is equal to the intersection of the bend loci of a set of generators of the ideal.

Proposition 8.1.3. *Let A be a torsion-free integral monoid-with-zero, and suppose S is totally ordered. If $f = ax + by$ is a binomial ($a,b \in R$, and $x,y \in A$) then $\mathcal{B}\,trop(f) = \langle \mathcal{B}(v(f)) \rangle$.*

Proof. We must show that $\langle \mathcal{B}(\nu(f)) \rangle$ implies $\langle \mathcal{B}(\nu(fg)) \rangle$ for any $g \in A \otimes R$. Since f is a binomial, $\langle \mathcal{B}(\nu(f)) \rangle$ is generated by the single relation $\nu(a)x \sim \nu(b)y$.

We define a binary relation '$\rightarrow$' on the set $\mathrm{supp}(g)$ as follows: $z_1 \rightarrow z_2$ if $z_1 x = z_2 y$. This generates an equivalence relation; let $\{C_i\}$ be the set of equivalence classes. Note that $C_i x \cup C_i y$ is necessarily disjoint from $C_j x \cup C_j y$ if $i \neq j$. Hence we can, without loss of generality, assume that $\mathrm{supp}(g)$ consists of just a single equivalence class C. If C consists of a single element then the claim holds trivially, so we assume that C consists of at least 2 elements.

Since A is integral and torsion free, C must consist of a sequence of elements $z_1, \ldots, z_n$ such that $z_i x = z_{i+1} y$ (having a loop would imply that xy^{-1} is a torsion element in the group completion of A, and the integral condition implies that if $x \rightarrow y$ and $x \rightarrow y'$ then $y = y'$).

Let c_i be the coefficient of z_i in g. We then have

$$\nu(fg) = \nu(ac_n)z_n x + \nu(ac_{n-1} + bc_n)z_{n-1}x + \cdots + \nu(ac_1 + bc_2)z_1 x + \nu(bc_1)z_1 y.$$

We first show that the relation $\nu(a)x \sim \nu(b)y$ allows the first term, $\nu(ac_n)z_n x$, to be absorbed into one of the terms to its right. First,

$$\nu(ac_n)z_n x \sim \nu(bc_n)z_n y = \nu(bc_n)z_{n-1}x.$$

Either $\nu(ac_{n-1} + bc_n) = \nu(ac_{n-1}) + \nu(bc_n)$, in which case we are done, or $\nu(ac_{n-1}) = \nu(bc_n)$, in which case $\nu(bc_n)z_{n-1}x = \nu(ac_{n-1})z_{n-1}x \sim \nu(bc_{n-1})z_{n-2}x$. We continue in this fashion until either the term $\nu(ac_n)z_n x$ absorbs or we reach the end of the chain, at which point it will be absorbed into the final term $\nu(bc_1)z_1 y$. Working from right to left instead, the final term can be absorbed into the terms to its left by the same argument.

Finally, given a middle term, $\nu(ac_{i-1} + bc_i)z_{i-1}x$, we have that $\nu(ac_{i-1})z_{i-1}x$ and $\nu(bc_i)z_{i-1}x$ are both larger, and so the above argument in reverse allows us to replace the term $\nu(ac_{i-1} + bc_i)z_{i-1}x$ with $\nu(ac_{i-1})z_{i-1}x + \nu(bc_i)z_{i-1}x$. Then the above argument in the forward direction allows these two terms to be absorbed into the terms to the right and left respectively. $\square$

Lemma 8.1.4. *Let A be an integral $\mathbb{F}_1$-algebra, $\nu : R \rightarrow S$ a valued ring, and I an ideal in $A \otimes R$ generated by elements $f_1, \ldots, f_n$. If $\mathcal{B}\mathrm{trop}(I) = \langle \mathcal{B}(\nu(f_i)) \rangle_{i=1\ldots n}$, and J is the ideal generated by I together with a monomial $f_0 \in A$, then*

$$\mathcal{B}\mathrm{trop}(J) = \langle \mathcal{B}(\nu(f_i)) \rangle_{i=0\ldots n}.$$

Proof. We will show that the generating relations of $\mathcal{B}\mathrm{trop}(J)$ are all contained in the subcongruence $\langle \mathcal{B}(\nu(f_i)) \rangle_{i=0\ldots n}$. Let $g = \sum_{i=0}^{n} h_i f_i \in J$, with $h_i \in A \otimes R$. Since $\langle \mathcal{B}(f_0) \rangle = \langle f_0 \sim 0_S \rangle$, for any $F \in A \otimes S$, the congruence $\langle \mathcal{B}(f_0) \rangle$ contains the relation $F \cdot f_0 \sim 0_S$. This means that if $F, F' \in A \otimes S$ have identical coefficients away from the set of monomials $f_0 \cdot A$, then the relation $F \sim F'$ is contained in $\langle \mathcal{B}(f_0) \rangle$.

Consider the polynomials $F = \nu(g)$ and $F' = \nu(\sum_{i=1}^{n} h_i f_i)$ in $A \otimes S$; they differ only outside of $f_0 \cdot A$, as do $F_{\widehat{j}}$ and $F'_{\widehat{j}}$. In $\langle \mathcal{B}(\nu(f_i)) \rangle_{i=0\ldots n}$ we thus have the relations

$$\begin{aligned}
F &\sim F' & &\text{from } \mathcal{B}(f_0) \\
&\sim F'_{\widehat{j}} & &\text{from } \mathcal{B}\mathrm{trop}(I) = \langle \mathcal{B}(\nu(f_i)) \rangle_{f=1\ldots n} \\
&\sim F_{\widehat{j}} & &\text{from } \mathcal{B}(f_0),
\end{aligned}$$

This completes the proof. $\square$

If f_0 is instead a binomial then the analogue of the above lemma can fail.

Example 8.1.5. Consider $f_1 = x - y$ and $f_0 = x + y$ in $R[x, y]$ and the trivial valuation $v : R \to \mathbb{B}$. By Proposition 8.1.3, $\mathcal{B}trop(f_1) = \langle \mathcal{B}(v(f_1)) \rangle$. However, $\langle \mathcal{B}(v(f_0)), \mathcal{B}(v(f_1)) \rangle = \langle x \sim y \rangle$ is not the tropicalization of the ideal (f_0, f_1), since the latter contains the bend relation of $v(f_0 + f_1) = x$, namely $\langle x \sim -\infty \rangle$, which is not implied by the former.

8.2. Tropical bases.

It is well-known that the set-theoretic tropicalization of the variety defined by an ideal is not necessarily equal to the intersection of the (set-theoretic) tropical hypersurfaces associated with a set of generators of this ideal. A set of generators for which this holds is called a *tropical basis* in [MS15, §2.5], where this notion is studied and related to Gröbner theory. We use the term *set-theoretic tropical basis* for this concept to distinguish it from the following notion of tropical basis that arises when considering scheme-theoretic tropicalization.

Definition 8.2.1. Let $v : R \to S$ be a valued ring, X a locally integral $\mathbb{F}_1$-scheme, and $Z \subset X_R$ a closed subscheme. A *scheme-theoretic tropical basis* for Z is a set $\{Y_1, Y_2, \ldots\}$ of hypersurfaces in X_R containing Z such that the following scheme-theoretic intersections hold:

$$Z = \bigcap_i Y_i \quad \text{and} \quad \mathit{Trop}(Z) = \bigcap_i \mathit{Trop}(Y_i).$$

In the affine case, say $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} A \otimes R/I$, a scheme-theoretic tropical basis is a generating set $\{f_1, f_2, \ldots\}$ for the ideal I such that the corresponding congruences $\mathcal{B}trop(f_i)$, obtained by tropicalizing the principal ideals (f_i), generate the congruence $\mathcal{B}trop(I)$. Note that this is generally a weaker requirement than the requirement that the bend relations of the f_i generate $\mathcal{B}trop(I)$. For instance, for a principal ideal $I = (f)$ it is automatic that $\{f\}$ is a scheme-theoretic tropical basis, whereas it is not always the case, as discussed above in §8.1 and Example 8.1.1, that $\mathcal{B}trop(I) = \langle \mathcal{B}(v(f)) \rangle$.

Not surprisingly, being a scheme-theoretic tropical basis is a stronger requirement than being a set-theoretic tropical basis.

Example 8.2.2. Let $R = k[x, y, z]$ with the trivial valuation. As discussed in [MS15, Example 3.2.2], the elements $x + y + z$ and $x + 2y$ do not form a tropical basis for the ideal I they generate, since $y - z \in I$ tropically yields the relation $y \sim z$ which is not contained in $\langle \mathcal{B}(x + y + z), \mathcal{B}(x + y) \rangle$. This can be rectified by adding the element $y - z$, and indeed these three polynomials form a set-theoretic tropical basis for I. However, if we instead add the element $(y - z)^2 \in I$ then the corresponding congruence has the same $\mathbb{T}$-points, so this is still a set-theoretic tropical basis, but it is no longer a scheme-theoretic tropical basis since the relation $y \sim z$ is still missing.

Remark 8.2.3. It is known that subvarieties of affine space defined over an algebraically closed field with non-trivial valuation admit *finite* set-theoretic tropical bases (see [SS04, Corollary 2.3] and [MS15, Corollary 3.2.3]). It would be interesting to see if this also holds scheme-theoretically.

TABLE OF NOTATION

Symbol	Description	Reference
$\ker \varphi$	the congruence kernel $S \times_R S$ of a semiring homomorphism $\varphi : S \to R$	Definition 2.4.3
$\langle J \rangle$	semiring congruence on S generated by a set of pairs $J \subset S \times S$	Lemma 2.4.5
$\mathcal{B}(f)$	module congruence of "bend relations" of a tropical polynomial f	Definition 5.1.1
$bend(f)$	set-theoretic bend locus ("tropical hypersurface") of a tropical polynomial f	Definition 5.1.1
$\mathcal{B}end(f)$	scheme-theoretic bend locus of a tropical polynomial f	Definition 5.4.2
$\nu(f)$	tropical polynomial obtained by coefficient-wise valuation of a polynomial f	§6.1
$trop(I)$	tropical ideal associated to an ideal I in a valued ring	Proposition 6.1.1
$trop(Z)$	set-theoretic tropicalization of a subvariety Z	Eq. (6.0.1)
$\mathcal{T}rop(Z)$	scheme-theoretic tropicalization of a subscheme Z	Definition 6.1.3
$\dim_S M$	maximal cardinality of an independent set in an S-module M	Definition 7.1.1
$M^\vee$	S-linear dual $\mathrm{Hom}_S(M, S)$ of an S-module	§5.2
ν_{univ}^R	universal valuation on a ring R, with values in semiring S_{univ}^R	§2.5
$\mathcal{V}al(R)$	moduli space of valuations on a ring R	§6.5

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