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RESEARCH ARTICLE

A comparison of two no-arbitrage conditions

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Abstract We give a comparison of two no-arbitrage conditions for the fundamental theorem of asset pricing. The first condition is named as the no free lunch with vanishing risk condition and the second the no good deal condition. We aim to derive a relationship between these two conditions.

Keywords No free lunch with vanishing risk condition, no good deal condition, extension theorem, fundamental theorem, equivalent martingale measures
MSC 60H30, 91G10

1 Introduction

Due to the seminal work [5,6] by Delbaen and Schachermayer, the fundamental theorem of asset pricing became pivotal in mathematical finance, which is a key result in establishing a mathematical framework for pricing and the key condition in the so-called the *no free lunch with vanishing risk condition* [7]. Since then, many investigations are devoted to generalize this remarkable condition to cover more general situations in the mathematical modelings, cf., e.g., [1,3,8,9,11] and references therein. Most recently, Bion-Nadal and Di Nunno [2] proposed a new condition for pricing in incomplete markets. This condition is named as the *no good deal condition*, which should be linked to the celebrated the no free lunch with vanishing risk condition. The objective of the present paper is to compare these two conditions in some simplified models. We aim to seek certain links between the free lunch with vanishing risk condition and the no good deal condition by explicating them into several simple models so that one can compare them more concretely. Our discussion reveal (theoretically) the essential properties of these models from stochastic analysis viewpoint.

The rest of the paper is organized as follows. In the next section, we start with the basic concepts of the First and Second Fundamental Theorems of asset

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pricing in the continuous model. Then a general continuous market model is defined and the fundamental theorems of asset pricing are proved in this setting. In the latter scenario, we focus on conditions of the model which satisfies no free lunch with vanishing risk condition. Tools from probability such as martingale, equivalent martingale measure, stochastic integrals, and Girsanov transformation are all used in this framework. In Section 3, we present a complete comparison with a thorough derivation.

2 Preliminaries on two no-arbitrage conditions

This section is devoted to explicate the two no-arbitrage conditions in a unified and convenient framework.

2.1 No free lunch with vanishing risk condition

Throughout this paper, we fix any $T>0$. Let $(\Omega,\mathscr{F},\{\mathscr{F}_t\}_{t\in[0,T]},P)$ be a given (complete) filtered probability space. Here as usual, the filtration $(\mathscr{F}_t)_{t\in[0,T]}$ is assumed to be right-continuous and $\mathscr{F}_0$ contains all P-null sets. We are concerned with a market model in which $d+1$ assets are priced at time $t\in[0,T]$ with $d\in\mathbb{N}$ and T being interpreted as the terminal time. The assets are classified into two categories—risky stocks and riskless bonds. Here in our model, we consider d risky stocks and denote their price dynamics by a d-dimensional stochastic process

$$\boldsymbol{S}_t=(S_t^1,\dots,S_t^d)_{t\in[0,T]}.$$

For the riskless bonds, for the simplicity, we only consider one bond which is denoted by S_t^0 with a fixed interest rate $r>0$, namely, $S_t^0=S_0^0\mathrm{e}^{rt}$, $t\geqslant 0$ with given initial capital $S_0^0>0$.

Let

$$\overline{\boldsymbol{S}}_t=(S_t^0,S_t^1,\dots,S_t^d)_{t\in[0,T]}$$

denote the corresponding price processes for this multi asset, which can be viewed as a vector-valued stochastic process. In general, we take $\overline{\boldsymbol{S}}_t$ to be a semi-martingale on the given filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\in[0,T]},P)$. The price of the i^{th} asset at time t is modeled as non-negative random variable S_t^i. We assume that $(S_t^1,\dots,S_t^d)_{t\in[0,T]}$ is $\{\mathscr{F}_t\}$-adapted.

Recall a trading strategy is an $\{\mathscr{F}_t\}$-predictable $\mathbb{R}^{d+1}$-valued process

$$\overline{\boldsymbol{\xi}}_t=(\xi_t^0,\boldsymbol{\xi}_t)=(\xi_t^0,\xi_t^1,\dots,\xi_t^d)_{t\in[0,T]}.$$

The value ξ_t^i of a trading strategy $\overline{\boldsymbol{\xi}}_t$ corresponds to the quantity of assets of the i^{th} asset held at time t. We simplify computation to use discounted asset price processes. For $i=0,1,\dots,d$, we define

$$S_t^{*,i}:=\frac{S_t^i}{S_t^0},\quad t\in[0,T].$$

Then

$$\overline{\boldsymbol{S}}_t^* = (S_t^{*,0}, S_t^{*,1}, \dots, S_t^{*,d})$$

is the value of the vector of discounted assets prices at time t. Note that we have

$$S_t^{*,0} \equiv 1, \quad \forall\, t \in [0,T].$$

Next, let us give some definitions.

Definition 1 [14, Lemma 4.2.1] A trading strategy

$$\overline{\boldsymbol{\xi}}_t = (\xi_t^0, \xi_t^1, \dots, \xi_t^d)_{t\in[0,T]}$$

is called self-financing if the discounted value process

$$V_t^* := \overline{\boldsymbol{\xi}}_t \cdot \overline{\boldsymbol{S}}_t^* = \sum_{i=0}^{d} \xi_t^i S_t^{*,i}$$

is a continuous, $\{\mathscr{F}_t\}$-adapted process such that for each $t \in [0,T]$,

$$V_t^* = \xi_0^0 + \sum_{i=1}^{d} \xi_0^i S_0^{*,i} + \sum_{i=1}^{d} \int_0^t \xi_s^i \mathrm{d}S_s^{*,i} =: V_0^* + G_t^*, \quad P\text{-a.s.} \tag{1}$$

or equivalently in stochastic differential formulation

$$\mathrm{d}V_t^* = \mathrm{d}G_t^* = \sum_{i=1}^{d} \xi_t^i \mathrm{d}S_t^{*,i}, \quad P\text{-a.s.}$$

with initial data

$$V_0^* := \xi_0^0 + \sum_{i=1}^{d} \xi_0^i S_0^{*,i}.$$

Here,

$$G_t^* := \sum_{i=1}^{d} \int_0^t \xi_s^i \mathrm{d}S_s^{*,i}$$

is the corresponding discounted gain process.

Definition 2 A self-financing trading strategy $\overline{\boldsymbol{\xi}}_t$ is called an arbitrage opportunity if V_t^* satisfies the following conditions:

(i) $V_0^* = 0$;

(ii) $\exists$ a constant a_0 such that $P(\{\omega \in \Omega \colon V_t^*(\omega) \geqslant a_0, \forall\, t \in [0,T]\}) = 1$;

(iii) $V_T^* \geqslant 0$, P-a.s.;

(iv) $P(V_T^* > 0) > 0$.

Moreover, a model satisfies the no-arbitrage condition if such a strategy does not exist.

It turns out that in the continuous-time setting, the no-arbitrage condition does not guarantee the existence of an equivalent local martingale measure (see [5, Example 7.7]). A stronger condition is needed. The following modification of the no-arbitrage condition was introduced by Delbaen and Schachermayer [5,6] and further considered by Shiryaev and Cherny [12].

Definition 3 [12, Definition 1.6] We say that a sequence of self-financing trading strategies $\{\overline{\boldsymbol{\xi}}_t^k : t \in [0,T]\}_{k\in\mathbb{N}}$ realises free lunch with vanishing risk condition, if the corresponding sequence of value processes $\{V_t^{*,k} : t \in [0,T]\}_{k\in\mathbb{N}}$ fulfills that for each $k \in \mathbb{N}$,

(i) $V_0^{*,k} = 0$;

(ii) there exists a constant a_k such that

$$P(\{\omega \in \Omega : V_t^{*,k}(\omega) := \overline{\boldsymbol{\xi}}_t(\omega) \cdot \overline{\boldsymbol{S}}_t^{*,k}(\omega) \geqslant a_k, \forall\, t \in [0,T]\}) = 1;$$

(iii) $V_T^{*,k} \geqslant -1/k$, P-a.s.;

(iv) $\exists$ constants $\delta_1, \delta_2 > 0$ (independent of k) such that, $\forall\, k \in \mathbb{N}$,

$$P\{V_T^{*,k} > \delta_1\} > \delta_2.$$

Moreover, a model satisfies the no free lunch with vanishing risk condition if such a sequence of strategies does not exist.

Furthermore, we introduce the following definition.

Definition 4 We say that a sequence of self-financing trading strategies $\{\overline{\boldsymbol{\xi}}_t^k : t \in [0,T]\}_{k\in\mathbb{N}}$ fulfills the free lunch with bounded risk if it satisfies conditions (i) and (ii) of Definition 3 as well as the following two conditions:

(a) there exists a constant a such that, for each $k \in \mathbb{N}$,

$$P(\{\omega \in \Omega : V_t^{*,k}(\omega) \geqslant a, \forall\, t \in [0,T]\}) = 1;$$

(b) there exist constants $\delta_1, \delta_2 > 0$ such that, for each k,

$$P\{V_T^{*,k} > \delta_1\} > \delta_2,$$

and for any $\delta > 0$,

$$\lim_{k\to\infty} P\{V_T^{*,k} < -\delta\} = 0.$$

A model satisfies the no free lunch with bounded risk condition if such a sequence of strategies does not exist.

Theorem 5 [5, Theorem 1.1] (Fundamental Theorem of Asset Pricing) *Assume that the asset price process $\overline{\boldsymbol{S}}_t$ is a locally bounded, $(d+1)$-dimensional vector-valued semi-martingale. Then there exists an equivalent local martingale measure for $\overline{\boldsymbol{S}}_t$ if and only if the no free lunch with vanishing risk condition holds.*

2.2 No good deal condition

Here, we will focus on the no good deal condition. With the same preamble as before, we work on the same probability set-up $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, P)$. Following Biog-Nasal and Di Nunno [2], we assume that the given $\{\mathscr{F}_t\}_{t\in[0,T]}$ satisfies that $\mathscr{F}_T = \mathscr{F}$. We work in an L_∞-framework and consider claims as elements of the space

$$L_\infty(\mathscr{F}_t) := L_\infty(\Omega, \mathscr{F}_t, P)$$

of random variables with finite norm

$$\|X\|_\infty := \operatorname{ess\,sup}|X|, \quad X \in L_\infty(\mathscr{F}_t).$$

For any time $t \in [0,T]$, let $L_t \subseteq L_\infty(\mathscr{F}_t)$ denote the linear subspace representing all market claims that are payable at time t. Note that on a complete market, $L_t = L_\infty(\mathscr{F}_t)$. For a given asset $X \in L_t$, we denote the systems of prices by x_{st}, $0 \leqslant s \leqslant t \leqslant T$. We assume that price $x_{st}(X)$, $0 \leqslant s \leqslant t \leqslant T$, for marked assets $X \in L_t$ are given and we describe them in axiomatic form, where $x_{st}(X)$ denote the price of asset X from s to t . Here, we set the bounds on prices:

$$m_{st}(X) \leqslant x_{st}(X) \leqslant M_{st}(X),$$

and we study the existence of a pricing measures P_0 that allows a linear representation

$$x_{st}(X) = E_{P_0}[X|\mathscr{F}_s], \quad X \in L_t,$$

fulfilling the given bounds. The pricing measure P_0 will reflect the choices of bounds. When we use + in the notation of space, we refer to the corresponding cone of the non-negative elements.

Next, we consider no good deal pricing measures. The good deal bound is a way to restrict the choice of equivalent martingale measures, usually denoted by Q, in incomplete markets. The idea is to consider martingale measures that not only rule out arbitrage possibilities, but also deals with 'too good to be true'. As usual, we work with general price systems and not with specific price dynamics.

Following Chicharee and Sa Requejo [4], a good deal of level $\delta > 0$ is a non-negative $\mathscr{F}_T$-measurable payoff X such that

$$\frac{E(X) - E_Q(X)}{\sqrt{\operatorname{Var}(X)}} \geqslant \delta.$$

Accordingly, a probability measure Q is equivalent to P is a no good deal pricing measure if there are no good deals of level δ under Q, i.e.,

$$E_Q[X] \geqslant E[X] - \delta\sqrt{\operatorname{Var}(X)}, \quad X \geqslant 0. \tag{2}$$

Note that (2) holds for all $X \in L_\infty(\mathscr{F}_T)$ as we have

$$X + \|X\|_\infty \geqslant 0.$$

Hence, also the relation

$$E_Q[X] \leqslant E[X] + \delta\sqrt{\mathrm{Var}(X)}, \quad X \geqslant 0$$

holds true for all $X \in L_\infty(\mathscr{F}_T)$. This motivates the following extended general definition of no good deals pricing measure.

Definition 6 [2, Definition 6.1] A probability measure Q (equivalent to P) is called a no good deal pricing measure if there exists a $\delta > 0$ such that there is no good deals of level $\delta > 0$ under Q (equivalently, for any $\delta > 0$, there is no good deal of level δ), i.e.,

$$-\delta \leqslant \frac{E(X) - E_Q(X)}{\sqrt{\mathrm{Var}(X)}} \leqslant \delta$$

for all $X \in L_2(\mathscr{F}_T, P) \cap L_1(\mathscr{F}_T, Q)$.

3 Comparing two no-abitrage conditions and further discussion

Let $d, m \in \mathbb{N}$ be fixed. We consider the following stochastic differential equation on $[0,T] \times \mathbb{R}^d$:

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t, \tag{3}$$

where

$$\mu\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d, \quad \sigma\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^{d\otimes m},$$

and W_t is an m-dimensional Brownian motion. Under the usual linear growth condition and the following Lipschitz condition:

$$|\mu(t,x)| + \|\sigma(t,x)\| \leqslant C_t(1 + |x|), \quad x \in \mathbb{R}^d,\ t \in [0,T],$$
$$|\mu(t,x) - \mu(t,y)| + \|\sigma(t,x) - \sigma(t,y)\| \leqslant C_t|x - y|, \quad x, y \in \mathbb{R}^d,\ t \in [0,T],$$

for some function $C_t > 0$ on $t \in [0,T]$, (3) admits a unique strong solution $(X_t)_{t\geqslant 0}$ for a given initial data $X_0 \in \mathbb{R}^d$. In the sequel, we will also use the following integral formulation for the process X_t:

$$X_t = X_0 + \int_0^t \mu(u, X_u)\mathrm{d}u + \int_0^t \sigma(u, X_u)\mathrm{d}W_u, \quad t \in [0,T].$$

In the special case, let $(S_t)_{t\in[0,T]}$ be the (one-dimensional) risky asset price process satisfying the following Black-Scholes pricing dynamics:

$$\frac{\mathrm{d}S_t}{S_t} = \mu\mathrm{d}t + \sigma\mathrm{d}W_t$$

along with a bank account $\mathrm{d}S_t^0 = rS_t^0\mathrm{d}t$ with interest rate $r > 0$ and the regularised initial capital $S_0^0 = 1$, where μ and σ are positive constants. Given initial data $S_0 > 0$, the risky asset S_t is determined uniquely by the above equation, and S_t is given explicitly by

$$S_t = S_0 \exp\Big[\Big(\mu - \frac{1}{2}\sigma^2\Big)t + \sigma W_t\Big].$$

The discounted price process is given by $S_t^* = S_t/S_t^0$. We then have discounted price

$$S_t^* = S_0 \exp\Big[\Big(\mu - r - \frac{1}{2}\sigma^2\Big)t + \sigma W_t\Big]. \tag{4}$$

Now, applying the Itô formula to (4), we have

$$\mathrm{d}S_t^* = (\mu - r)S_t^*\mathrm{d}t + \sigma S_t^*\mathrm{d}W_t. \tag{5}$$

Let us clarify a bit here that our asset price processes here is $\overline{\boldsymbol{S}}_t = (S_t^0, S_t)$ and according to Definition 1 (with $d = 1$), the discounted value process for a self-financing trading strategy $\overline{\boldsymbol{\xi}}_t = (\xi_t^0, \xi_t)$ is

$$V_t^* = V_0^* + \int_0^t \xi_s \mathrm{d}S_s^*, \quad t \in [0, T].$$

Since we are concerned with comparing the two no-arbitrage conditions, we simply take $V_0^* = \mathrm{e}^{rT}$ in our later derivations. This is just for the sake to make the comparison conditions neat in our conclusions. Of course, one can take any non-negative constant as the initial V_0^* and then by a constant shift to e^{rT}. By (5), the terminal value for self-financing trading strategy $\overline{\boldsymbol{\xi}}_t$ is then

$$\begin{aligned}
V_T^* &= \mathrm{e}^{rT} + \int_0^T \xi_t \mathrm{d}S_t^* \\
&= \mathrm{e}^{rT} + \int_0^T \xi_t S_t^* \{(\mu - r)\mathrm{d}t + \sigma \mathrm{d}W_t\} \\
&= \mathrm{e}^{rT} + (\mu - r)\int_0^T \xi_t S_t^* \mathrm{d}t + \sigma \int_0^T \xi_t S_t^* \mathrm{d}W_t \\
&= \mathrm{e}^{rT} + (\mu - r)S_0 \int_0^T \xi_t \mathrm{e}^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma W_t}\mathrm{d}t \\
&\quad + \sigma S_0 \int_0^T \xi_t \mathrm{e}^{(\mu - r - \frac{1}{2}\sigma^2)t + \sigma W_t}\mathrm{d}W_t.
\end{aligned}$$

Next, let

$$f(t, x) := \frac{1}{\sigma}\,\xi_t \mathrm{e}^{(\mu - r - \frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma x}.$$

Then we have

$$\frac{\partial f}{\partial x} = \xi_t \mathrm{e}^{(\mu - r - \frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma x}, \quad \frac{\partial^2 f}{\partial x^2} = \sigma \xi_t \mathrm{e}^{(\mu - r - \frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma x},$$

$$\frac{\partial f}{\partial t} = \frac{1}{\sigma}\,(\mu - r - \frac{1}{2}\,\sigma^2)\xi_t \mathrm{e}^{(\mu - r - \frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma x} + \frac{1}{\sigma}\,\mathrm{e}^{(\mu - r - \frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma x}\frac{\partial \xi_t}{\partial t},$$

where we assumed that ξ_t is differentiable with respect to t. Clearly, in terms of the Itô formula [10, Theorem 4.2.1],

$$\begin{aligned}
&\int_0^T \frac{\partial f}{\partial x}(t, W_t)\mathrm{d}W_t \\
&= f(T, W_T) - f(0, W_0) - \int_0^T \Big[\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, W_t) + \frac{\partial f}{\partial t}(t, W_t)\Big]\mathrm{d}t \\
&= \frac{1}{\sigma}\xi_T \mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)T}\mathrm{e}^{\sigma W_T} - \frac{1}{\sigma}\xi_0 - \int_0^T \Big[\frac{1}{2}\sigma\xi_t \mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma W_t} \\
&\quad + \frac{1}{\sigma}(\mu - r - \frac{1}{2}\sigma^2)\xi_t \mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma W_t} + \frac{1}{\sigma}\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma W_t}\frac{\partial \xi_t}{\partial t}\Big]\mathrm{d}t \\
&= \frac{1}{\sigma}\xi_T \mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)T}\mathrm{e}^{\sigma W_T} - \frac{1}{\sigma}\xi_0 \\
&\quad - \int_0^T \Big[\frac{1}{\sigma}(\mu - r)\xi_t + \frac{1}{\sigma}\frac{\partial \xi_t}{\partial t}\Big]\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma W_t}\mathrm{d}t.
\end{aligned}$$

Thus, we get

$$\begin{aligned}
V_T^* &= \mathrm{e}^{rT} + (\mu - r)S_0\int_0^T \xi_t \mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t+\sigma W_t}\mathrm{d}t + \sigma S_0\Big\{\frac{1}{\sigma}\xi_T \mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)T}\mathrm{e}^{\sigma W_T} \\
&\quad -\frac{1}{\sigma}\xi_0 - \int_0^T \Big[\frac{1}{\sigma}(\mu - r)\xi_t + \frac{1}{\sigma}\frac{\partial \xi_t}{\partial t}\Big]\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma W_t}\mathrm{d}t\Big\} \\
&= \mathrm{e}^{rT} + S_0\xi_T \mathrm{e}^{(\mu-\frac{1}{2}\sigma^2)T}\mathrm{e}^{\sigma W_T} - S_0\int_0^T \frac{\partial \xi_t}{\partial t}\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma W_t}\mathrm{d}t \\
&= \int_0^T \Big(\frac{1}{T}\mathrm{e}^{rT} + \frac{1}{T}S_0\xi_T \mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)T}\mathrm{e}^{\sigma W_T} - S_0\frac{\partial \xi_t}{\partial t}\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma W_t}\Big)\mathrm{d}t. \quad (6)
\end{aligned}$$

Now, we set

$$X(t) := \frac{1}{T}\mathrm{e}^{rT} + \frac{1}{T}S_0\xi_T \mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)T}\mathrm{e}^{\sigma W_T} - S_0\frac{\partial \xi_t}{\partial t}\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma W_t}.$$

Then by Definition 3 (iv), there are constants $\delta_1, \delta_2 > 0$ such that

$$P\Big\{\int_0^T X(t)\mathrm{d}t > \delta_1\Big\} > \delta_2.$$

For any $p > 0$, by Chebyshev's inequality,

$$\delta_2 \leqslant P\Big\{\Big|\int_0^T X(t)\mathrm{d}t\Big| > \delta_1\Big\} \leqslant \frac{E|\int_0^T X(t)\mathrm{d}t|^p}{\delta_1^p}.$$

For any $p, q > 1$ and $\frac{1}{p} + \frac{1}{p} = 1$,

$$E\Big(\Big|\int_0^T X(t)\mathrm{d}t\Big|^p\Big) \leqslant T^{p/q}\int_0^T E(|X(t)|^p)\mathrm{d}t.$$

More generally, when $p = q = 2$, in that case, we can get

$$\frac{\delta_2 \delta_1^2}{T} \leqslant \int_0^T E(X(t)^2)\mathrm{d}t.$$

Let us calculate $\int_0^T E(X(t)^2)\mathrm{d}t$. To make our calculation easier, we simply assume that $\xi_t = Ct$ for some constant $C > 0$. By the identity

$$E(\mathrm{e}^{\sigma W_t - \frac{1}{2}\sigma^2 t}) = 1,$$

we then have

$$\begin{aligned}
&\int_0^T E(X(t)^2)\mathrm{d}t \\
&= \int_0^T E\Big(\frac{1}{T}\,\mathrm{e}^{rT} + \Big(\frac{1}{T}\,S_0 \xi_T \mathrm{e}^{(\mu - r - \frac{1}{2}\sigma^2)T}\mathrm{e}^{\sigma W_T} - S_0 C \mathrm{e}^{(\mu - r - \frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma W_t}\Big)\Big)^2 \mathrm{d}t \\
&= \frac{1}{T}\,\mathrm{e}^{2rT} + S_0^2 C^2 \mathrm{e}^{(2(\mu - r) + \sigma^2)T}\Big(T - \frac{2}{\sigma^2 + (\mu - r)} + \frac{1}{\sigma^2 + 2(\mu - r)}\Big) \\
&\quad + 2S_0^2 C^2\,\frac{\mathrm{e}^{(\mu - r)T}}{\sigma^2 + (\mu - r)} - \frac{C^2 S_0^2}{\sigma^2 + 2(\mu - r)} \\
&\quad + 2S_0 C \mathrm{e}^{\mu T} - \frac{2S_0 C}{T(\mu - r)}\,\mathrm{e}^{\mu T} + \frac{2S_0 C}{T(\mu - r)}\,\mathrm{e}^{rT} \\
&=: I \\
&\geqslant \frac{\delta_1^2 \delta_2}{T}.
\end{aligned} \tag{7}$$

Now, we let

$$J := \mathrm{e}^{(2(\mu - r) + \sigma^2)T}\Big(T - \frac{3}{2(\sigma^2 + (\mu - r))}\Big) + \frac{2(\mu - r)T + 1}{\sigma^2 + 2(\mu - r)} + \frac{\mathrm{e}^{2rT}}{T S_0^2 C^2} + \frac{2\mathrm{e}^{\mu T}}{S_0 C}.$$

Clearly,

$$I \geqslant S_0^2 C^2 J.$$

Thus, if

$$J \geqslant \frac{\delta_1^2 \delta_2}{T C^2 S_0^2}, \tag{8}$$

then

$$\int_0^T E\left(X(t)\right)^2 \mathrm{d}t \geqslant \frac{\delta_2 \delta_1^2}{T}.$$

Obviously, (8) is stronger than (7).

On the other hand, towards the no good deal condition, we recall that the price is given by the system $\{x_{s,t}\}_{0 \leqslant s < t \leqslant T}$, instead of price process itself, $X := \{X_t\}_{0 \leqslant t \leqslant T}$, which is determined by (3). Therefore, with $d = m = 1$, for the coefficients $\mu\colon [0,T]\times\mathbb{R}\to\mathbb{R}$ and the non-degenerate $\sigma\colon [0,T]\times\mathbb{R}\to\mathbb{R}\backslash\{0\}$, we define

$$x_{s,t}(X) := \int_s^t \mu(u,X_u)\mathrm{d}u + \int_s^t \sigma(u,X_u)\mathrm{d}W_u. \tag{9}$$

Furthermore, we have (cf., e.g., [13])

$$X_t = X_0 + \int_0^t \sigma(u,X_u)\mathrm{d}\widetilde{W}_u, \quad t\in[0,T], \tag{10}$$

with

$$\widetilde{W}_t := W_t + \int_0^t \sigma^{-1}(u,X_u)\mu(u,X_u)\mathrm{d}u, \quad t\in[0,T].$$

Next, define Q via

$$\mathrm{d}Q := \mathrm{e}^{\int_0^T \sigma^{-1}(u,X_u)\mu(u,X_u)\mathrm{d}W_u - \frac{1}{2}\int_0^T[\sigma^{-1}(u,X_u)\mu(u,X_u)]^2\mathrm{d}u}\mathrm{d}P.$$

Then, by the Girsanov theorem, $\widetilde{W}_t$, $t\in[0,T]$, is a Q-Brownian motion.

Taking expectation on both sides of (10) yields

$$E_Q(X_t) = E_Q(X_0) + E_Q\int_0^t \sigma(s,X_s)\mathrm{d}\widetilde{W}_s = X_0$$

and

$$E(X_t) = X_0 + E\int_0^t \mu(s,X_s)\mathrm{d}s.$$

Therefore,

$$E(X_t) - E_Q(X_t) = E\int_0^t \mu(s,X_s)\mathrm{d}s. \tag{11}$$

Next, we turn to the special case of the (previously introduced) asset price process $\overline{\boldsymbol{S}}_t = (S_t^0, S_t)$ for

$$\mathrm{d}S_t^0 = rS_t^0\mathrm{d}t, \quad S_0^0 = 1,$$

and

$$\mathrm{d}S_t = \mu S_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \quad S_0 > 0,$$

with solutions $S_t^0 = \mathrm{e}^{rt}$ and the Black-Scholes price process

$$S_t = S_0\exp\Big[\Big(\mu - \frac{1}{2}\sigma^2\Big)t + \sigma W_t\Big], \quad t\in[0,T].$$

We have the associated $x_{s,t}$ defined as follows:

$$x_{s,t} := \int_s^t rS_u^0\mathrm{d}u + \int_s^t \mu S_u\mathrm{d}u + \int_s^t \sigma S_u\mathrm{d}W_u, \quad 0\leqslant s<t\leqslant T.$$

Recall from (4) and (5) that the discounted price process

$$S_t^* = \frac{S_t}{S_t^0} = S_0 \exp\Big[\Big(\mu - r - \frac{1}{2}\,\sigma^2\Big)t + \sigma W_t\Big]$$

and its equation

$$\mathrm{d}S_t^* = (\mu - r)S_t^*\mathrm{d}t + \sigma S_t^*\mathrm{d}W_t.$$

Recall further that, by (6), we have the discounted terminal value V_T^* for $\xi_t = Ct$:

$$\begin{aligned} V_T^* &= \int_0^T \Big(\frac{1}{T}\,\mathrm{e}^{rT} + CS_0\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)T}\mathrm{e}^{\sigma W_T} - CS_0\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma W_t}\Big)\mathrm{d}t \\ &= \mathrm{e}^{rT} + TCS_0\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)T}\mathrm{e}^{\sigma W_T} - \int_0^T CS_0\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t}\mathrm{e}^{\sigma W_t}\mathrm{d}t. \end{aligned}$$

It is clear that

$$V_T^* \in L_2(\mathscr{F}_T, P) \cap L_1(\mathscr{F}_T, Q).$$

Therefore, for the payoff $X := V_T^*$, by utilising the identity

$$E(\mathrm{e}^{\sigma W_t - \frac{1}{2}\sigma^2 t}) = 1,$$

we have

$$E(X) - E_Q(X) = E\Big(\int_0^T rS_t^0\mathrm{d}t + \int_0^T \mu S_t^*\mathrm{d}t\Big) \tag{12}$$

and

$$\mathrm{Var}(X) = E\Big(\int_0^T r^2(S_t^0)^2\mathrm{d}t + \int_0^T \sigma^2 S_t^{*2}\mathrm{d}t\Big). \tag{13}$$

Then, according to Definition 6 and by (12) and (13), the no good deal condition is fulfilled if

$$-\delta \leqslant \frac{E(\int_0^T rS_t^0\mathrm{d}t + \int_0^T \mu S_t^*\mathrm{d}t)}{\sqrt{E(\int_0^T r^2(S_t^0)^2\mathrm{d}t + \int_0^T \sigma^2 S_t^{*2}\mathrm{d}t)}} \leqslant \delta. \tag{14}$$

Compute (14),

$$\begin{aligned} E\Big(\int_0^T rS_t^0\mathrm{d}t + \int_0^T \mu S_t^*\mathrm{d}t\Big) &= E\Big(\int_0^T r\mathrm{e}^{rt}\mathrm{d}t + \mu\int_0^T S_0\mathrm{e}^{(\mu-r-\frac{1}{2}\sigma^2)t+\sigma W_t}\mathrm{d}t\Big) \\ &= (\mathrm{e}^{rT} - 1) + \frac{\mu S_0}{\mu - r}\,(\mathrm{e}^{(\mu-r)T} - 1) \end{aligned}$$

and

$$\begin{aligned} &\sqrt{E\Big(\int_0^T r^2(S_t^0)^2\mathrm{d}t + \int_0^T \sigma^2 S_t^{*2}\mathrm{d}t\Big)} \\ &\quad = \sqrt{\frac{r}{2}\,(\mathrm{e}^{2rT} - 1) + \sigma^2 S_0^2\,\frac{\mathrm{e}^{(2(\mu-r)+\sigma^2)T} - 1}{2(\mu - r) + \sigma^2}}. \end{aligned}$$

Then we can get

$$\frac{E(\int_0^T rS_t^0\mathrm{d}t+\int_0^T \mu S_t^*\mathrm{d}t)}{\sqrt{E(\int_0^T r^2(S_t^0)^2\mathrm{d}t+\int_0^T \sigma^2 S_t^{*2}\mathrm{d}t)}}=\frac{(\mathrm{e}^{rT}-1)+\frac{\mu S_0}{\mu-r}(\mathrm{e}^{(\mu-r)T}-1)}{\sqrt{\frac{r}{2}(\mathrm{e}^{2rT}-1)+\sigma^2S_0^2\frac{\mathrm{e}^{(2(\mu-r)+\sigma^2)T}-1}{2(\mu-r)+\sigma^2}}}.$$

Now, substituting the result above into (14), we derive

$$-\delta\leqslant\frac{(\mathrm{e}^{rT}-1)+\frac{\mu S_0}{\mu-r}(\mathrm{e}^{(\mu-r)T}-1)}{\sqrt{\frac{r}{2}(\mathrm{e}^{2rT}-1)+\sigma^2S_0^2\frac{\mathrm{e}^{(2(\mu-r)+\sigma^2)T}-1}{2(\mu-r)+\sigma^2}}}\leqslant\delta. \tag{15}$$

Now, we can summarize the above derivation as the following main result.

Theorem 7 *Assume*

$$T\geqslant\frac{3}{2(\sigma^2+(\mu-r))},\quad \delta\leqslant\sqrt{\frac{2}{r}},\quad r\geqslant(\mu-\mu S_0)\vee 0. \tag{16}$$

Then condition (15) *can imply condition* (7)*, which indicates that the no good deal condition for fundamental theorem is stronger than the no free lunch with vanishing risk condition.*

Proof We first note that (15) can be reduced to

$$\begin{aligned}\mathrm{e}^{(2(\mu-r)+\sigma^2)T}\geqslant\Big(&\Big(1-\frac{r\delta^2}{2}\Big)\mathrm{e}^{2rT}+2(\mu S_0+1)\mathrm{e}^{rT}\\&+\Big[\frac{\mu S_0}{\mu-r}\mathrm{e}^{(\mu-r)T}-\frac{\mu S_0(\mu-r)+\mu^2S_0^2}{\mu^2S_0^2}\Big]^2\\&-\frac{(\mu-r)^2}{\mu^2S_0^2}-\frac{2(\mu-r)}{\mu S_0}+\frac{2\mu S_0}{\mu-r}+\frac{\mu^2S_0^2}{(\mu-r)^2}\\&+\frac{r\delta^2}{2}+\frac{\delta^2\sigma^2S_0^2}{2(\mu-r)+\sigma^2}+1\Big)\frac{2(\mu-r)+\sigma^2}{\delta^2\sigma^2S_0^2}.\end{aligned} \tag{17}$$

If (15) is true, then putting (17) into the left-hand side (LHS) of (8) yields that

$$\begin{aligned}\text{LHS of }(8)\geqslant\Big(&\Big(1-\frac{r\delta^2}{2}\Big)\mathrm{e}^{2rT}+2(\mu S_0+1)\mathrm{e}^{rT}\\&+\Big[\frac{\mu S_0}{\mu-r}\mathrm{e}^{(\mu-r)T}-\frac{\mu S_0(\mu-r)+\mu^2S_0^2}{\mu^2S_0^2}\Big]^2\\&-\frac{(\mu-r)^2}{\mu^2S_0^2}-\frac{2(\mu-r)}{\mu S_0}+\frac{2\mu S_0}{\mu-r}+\frac{\mu^2S_0^2}{(\mu-r)^2}+\frac{r\delta^2}{2}\\&+\frac{\delta^2\sigma^2S_0^2}{2(\mu-r)+\sigma^2}+1\Big)\frac{2(\mu-r)+\sigma^2}{\delta^2\sigma^2S_0^2}\Big(T-\frac{3}{2(\sigma^2+(\mu-r))}\Big)\\&+\frac{2(\mu-r)T+1}{\sigma^2+2(\mu-r)}+\frac{\mathrm{e}^{2rT}}{TS_0^2C^2}+\frac{2\mathrm{e}^{\mu T}}{S_0C}.\end{aligned} \tag{18}$$

Let us assume that the right-hand side (RHS) of (18) $\geqslant 0$. Then it is easy to find

$$1-\frac{r\delta^2}{2}\geqslant 0,\quad T-\frac{3}{2(\sigma^2+(\mu-r))}\geqslant 0,$$
$$-\frac{(\mu-r)^2}{\mu^2S_0^2}-\frac{2(\mu-r)}{\mu S_0}+\frac{2\mu S_0}{\mu-r}+\frac{\mu^2S_0^2}{(\mu-r)^2}\geqslant 0.$$

Therefore, we get that under conditions (16), condition (15) can imply condition (8). As condition (8) is stronger than condition (7), we then verify that condition (15) implies condition (7). □

Remark 8 When

$$T-\frac{3}{2(\sigma^2+(\mu-r))}\leqslant 0,$$

it might not be feasible to compare. We cannot compare them in a short time.

On the other hand, we would like to examine whether (7) implies (15). Assume that (7) is true. Then we have

$$\begin{aligned}&\mathrm{e}^{(2(\mu-r)+\sigma^2)T}\Big(T-\frac{2}{\sigma^2+(\mu-r)}+\frac{1}{\sigma^2+2(\mu-r)}\Big)\\&\geqslant\frac{\delta_1^2\delta_2}{TS_0^2C^2}-\frac{1}{TS_0^2C^2}\,\mathrm{e}^{2rT}-\frac{2\mathrm{e}^{(\mu-r)T}}{\sigma^2+(\mu-r)}+\frac{1}{\sigma^2+2(\mu-r)}\\&\quad-\frac{2\mathrm{e}^{\mu}T}{S_0C}+\frac{2\mathrm{e}^{\mu T}}{TS_0C(\mu-r)}-\frac{2\mathrm{e}^{rT}}{TS_0C(\mu-r)}.\end{aligned}\tag{19}$$

Letting RHS of (19) $\geqslant 0$, we need

$$T-\frac{2}{\sigma^2+(\mu-r)}+\frac{1}{\sigma^2+2(\mu-r)}\geqslant 0,\quad 1-\frac{1}{T(\mu-r)}\geqslant 0.$$

We can show that

$$T\geqslant\frac{\sigma^2+3(\mu-r)}{[\sigma^2+(\mu-r)][\sigma^2+2(\mu-r)]},\quad T\geqslant\frac{1}{\mu-r},$$

so that

$$T\geqslant\frac{\sigma^2+3(\mu-r)}{[\sigma^2+(\mu-r)][\sigma^2+2(\mu-r)]}\vee\frac{1}{\mu-r}.\tag{20}$$

Consequently, we conclude that under condition (20), (7) implies (17). From (17), we work backwards

$$\Big[(\mathrm{e}^{rT}-1)+\frac{\mu S_0}{\mu-r}\,(\mathrm{e}^{(\mu-r)T}-1)\Big]^2\leqslant\delta^2\Big(\frac{r}{2}\,(\mathrm{e}^{2rT}-1)+\sigma^2S_0^2\frac{\mathrm{e}^{(2(\mu-r)+\sigma^2)T}-1}{2(\mu-r)+\sigma^2}\Big),$$

then we obtain

$$0\leqslant(\mathrm{e}^{rT}-1)+\frac{\mu S_0}{\mu-r}\,(\mathrm{e}^{(\mu-r)T}-1)\leqslant\delta\sqrt{\frac{r}{2}\,(\mathrm{e}^{2rT}-1)+\sigma^2S_0^2\frac{\mathrm{e}^{(2(\mu-r)+\sigma^2)T}-1}{2(\mu-r)+\sigma^2}}.$$

The above is equivalent to (15). We summarize our discussion by the following theorem.

Theorem 9 *Under conditions*

$$T \geqslant \frac{\sigma^2 + 3(\mu - r)}{[\sigma^2 + (\mu - r)][\sigma^2 + 2(\mu - r)]} \vee \frac{1}{\mu - r},$$

(7) *implies* (15)*, which means that the no good deal condition for fundamental theorem is weaker than the no free lunch with vanishing risk condition.*

We now turn to the situation in higher dimensions. We consider a finite time interval $[0,T]$ as the interval during which trading may take place. Let $(\Omega, \mathscr{F}, P; \mathscr{F}_t)$ be a given complete probability space on which defined a stranded m-dimensional Browning motion $W = \{W_t,\, t \in [0,T]\}$. In particular, $W = (W^1, W^2, \ldots, W^m)$ is an m-dimensional process defined on the time interval $[0,T]$. Our multi-dimensional model has $d+1$ assets, where d is a positive integer.

Sometimes, the money market asset is referred to as a riskless asset, denoted by S_t^0, which is given by

$$\frac{\mathrm{d}S_t^0}{S_t^0} = r\mathrm{d}t.$$

We assume that there are d stock with continuous, adapted price process $\boldsymbol{S}_t = (S_t^1, \ldots, S_t^d)$, which satisfying the following higher-dimensional Black-Scholes pricing dynamics:

$$\frac{\mathrm{d}S_t^i}{S_t^i} = \mu^i \mathrm{d}t + \sum_{j=1}^m \sigma^{ij} \mathrm{d}W_t^j, \quad 1 \leqslant i \leqslant d.$$

Here, the solution can be explicitly given as follows:

$$S_t^i = S_0^i \exp\left(\left(\mu^i - \frac{1}{2}\sum_{j=1}^m (\sigma\sigma^{\mathrm{T}})^{ij}\right)t + \sum_{j=1}^m \sigma^{ij} W_t^j\right), \tag{21}$$

where S_0^i is a positive constant and μ^i is the i^{th} component of a d-dimensional drift vector, and $\sigma = (\sigma^{ij})_{1\leqslant i\leqslant d, 1\leqslant j\leqslant m}$ is a $(d\times m)$-matrix.

Now, applying the Itô formula yields

$$\mathrm{d}S_t^{*i} = S_t^{*i}(\mu^i - r)\mathrm{d}t + S_t^{*i}\sum_{j=1}^m \sigma^{ij}\mathrm{d}W_t^j. \tag{22}$$

A self-financing trading strategy in this case is a $(d+1)$-dimensional process

$$\overline{\boldsymbol{\xi}}_t = (\xi_t^0, \xi_t^1, \ldots, \xi_t^d), \quad t \in [0,T].$$

The value at t of the portfolio associated with $\overline{\boldsymbol{\xi}}_t$ is given by

$$V_t^* = \overline{\boldsymbol{\xi}}_t \cdot \overline{\boldsymbol{S}}_t^* = V_0^* + \sum_{i=1}^{d} \int_0^t \xi_s^i \mathrm{d}S_s^{*,i}, \quad t \in [0, T].$$

Putting (22) into above, we get

$$\begin{aligned} V_T^* - V_0^* &= \sum_{i=1}^{d} \int_0^T \xi_t^i \mathrm{d}S_t^{*i} \\ &= \sum_{i=1}^{d} \int_0^T \xi_t^i \Big(S_t^{*i}(\mu^i - r)\mathrm{d}t + S_t^{*i} \sum_{j=1}^{m} \sigma^{ij} \mathrm{d}W_t^j \Big) \\ &= \sum_{i=1}^{d} \int_0^T \xi_t^i S_t^{*i}(\mu^i - r)\mathrm{d}t + \sum_{i=1}^{d} \sum_{j=1}^{m} \int_0^T \xi_t^i S_t^{*i} \sigma^{ij} \mathrm{d}W_t^j. \end{aligned}$$

Next, let

$$\xi_T^0 = 1, \quad S_T^0 = \mathrm{e}^{rT}.$$

Then

$$V_0^* = S_0^T = \mathrm{e}^{rT}$$

and

$$\begin{aligned} V_T^* = \mathrm{e}^{rT} &+ \sum_{i=1}^{d} \int_0^T (\mu^i - r)\xi_t^i S_0^i \exp\Bigg(\Bigg(\mu^i - r - \frac{1}{2} \sum_{j=1}^{m} (\sigma\sigma^{\mathrm{T}})^{ij} \Bigg) t + \sum_{j=1}^{m} \sigma^{ij} W_t^j \Bigg) \mathrm{d}t \\ &+ \sum_{i=1}^{d} \sum_{j=1}^{m} \sigma^{ij} \int_0^T \exp\Bigg(\Bigg(\mu^i - r - \frac{1}{2} \sum_{l=1}^{m} (\sigma\sigma^{\mathrm{T}})^{il} \Bigg) t + \sum_{l=1}^{m} \sigma^{il} W_t^l \Bigg) \mathrm{d}W_t^j. \end{aligned}$$

Remark 10 For fixed i, j, $(\sigma\sigma^{\mathrm{T}})^{ij}$ is given by

$$(\sigma\sigma^{\mathrm{T}})^{ij} = \sum_{k=1}^{m} \sigma^{ik}\sigma^{jk}.$$

Proposition 11 *For each $i = 1, 2, ..., d$, the price process S_t^i can be represented as*

$$S_0^i \mathrm{e}^{(\mu^i - r - \frac{1}{2}(\sigma^i)^2)t + \sigma^i B_t},$$

where

$$(\sigma^i)^2 = \sum_{j=1}^{m} (\sigma^{ij})^2 = (\sigma\sigma^{\mathrm{T}})^{ii} > 0,$$

and B_t, $t \in [0, T]$, is a one-dimensional Brownian motion such that

$$\sum_{j=1}^{m} \sigma^{ij} \mathrm{d}W_t^j = \sigma^i \mathrm{d}B_t. \tag{23}$$

Proof From (23), we have

$$\mathrm{d}B_t = \frac{\sum_{j=1}^m \sigma^{ij}\mathrm{d}W_t^j}{\sigma^i}.$$

According to definition of quadratic variation, we have

$$\mathrm{d}\langle B,B\rangle(t) = \frac{1}{(\sigma^i)^2}\left\langle \sum_{j=1}^m \sigma^{ij}\mathrm{d}W_t^j, \sum_{j=1}^m \sigma^{ij}\mathrm{d}W_t^j\right\rangle = \frac{1}{(\sigma^i)^2}\sum_{j=1}^m (\sigma^{ij})^2\mathrm{d}t = \mathrm{d}t.$$

Therefore, B_t is a one-dimensional Browning motion and

$$S_t^i = S_0^i \mathrm{e}^{(\mu^i - r - \frac{1}{2}(\sigma^i)^2)t + \sigma^i B_t}.$$

We are done. □

Proposition 11 shows that the result from multi-dimensional model is essentially similar to the one-dimensional case. In other words, one-dimensional model is a special case in higher-dimensional model. By Definition 3 (iv) and Proposition 11, if there are constants $\delta_1, \delta_2 > 0$, then free lunch with vanishing risk exists:

$$P\bigg\{\int_0^T \Big(\frac{1}{T}\mathrm{e}^{rT} + \frac{1}{T} S_0^i \xi_T^i \mathrm{e}^{(\mu^i - r - \frac{1}{2}\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij})T}\mathrm{e}^{\sum_{j=1}^m \sigma^{ij}W_T^j} - S_0^i \frac{\partial \xi_t}{\partial t}\mathrm{e}^{(\mu^i - r - \frac{1}{2}\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij})t}\mathrm{e}^{\sum_{j=1}^m \sigma^{ij}W_t^j}\Big)\mathrm{d}t > \delta_1\bigg\} > \delta_2. \tag{24}$$

Set

$$X(t) := \frac{\mathrm{e}^{rT} + S_0^i \xi_T^i \mathrm{e}^{(\mu^i - r - \frac{1}{2}\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij})T}\mathrm{e}^{\sum_{j=1}^m \sigma^{ij}W_T^j}}{T} - S_0^i \frac{\partial \xi_t}{\partial t}\mathrm{e}^{(\mu^i - r - \frac{1}{2}\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij})t}\mathrm{e}^{\sum_{j=1}^m \sigma^{ij}W_t^j}.$$

Then (24) reduce to

$$P\left\{\int_0^T X(t)\mathrm{d}t > \delta_1\right\} > \delta_2.$$

For any $p \geqslant 1$, by Markov's inequality, we can get

$$\delta_2 \leqslant P\left\{\Big|\int_0^T X(t)\mathrm{d}t\Big| > \delta_1\right\} \leqslant \frac{E|\int_0^T X(t)\mathrm{d}t|^p}{\delta_1^p}.$$

More specifically, taking $p = 2$ and by Cauchy's inequality, we get

$$\frac{\delta_2\delta_1^2}{T} \leqslant \int_0^T E\,(X(t))^2\,\mathrm{d}t.$$

We calculate $\int_0^T E\left(X(t)\right)^2 \mathrm{d}t$. By

$$E(\mathrm{e}^{-\frac{1}{2}\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij}t+\sum_{j=1}^m\sigma^{ij}W_t^j}) = 1, \quad \xi_t^i = C^i t,$$

where $i = 1, \ldots, d$, $j = 1, \ldots, m$. We can get the following inequality:

$$\begin{aligned}
&\frac{1}{T}\,\mathrm{e}^{2rT} + {S_0^i}^2{C^i}^2\mathrm{e}^{2((\mu^i-r)+\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij})T} \\
&\times\left(T - \frac{2}{\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij}+(\mu-r)} + \frac{1}{\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij}+2(\mu^i-r)}\right) \\
&+2{S_0^i}^2{C^i}^2\frac{\mathrm{e}^{(\mu^i-r)T}}{\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij}+(\mu^i-r)} - \frac{{C^i}^2{S_0^i}^2}{\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij}+2(\mu^i-r)} \\
&+2S_0^iC^i\mathrm{e}^{\mu^iT} - \frac{2S_0^iC^i}{T(\mu^i-r)}\,\mathrm{e}^{\mu^iT} + \frac{2S_0^iC^i}{T(\mu^i-r)}\,\mathrm{e}^{rT} \geqslant \frac{\delta_1^2\delta_2}{T}.
\end{aligned} \tag{25}$$

In the no good deal condition, the price is $x_{s,t}^i$, $0 \leqslant s < t < T$. We define

$$\mu\colon (0,T)\times\mathbb{R}\to\mathbb{R}^d, \quad \sigma\colon (0,T)\times\mathbb{R}\to\mathbb{R}^{d\otimes m},$$

such that

$$x_{s,t}^i(X) := \int_s^t \mu^i(u,X_u)\mathrm{d}u + \sum_{j=1}^m\int_s^t \sigma^{ij}(u,X_u)\mathrm{d}W_u^j. \tag{26}$$

Considering a special case of (26) with the riskiness bond, for each $i = 1, ..., d$ we have

$$x_{s,t}^i(S_t) := \int_s^t rS_u^0\mathrm{d}u + \int_s^t \mu^iS_u^{i,*}\mathrm{d}u + \sum_{j=1}^m\int_s^t \sigma^{ij}S_u^{i,*}\mathrm{d}W_u^j. \tag{27}$$

By the definition, the no good deal condition can be satisfied if

$$-\delta \leqslant \frac{E(\int_0^T rS_t^0\mathrm{d}t + \int_0^T \mu^iS_t^{*i}\mathrm{d}t)}{\sqrt{E(\int_0^T r^2(S_t^0)^2\mathrm{d}t + \sum_{j=1}^m\int_0^T(\sigma\sigma^{\mathrm{T}})^{ij}(S_t^{*i})^2\mathrm{d}t)}} \leqslant \delta. \tag{28}$$

We can then get the following explicit expression:

$$-\delta \leqslant \frac{(\mathrm{e}^{rT}-1) + \frac{\mu^iS_0}{\mu^i-r}\,(\mathrm{e}^{(\mu^i-r)T}-1)}{\sqrt{\frac{r}{2}\,(\mathrm{e}^{2rT}-1) + \sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij}{S_0^i}^2\,\frac{\mathrm{e}^{(2(\mu^i-r)+\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij})T}-1}{2(\mu^i-r)+\sum_{j=1}^m(\sigma\sigma^{\mathrm{T}})^{ij}}}} \leqslant \delta. \tag{29}$$

Let us finish our paper by summarising our above discussion as the following results.

Theorem 12 *Under assumptions*

$$T \geqslant \frac{3}{2(\sum_{j=1}^{m}(\sigma\sigma^{\mathrm{T}})^{ij} + (\mu^i - r))}, \quad \delta \leqslant \sqrt{\frac{2}{r}}, \quad r \geqslant (\mu^i - \mu^i S_0^i) \vee 0,$$

we conclude that condition (29) *can imply condition* (25)*, which shows that in financial markets with multi-assets, the no good deal condition for fundamental theorem is more general than the no free lunch with vanishing risk condition.*

Theorem 13 *Assume that*

$$T \geqslant \frac{\sum_{j=1}^{m}(\sigma\sigma^*)^{ij} + 3(\mu^i - r)}{[\sum_{j=1}^{m}(\sigma^{ij})^2 + (\mu^i - r)][\sum_{j=1}^{m}(\sigma\sigma^{\mathrm{T}})^{ij} + 2(\mu^i - r)]} \vee \frac{1}{\mu - r}.$$

Then condition (25) *implies condition* (29)*, which indicates that the no good deal condition for fundamental theorem is weaker than the no free lunch with vanishing risk condition.*

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