Paper:
Ledger, P. & Lionheart, W. (2015). The perturbation of electromagnetic fields at distances that are large compared with the object's size. *IMA Journal of Applied Mathematics, 80*(3), 865-892.

http://dx.doi.org/10.1093/imamat/hxu009

IMA Journal of Applied Mathematics Advance Access published May 14, 2014

IMA Journal of Applied Mathematics (2014) Page 1 of 28
doi:10.1093/imamat/hxu009

The perturbation of electromagnetic fields at distances that are large compared with the object's size

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[Received on 28 November 2012; revised on 12 November 2013; accepted on 20 March 2014]

We rigorously derive the leading-order terms in asymptotic expansions for the scattered electric and magnetic fields in the presence of a small object at distances that are large compared with its size. Our expansions hold for fixed wavenumber when the scatterer is a (lossy) homogeneous dielectric object with constant material parameters or a perfect conductor. We also derive the corresponding leading-order terms in expansions for the fields for a low-frequency problem when the scatterer is a non-lossy homogeneous dielectric object with constant material parameters or a perfect conductor. In each case, we express our results in terms of polarization tensors.

Keywords: polarization tensors; low-frequency scattering; asymptotic expansions.

1. Introduction

The scattering of electromagnetic fields at low frequencies that occurs due to the presence of a small object has previously received considerable attention; see, for example, Kleinman (1965, 1967, 1973), Kleinman & Senior (1982), Dassios & Kleinman (2000) and references therein. A review of work prior to 1965 was conducted by Kleinman (1965) and, more recently, Dassios & Kleinman (2000) have written a monograph. Kleinman (1967, 1973) has shown, by making the Rayleigh approximation, that the first non-trivial term in the low-frequency far field scattering expansion of the electric and magnetic fields involves k^2/r, where k is the wavenumber and r is the distance from the object to the point of observation. Furthermore, he shows that this Rayleigh term can be written in terms of electric and magnetic polarization moments[1] that can be identified with the corresponding dipole moments in the leading-order k^2/r term for the far field patterns of radiating dipoles. Later, it was shown how the polarization moments in these expansions can be expressed in terms of polarization tensors (Kleinman & Senior, 1982). In particular, in the case of a perfectly conducting scatterer, the electric moment can be expressed in terms of a polarization tensor that is related to the isolated conductor polarization tensor of Schiffer & Szegö (1949) and the magnetic moment can be expressed in terms of a polarization tensor, which is related to their added mass tensor (Keller *et al.*, 1972). Kleinman and Senior claim that, for the case of lossy and non-lossy dielectrics, the polarization tensor, associated with the electric and magnetic

[1] Polarization (moments and tensors) should be understood in the sense of the definitions in, e.g., Dassios & Kleinman (2000), Ammari & Kang (2007) and is not related to the polarization of electromagnetic waves.

moments, is a generalized form of the isolated conductor and added mass tensors (Kleinman & Senior, 1982), which Dassios & Kleinman (2000) call the general polarization tensor.

Ammari *et al.* (2001) have revisited the problem of determining the perturbation in the electromagnetic fields for a small object. For a bounded domain, where the tangential traces of the electric and magnetic field are available on the boundary, they obtain the leading-order terms in asymptotic expansions for the perturbation in the near fields due to the presence of an object as $\delta \to 0$, where δ is the object's size. Ammari & Volkov (2005) have also considered the unbounded problem and obtained the leading-order terms in asymptotic expansions for the perturbation of the far fields as $\delta \to 0$. Their expansions are expressed in terms of a polarization tensor that is related to the general polarization tensor defined in Dassios & Kleinman (2000). In the related problem of electric impedance tomography, Ammari & Kang (2007) define a generalized polarization tensor that, in the lowest order case, agrees with the earlier general polarization tensor. They are able to show that the complete far-field expansion of a potential field satisfying a scalar Laplace transmission problem can be expressed in terms of this new tensor (Ammari & Kang, 2007). Similar results have also been obtained for the near-field expansion of a potential field satisfying a Laplace transmission problem on a bounded domain as well as for the Helmholtz (Ammari & Kang, 2004a,b) and linear elasticity (Ammari & Kang, 2004b) transmission problems. By considering the low-frequency eddy current problem, Ammari *et al.* (2014) have recently shown, when the skin depth is of the same order as the size of the object, that the leading-order term in an asymptotic expansion of the perturbed magnetic field as $\delta \to 0$ can be expressed in terms of a new form of polarization tensor for conducting objects.

The ability to describe the perturbation in the electric and magnetic fields, caused by the presence of a small object, in terms of polarization tensors has recently attracted considerable interest as it offers possibilities for determining the shape, location and material properties of the inclusion from measurement of the perturbed fields (Ammari & Kang, 2004b, 2007; Ammari *et al.*, 2014). Indeed, a number of inverse algorithms for determining the entries of the generalized polarization tensors from measured data, and henceforth determining the shape, location and material properties of an object, have been proposed and are discussed in Ammari & Kang (2004b, 2007) and Ammari *et al.* (2014). Such approaches could potentially have important applications in medical imaging including the location of tumours and the detection of land mines.

Before stating the contributions made in this work, we make the notions of low- and high-frequency problems and near and far field, which we plan to use, precise. In physics it is common to call the situation of $k\delta \ll 1$, where δ is a characteristic length scale of the object, a low-frequency problem and that of $k\delta \gg 1$ a high-frequency problem. It is also common to call $kr \ll 1$ the near field and the case of $kr \gg 1$ the far field. In this work, we are interested in developing asymptotic expansions and we will therefore call $k\delta \to 0$ a low-frequency problem and denote by $\delta/r \to 0$ distances that are large compared with the object's size. We have included the object's size in these definitions as it is crucial to the derivation and understanding of asymptotic expansions for the perturbations in the electric and magnetic fields at distances relative to this length scale. In the earlier work of Kleinman the object size was not explicitly included and instead it was assumed to be small. In addition, he used the alternative characterization of $k \to 0$ to denote a low-frequency problem. The k^2/r Rayleigh term he obtains corresponds to the leading-order term for the perturbation of the fields when $k \to 0$ and $r \to \infty$. However, there is also interest in understanding how the perturbation of the fields can be interpreted in terms of polarization tensors at other locations and, in particular, at distances that are large compared with the object's size. This, in turn, is important for practical systems attempting to distinguish between different objects by using their polarization tensors, which might occur at different distances from the measurement system.

Our novel contributions are twofold: First, we derive the leading-order terms in asymptotic expansions for the perturbation in the electric and magnetic fields due to the presence of a small (lossy) homogeneous dielectric object for fixed k as $\max(\delta/r, \delta) \to 0$: such that it describes the fields at distances that are large compared with the object's size for a small object. Secondly, we derive the leading-order terms in asymptotic expansions for the perturbation in the fields due to the presence of a non-lossy homogeneous dielectric object as $\max(\delta/r, k\delta) \to 0$: such that it describes the fields at distances that are large compared with the object's size for a low-frequency problem. Our main results are expressed in terms of polarization tensors, although we also give an intermediate step where the asymptotic expansions we obtain are expressed in terms of polarization moments, which enables easy comparison with radiating dipoles. The resulting expansions for the exact fields contain the same terms obtained previously by Baum (1971), although our derivation is radically different, includes the object's size and uses, as its starting point, the same representation formulae as Kleinman (1967). The methodology employed in our paper greatly extends that presented in Kleinman (1967, 1973) through the use of multi-index Taylor series expansions and involves the derivation and application of new sets of integral identities to handle the new aforementioned cases. The polarization moments that arise are expressed in terms of tensors that are related to the general polarization tensors defined in Dassios & Kleinman (2000) and to a subset of the generalized polarization tensors of Ammari & Kang (2007). We make comparisons with (Ammari *et al.*, 2001; Ammari & Volkov, 2005), which considers the perturbation in the fields when $r \to \infty$ and $\delta \to 0$.

The presentation of the material proceeds as follows: In Section 2, we present the governing equations and then, in Section 3, we introduce the scattering problem. In Section 4, we present a summary of our main results. Then, in Section 5, we present asymptotic expansions for the perturbation in the electric and magnetic fields in terms of polarization moments and prove our main results. The appendices define a series of integral identities that are required for the derivations in Section 5 and make explicit the connection between the different polarization tensors.

2. Governing equations

Under the assumption of linear materials, the time-harmonic Maxwell equations for a source-free medium are

$$\begin{aligned} \nabla \times \boldsymbol{\mathcal{E}} &= \mathrm{i}\omega\mu\boldsymbol{\mathcal{H}}, \\ \nabla \times \boldsymbol{\mathcal{H}} &= \sigma\boldsymbol{\mathcal{E}} - \mathrm{i}\omega\epsilon\boldsymbol{\mathcal{E}}, \\ \nabla \cdot (\epsilon\boldsymbol{\mathcal{E}}) &= \frac{1}{\mathrm{i}\omega}\nabla \cdot (\sigma\boldsymbol{\mathcal{E}}), \\ \nabla \cdot (\mu\boldsymbol{\mathcal{H}}) &= 0, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$ denote the complex amplitudes of the electric and magnetic field intensity vectors, respectively, for an assumed $\mathrm{e}^{-\mathrm{i}\omega t}$ time variation with angular frequency ω. The parameters ϵ, μ and σ denote the permittivity, permeability and conductivity, respectively, and satisfy

$$0 < \epsilon^{\min} \leqslant \epsilon \leqslant \epsilon^{\max} < \infty, \quad 0 < \mu^{\min} \leqslant \mu \leqslant \mu^{\max} < \infty, \quad 0 \leqslant \sigma \leqslant \sigma^{\max} < \infty.$$

Following Monk (2003), the scaled electric and magnetic fields are introduced as $\boldsymbol{E} = \epsilon_0^{1/2}\boldsymbol{\mathcal{E}}$ and $\boldsymbol{H} = \mu_0^{1/2}\boldsymbol{\mathcal{H}}$, where $\epsilon_0 \approx 8.854 \times 10^{-12} F/m$ and $\mu_0 = 4\pi \times 10^{-7} H/m$ are the free space values of the permittivity and permeability, respectively, leading to

$$\begin{aligned} \nabla \times \boldsymbol{E} &= \mathrm{i}k\tilde{\mu}_r\boldsymbol{H}, \\ \nabla \times \boldsymbol{H} &= -\mathrm{i}k\tilde{\epsilon}_r\boldsymbol{E}, \\ \nabla \cdot (\tilde{\epsilon}_r\boldsymbol{E}) &= 0, \\ \nabla \cdot (\tilde{\mu}_r\boldsymbol{H}) &= 0, \end{aligned} \tag{2.2}$$

where

$$\tilde{\epsilon}_r = \frac{1}{\epsilon_0}\left(\epsilon + \frac{\mathrm{i}\sigma}{\omega}\right), \quad \tilde{\mu}_r = \frac{\mu}{\mu_0}, \quad k = \omega\sqrt{\epsilon_0\mu_0},$$

and $\tilde{\epsilon}_r$, $\tilde{\mu}_r$ are, in general, functions of position.

3. Scattering formulation

We consider a smooth closed object D with boundary ∂D equipped with unit outward normal $\hat{\boldsymbol{n}}$ that is homogeneous with material coefficients ϵ_*, σ_* and μ_* and, therefore, is characterized by the constants $\tilde{\epsilon}_r(\boldsymbol{r}) = \epsilon_r := 1/\epsilon_0(\epsilon_* - \mathrm{i}\sigma_*/\omega)$ and $\tilde{\mu}_r(\boldsymbol{r}) = \mu_r := \mu_*/\mu_0$ for $\boldsymbol{r} \in D$. The object is surrounded by an unbounded region of free space $\mathbb{R}^3 \setminus D$ and the complete configuration is illustrated in Fig. 1. The electric and magnetic fields satisfy (2.2) in D and in $\mathbb{R}^3 \setminus D$, with $\tilde{\epsilon}_r(\boldsymbol{r}) = \tilde{\mu}_r(\boldsymbol{r}) = 1$ for $\boldsymbol{r} \in \mathbb{R}^3 \setminus D$. The fields also obey the standard transmission conditions

$$\begin{aligned} \hat{\boldsymbol{n}} \times \boldsymbol{E}|_+ - \hat{\boldsymbol{n}} \times \boldsymbol{E}|_- &= \boldsymbol{0}, \\ \hat{\boldsymbol{n}} \times \boldsymbol{H}|_+ - \hat{\boldsymbol{n}} \times \boldsymbol{H}|_- &= \boldsymbol{0}, \\ \hat{\boldsymbol{n}} \cdot \boldsymbol{E}|_+ - \hat{\boldsymbol{n}} \cdot \epsilon_r\boldsymbol{E}|_- &= 0, \\ \hat{\boldsymbol{n}} \cdot \boldsymbol{H}|_+ - \hat{\boldsymbol{n}} \cdot \mu_r\boldsymbol{H}|_- &= 0 \end{aligned}$$

on ∂D, where $+$ indicates the evaluation just outside D and $-$ just inside D. It is convenient to split the total electric and magnetic fields into incident and scattered components, $\boldsymbol{E} = \boldsymbol{E}^{\mathrm{in}} + \boldsymbol{E}^{\mathrm{sc}}$ and $\boldsymbol{H} = \boldsymbol{H}^{\mathrm{in}} + \boldsymbol{H}^{\mathrm{sc}}$, where the analytic incident fields are chosen to satisfy (2.2) with $\tilde{\epsilon}_r = \tilde{\mu}_r = 1$ for all of

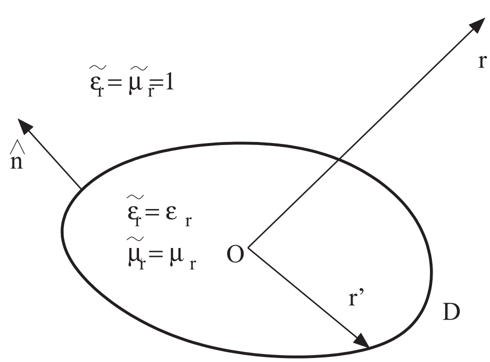

FIG. 1. Illustration of a small object D located in an unbounded region of free space.

$\mathbb{R}^3$, which, by linearity, the scattered components also satisfy outside the object. The scattered fields represent the perturbation in the electric and magnetic field due to the presence of the object and satisfy the radiation conditions

$$\lim_{r\to\infty}(\boldsymbol{r}\times(\nabla\times\boldsymbol{E}^{\mathrm{sc}})+\mathrm{i}kr\boldsymbol{E}^{\mathrm{sc}})=\boldsymbol{0},$$
$$\lim_{r\to\infty}(\boldsymbol{r}\times(\nabla\times\boldsymbol{H}^{\mathrm{sc}})+\mathrm{i}kr\boldsymbol{H}^{\mathrm{sc}})=\boldsymbol{0},$$

where $r=|\boldsymbol{r}|$ and $\hat{\boldsymbol{r}}=\boldsymbol{r}/r$.

Rather than the usual Stratton–Chu type formulae we follow Kleinman (1967) and use the alternative expressions

$$\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r})=\frac{\mathrm{i}}{k}\nabla_r\times\left(\nabla_r\times\int_{\partial D}u_k(\boldsymbol{r}-\boldsymbol{r}')\hat{\boldsymbol{n}}'\times\boldsymbol{H}^{\mathrm{sc}}|_+\,\mathrm{d}S'\right)+\nabla_r\times\int_{\partial D}u_k(\boldsymbol{r}-\boldsymbol{r}')\hat{\boldsymbol{n}}'\times\boldsymbol{E}^{\mathrm{sc}}|_+\,\mathrm{d}S' \tag{3.1}$$

and

$$\boldsymbol{H}^{\mathrm{sc}}(\boldsymbol{r})=-\frac{\mathrm{i}}{k}\nabla_r\times\left(\nabla_r\times\int_{\partial D}u_k(\boldsymbol{r}-\boldsymbol{r}')\hat{\boldsymbol{n}}'\times\boldsymbol{E}^{\mathrm{sc}}|_+\,\mathrm{d}S'\right)+\nabla_r\times\int_{\partial D}u_k(\boldsymbol{r}-\boldsymbol{r}')\hat{\boldsymbol{n}}'\times\boldsymbol{H}^{\mathrm{sc}}|_+\,\mathrm{d}S', \tag{3.2}$$

which are valid for $\boldsymbol{r}\in\mathbb{R}^3\setminus D$, for the representation of the scattered fields (see also Jones, 1964, p. 55; Monk, 2003, p. 230). In the above

$$u_k(\boldsymbol{r}-\boldsymbol{r}')=\frac{\mathrm{e}^{\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{3.3}$$

is the standard free space Green's function for the scalar Helmholtz equation with wavenumber k and, to avoid confusion, we make explicit that ∇_r denotes differentiation with respect to $\boldsymbol{r}$.

We restrict consideration to $D=\delta B+\boldsymbol{z}$ where B denotes a smooth closed object with known (unit) shape, δ is the object's size and $\boldsymbol{z}$ represents the translation from the origin. This means that the centre of the object is now located at a constant distance $z=|\boldsymbol{z}|$ from the origin and the distance from the centre of the scaled object to its surface becomes $|\boldsymbol{r}'_\delta|=\delta|\boldsymbol{r}'|$. The configuration is illustrated in Fig. 2.

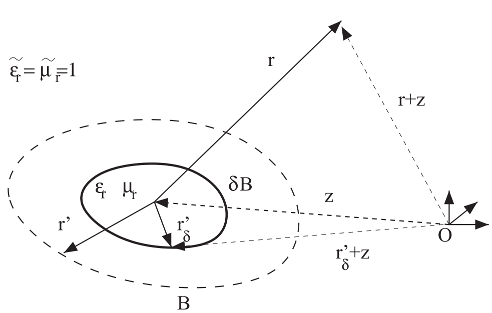

FIG. 2. Illustration of an object B scaled by small δ located in an unbounded region of free space.

The scattered electric and magnetic fields external to the object are now given by

$$\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z}) = \frac{\mathrm{i}\delta^2}{k}\nabla_r \times \left(\nabla_r \times \int_{\partial B} u_k(\boldsymbol{r}-\delta\boldsymbol{r}')\hat{\boldsymbol{n}}' \times \boldsymbol{h}^{\mathrm{sc}}(\boldsymbol{r}')|_+ \,\mathrm{d}S'\right) + \delta^2 \nabla_r \times \int_{\partial B} u_k(\boldsymbol{r}-\delta\boldsymbol{r}')\hat{\boldsymbol{n}}' \times \boldsymbol{e}^{\mathrm{sc}}(\boldsymbol{r}')|_+ \,\mathrm{d}S', \tag{3.4}$$

$$\boldsymbol{H}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z}) = -\frac{\mathrm{i}\delta^2}{k}\nabla_r \times \left(\nabla_r \times \int_{\partial B} u_k(\boldsymbol{r}-\delta\boldsymbol{r}')\hat{\boldsymbol{n}}' \times \boldsymbol{e}^{\mathrm{sc}}(\boldsymbol{r}')|_+ \,\mathrm{d}S'\right) + \delta^2 \nabla_r \times \int_{\partial B} u_k(\boldsymbol{r}-\delta\boldsymbol{r}')\hat{\boldsymbol{n}}' \times \boldsymbol{h}^{\mathrm{sc}}(\boldsymbol{r}')|_+ \,\mathrm{d}S', \tag{3.5}$$

where we have used the notation $\boldsymbol{e}(\boldsymbol{r}') = \boldsymbol{E}(\delta\boldsymbol{r}'+\boldsymbol{z})$ and $\boldsymbol{h}(\boldsymbol{r}') = \boldsymbol{H}(\delta\boldsymbol{r}'+\boldsymbol{z})$. These scaled total fields satisfy the system

$$\begin{aligned} \nabla_{r'} \times \boldsymbol{e} &= \mathrm{i}k\delta\tilde{\mu}_r\boldsymbol{h}, \\ \nabla_{r'} \times \boldsymbol{h} &= -\mathrm{i}k\delta\tilde{\epsilon}_r\boldsymbol{e}, \\ \nabla_{r'} \cdot (\tilde{\epsilon}_r\boldsymbol{e}) &= 0, \\ \nabla_{r'} \cdot (\tilde{\mu}_r\boldsymbol{h}) &= 0 \end{aligned} \tag{3.6}$$

in B, with $\tilde{\epsilon}_r = \epsilon_r$, $\tilde{\mu}_r = \mu_r$, and the incident and scattered components satisfy a similar system in $\mathbb{R}^3 \setminus B$, with $\tilde{\epsilon}_r = \tilde{\mu}_r = 1$. The fields also satisfy the transmission conditions

$$\begin{aligned} \hat{\boldsymbol{n}} \times \boldsymbol{e}^{\mathrm{in}}|_+ - \hat{\boldsymbol{n}} \times \boldsymbol{e}^{\mathrm{in}}|_- &= \hat{\boldsymbol{n}} \times \boldsymbol{e}^{\mathrm{sc}}|_+ - \hat{\boldsymbol{n}} \times \boldsymbol{e}^{\mathrm{sc}}|_- = \boldsymbol{0}, \\ \hat{\boldsymbol{n}} \times \boldsymbol{h}^{\mathrm{in}}|_+ - \hat{\boldsymbol{n}} \times \boldsymbol{h}^{\mathrm{in}}|_- &= \hat{\boldsymbol{n}} \times \boldsymbol{h}^{\mathrm{sc}}|_+ - \hat{\boldsymbol{n}} \times \boldsymbol{h}^{\mathrm{sc}}|_- = \boldsymbol{0}, \\ \hat{\boldsymbol{n}} \cdot \boldsymbol{e}^{\mathrm{sc}}|_+ - \hat{\boldsymbol{n}} \cdot \epsilon_r\boldsymbol{e}^{\mathrm{sc}}|_- &= (\epsilon_r - 1)\hat{\boldsymbol{n}} \cdot \boldsymbol{e}^{\mathrm{in}}, \\ \hat{\boldsymbol{n}} \cdot \boldsymbol{h}^{\mathrm{sc}}|_+ - \hat{\boldsymbol{n}} \cdot \mu_r\boldsymbol{h}^{\mathrm{sc}}|_- &= (\mu_r - 1)\hat{\boldsymbol{n}} \cdot \boldsymbol{h}^{\mathrm{in}} \end{aligned}$$

on ∂B.

In Sections 3.1 and 3.2, we introduce two contrasting expansions for the scaled fields that we apply in the case of a small (lossy) dielectric scatterer and a non-lossy dielectric scatterer at low frequencies.

3.1 *Scattering by a small (lossy) dielectric scatterer*

In the case of a small (lossy) dielectric scatterer, we assume that the fields $\boldsymbol{e}$ and $\boldsymbol{h}$ for fixed k can be expanded as

$$\boldsymbol{e} = \boldsymbol{e}|_{\delta=0} + \delta \left.\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}\delta}\right|_{\delta=0} + O(\delta^2) = \tilde{\boldsymbol{e}}_0 + \delta\tilde{\boldsymbol{e}}_1 + O(\delta^2), \tag{3.7}$$

$$\boldsymbol{h} = \boldsymbol{h}|_{\delta=0} + \delta \left.\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}\delta}\right|_{\delta=0} + O(\delta^2) = \tilde{\boldsymbol{h}}_0 + \delta\tilde{\boldsymbol{h}}_1 + O(\delta^2) \tag{3.8}$$

in B and in $\mathbb{R}^3 \setminus B$ and that they are convergent as $\delta \to 0$. We also assume that similar expansions hold for the incident and scattered components and that the incident field, being analytic, admits a complete

Taylor's series expansion in $\mathbb{R}^3$. Substituting (3.7) and (3.8) into (3.6), and equating coefficients of δ^0, it follows that

$$\begin{aligned}\nabla_{r'} \times \tilde{\boldsymbol{h}}_0^{\text{in}} &= \nabla_{r'} \times \tilde{\boldsymbol{h}}_0^{\text{sc}} = \boldsymbol{0},\\ \nabla_{r'} \times \tilde{\boldsymbol{e}}_0^{\text{in}} &= \nabla_{r'} \times \tilde{\boldsymbol{e}}_0^{\text{sc}} = \boldsymbol{0},\\ \nabla_{r'} \cdot \tilde{\boldsymbol{h}}_0^{\text{in}} &= \nabla_{r'} \cdot \tilde{\boldsymbol{h}}_0^{\text{sc}} = 0,\\ \nabla_{r'} \cdot \tilde{\boldsymbol{e}}_0^{\text{in}} &= \nabla_{r'} \cdot \tilde{\boldsymbol{e}}_0^{\text{sc}} = 0\end{aligned} \tag{3.9}$$

in $\mathbb{R}^3 \setminus B$ and in B. Equating coefficients of δ^1 yields

$$\begin{aligned}\nabla_{r'} \times \tilde{\boldsymbol{e}}_1 &= \mathrm{i}k\tilde{\mu}_r\tilde{\boldsymbol{h}}_0,\\ \nabla_{r'} \times \tilde{\boldsymbol{h}}_1 &= -\mathrm{i}k\tilde{\epsilon}_r\tilde{\boldsymbol{e}}_0,\\ \nabla_{r'} \cdot \tilde{\boldsymbol{e}}_1 &= \nabla_{r'} \cdot \tilde{\boldsymbol{h}}_1 = 0\end{aligned} \tag{3.10}$$

in B, with $\tilde{\epsilon}_r = \epsilon_r$, $\tilde{\mu}_r = \mu_r$. By equating δ^1 coefficients, and considering the incident and scattered components, a similar result can be obtained in $\mathbb{R}^3 \setminus B$, with $\tilde{\epsilon}_r = \tilde{\mu}_r = 1$. For $m = 0, 1$ the δ^m coefficients of the fields satisfy the transmission conditions

$$\begin{aligned}\hat{\boldsymbol{n}} \times \tilde{\boldsymbol{e}}_m^{\text{in}}|_+ - \hat{\boldsymbol{n}} \times \tilde{\boldsymbol{e}}_m^{\text{in}}|_- &= \hat{\boldsymbol{n}} \times \tilde{\boldsymbol{e}}_m^{\text{sc}}|_+ - \hat{\boldsymbol{n}} \times \tilde{\boldsymbol{e}}_m^{\text{sc}}|_- = \boldsymbol{0},\\ \hat{\boldsymbol{n}} \times \tilde{\boldsymbol{h}}_m^{\text{in}}|_+ - \hat{\boldsymbol{n}} \times \tilde{\boldsymbol{h}}_m^{\text{in}}|_- &= \hat{\boldsymbol{n}} \times \tilde{\boldsymbol{h}}_m^{\text{sc}}|_+ - \hat{\boldsymbol{n}} \times \tilde{\boldsymbol{h}}_m^{\text{sc}}|_- = \boldsymbol{0},\\ \hat{\boldsymbol{n}} \cdot \tilde{\boldsymbol{e}}_m^{\text{sc}}|_+ - \hat{\boldsymbol{n}} \cdot \epsilon_r\tilde{\boldsymbol{e}}_m^{\text{sc}}|_- &= (\epsilon_r - 1)\hat{\boldsymbol{n}} \cdot \tilde{\boldsymbol{e}}_m^{\text{in}},\\ \hat{\boldsymbol{n}} \cdot \tilde{\boldsymbol{h}}_m^{\text{sc}}|_+ - \hat{\boldsymbol{n}} \cdot \mu_r\tilde{\boldsymbol{h}}_m^{\text{sc}}|_- &= (\mu_r - 1)\hat{\boldsymbol{n}} \cdot \tilde{\boldsymbol{h}}_m^{\text{in}}\end{aligned} \tag{3.11}$$

on ∂B.

3.2 *Scattering by a non-lossy dielectric scatterer at low frequencies*

In the case of scattering by a non-lossy dielectric scatterer, where $\epsilon_r = \epsilon_*/\epsilon_0$, at low frequencies, we assume that the fields $\boldsymbol{e}$ and $\boldsymbol{h}$ can be expanded as

$$\boldsymbol{e} = \boldsymbol{e}|_{k\delta=0} + k\delta \left.\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}(k\delta)}\right|_{k\delta=0} + O((k\delta)^2) = \tilde{\tilde{\boldsymbol{e}}}_0 + k\delta\tilde{\tilde{\boldsymbol{e}}}_1 + O((k\delta)^2), \tag{3.12}$$

$$\boldsymbol{h} = \boldsymbol{h}|_{k\delta=0} + k\delta \left.\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}(k\delta)}\right|_{k\delta=0} + O((k\delta)^2) = \tilde{\tilde{\boldsymbol{h}}}_0 + k\delta\tilde{\tilde{\boldsymbol{h}}}_1 + O((k\delta)^2) \tag{3.13}$$

in B and in $\mathbb{R}^3 \setminus B$ and that they are convergent as $k\delta \to 0$. We also assume that similar expansions hold for the incident and scattered components and that the incident field, being analytic, admits a complete Taylor's series expansion in $\mathbb{R}^3$. Substituting (3.12) and (3.13) into (3.6) and equating coefficients of $(k\delta)^0$, it follows that the fields $\tilde{\tilde{\boldsymbol{e}}}_0^{\text{sc}}$, $\tilde{\tilde{\boldsymbol{h}}}_0^{\text{sc}}$, $\tilde{\tilde{\boldsymbol{e}}}_0^{\text{in}}$, $\tilde{\tilde{\boldsymbol{h}}}_0^{\text{in}}$ satisfy a similar system to (3.9) in B and in $\mathbb{R}^3 \setminus B$.

Equating coefficients of $(k\delta)^1$ yields

$$\begin{aligned}\nabla_{r'} \times \tilde{\tilde{\boldsymbol{e}}}_1 &= \mathrm{i}\tilde{\mu}_r \tilde{\tilde{\boldsymbol{h}}}_0,\\ \nabla_{r'} \times \tilde{\tilde{\boldsymbol{h}}}_1 &= -\mathrm{i}\tilde{\epsilon}_r \tilde{\tilde{\boldsymbol{e}}}_0,\\ \nabla_{r'} \cdot \tilde{\tilde{\boldsymbol{e}}}_1 &= \nabla_{r'} \cdot \tilde{\tilde{\boldsymbol{h}}}_1 = \boldsymbol{0}\end{aligned} \tag{3.14}$$

in B, with $\tilde{\epsilon}_r = \epsilon_r$, $\tilde{\mu}_r = \mu_r$, and the corresponding coefficients of the incident and scattered fields satisfy a similar system in $\mathbb{R}^3 \setminus B$ with $\tilde{\epsilon}_r = \tilde{\mu}_r = 1$. For $m = 0, 1$ the $(k\delta)^m$ coefficients of the fields satisfy similar transmission conditions to those given in (3.11).

4. Main results

In this section, we summarize the main results of the paper that will later be proved in Section 5. We reiterate that we call the situation of $k\delta \to 0$ a low-frequency problem and that, by $\delta/r \to 0$, we denote distances that are large compared with the object's size. This is different from the characterization in Kleinman (1967, 1973), which omits the object's size in the definition of a low-frequency problem. The leading-order k^2/r term he obtains corresponds to the case where $k \to 0$ and $r \to \infty$. Our results relate to the perturbation in the fields at locations where the distance from the object to the point of observation can be characterized in terms of the object's size.

The case of the scattered fields at distances that are large compared with the object's size due to the presence of a small (*lossy*) dielectric scatterer, *for fixed k*, is described by the following theorem.

THEOREM 4.1 For a smooth closed (lossy) object B located at a distance $|z|$ from the origin, lying in free space and scaled by δ, the scattered electric and magnetic fields admitting expansions of the form described in Section 3.1 at a distance $r = |\boldsymbol{r}|$ from the centre of the object and $|\boldsymbol{r} + z|$ from the origin for fixed k take the form

$$\begin{aligned}\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r}+z) = \frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi}\Bigg(&\frac{1}{r^3}(3(\hat{\boldsymbol{r}}\cdot([\![\mathcal{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\mathrm{in}}(z)))\hat{\boldsymbol{r}} - [\![\mathcal{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\mathrm{in}}(z))\\ &-\frac{\mathrm{i}k}{r^2}(3(\hat{\boldsymbol{r}}\cdot([\![\mathcal{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\mathrm{in}}(z)))\hat{\boldsymbol{r}} - [\![\mathcal{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\mathrm{in}}(z) + \hat{\boldsymbol{r}}\times([\![\mathcal{M}_B(\mu_r)]\!]\boldsymbol{H}^{\mathrm{in}}(z)))\\ &-\frac{k^2}{r}(\hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times([\![\mathcal{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\mathrm{in}}(z))) + \hat{\boldsymbol{r}}\times([\![\mathcal{M}_B(\mu_r)]\!]\boldsymbol{H}^{\mathrm{in}}(z)))\Bigg) + O(\varGamma^4)\end{aligned} \tag{4.1}$$

and

$$\begin{aligned}\boldsymbol{H}^{\mathrm{sc}}(\boldsymbol{r}+z) = \frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi}\Bigg(&\frac{1}{r^3}(3(\hat{\boldsymbol{r}}\cdot([\![\mathcal{M}_B(\mu_r)]\!]\boldsymbol{H}^{\mathrm{in}}(z)))\hat{\boldsymbol{r}} - [\![\mathcal{M}_B(\mu_r)]\!]\boldsymbol{H}^{\mathrm{in}}(z))\\ &-\frac{\mathrm{i}k}{r^2}(3(\hat{\boldsymbol{r}}\cdot([\![\mathcal{M}_B(\mu_r)]\!]\boldsymbol{H}^{\mathrm{in}}(z)))\hat{\boldsymbol{r}} - [\![\mathcal{M}_B(\mu_r)]\!]\boldsymbol{H}^{\mathrm{in}}(z) - \hat{\boldsymbol{r}}\times([\![\mathcal{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\mathrm{in}}(z)))\\ &-\frac{k^2}{r}(\hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times([\![\mathcal{M}_B(\mu_r)]\!]\boldsymbol{H}^{\mathrm{in}}(z))) - \hat{\boldsymbol{r}}\times([\![\mathcal{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\mathrm{in}}(z)))\Bigg) + O(\varGamma^4)\end{aligned} \tag{4.2}$$

as $\Gamma = \max(\delta/r, \delta) \to 0$ where r is independent of δ. In the above

$$[\![\mathscr{M}_B(c)]\!]_{ij} = (c-1)|B|[\![\mathbb{I}]\!]_{ij} + (c-1)^2 \int_{\partial B} r_i' \hat{\boldsymbol{n}}' \cdot \nabla_{r'} \vartheta_j^{\mathrm{sc}}(c)|_- \, \mathrm{d}S' \tag{4.3}$$

is a symmetric polarization tensor associated with the material contrast c where $[\![\mathbb{I}]\!]$ denotes the identity matrix and $|B|$ the size of B, and $\vartheta_i^{\mathrm{sc}}(c)$ satisfies the scalar transmission problem

$$\begin{aligned} \nabla_{r'} \cdot c \nabla_{r'} \vartheta_i^{\mathrm{sc}} &= 0 \quad \text{in } B, \\ \nabla_{r'}^2 \vartheta_i^{\mathrm{sc}} &= 0 \quad \text{in } \mathbb{R}^3 \setminus B, \\ \vartheta_i^{\mathrm{sc}}|_+ - \vartheta_i^{\mathrm{sc}}|_- &= 0 \quad \text{on } \partial B, \\ \hat{\boldsymbol{n}}' \cdot \nabla_{r'} \vartheta_i^{\mathrm{sc}}|_+ - \hat{\boldsymbol{n}}' \cdot c \nabla_{r'} \vartheta_i^{\mathrm{sc}}|_- &= \hat{\boldsymbol{n}}' \cdot \nabla_{r'} r_i' \quad \text{on } \partial B, \\ \vartheta_i^{\mathrm{sc}} &\to 0 \quad \text{as } r \to \infty. \end{aligned}$$

COROLLARY 4.1 In the case of a small perfect conductor the corresponding asymptotic expansions for $\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z})$ and $\boldsymbol{H}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z})$, for fixed k as $\Gamma \to 0$, are given by replacing $[\![\mathscr{M}_B(\epsilon_r)]\!]$ with $[\![\mathscr{M}_B(\infty)]\!]$ and $[\![\mathscr{M}_B(\mu_r)]\!]$ with $[\![\mathscr{M}_B(0)]\!]$ in (4.1) and (4.2) in Theorem 4.1.

REMARK 4.1 If we set $\boldsymbol{x} = \boldsymbol{r} + \boldsymbol{z}$, $x = |\boldsymbol{x}|$ and $\hat{\boldsymbol{x}} = \boldsymbol{x}/x$ in (4.1) in Theorem 4.1 we can show, for $x \to \infty$, with fixed $\boldsymbol{z}$, that

$$\begin{aligned} \boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{x}) = -\frac{\delta^3 k^2 \, \mathrm{e}^{\mathrm{i}k(x - \hat{\boldsymbol{x}} \cdot \boldsymbol{z})}}{4\pi x} &((\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times ([\![\mathscr{M}_B(\epsilon_r)]\!] \boldsymbol{E}^{\mathrm{in}}(\boldsymbol{z}))) \\ &+ \hat{\boldsymbol{x}} \times ([\![\mathscr{M}_B(\mu_r)]\!] \boldsymbol{H}^{\mathrm{in}}(\boldsymbol{z})))) + O(\delta^4) \end{aligned} \tag{4.4}$$

as $\delta \to 0$, with a similar expression for $\boldsymbol{H}^{\mathrm{sc}}(\boldsymbol{x})$. By noting that $\hat{\boldsymbol{x}} \times (\hat{\boldsymbol{x}} \times \boldsymbol{A}) = -([\![\mathbb{I}]\!] - \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}^\top)\boldsymbol{A}$ for any vector field $\boldsymbol{A}$ and applying simple manipulations, we can show that (4.4) is the same as Ammari & Volkov (2005, Theorem 1.1) for a single scatterer. Note that the polarization tensor that appears in Ammari & Volkov (2005, Theorem 1.1) is equivalent (apart from a minus sign) to (4.3). This is easily seen by considering the definition of the potential, the application of the divergence theorem, the symmetry of the polarization tensor and considering the scaling that appears in their Theorem 1.1.

The corresponding result for the scattered fields due to the presence of a *non-lossy* dielectric scatterer at distances that are large compared with the object's size, *for a low-frequency problem*, is described by the following theorem.

THEOREM 4.2 For a smooth closed non-lossy object B located at a distance $|\boldsymbol{z}|$ from the origin, lying in free space and scaled by δ, the scattered electric and magnetic fields admitting expansions of the form described in Section 3.2 at a distance $r = |\boldsymbol{r}|$ from the centre of the object and $|\boldsymbol{r} + \boldsymbol{z}|$ from the origin take the form

$$\boldsymbol{E}^{\text{sc}}(\boldsymbol{r}+\boldsymbol{z}) = \frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi}\left(\frac{1}{r^3}(3(\hat{\boldsymbol{r}}\cdot([\![\mathscr{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\text{in}}(\boldsymbol{z})))\hat{\boldsymbol{r}} - [\![\mathscr{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\text{in}}(\boldsymbol{z})) \right.$$
$$- \frac{\mathrm{i}k}{r^2}(3(\hat{\boldsymbol{r}}\cdot([\![\mathscr{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\text{in}}(\boldsymbol{z})))\hat{\boldsymbol{r}} - [\![\mathscr{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\text{in}}(\boldsymbol{z}) + \hat{\boldsymbol{r}}\times([\![\mathscr{M}_B(\mu_r)]\!]\boldsymbol{H}^{\text{in}}(\boldsymbol{z})))$$
$$\left. - \frac{k^2}{r}(\hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times([\![\mathscr{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\text{in}}(\boldsymbol{z}))) + \hat{\boldsymbol{r}}\times([\![\mathscr{M}_B(\mu_r)]\!]\boldsymbol{H}^{\text{in}}(\boldsymbol{z})))\right) + O(\varXi^4) \tag{4.5}$$

and

$$\boldsymbol{H}^{\text{sc}}(\boldsymbol{r}+\boldsymbol{z}) = \frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi}\left(\frac{1}{r^3}(3(\hat{\boldsymbol{r}}\cdot([\![\mathscr{M}_B(\mu_r)]\!]\boldsymbol{H}^{\text{in}}(\boldsymbol{z})))\hat{\boldsymbol{r}} - [\![\mathscr{M}_B(\mu_r)]\!]\boldsymbol{H}^{\text{in}}(\boldsymbol{z})) \right.$$
$$- \frac{\mathrm{i}k}{r^2}(3(\hat{\boldsymbol{r}}\cdot([\![\mathscr{M}_B(\mu_r)]\!]\boldsymbol{H}^{\text{in}}(\boldsymbol{z})))\hat{\boldsymbol{r}} - [\![\mathscr{M}_B(\mu_r)]\!]\boldsymbol{H}^{\text{in}}(\boldsymbol{z}) - \hat{\boldsymbol{r}}\times([\![\mathscr{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\text{in}}(\boldsymbol{z})))$$
$$\left. - \frac{k^2}{r}(\hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times([\![\mathscr{M}_B(\mu_r)]\!]\boldsymbol{H}^{\text{in}}(\boldsymbol{z}))) - \hat{\boldsymbol{r}}\times([\![\mathscr{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\text{in}}(\boldsymbol{z})))\right) + O(\varXi^4) \tag{4.6}$$

as $\varXi = \max(\delta/r, k\delta) \to 0$, where k and r are independent of δ. In the above $[\![\mathscr{M}_B(c)]\!]$ is the symmetric polarization tensor given in (4.3).

COROLLARY 4.2 In the case of a small perfect conductor at low frequencies the corresponding asymptotic expansions for $\boldsymbol{E}^{\text{sc}}(\boldsymbol{r}+\boldsymbol{z})$ and $\boldsymbol{H}^{\text{sc}}(\boldsymbol{r}+\boldsymbol{z})$ as $\varXi \to 0$ are given by replacing $[\![\mathscr{M}_B(\epsilon_r)]\!]$ with $[\![\mathscr{M}_B(\infty)]\!]$ and $[\![\mathscr{M}_B(\mu_r)]\!]$ with $[\![\mathscr{M}_B(0)]\!]$ in (4.5) and (4.6) in Theorem 4.2.

REMARK 4.2 Theorems 4.1 and 4.2 are the same, apart from the remainders, $\boldsymbol{E}^s(\boldsymbol{r}+\boldsymbol{z})$ and $\boldsymbol{H}^s(\boldsymbol{r}+\boldsymbol{z})$, and take a similar form to that of a radiating dipole (Jones, 1964), when the products of polarization tensor and incident fields evaluated at the centre of the object are replaced by appropriate electric or magnetic dipole moments. Note that Theorem 4.1 is not a consequence of Theorem 4.2 as the former also holds in the case of a lossy dielectric object. However, by fixing k in Theorem 4.2 the results agree for a non-lossy object. By fixing δ, letting $r \to \infty$ and setting $\boldsymbol{z} = \boldsymbol{0}$ in Theorem 4.2, then the result of Keller *et al.* (1972) for $\boldsymbol{E}^s(\boldsymbol{r})$ and $\boldsymbol{H}^s(\boldsymbol{r})$ as $k \to 0$ can be recovered. However, importantly, Theorem 4.2, and hence the aforementioned simplification, is not applicable to lossy objects, as is sometimes advocated e.g. Kleinman & Senior (1982), since ϵ_r would be a function of k in this case (see also Remark 5.1). Furthermore, our results also confirm that the scattering from a small (lossy) object, or a non-lossy object at low frequencies, behaves like a radiating dipole with appropriate moments, not only in the far field (as described in, e.g., Keller *et al.*, 1972), but also at distances that are large compared with the object's size.

5. Asymptotic expansions of the fields in terms of polarization moments and proofs of the main results

The leading-order terms in asymptotic expansions of the scattered fields, in terms of polarization moments, for the exact scaled fields, the scaled fields expanded in terms of δ and in terms of $k\delta$, at distances that are large compared with the object's size, are described by the following lemma.

LEMMA 5.1 For a smooth closed object B located at a distance $|z|$ from the origin, lying in free space and scaled by δ, the scattered electric and magnetic fields at a distance $r=|\boldsymbol{r}|$ from the centre of the object and $|\boldsymbol{r}+z|$ from the origin take the form

$$\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r}+z)=\frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi}\left(\frac{1}{r^3}(3(\hat{\boldsymbol{r}}\cdot\boldsymbol{p}_B)\hat{\boldsymbol{r}}-\boldsymbol{p}_B)-\frac{\mathrm{i}k}{r^2}(3(\hat{\boldsymbol{r}}\cdot\boldsymbol{p}_B)\hat{\boldsymbol{r}}-\boldsymbol{p}_B+\hat{\boldsymbol{r}}\times\boldsymbol{m}_B)\right.$$
$$\left.-\frac{k^2}{r}(\hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times\boldsymbol{p}_B)+\hat{\boldsymbol{r}}\times\boldsymbol{m}_B)\right)+\Delta \tag{5.1}$$

and

$$\boldsymbol{H}^{\mathrm{sc}}(\boldsymbol{r}+z)=\frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi}\left(\frac{1}{r^3}(3(\hat{\boldsymbol{r}}\cdot\boldsymbol{m}_B)\hat{\boldsymbol{r}}-\boldsymbol{m}_B)-\frac{\mathrm{i}k}{r^2}(3(\hat{\boldsymbol{r}}\cdot\boldsymbol{m}_B)\hat{\boldsymbol{r}}-\boldsymbol{m}_B-\hat{\boldsymbol{r}}\times\boldsymbol{p}_B)\right.$$
$$\left.-\frac{k^2}{r}(\hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times\boldsymbol{m}_B)-\hat{\boldsymbol{r}}\times\boldsymbol{p}_B)\right)+\Delta, \tag{5.2}$$

where

$$\boldsymbol{p}_B(\boldsymbol{u}^{\mathrm{sc}})=\int_{\partial B}\boldsymbol{r}'\hat{\boldsymbol{n}}'\cdot\boldsymbol{u}^{\mathrm{sc}}|_+ +\frac{1}{2}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{u}^{\mathrm{sc}})|_+\,\mathrm{d}S', \tag{5.3}$$

$$\boldsymbol{m}_B(\boldsymbol{v}^{\mathrm{sc}})=\int_{\partial B}\boldsymbol{r}'\hat{\boldsymbol{n}}'\cdot\boldsymbol{v}^{\mathrm{sc}}|_+ +\frac{1}{2}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{v}^{\mathrm{sc}})|_+\,\mathrm{d}S' \tag{5.4}$$

are the electric and magnetic polarization moments (or their leading-order term, depending on the argument) for the object B. Depending on whether the exact scaled fields are used, or if an appropriate expansion is chosen, we have the following:

(a) $\Delta=O(\Xi^4)f(k\delta)$, for some $f(k\delta)$ that has the property that $f(k\delta)\to 0$ as $k\delta\to 0$, where $\Xi=\max(\delta/r,k\delta)$ when $\boldsymbol{u}^{\mathrm{sc}}=\boldsymbol{e}^{\mathrm{sc}}$, $\boldsymbol{v}^{\mathrm{sc}}=\boldsymbol{h}^{\mathrm{sc}}$ and r, k are independent of δ. In this case (5.1) and (5.2) are not asymptotic expansions as $f(k\delta)$ is used to denote the fact that the higher-order terms involve the exact scaled fields, but nevertheless the result is useful to make comparisons with those available in the literature.

(b) $\Delta=O(\Gamma^4)$, $\boldsymbol{u}^{\mathrm{sc}}=\tilde{\boldsymbol{e}}_0^{\mathrm{sc}}$ and $\boldsymbol{v}^{\mathrm{sc}}=\tilde{\boldsymbol{h}}_0^{\mathrm{sc}}$ assuming that an expansion of the form (3.7) is valid for $\boldsymbol{e}$ and (3.8) is valid for $\boldsymbol{h}$. In this case (5.1) and (5.2) are asymptotic expansions for the fields in the presence of a (lossy) dielectric scatterer for fixed k as $\Gamma=\max(\delta/r,\delta)\to 0$, where r is independent of δ.

(c) $\Delta=O(\Xi^4)$, $\boldsymbol{u}^{\mathrm{sc}}=\tilde{\tilde{\boldsymbol{e}}}_0^{\mathrm{sc}}$ and $\boldsymbol{v}^{\mathrm{sc}}=\tilde{\tilde{\boldsymbol{h}}}_0^{\mathrm{sc}}$ assuming that an expansion of the form (3.12) is valid for $\boldsymbol{e}$ and (3.13) is valid for $\boldsymbol{h}$. In this case (5.1) and (5.2) are asymptotic expansions for the fields in the presence of a non-lossy dielectric scatterer as $\Xi=\max(\delta/r,k\delta)\to 0$, where r and k are independent of δ.

Proof.
Case (a)

We focus on the expansion for $\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r}+z)$; the result for $\boldsymbol{H}^{\mathrm{sc}}(\boldsymbol{r}+z)$ can be obtained analogously. Noting that the differentiation in (3.4) is with respect to $\boldsymbol{r}$, and that the tangential components of $\boldsymbol{e}^{\mathrm{sc}}$ and

$\boldsymbol{h}^{\text{sc}}$ in the integrand are functions of $\boldsymbol{r}'$ only, we can rewrite this expression as

$$\boldsymbol{E}^{\text{sc}}(\boldsymbol{r}+\boldsymbol{z}) = \frac{\mathrm{i}\delta^2}{k}\int_{\partial B}(k^2 u_k(\boldsymbol{r}-\delta\boldsymbol{r}')\boldsymbol{a} + \nabla_r(\nabla_r u_k(\boldsymbol{r}-\delta\boldsymbol{r}'))\cdot\boldsymbol{a})\,\mathrm{d}S' + \delta^2\int_{\partial B}(\nabla_r u_k(\boldsymbol{r}-\delta\boldsymbol{r}')\times\boldsymbol{b})\,\mathrm{d}S', \tag{5.5}$$

where $\boldsymbol{a}(\boldsymbol{r}')=\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\text{sc}}$, $\boldsymbol{b}(\boldsymbol{r}')=\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\text{sc}}$ and the results $\nabla_r\times(u_k(\boldsymbol{b}))=\nabla_r u_k\times\boldsymbol{b}$ and $\nabla_r\times(\nabla_r\times(u_k\boldsymbol{a}))=k^2u\boldsymbol{a}+\nabla_r(\boldsymbol{a}\cdot\nabla_r u_k)=k^2u\boldsymbol{a}+\nabla_r(\nabla_r u_k)\cdot\boldsymbol{a}$ (Stratton, 1941, page 465) have been applied. Throughout the proof we assume that the integrands are to be evaluated in the limit as ∂B is approached from outside the object.

Let $\alpha=(\alpha_1,\ldots,\alpha_d)$ be a multi-index, such that $\alpha!=\alpha_1!\ldots\alpha_d!$, $\boldsymbol{r}^\alpha=r_1^{\alpha_1}\ldots r_d^{\alpha_d}$ and $\partial_r^\alpha(u)=\partial_r^{\alpha_1}\ldots\partial_r^{\alpha_d}(u)$. Then, by assuming that $\delta/r\to 0$, the terms that involve the Green's function evaluated at $\boldsymbol{r}-\delta\boldsymbol{r}'$ in $k^2u_k(\boldsymbol{r}-\delta\boldsymbol{r}')\boldsymbol{a}+\nabla_r(\nabla_r u_k(\boldsymbol{r}-\delta\boldsymbol{r}'))\cdot\boldsymbol{a}$ and $\nabla_r u_k(\boldsymbol{r}-\delta\boldsymbol{r}')$ can be expanded using a Taylor's series expansion about $\boldsymbol{r}$ so that

$$\nabla_r u_k(\boldsymbol{r}-\delta\boldsymbol{r}') = \sum_{\alpha,|\alpha|=0}^{\infty}\delta^{|\alpha|}\frac{(-\boldsymbol{r}')^\alpha}{\alpha!}\partial_r^\alpha(\nabla_r u_k(\boldsymbol{r})), \tag{5.6}$$

$$k^2u_k(\boldsymbol{r}-\delta\boldsymbol{r}')\boldsymbol{a}+\nabla_r(\nabla_r u_k(\boldsymbol{r}-\delta\boldsymbol{r}'))\cdot\boldsymbol{a} = k^2\boldsymbol{a}\sum_{\alpha,|\alpha|=0}^{\infty}\delta^{|\alpha|}\frac{(-\boldsymbol{r}')^\alpha}{\alpha!}\partial_r^\alpha(u_k(\boldsymbol{r})) + \left(\sum_{\alpha,|\alpha|=0}^{\infty}\delta^{|\alpha|}\frac{(-\boldsymbol{r}')^\alpha}{\alpha!}\partial_r^\alpha(\nabla_r(\nabla_r u_k(\boldsymbol{r})))\right)\cdot\boldsymbol{a}. \tag{5.7}$$

Focusing on terms with $|\alpha|=0$ or $|\alpha|=1$, these can be explicitly computed as

$$\sum_{|\alpha|=0}\delta^{|\alpha|}\frac{(-\boldsymbol{r}')^\alpha}{\alpha!}\partial_r^\alpha(\nabla_r u_k(\boldsymbol{r})) = \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi}\left(\frac{\mathrm{i}k}{r}-\frac{1}{r^2}\right)\hat{\boldsymbol{r}}, \tag{5.8}$$

$$\sum_{|\alpha|=1}\delta^{|\alpha|}\frac{(-\boldsymbol{r}')^\alpha}{\alpha!}\partial_r^\alpha(\nabla_r u_k(\boldsymbol{r})) = \frac{\delta\,\mathrm{e}^{\mathrm{i}kr}}{4\pi}\left(\frac{k^2}{r}(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}+\frac{\mathrm{i}k}{r^2}(3(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}-\boldsymbol{r}')+\frac{1}{r^3}(\boldsymbol{r}'-3(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}})\right), \tag{5.9}$$

$$k^2\boldsymbol{a}\sum_{|\alpha|=0}\delta^{|\alpha|}\frac{(-\boldsymbol{r}')^\alpha}{\alpha!}\partial_r^\alpha(u_k(\boldsymbol{r})) + \left(\sum_{|\alpha|=0}\delta^{|\alpha|}\frac{(-\boldsymbol{r}')^\alpha}{\alpha!}\partial_r^\alpha(\nabla_r(\nabla_r u_k(\boldsymbol{r})))\right)\cdot\boldsymbol{a}$$
$$= \frac{\mathrm{e}^{\mathrm{i}kr}}{4\pi}\left(\frac{k^2}{r}(\boldsymbol{a}-(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}})+\frac{\mathrm{i}k}{r^2}(\boldsymbol{a}-3(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}})+\frac{1}{r^3}(3(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}-\boldsymbol{a})\right), \tag{5.10}$$

$$k^2\boldsymbol{a}\sum_{|\alpha|=1}\delta^{|\alpha|}\frac{(-\boldsymbol{r}')^\alpha}{\alpha!}\partial_r^\alpha(u_k(\boldsymbol{r})) + \left(\sum_{|\alpha|=1}\delta^{|\alpha|}\frac{(-\boldsymbol{r}')^\alpha}{\alpha!}\partial_r^\alpha(\nabla_r(\nabla_r u_k(\boldsymbol{r})))\right)\cdot\boldsymbol{a}$$
$$= \frac{\delta\,\mathrm{e}^{\mathrm{i}kr}}{4\pi}\left(-\frac{\mathrm{i}k^3}{r}((\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\boldsymbol{a}-(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}})\right.$$

$$+\frac{k^2}{r^2}(2(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\boldsymbol{a}+(\boldsymbol{a}\cdot\boldsymbol{r}')\hat{\boldsymbol{r}}+(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})\boldsymbol{r}'-6(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}})$$
$$+\frac{\mathrm{i}k}{r^3}(3(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\boldsymbol{a}+3(\boldsymbol{a}\cdot\boldsymbol{r}')\hat{\boldsymbol{r}}+3(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})\boldsymbol{r}'-12(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}})\Bigg). \tag{5.11}$$

To obtain the asymptotic expansion for $\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z})$, we insert (5.6) and (5.7) into (5.5) and then use equations (5.8–5.11) and a series of integral identities. To simply the presentation, we gather terms in powers of r and work towards obtaining an expression of the form

$$\begin{aligned}\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z})&=\boldsymbol{t}_{k^2\delta^3/r}+\boldsymbol{t}_{k\delta^3/r^2}+\boldsymbol{t}_{\delta^3/r^3}+\Delta,\\ \Delta&=(\Delta_{k^3\delta^4/r}+\Delta_{k^2\delta^4/r^2}+\Delta_{k\delta^4/r^3})=O(\varXi^4)f(k\delta),\end{aligned} \tag{5.12}$$

where $\varXi=\max(\delta/r,k\delta)$ and $f(k\delta)$, which has the property that $f(k\delta)\to 0$ as $k\delta\to 0$, is used to denote that the higher-order terms involve the exact fields that are functions of $k\delta$. The notation $\boldsymbol{t}_{k^2\delta^3/r}$ is used to denote the leading-order $O(k^2\delta^3/r)f(k\delta)$ term with similar meanings for $\boldsymbol{t}_{k\delta^3/r^2}$ and $\boldsymbol{t}_{\delta^3/r^3}$. Note that these leading order terms contain $(\delta/r)^{\beta_1}(k\delta)^{\beta_2}f(k\delta)$ with the multi-index $\beta=(\beta_1,\beta_2)$ such that $|\beta|\leqslant 3$. The remainder Δ is a sum of three terms, each involving $f(k\delta)$, $\Delta_{k^3\delta^4/r}$, indicating a term of order $O(k^3\delta^4/r)f(k\delta)$ with similar meanings for $\Delta_{k^2\delta^4/r^2}$ and $\Delta_{k\delta^4/r^3}$. In our derivation the terms involving different powers of r are considered separately, and, for a fixed power of r, the term involving the lowest power of k is sought, with k and r independent of δ. Higher-order terms, with $|\alpha|\geqslant 2$, are such that they will be absorbed into Δ.

Terms making up the $\boldsymbol{t}_{k^2\delta^3/r}$ *contribution*

Considering (5.8), we obtain

$$\begin{aligned}\frac{\mathrm{i}k\delta^2\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\hat{\boldsymbol{r}}\times\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}}\,\mathrm{d}S'&=\frac{\mathrm{i}k\delta^2\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\nabla_{r'}\times\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\\&=-\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\boldsymbol{h}^{\mathrm{sc}})\,\mathrm{d}S',\end{aligned} \tag{5.13}$$

where the last two equalities follow by using the identity (A.1) and applying (3.6) for the scattered fields in $\mathbb{R}^3\setminus B$ since the integrand is evaluated just outside of ∂B. From (5.9), we obtain

$$\begin{aligned}&\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\hat{\boldsymbol{r}}\times\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}}(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\,\mathrm{d}S'\\&=\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\hat{\boldsymbol{r}}\times\left(-\frac{1}{2}\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'+\frac{1}{2}\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\nabla_{r'}\times\boldsymbol{e}^{\mathrm{sc}})(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\,\mathrm{d}S'\right)\\&=-\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4(2)\pi r}\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'-\mathrm{i}k\delta\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\boldsymbol{h}^{\mathrm{sc}})(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\,\mathrm{d}S'\right)\\&=-\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4(2)\pi r}\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right)+O\left(\frac{\delta}{r}k^3\delta^3\right)f(k\delta),\end{aligned} \tag{5.14}$$

which follows by using the identity (A.2) and (3.6). Considering (5.10), we obtain

$$\frac{\mathrm{i}k\delta^2\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\left(\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\,\mathrm{d}S'-\int_{\partial B}(\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\,\mathrm{d}S'\right)=-\frac{\mathrm{i}k\delta^2\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\,\mathrm{d}S'\right)$$
$$=-\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right),\tag{5.15}$$

by application of identities (A.3) and (A.1) and using (3.6). Finally, considering (5.11), we obtain

$$\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\left(\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}(\boldsymbol{r}'\cdot\boldsymbol{r})\,\mathrm{d}S'-\int_{\partial B}(\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\cdot\hat{\boldsymbol{r}})(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\,\mathrm{d}S'\right)$$
$$=-\frac{1}{2}\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}})\,\mathrm{d}S'+O\left(\frac{\delta}{r}k^3\delta^3\right)f(k\delta),\tag{5.16}$$

which follows from the identities (A.2) and (A.4).

Summing (5.13–5.16), and using the definitions of $\boldsymbol{p}_B=\boldsymbol{p}_B(\boldsymbol{e}^{\mathrm{sc}})$ and $\boldsymbol{m}_B=\boldsymbol{m}_B(\boldsymbol{h}^{\mathrm{sc}})$, we find that the order $\boldsymbol{t}_{k^2\delta^3/r}$ contribution to (5.5) is

$$.\boldsymbol{t}_{k^2\delta^3/r}=-\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}(\hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times\boldsymbol{p}_B)+\hat{\boldsymbol{r}}\times\boldsymbol{m}_B),\quad \Delta_{k^3\delta^4/r}=O\left(\frac{\delta}{r}k^3\delta^3\right)f(k\delta).\tag{5.17}$$

Terms making up the $\boldsymbol{t}_{k\delta^3/r^2}$ *contribution*
Considering (5.8), we obtain

$$-\frac{\delta^2\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}\hat{\boldsymbol{r}}\times\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}}\,\mathrm{d}S'=-\frac{\mathrm{i}k\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\boldsymbol{h}^{\mathrm{sc}})\,\mathrm{d}S',\tag{5.18}$$

which is obtained in a similar manner to (5.13). From (5.9), we obtain

$$\frac{\mathrm{i}k\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}\left(3\hat{\boldsymbol{r}}\times\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}}(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\,\mathrm{d}S'-\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right)$$
$$=-\frac{\mathrm{i}k\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}\left(\frac{3}{2}\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right)+\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right)+O\left(\frac{\delta^2}{r^2}k^2\delta^2\right)f(k\delta),\tag{5.19}$$

by using identity (A.2). From (5.10), we obtain

$$-\frac{\delta^2\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}\left(\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\,\mathrm{d}S'-3\int_{\partial B}(\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\,\mathrm{d}S'\right)$$
$$=\frac{\delta^2\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}\left(2\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\,\mathrm{d}S'+3\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\,\mathrm{d}S'\right)\right)$$
$$=-\frac{\mathrm{i}k\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}\left(2\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'+3\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right)\right),\tag{5.20}$$

where the last two equalities follow by using the identities (A.1) and (A.3) and (3.6). Finally, considering (5.11), we obtain

$$\frac{\mathrm{i}k\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}\left(2\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\,\mathrm{d}S'+\int_{\partial B}(\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\cdot\boldsymbol{r}')\hat{\boldsymbol{r}}\,\mathrm{d}S'+\int_{\partial B}(\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\cdot\hat{\boldsymbol{r}})\boldsymbol{r}'\,\mathrm{d}S'\right.$$
$$\left.-6\int_{\partial B}(\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\cdot\hat{\boldsymbol{r}})(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\,\mathrm{d}S'\right)=-\frac{1}{2}\frac{\mathrm{i}k\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}})\,\mathrm{d}S'+O\left(\frac{\delta^2}{r^2}k^2\delta^2\right)f(k\delta), \tag{5.21}$$

which follows from the identities (A.2), (A.5), (A.9) and (A.4).

Summing (5.18–5.21), and using the definitions of $\boldsymbol{p}_B$ and $\boldsymbol{m}_B$, we find that the $\boldsymbol{t}_{k\delta^3/r^2}$ contribution to (5.5) is

$$\begin{aligned}\boldsymbol{t}_{k\delta^3/r^2}&=-\frac{\mathrm{i}k\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}(3\hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times\boldsymbol{p}_B)+2\boldsymbol{p}_B+\hat{\boldsymbol{r}}\times\boldsymbol{m}_B)\\&=-\frac{\mathrm{i}k\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}(3(\hat{\boldsymbol{r}}\cdot\boldsymbol{p}_B)\hat{\boldsymbol{r}}-\boldsymbol{p}_B+\hat{\boldsymbol{r}}\times\boldsymbol{m}_B),\quad \Delta_{k^2\delta^4/r^2}=O\left(\frac{\delta^2}{r^2}k^2\delta^2\right)f(k\delta).\end{aligned} \tag{5.22}$$

Terms making up the $\boldsymbol{t}_{\delta^3/r^3}$ *contribution*

There is no contribution from (5.8) to $\boldsymbol{t}_{\delta^3/r^3}$. From (5.9), we obtain

$$\begin{aligned}&\frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^3}\left(-3\hat{\boldsymbol{r}}\times\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}}(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\,\mathrm{d}S'+\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right)\\&=\frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^3}\left(\frac{3}{2}\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right)+\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right)+O\left(\frac{\delta^3}{r^3}k\delta\right)f(k\delta),\end{aligned} \tag{5.23}$$

by using identity (A.2). From (5.10), we obtain

$$\begin{aligned}&\frac{\delta^2\mathrm{i}\,\mathrm{e}^{\mathrm{i}kr}}{4k\pi r^3}\left(3\int_{\partial B}(\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\,\mathrm{d}S'-\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\,\mathrm{d}S'\right)\\&=\frac{\mathrm{i}\delta^2\,\mathrm{e}^{\mathrm{i}kr}}{4k\pi r^3}\left(3\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\,\mathrm{d}S'\right)+2\int_{\partial B}\hat{\boldsymbol{n}}'\times\boldsymbol{h}^{\mathrm{sc}}\,\mathrm{d}S'\right)\\&=\frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^3}\left(3\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right)+2\int\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right),\end{aligned} \tag{5.24}$$

where the last two equalities follow by using the identity (A.3) and (3.6). Finally, considering (5.11), we obtain

$$
\begin{aligned}
-&\frac{\delta^3 \mathrm{e}^{\mathrm{i}kr}}{4\pi r^3}\left(3\int_{\partial B} \hat{\boldsymbol{n}}' \times \boldsymbol{h}^{\mathrm{sc}}(\boldsymbol{r}'\cdot\boldsymbol{r})\,\mathrm{d}S' + 3\int_{\partial B}(\hat{\boldsymbol{n}}' \times \boldsymbol{h}^{\mathrm{sc}}\cdot\boldsymbol{r}')\hat{\boldsymbol{r}}\,\mathrm{d}S' + 3\int_{\partial B}(\hat{\boldsymbol{n}}' \times \boldsymbol{h}^{\mathrm{sc}}\cdot\hat{\boldsymbol{r}})\boldsymbol{r}'\,\mathrm{d}S'\right.\\
&\left.-12\int_{\partial B}(\hat{\boldsymbol{n}}' \times \boldsymbol{h}^{\mathrm{sc}}\cdot\hat{\boldsymbol{r}})(\boldsymbol{r}'\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\,\mathrm{d}S'\right)\\
&= -\frac{\delta^3 \mathrm{e}^{\mathrm{i}kr}}{4\pi r^3}\left(-\frac{3}{2}\hat{\boldsymbol{r}} \times \int_{\partial B}\boldsymbol{r}' \times (\hat{\boldsymbol{n}}' \times \boldsymbol{h}^{\mathrm{sc}})\,\mathrm{d}S' + \frac{3}{2}\hat{\boldsymbol{r}} \times \int_{\partial B}\boldsymbol{r}' \times (\hat{\boldsymbol{n}}' \times \boldsymbol{h}^{\mathrm{sc}})\,\mathrm{d}S' + O(k\delta)f(k\delta)\right)\\
&= O\left(\frac{\delta^3}{r^3}k\delta\right)f(k\delta),
\end{aligned}
\tag{5.25}
$$

which follows from the identities (A.2), (A.5), (A.9) and (A.4).

Summing (5.23–5.25), and using the definition of $\boldsymbol{p}_B$, we find that the $\boldsymbol{t}_{\delta^3/r^3}$ contribution to (5.5) is

$$
\boldsymbol{t}_{\delta^3/r^3} = \frac{\delta^3 \mathrm{e}^{\mathrm{i}kr}}{4\pi r^3}(3\hat{\boldsymbol{r}} \times (\hat{\boldsymbol{r}} \times \boldsymbol{p}_B) + 2\boldsymbol{p}_B) = \frac{\delta^3 \mathrm{e}^{\mathrm{i}kr}}{4\pi r^3}(3(\hat{\boldsymbol{r}}\cdot\boldsymbol{p}_B)\hat{\boldsymbol{r}} - \boldsymbol{p}_B), \quad \Delta_{k\delta^4/r^3} = O\left(\frac{\delta^3}{r^3}k\delta\right)f(k\delta). \tag{5.26}
$$

Summing (5.17), (5.22) and (5.26), we realize that the each of the higher-order terms will involve the exact fields, as denoted by the presence of $f(k\delta)$ in Δ. We also realize that $\Delta_{k^3\delta^4/r} + \Delta_{k^2\delta^4/r^2} + \Delta_{k\delta^4/r^3}$ can be written as $O(\Xi^4)f(k\delta)$, since the sum of the contributions can be bounded as $\Delta_{k^3\delta^4/r} + \Delta_{k^2\delta^4/r^2} + \Delta_{k\delta^4/r^3} \leqslant C\Xi^4 f(k\delta)$, where $\Xi = \max(\delta/r, k\delta)$. This follows since $(\delta/r)^{\beta_1}(k\delta)^{\beta_2} \leqslant \Xi^{|\beta|}$ for $\delta/r < 1$ and $k\delta < 1$ with the multi-index $\beta = (\beta_1, \beta_2)$ and $|\beta| = 4$. This then completes the proof for part a).

Case (b)

We focus on the expansion for $\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z})$; the result for $\boldsymbol{H}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z})$ can be obtained analogously. We proceed in a similar manner to case (a), again assuming that $\delta/r \to 0$ and expanding the terms that involve the Green's function evaluated at $\boldsymbol{r} - \delta\boldsymbol{r}'$ in $k^2 u_k(\boldsymbol{r} - \delta\boldsymbol{r}')\boldsymbol{a} + \nabla_r(\nabla_r u_k(\boldsymbol{r} - \delta\boldsymbol{r}'))\cdot\boldsymbol{a}$ and $\nabla_r u_k(\boldsymbol{r} - \delta\boldsymbol{r}')$ about $\boldsymbol{r}$ for small $\delta\boldsymbol{r}'$. In this case, we look for an expansion of the form

$$
\begin{aligned}
\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z}) &= \tilde{\boldsymbol{t}}_{k^2\delta^3/r} + \tilde{\boldsymbol{t}}_{k\delta^3/r^2} + \tilde{\boldsymbol{t}}_{\delta^3/r^3} + \Delta = \sum_{m=0}^{1}(\tilde{\boldsymbol{t}}_{m,k^2\delta^3/r} + \tilde{\boldsymbol{t}}_{m,k\delta^3/r^2} + \tilde{\boldsymbol{t}}_{m,\delta^3/r^3}) + \Delta,\\
\Delta &= \tilde{\Delta}_{k^3\delta^4/r} + \tilde{\Delta}_{k^2\delta^4/r^2} + \tilde{\Delta}_{k\delta^4/r^3} = O(\Gamma^4),
\end{aligned}
$$

where $\Gamma = \max(\delta/r, \delta)$. The notation $\tilde{\boldsymbol{t}}_{k^2\delta^3/r} = \tilde{\boldsymbol{t}}_{0,k^2\delta^3/r} + \tilde{\boldsymbol{t}}_{1,k^2\delta^3/r}$ will be used to denote the leading-order $O(k^2\delta^3/r) = O(\delta^3/r)$ term, which simplifies since k is a constant in this case, with similar meanings for $\tilde{\boldsymbol{t}}_{k\delta^3/r^2}$ and $\tilde{\boldsymbol{t}}_{\delta^3/r^3}$. We will show that these leading-order terms contain $(\delta/r)^{\beta_1}(\delta)^{\beta_2}$ with the multi-index $\beta = (\beta_1, \beta_2)$ such that $|\beta| \leqslant 3$. The remainder Δ is a sum of three terms, $\tilde{\Delta}_{k^3\delta^4/r}$ indicating a term of order $O(k^3\delta^4/r) = O(\delta^4/r)$ with similar meanings for $\tilde{\Delta}_{k^2\delta^4/r^2}$ and $\tilde{\Delta}_{k\delta^4/r^3}$. In a similar manner to case (a) we consider only those terms with $|\alpha| < 2$ since we will see that terms with $|\alpha| \geqslant 2$ will be absorbed in to Δ.

Terms making up the $\tilde{\boldsymbol{t}}_{k^2\delta^3/r}$ contribution

By replacing $\boldsymbol{e}^{\mathrm{sc}}$ by $\tilde{\boldsymbol{e}}_0^{\mathrm{sc}} + O(\delta)$ and $\boldsymbol{h}^{\mathrm{sc}}$ by $\tilde{\boldsymbol{h}}_0^{\mathrm{sc}} + O(\delta)$, and adding contributions from (5.14) and (5.16) under the assumption that the expansions also hold for $\boldsymbol{e}^{\mathrm{sc}} = \lim_{\boldsymbol{r}'\to\partial B^+}\boldsymbol{e}^{\mathrm{sc}}(\boldsymbol{r}')$ and $\boldsymbol{h}^{\mathrm{sc}} =$

$\lim_{\boldsymbol{r}'\to\partial B^+}\boldsymbol{h}^{\mathrm{sc}}(\boldsymbol{r}')$, we find that

$$\tilde{\boldsymbol{t}}_{0,k^2\delta^3/r}=-\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4(2)\pi r}\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\tilde{\boldsymbol{e}}_0^{\mathrm{sc}})\,\mathrm{d}S'+\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\tilde{\boldsymbol{h}}_0^{\mathrm{sc}})\,\mathrm{d}S'\right),\tag{5.27}$$

plus higher-order terms that contribute to $\tilde{\varDelta}_{k^3\delta^4/r}=O((\delta/r)\delta^3)$. By replacing $\boldsymbol{e}^{\mathrm{sc}}$ by $\tilde{\boldsymbol{e}}_0^{\mathrm{sc}}+\delta\tilde{\boldsymbol{e}}_1^{\mathrm{sc}}+O(\delta^2)$ and $\boldsymbol{h}^{\mathrm{sc}}$ by $\tilde{\boldsymbol{h}}_0^{\mathrm{sc}}+\delta\tilde{\boldsymbol{h}}_1^{\mathrm{sc}}+O(\delta^2)$, adding contributions from (5.13) and (5.15), and noting that the scattered fields outside B satisfy a similar system to (3.10), we obtain

$$\tilde{\boldsymbol{t}}_{1,k^2\delta^3/r}=-\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}\hat{\boldsymbol{r}}\times\left(\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\tilde{\boldsymbol{h}}_0^{\mathrm{sc}})\,\mathrm{d}S'+\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\tilde{\boldsymbol{e}}_0^{\mathrm{sc}})\,\mathrm{d}S'\right),\tag{5.28}$$

plus higher-order terms that contribute to $\tilde{\varDelta}_{k^3\delta^4/r}$. Thus, we have that

$$\tilde{\boldsymbol{t}}_{k^2\delta^3/r}=\sum_{m=0}^{1}\tilde{\boldsymbol{t}}_{m,k^2\delta^3/r}=-\frac{k^2\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r}(\hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times\boldsymbol{p}_B)+\hat{\boldsymbol{r}}\times\boldsymbol{m}_B),\quad\tilde{\varDelta}_{k^3\delta^4/r}=O\left(\frac{\delta}{r}\delta^3\right),\tag{5.29}$$

where $\boldsymbol{p}_B=\boldsymbol{p}_B(\tilde{\boldsymbol{e}}_0^{\mathrm{sc}})$ and $\boldsymbol{m}_B=\boldsymbol{m}_B(\tilde{\boldsymbol{h}}_0^{\mathrm{sc}})$ are leading-order terms of the polarization moments in this case.
Terms making up the $\tilde{\boldsymbol{t}}_{k\delta^3/r^2}$ contribution
Again, following similar steps to part (a) and above

$$\begin{aligned}\tilde{\boldsymbol{t}}_{0,k\delta^3/r^2}=&-\frac{\mathrm{i}k\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4(2)\pi r^2}\left(3\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\tilde{\boldsymbol{e}}_0^{\mathrm{sc}})\,\mathrm{d}S'\right)+2\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\tilde{\boldsymbol{e}}_0^{\mathrm{sc}})\,\mathrm{d}S'\right.\\&\left.+\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\tilde{\boldsymbol{h}}_0^{\mathrm{sc}})\,\mathrm{d}S'\right),\\\tilde{\boldsymbol{t}}_{1,k\delta^3/r^2}=&-\frac{\mathrm{i}k\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\boldsymbol{h}_m^{\mathrm{sc}})\,\mathrm{d}S'+2\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'+3\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\boldsymbol{e}^{\mathrm{sc}})\,\mathrm{d}S'\right)\right),\end{aligned}$$

so that

$$\tilde{\boldsymbol{t}}_{k\delta^3/r^2}=\sum_{m=0}^{1}\tilde{\boldsymbol{t}}_{m,k\delta^3/r^2}=-\frac{\mathrm{i}k\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^2}(3(\hat{\boldsymbol{r}}\cdot\boldsymbol{p}_B)\hat{\boldsymbol{r}}-\boldsymbol{p}_B+\hat{\boldsymbol{r}}\times\boldsymbol{m}_B),\quad\tilde{\varDelta}_{k^2\delta^4/r^2}=O\left(\frac{\delta^2}{r^2}\delta^2\right).\tag{5.30}$$

Terms making up the $\tilde{\boldsymbol{t}}_{\delta^3/r^3}$ contribution
Again similar steps yield

$$\begin{aligned}\tilde{\boldsymbol{t}}_{0,\delta^3/r^3}&=\frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^3}\left(\frac{3}{2}\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\tilde{\boldsymbol{e}}_0^{\mathrm{sc}})\,\mathrm{d}S'\right)+\int_{\partial B}\boldsymbol{r}'\times(\hat{\boldsymbol{n}}'\times\tilde{\boldsymbol{e}}_0^{\mathrm{sc}})\,\mathrm{d}S'\right),\\\tilde{\boldsymbol{t}}_{1,\delta^3/r^3}&=\frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi r^3}\left(3\hat{\boldsymbol{r}}\times\left(\hat{\boldsymbol{r}}\times\int_{\partial B}\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\tilde{\boldsymbol{e}}_0^{\mathrm{sc}})\,\mathrm{d}S'\right)+2\int\boldsymbol{r}'(\hat{\boldsymbol{n}}'\cdot\tilde{\boldsymbol{e}}_0^{\mathrm{sc}})\,\mathrm{d}S'\right),\end{aligned}$$

so that

$$\tilde{\boldsymbol{t}}_{\delta^3/r^3} = \sum_{m=0}^{1} \tilde{\boldsymbol{t}}_{m,\delta^3/r^3} = \frac{\delta^3 \, \mathrm{e}^{\mathrm{i}kr}}{4\pi r^3}(3(\hat{\boldsymbol{r}} \cdot \boldsymbol{p}_B)\hat{\boldsymbol{r}} - \boldsymbol{p}_B), \quad \tilde{\Delta}_{k\delta^4/r^3} = O\left(\frac{\delta^3}{r^3}\delta\right). \tag{5.31}$$

Summing (5.29–5.31) gives the desired result. Note that, in a similar manner to case (a), we have $\tilde{\Delta}_{k^3\delta^4/r} + \tilde{\Delta}_{k^2\delta^4/r^2} + \tilde{\Delta}_{k\delta^4/r^3} = O(\Gamma^4)$, since the contributions can be bounded as $\tilde{\Delta}_{k^3\delta^4/r} + \tilde{\Delta}_{k^2\delta^4/r^2} + \tilde{\Delta}_{k\delta^4/r^3} \leqslant C\Gamma^4$, where $\Gamma = \max(\delta/r, \delta)$. This follows since $(\delta/r)^{\beta_1}\delta^{\beta_2} \leqslant \Gamma^{|\beta|}$ for $\delta/r < 1$ and $\delta < 1$ with the multi-index $\beta = (\beta_1, \beta_2)$ and $|\beta| = 4$. In this case, $\boldsymbol{p}_B = \boldsymbol{p}_B(\tilde{\boldsymbol{e}}_0^{\mathrm{sc}})$ and $\boldsymbol{m}_B = \boldsymbol{m}_B(\tilde{\boldsymbol{h}}_0^{\mathrm{sc}})$ are the leading-order terms for the polarization moments, which are not functions of δ, so we obtain asymptotic expansions for the fields in the presence of a (lossy) dielectric scatterer as $\Gamma \to 0$. This completes the proof of case (b).

Case (c)

We focus on the expansion for $\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z})$, the result for $\boldsymbol{H}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z})$ can be obtained analogously. We look for an expansion of the form

$$\boldsymbol{E}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z}) = \tilde{\tilde{\boldsymbol{t}}}_{k^2\delta^3/r} + \tilde{\tilde{\boldsymbol{t}}}_{k\delta^3/r^2} + \tilde{\tilde{\boldsymbol{t}}}_{\delta^3/r^3} + \Delta = \sum_{m=0}^{1}(\tilde{\tilde{\boldsymbol{t}}}_{m,k^2\delta^3/r} + \tilde{\tilde{\boldsymbol{t}}}_{m,k\delta^3/r^2} + \tilde{\tilde{\boldsymbol{t}}}_{m,\delta^3/r^3}) + \Delta,$$

$$\Delta = \tilde{\tilde{\Delta}}_{k^3\delta^4/r} + \tilde{\tilde{\Delta}}_{k^2\delta^4/r^2} + \tilde{\tilde{\Delta}}_{k\delta^4/r^3} = O(\Xi^4),$$

where $\Xi = \max(\delta/r, k\delta)$, which we obtain by replacing $\boldsymbol{e}^{\mathrm{sc}}$ by $\tilde{\tilde{\boldsymbol{e}}}_0^{\mathrm{sc}} + k\delta\tilde{\tilde{\boldsymbol{e}}}_1^{\mathrm{sc}} + O((k\delta)^2)$ and $\boldsymbol{h}^{\mathrm{sc}}$ by $\tilde{\tilde{\boldsymbol{h}}}_0^{\mathrm{sc}} + k\delta\tilde{\tilde{\boldsymbol{h}}}_1^{\mathrm{sc}} + O((k\delta)^2)$ in a similar way to case (b). Following the two previous cases, the notation $\tilde{\tilde{\boldsymbol{t}}}_{k^2\delta^3/r} = \tilde{\tilde{\boldsymbol{t}}}_{0,k^2\delta^3/r} + \tilde{\tilde{\boldsymbol{t}}}_{1,k^2\delta^3/r}$ is used to denote the leading order $O(k^2\delta^3/r)$ term with similar meanings for $\tilde{\tilde{\boldsymbol{t}}}_{k\delta^3/r^2}$ and $\tilde{\tilde{\boldsymbol{t}}}_{\delta^3/r^3}$. Note that these leading-order terms contain $(\delta/r)^{\beta_1}(k\delta)^{\beta_2}$ with the multi-index $\beta = (\beta_1, \beta_2)$ such that $|\beta| \leqslant 3$. The remainder Δ is a sum of three terms, $\tilde{\Delta}_{k^3\delta^4/r}$ indicating a term of order $O(k^3\delta^4/r)$ with similar meanings for $\tilde{\Delta}_{k^2\delta^4/r^2}$ and $\tilde{\Delta}_{k\delta^4/r^3}$. The resulting $\boldsymbol{p}_B = \boldsymbol{p}_B(\tilde{\tilde{\boldsymbol{e}}}_0^{\mathrm{sc}})$ and $\boldsymbol{m}_B = \boldsymbol{m}_B(\tilde{\tilde{\boldsymbol{h}}}_0^{\mathrm{sc}})$ are the leading-order terms for the polarization moments, which are not functions of $k\delta$ in this case. The higher-order terms $\tilde{\tilde{\Delta}}_{k^3\delta^4/r} + \tilde{\tilde{\Delta}}_{k^2\delta^4/r^2} + \tilde{\tilde{\Delta}}_{k\delta^4/r^3}$ can be written as $O(\Xi^4)$, using similar arguments to case (a) and (b), and the final results are asymptotic expansions of the fields in the presence of a non-lossy dielectric scatterer as $\Xi \to 0$. This completes the proof of part (c). □

REMARK 5.1 To make comparisons with available results in the literature, we first consider case (a) in Lemma 5.1. By setting $\boldsymbol{z} = \boldsymbol{0}$ and identifying $\delta^3\boldsymbol{p}_B(\boldsymbol{e}^{\mathrm{sc}}) = \boldsymbol{p}$ and $\delta^3\boldsymbol{m}_B(\boldsymbol{h}^{\mathrm{sc}}) = \boldsymbol{m}$ as the electric and magnetic dipole moments, respectively, then the scattered fields are exactly those of radiating electric and magnetic dipoles placed at the origin (Jones, 1964, pp. 152–154). If we also fix r, the expansion agrees with those obtained by Baum (1971) by different means. Baum obtained his result for the unscaled fields $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$. To transform to the result of Baum, we additionally set $\boldsymbol{E} = \epsilon_0^{1/2}\boldsymbol{\mathcal{E}}$, $\boldsymbol{H} = \mu_0^{1/2}\boldsymbol{\mathcal{H}}$, $\boldsymbol{p} = \epsilon_0^{-1/2}\tilde{\boldsymbol{p}}$ and $\boldsymbol{m} = \mu_0^{1/2}\tilde{\boldsymbol{m}}$ and take the complex conjugate. Note that the scaling for $\boldsymbol{p}$ is different from $\boldsymbol{E}$ since Baum scales his electric moment $\tilde{\boldsymbol{p}}$ with ϵ_0, but does not scale the corresponding magnetic moment $\tilde{\boldsymbol{m}}$. The complex conjugate must be applied since Baum assumes the time variation $\mathrm{e}^{\mathrm{i}\omega t}$. Compared with the result of Baum, ours benefits from the explicit inclusion of the object size, δ, and higher-order terms which make the dependence on k, δ and r explicit.

If we consider case (c), and set $\delta = 1$ and $\boldsymbol{z} = \boldsymbol{0}$, the form of the order k^2/r term agrees with the Rayleigh term obtained by Kleinman (1973) and Dassios & Kleinman (2000) for the limiting case of $k \to 0$, $r \to \infty$. However, as δ is not taken into account in their work, it cannot be rigorously applied to problems described by $k\delta \to 0$, and may encounter problems if the scatterer is lossy so that ϵ_r is a function of k. Our result for case (c) benefits from the inclusion of the object size, is valid as $\max(\delta/r, k\delta) \to 0$, such that it holds for distances that are large compared with the object size and for low frequencies, and applies if the scatterer is a non-lossy dielectric or a perfect conductor. By fixing k in case (c) we recover case (b) for non-lossy dielectric and perfect conductors. However, it is important to remark that case (b) is not just a consequence of case (c) since the former also holds if the scatterer is a lossy dielectric object. It is also important to note that case (b) does not describe the perturbed fields for a lossy scatterer at low frequencies as the expansion is valid for fixed k when $\max(\delta/r, \delta) \to 0$.

By considering a scatterer, which is a (lossy) dielectric or a non-lossy dielectric scatterer at low frequencies, as described by cases (b) and (c) in Lemma 5.1, we are able to express the leading-order terms in asymptotic expansions of the scattered fields at distances that are large compared with the object's size in terms of polarization tensors and, hence, prove the main results of the paper. We also consider the polarization tensors that result when the scatterer is a perfect conductor as a limiting case of a dielectric object.

The zeroth-order coefficients of the scaled fields satisfy transmission conditions of the form (3.11) on ∂B, which allows the leading-order terms for the polarization moments appearing in cases (b) and (c) of Lemma 5.1 to be expressed as

$$\boldsymbol{p}_B = \int_{\partial B} \boldsymbol{r}'\hat{\boldsymbol{n}}' \cdot (\epsilon_r - 1)\boldsymbol{u}^{\text{in}} + \boldsymbol{r}'\hat{\boldsymbol{n}}' \cdot \epsilon_r \boldsymbol{u}^{\text{sc}}|_- + \frac{1}{2}\boldsymbol{r}' \times (\hat{\boldsymbol{n}}' \times \boldsymbol{u}^{\text{sc}})|_- \, \mathrm{d}S', \tag{5.32}$$

$$\boldsymbol{m}_B = \int_{\partial B} \boldsymbol{r}'\hat{\boldsymbol{n}}' \cdot (\mu_r - 1)\boldsymbol{v}^{\text{in}} + \boldsymbol{r}'\hat{\boldsymbol{n}}' \cdot \epsilon_r \boldsymbol{v}^{\text{sc}}|_- + \frac{1}{2}\boldsymbol{r}' \times (\hat{\boldsymbol{n}}' \times \boldsymbol{v}^{\text{sc}})|_- \, \mathrm{d}S', \tag{5.33}$$

where $\boldsymbol{u}^{\text{sc}} = \tilde{\boldsymbol{e}}_0^{\text{sc}}$, $\boldsymbol{u}^{\text{in}} = \tilde{\boldsymbol{e}}_0^{\text{in}}$, $\boldsymbol{v}^{\text{sc}} = \tilde{\boldsymbol{h}}_0^{\text{sc}}$ and $\boldsymbol{v}^{\text{in}} = \tilde{\boldsymbol{h}}_0^{\text{in}}$ for case (b) and $\boldsymbol{u}^{\text{sc}} = \tilde{\tilde{\boldsymbol{e}}}_0^{\text{sc}}$, $\boldsymbol{u}^{\text{in}} = \tilde{\tilde{\boldsymbol{e}}}_0^{\text{in}}$, $\boldsymbol{v}^{\text{sc}} = \tilde{\tilde{\boldsymbol{h}}}_0^{\text{sc}}$ and $\boldsymbol{v}^{\text{in}} = \tilde{\tilde{\boldsymbol{h}}}_0^{\text{in}}$ for case (c).

Proof of Theorem 4.1. We recall the expansions of the scattered and incident scaled fields in terms of the coefficients $\tilde{\boldsymbol{e}}_m^{\text{sc}}$, $\tilde{\boldsymbol{h}}_m^{\text{sc}}$, $\tilde{\boldsymbol{e}}_m^{\text{in}}$ and $\tilde{\boldsymbol{h}}_m^{\text{in}}$, $m = 0, 1$, as described by (3.7) and (3.8), and note that the scaled incident fields can also be expressed in terms of the multi-index Taylor series expansions

$$\boldsymbol{e}^{\text{in}}(\boldsymbol{r}') = \boldsymbol{E}^{\text{in}}(\delta\boldsymbol{r}' + \boldsymbol{z}) = \sum_{\alpha, |\alpha|=0}^{\infty} \frac{\delta^{|\alpha|}}{\alpha!}(\boldsymbol{r}')^{\alpha}\partial^{\alpha}(\boldsymbol{E}^{\text{in}})(\boldsymbol{z}), \tag{5.34}$$

$$\boldsymbol{h}^{\text{in}}(\boldsymbol{r}') = \boldsymbol{H}^{\text{in}}(\delta\boldsymbol{r}' + \boldsymbol{z}) = \sum_{\alpha, |\alpha|=0}^{\infty} \frac{\delta^{|\alpha|}}{\alpha!}(\boldsymbol{r}')^{\alpha}\partial^{\alpha}(\boldsymbol{H}^{\text{in}})(\boldsymbol{z}), \tag{5.35}$$

for $\boldsymbol{r}' \in \partial B$ as $\delta \to 0$, so that on comparison of these expansions we arrive at the conclusion that $\tilde{\boldsymbol{e}}_0^{\text{in}}(\boldsymbol{r}') = \boldsymbol{E}^{\text{in}}(\boldsymbol{z})$ and $\tilde{\boldsymbol{h}}_0^{\text{in}}(\boldsymbol{r}') = \boldsymbol{H}^{\text{in}}(\boldsymbol{z})$.

We also recall that $\tilde{\boldsymbol{e}}_0^{\rm sc}$ and $\tilde{\boldsymbol{h}}_0^{\rm sc}$ are curl- and divergence-free in B and in $\mathbb{R}^3 \setminus B$, and thus, since B is closed, we can write $\tilde{\boldsymbol{e}}_0^{\rm sc} = \nabla_{r'}\tilde{\phi}^{\rm sc}$ and $\tilde{\boldsymbol{h}}_0^{\rm sc} = \nabla_{r'}\tilde{\psi}^{\rm sc}$. By expressing $\tilde{\phi} = \sum_{i=1}^3 (\epsilon_r - 1)(\boldsymbol{E}^{\rm in}(z) \cdot \nabla_{r'} r_i')\vartheta_i^{\rm sc}(\epsilon_r)$ and $\tilde{\psi} = \sum_{i=1}^3 (\mu_r - 1)(\boldsymbol{H}_0^{\rm in}(z) \cdot \nabla_{r'} r_i')\vartheta_i^{\rm sc}(\mu_r)$ we see that $\vartheta_i^{\rm sc}(c)$ satisfies

$$\begin{aligned} \nabla_{r'} \cdot c\nabla_{r'}\vartheta_i^{\rm sc} &= 0 \quad \text{in } B, \\ \nabla_{r'}^2 \vartheta_i^{\rm sc} &= 0 \quad \text{in } \mathbb{R}^3 \setminus B, \\ \vartheta_i^{\rm sc}|_+ - \vartheta_i^{\rm sc}|_- &= 0 \quad \text{on } \partial B, \\ \hat{\boldsymbol{n}}' \cdot \nabla_{r'}\vartheta_i^{\rm sc}|_+ - \hat{\boldsymbol{n}}' \cdot c\nabla_{r'}\vartheta_i^{\rm sc}|_- &= \hat{\boldsymbol{n}}' \cdot \nabla' r_i' \quad \text{on } \partial B, \\ \vartheta_i^{\rm sc} &\to 0 \quad \text{as } r \to \infty. \end{aligned} \tag{5.36}$$

Inserting $\boldsymbol{u}^{\rm sc} = \tilde{\boldsymbol{e}}_0^{\rm sc} = \nabla_{r'}\tilde{\phi}^{\rm sc}$, $\boldsymbol{u}^{\rm in} = \tilde{\boldsymbol{e}}_0^{\rm in} = \boldsymbol{E}^{\rm in}(z)$, $\boldsymbol{v}^{\rm sc} = \tilde{\boldsymbol{h}}_0^{\rm sc} = \nabla_{r'}\tilde{\psi}^{\rm sc}$ and $\boldsymbol{v}^{\rm in} = \tilde{\boldsymbol{h}}_0^{\rm in} = \boldsymbol{H}^{\rm in}(z)$ into (5.32) and (5.33), using the fact that $\frac{1}{2}\int_{\partial B} \boldsymbol{r}' \times (\hat{\boldsymbol{n}}' \times \nabla_{r'} f)|_- \,\mathrm{d}S' = -\int_{\partial B} \hat{\boldsymbol{n}}' f|_- \,\mathrm{d}S'$ for a continuous scalar function f (Kleinman & Senior, 1982) and that $\int_{\partial B} \hat{\boldsymbol{n}}' f|_- \,\mathrm{d}S' = \int_{\partial B} \boldsymbol{r}'(\hat{\boldsymbol{n}}' \cdot \nabla_{r'} f|_-) \,\mathrm{d}S'$, if $\nabla_{r'}^2 f = 0$ in B, we can finally obtain

$$\boldsymbol{p}_B = [\![\mathscr{M}_B(\epsilon_r)]\!]\boldsymbol{E}^{\rm in}(z), \quad \boldsymbol{m}_B = [\![\mathscr{M}_B(\mu_r)]\!]\boldsymbol{H}^{\rm in}(z), \tag{5.37}$$

where

$$[\![\mathscr{M}_B(c)]\!]_{ij} = (c-1)[\![\mathbb{I}]\!]_{ij}|B| + (c-1)^2 \int_{\partial B} r_i' \hat{\boldsymbol{n}}' \cdot \nabla_{r'}\vartheta_j^{\rm sc}(c)|_- \,\mathrm{d}S' \tag{5.38}$$

is a symmetric polarization tensor. Inserting (5.37) into case (b) of Lemma 5.1 completes the proof. □

The above expression for $[\![\mathscr{M}_B(c)]\!]$ is identical to the first-order generalized polarization tensor of Ammari & Kang (2007, p. 79). By expressing the solution of the transmission problem for $\vartheta_j^{\rm sc}$ in terms of single- and double-layer potentials Ammari and Kang show that this tensor can be computed by solving an integral equation on the surface of ∂B. In Appendix B, we discuss this further and show that the above polarization tensor is also identical (apart from a minus sign) to the general polarization presented by Kleinman & Senior (1982) and Dassios & Kleinman (2000).

Proof of Corollary 4.1. In the case of a perfectly conducting object the boundary conditions on ∂B are

$$\hat{\boldsymbol{n}}' \times \boldsymbol{e}^{\rm sc}|_+ = -\hat{\boldsymbol{n}}' \times \boldsymbol{e}^{\rm in}|_-, \quad \hat{\boldsymbol{n}}' \cdot \boldsymbol{h}^{\rm sc}|_+ = -\hat{\boldsymbol{n}}' \cdot \boldsymbol{h}^{\rm in}|_-.$$

Following similar steps to the proof of Theorem 4.1 and using the representations $\tilde{\boldsymbol{e}}_0^{\rm sc} = \nabla_{r'}\tilde{\phi}^{\rm sc} = \sum_{i=1}^3 (\boldsymbol{E}^{\rm in}(z) \cdot \nabla_{r'} r_i')\nabla_{r'}\tilde{\phi}_i^s$, and $\tilde{\boldsymbol{h}}_0^{\rm sc} = \nabla_{r'}\tilde{\psi}^{\rm sc} = \sum_{i=1}^3 (\boldsymbol{H}^{\rm in}(z) \cdot \nabla_{r'} r_i')\nabla_{r'}\tilde{\psi}_i^s$, it is seen that the electric potential $\tilde{\phi}_i^{\rm sc}$ satisfies

$$\begin{aligned} \nabla_{r'}^2 \tilde{\phi}_i^{\rm sc} &= 0 \quad \text{in } \mathbb{R}^3 \setminus B, \\ \tilde{\phi}_i^{\rm sc}|_+ &= -r_i' \quad \text{on } \partial B, \\ \tilde{\phi}_i^{\rm sc} &\to 0 \quad \text{as } r \to \infty, \end{aligned}$$

and the magnetostatic potential $\tilde{\psi}_i^{\text{sc}}$ satisfies

$$\nabla_{r'}^2 \tilde{\psi}_i^{\text{sc}} = 0 \quad \text{in } \mathbb{R}^3 \setminus B,$$
$$\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \tilde{\psi}_i^{\text{sc}}|_+ = -\hat{\boldsymbol{n}}' \cdot \nabla_{r'} r_i' \quad \text{on } \partial B,$$
$$\tilde{\psi}_i^{\text{sc}} \to 0 \quad \text{as } r \to \infty.$$

By substituting the representations $\boldsymbol{u}^{\text{sc}} = \tilde{\boldsymbol{e}}_0^{\text{sc}} = \nabla_{r'} \tilde{\phi}^{\text{sc}}$ and $\boldsymbol{v}^{\text{sc}} = \tilde{\tilde{\boldsymbol{h}}}_0^{\text{sc}} = \nabla_{r'} \tilde{\psi}^{\text{sc}}$ into (5.3) and (5.4), we immediately obtain

$$\boldsymbol{p}_B = [\![\mathscr{M}_B(\infty)]\!] \boldsymbol{E}^{\text{in}}(\boldsymbol{z}), \quad \boldsymbol{m}_B = [\![\mathscr{M}_B(0)]\!] \boldsymbol{H}^{\text{in}}(\boldsymbol{z}), \tag{5.39}$$

where

$$[\![\mathscr{M}_B(\infty)]\!]_{ij} = [\![\mathbb{I}]\!]_{ij} |B| + \int_{\partial B} r_i' \hat{\boldsymbol{n}}' \cdot \nabla_{r'} \tilde{\phi}_j^{\text{sc}}|_+ \, \mathrm{d}S' \tag{5.40}$$

is the electric polarizability tensor and

$$[\![\mathscr{M}_B(0)]\!]_{ij} = -[\![\mathbb{I}]\!]_{ij} |B| - \int_{\partial B} \hat{n}_i' \tilde{\psi}_j^{\text{sc}}|_+ \, \mathrm{d}S' \tag{5.41}$$

is the magnetic polarizability tensor (Keller *et al.*, 1972). □

Alternatively, a perfect electrical conductor can be understood as the limiting cases of $\epsilon_r \to +\infty$ and $\mu_r \to 0$. In this case, the integral equation approach for the computation of $[\![\mathscr{M}_B(c)]\!]$, proposed by Ammari and Kang and discussed in Appendix B, still holds for $c = +\infty$ and $c = 0$ although, in the former, it requires the solution of a singular system (Ammari & Kang, 2007, pp. 88-89) and cannot be computed independently from $\boldsymbol{E}^{\text{in}}(\boldsymbol{z})$ (see also Ammari & Kang, 2007, p. 37).

Proof of Theorem 4.2. We recall the expansion of the scattered and incident fields in terms of the coefficients $\tilde{\boldsymbol{e}}_m^{\text{sc}}$, $\tilde{\tilde{\boldsymbol{h}}}_m^{\text{sc}}$, $\tilde{\boldsymbol{e}}_m^{\text{in}}$ and $\tilde{\tilde{\boldsymbol{h}}}_m^{\text{in}}$, $m = 0, 1$, as described by (3.12) and (3.13). The incident fields $\boldsymbol{H}^{\text{in}}(\delta \boldsymbol{r}' + \boldsymbol{z})$ and $\boldsymbol{E}^{\text{in}}(\delta \boldsymbol{r}' + \boldsymbol{z})$ are functions not only of $\delta \boldsymbol{r}' + \boldsymbol{z}$, but also of $k(\delta \boldsymbol{r}' + \boldsymbol{z})$ and so they can be expanded about $k\boldsymbol{z}$ in terms of the multi–index expansions

$$\boldsymbol{e}^{\text{in}}(\boldsymbol{r}') = \boldsymbol{E}^{\text{in}}(\delta \boldsymbol{r}' + \boldsymbol{z}) = \sum_{\alpha, |\alpha|=0}^{\infty} \frac{(k\delta)^{|\alpha|}}{\alpha!} (\boldsymbol{r}')^{\alpha} \partial^{\alpha} (\boldsymbol{E}^{\text{in}})(\boldsymbol{z}),$$
$$\boldsymbol{h}^{\text{in}}(\boldsymbol{r}') = \boldsymbol{H}^{\text{in}}(\delta \boldsymbol{r}' + \boldsymbol{z}) = \sum_{\alpha, |\alpha|=0}^{\infty} \frac{(k\delta)^{|\alpha|}}{\alpha!} (\boldsymbol{r}')^{\alpha} \partial^{\alpha} (\boldsymbol{H}^{\text{in}})(\boldsymbol{z}),$$

for $\boldsymbol{r}' \in \partial B$ as $k\delta \to 0$. Thus, we can identify that $\tilde{\boldsymbol{e}}_0^{\text{in}}(\boldsymbol{r}') = \boldsymbol{E}^{\text{in}}(\boldsymbol{z})$ and $\tilde{\tilde{\boldsymbol{h}}}_0^{\text{in}}(\boldsymbol{r}') = \boldsymbol{H}^{\text{in}}(\boldsymbol{z})$.

We also recall that $\tilde{\boldsymbol{e}}_0^{\text{sc}}$ and $\tilde{\tilde{\boldsymbol{h}}}_0^{\text{sc}}$ are curl and divergence free in B and in $\mathbb{R}^3 \setminus B$, and thus, since B is closed, we can write $\tilde{\boldsymbol{e}}_0^{\text{sc}} = \nabla_{r'} \tilde{\tilde{\phi}}^{\text{sc}}$ and $\tilde{\tilde{\boldsymbol{h}}}_0^{\text{sc}} = \nabla_{r'} \tilde{\tilde{\psi}}^{\text{sc}}$. Choosing $\tilde{\tilde{\phi}}^{\text{sc}} = \sum_{i=1}^{3} (\epsilon_r - 1)(\boldsymbol{E}^{\text{in}}(\boldsymbol{z}) \cdot \nabla_{r'} r_i') \vartheta(\epsilon_r)$ and $\tilde{\tilde{\psi}}^{\text{sc}} = \sum_{i=1}^{3} (\mu_r - 1)(\boldsymbol{H}^{\text{in}}(\boldsymbol{z}) \cdot \nabla_{r'} r_i') \vartheta(\mu_r)$, with $\vartheta(c)$ satisfying the transmission problem (5.36), we can show that

$$\boldsymbol{p}_B = [\![\mathscr{M}_B(\epsilon_r)]\!] \boldsymbol{E}^{\text{in}}(\boldsymbol{z}), \quad \boldsymbol{m}_B = [\![\mathscr{M}_B(\mu)]\!] \boldsymbol{H}^{\text{in}}(\boldsymbol{z}). \tag{5.42}$$

Inserting (5.42) into case (c) of Lemma 5.1 completes the proof. □

Proof of Corollary 4.2. Following similar steps to the proof of Theorem 4.2 and Corollary 4.2, we arrive at

$$\boldsymbol{p}_B = [\![\mathscr{M}_B(\infty)]\!]\boldsymbol{E}^{\mathrm{in}}(\boldsymbol{z}), \quad \boldsymbol{m}_B = [\![\mathscr{M}_B(0)]\!]\boldsymbol{H}^{\mathrm{in}}(\boldsymbol{z}), \tag{5.43}$$

for a perfectly conducting scatterer at low frequencies. □

6. Discussion

Asymptotic expansions for the perturbation in the electric and magnetic (near) fields, due to the presence of an object, as $\delta \to 0$, have been obtained by Ammari *et al.* (2001). Their formulae are based on a bounded, rather than an unbounded, domain where the tangential traces of the electric and magnetic fields are known on a boundary placed at some finite distance from the object. The leading-order terms they obtain show a similar explicit dependence on the object size to that obtained in (4.1) and (4.2) and (4.5) and (4.6). However, unlike the results in Ammari *et al.* (2001) and Ammari & Volkov (2005) (discussed in the Remark 4.1), our asymptotic expansions in Theorem 4.1 contain additional terms since they hold as $\max(\delta/r, \delta) \to 0$ rather than for $r \to \infty$, as $\delta \to 0$. Additionally, in the case of a non-lossy dielectric, Theorem 4.2 describes the perturbation in the fields as $\max(\delta/r, k\delta) \to 0$, which are relevant for low-frequency problems at distances that are large compared with the object's size. The k^2/r term is a component in the leading-order terms we have obtained, and we expect our new expansions will be useful for understanding for low-frequency inverse problems where the interplay between the wavenumber and the distance from the object to the point of measurement becomes a critical factor.

As stated in Remark 5.1, by identifying the polarization moments $\delta^3\boldsymbol{p}_B(\boldsymbol{e}^{\mathrm{sc}})$ and $\delta^3\boldsymbol{m}_B(\boldsymbol{h}^{\mathrm{sc}})$ with the electric and magnetic dipole moments $\boldsymbol{p}$ and $\boldsymbol{m}$, respectively, the fields for case (a) of Lemma 5.1 are exactly those of radiating dipoles. In the case of low-frequency scattering by a non-lossy dielectric sphere, centred at $\boldsymbol{z}$, with radius δ and described by $k\delta \to 0$, the scattered fields, at sufficiently large distances, are the same as radiating dipoles centred at $\boldsymbol{z}$ with moments (e.g. Jackson, 1975, p. 413)

$$\boldsymbol{p} = 4\pi\delta^3\left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right)\boldsymbol{E}^{\mathrm{in}}(\boldsymbol{z}), \quad \boldsymbol{m} = 4\pi\delta^3\left(\frac{\mu_r - 1}{\mu_r + 2}\right)\boldsymbol{H}^{\mathrm{in}}(\boldsymbol{z}).$$

If the sphere is centred at the origin, then $\boldsymbol{E}^{\mathrm{in}}(\boldsymbol{z}) = \boldsymbol{E}^{\mathrm{in}}(\boldsymbol{0})$ and $\boldsymbol{H}^{\mathrm{in}}(\boldsymbol{z}) = \boldsymbol{H}^{\mathrm{in}}(\boldsymbol{0})$ are the amplitudes of the incident plane waves. Theorem 4.2 describes the scattered fields as $\max(\delta/r, k\delta) \to 0$, for an object B of unit size scaled by δ. If the object B is a sphere of unit radius, the polarization tensor has the known form $[\![\mathscr{M}_B(c)]\!] = 4\pi((c-1)/(c+2))[\![\mathbb{I}]\!]$ (Ammari & Kang, 2007). On consideration of the scaling δ, and the known form of the polarization tensor, we see that Theorem 4.2 is exact in the case of low-frequency scattering by a dielectric sphere. In the case of scattering by a perfectly conducting sphere the dipole moments are known as (e.g. Jackson, 1975, p. 415)

$$\boldsymbol{p} = 4\pi\delta^3\boldsymbol{E}^{\mathrm{in}}(\boldsymbol{z}), \quad \boldsymbol{m} = -2\pi\delta^3\boldsymbol{H}^{\mathrm{in}}(\boldsymbol{z}).$$

By noting that a perfectly conductor sphere of unit radius has polarization tensors $[\![\mathscr{M}_B(\infty)]\!] = 4\pi[\![\mathbb{I}]\!]$ and $[\![\mathscr{M}_B(0)]\!] = -2\pi[\![\mathbb{I}]\!]$, we see that Theorem 4.2 is also exact in the case of low frequency scattering by a perfectly conducting sphere.

Finally, to make comparisons with the results sometimes quoted in the engineering literature (e.g. Das & McFee, 1991; Braunisch *et al.*, 2001; Norton & Won, 2001; Marsh *et al.*, 2013), we take the component of $\boldsymbol{H}^s$ in the direction of $\hat{\boldsymbol{r}}$ for a purely magnetic object and, using the expansion (4.2), we find that

$$\boldsymbol{H}^{\mathrm{sc}}(\boldsymbol{r}+\boldsymbol{z})\cdot\hat{\boldsymbol{r}} = \frac{\delta^3\,\mathrm{e}^{\mathrm{i}kr}}{4\pi}\left(\frac{2}{r^3}\hat{\boldsymbol{r}}\cdot([\![\mathscr{M}_B(\mu_r)]\!]\boldsymbol{H}^{\mathrm{in}}(z)) - \frac{2\mathrm{i}k}{r^2}\hat{\boldsymbol{r}}\cdot([\![\mathscr{M}_B(\mu_r)]\!]\boldsymbol{H}^{\mathrm{in}}(z))\right) + O(\delta^4)$$
$$= -2\frac{\delta^3}{r}\nabla_r u_k\cdot[\![\mathscr{M}_B(\mu_r)]\!]\boldsymbol{H}^{\mathrm{in}}(z) + O(\delta^4), \tag{6.1}$$

where the leading-order term is of the form of a $\boldsymbol{H}_1\cdot[\![\mathscr{N}_B(\mu_r)]\!]\boldsymbol{H}_2$ sensitivity, with $[\![\mathscr{N}_B(\mu_r)]\!] = -(2\delta^3/r)[\![\mathscr{M}_B(\mu_r)]\!]$. As mentioned in the introduction, the perturbed magnetic field for a conducting object at low frequencies is described in terms of a different kind of polarization tensor (Ammari *et al.*, 2014).

Funding

The authors would also like to thank EPSRC for the financial support received from the grants EP/K00428X/1 and EP/K039865/1. Funding to pay the Open Access publication charges for this article was provided by the Engineering and Physical Sciences Research Council (EPSRC).

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Appendix A. Integral identities

Kleinman (1967) shows that

$$\int_{\partial B} \hat{\boldsymbol{n}}' \times \boldsymbol{F} \,\mathrm{d}S' = \int_{\partial B} \boldsymbol{r}'(\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times \boldsymbol{F}) \,\mathrm{d}S', \tag{A.1}$$

holds for a vector field $\boldsymbol{F}$ that is differentiable everywhere in the neighbourhood of ∂B for closed B. By choosing $\boldsymbol{F} = \boldsymbol{G}(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})$, we can use the above result to obtain

$$\begin{aligned}\int_{\partial B} \hat{\boldsymbol{n}}' \times \boldsymbol{G}(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}}) \,\mathrm{d}S' &= \int_{\partial B} \boldsymbol{r}'(\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times (\boldsymbol{G}(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}}))) \,\mathrm{d}S' \\ &= \int_{\partial B} \boldsymbol{r}'(\hat{\boldsymbol{n}}' \cdot \nabla_{r'}(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}}) \times \boldsymbol{G}) \,\mathrm{d}S' + \int_{\partial B} \boldsymbol{r}'(\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times \boldsymbol{G})(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}}) \,\mathrm{d}S' \\ &= -\int_{\partial B} \boldsymbol{r}'(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{n}}' \times \boldsymbol{G}) \,\mathrm{d}S' + \int_{\partial B} \boldsymbol{r}'(\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times \boldsymbol{G})(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}}) \,\mathrm{d}S'.\end{aligned}$$

Next, applying $\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}$ with $\boldsymbol{A} = \hat{\boldsymbol{r}}$, $\boldsymbol{B} = \hat{\boldsymbol{n}} \times \boldsymbol{G}$ and $\boldsymbol{C} = \boldsymbol{r}'$ gives

$$\int_{\partial B} \boldsymbol{r}'(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{n}}' \times \boldsymbol{G}) \,\mathrm{d}S' = \hat{\boldsymbol{r}} \times \int_{\partial B} \boldsymbol{r}' \times (\hat{\boldsymbol{n}}' \times \boldsymbol{G}) \,\mathrm{d}S' + \int_{\partial B} \hat{\boldsymbol{n}}' \times \boldsymbol{G}(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}}) \,\mathrm{d}S',$$

and thus

$$\int_{\partial B} \hat{\boldsymbol{n}}' \times \boldsymbol{G}(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}}) \,\mathrm{d}S' = -\frac{1}{2}\hat{\boldsymbol{r}} \times \int_{\partial B} \boldsymbol{r}' \times (\hat{\boldsymbol{n}}' \times \boldsymbol{G}) \,\mathrm{d}S' + \frac{1}{2}\int_{\partial B} \boldsymbol{r}'(\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times \boldsymbol{G})(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}}) \,\mathrm{d}S'. \tag{A.2}$$

The result

$$\int_{\partial B} (\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{n}}' \times \boldsymbol{F})\hat{\boldsymbol{r}} \,\mathrm{d}S' = \hat{\boldsymbol{r}} \times \left(\hat{\boldsymbol{r}} \times \int_{\partial B} \hat{\boldsymbol{n}}' \times \boldsymbol{F} \,\mathrm{d}S'\right) + \int_{\partial B} \hat{\boldsymbol{n}}' \times \boldsymbol{F} \,\mathrm{d}S', \tag{A.3}$$

follows from the application of the vector triple product identity with $\boldsymbol{A} = \boldsymbol{B} = \hat{\boldsymbol{r}}$ and $\boldsymbol{C} = \hat{\boldsymbol{n}}' \times \boldsymbol{F}$.

Next, consider

$$\int_{\partial B} (\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{n}}' \times \boldsymbol{F})(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\, \mathrm{d}S' = \int (\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{n}}' \times (\boldsymbol{F}(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})))\hat{\boldsymbol{r}}\, \mathrm{d}S'$$
$$= \hat{\boldsymbol{r}} \times \left(\hat{\boldsymbol{r}} \times \int_{\partial B} \hat{\boldsymbol{n}}' \times \boldsymbol{F}(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})\, \mathrm{d}S'\right) + \int_{\partial B} \hat{\boldsymbol{n}}' \times \boldsymbol{F}(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})\, \mathrm{d}S',$$

where the second equality follows from identity (A.3). Applying the vector triple product identity with $\boldsymbol{A} = \boldsymbol{B} = \hat{\boldsymbol{r}}$ and $\boldsymbol{C} = \hat{\boldsymbol{n}}' \times \boldsymbol{F}(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})$ to the first term on the right-hand side of the equality followed by applying identity (A.2) and using properties of the scalar triple product gives

$$\int_{\partial B} (\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{n}}' \times \boldsymbol{F})(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}}\, \mathrm{d}S' = \left(\hat{\boldsymbol{r}} \cdot \left\{-\frac{1}{2}\hat{\boldsymbol{r}} \times \int_{\partial B} \hat{\boldsymbol{n}}' \times \boldsymbol{F}\, \mathrm{d}S' + \int_{\partial B} \boldsymbol{r}'(\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times \boldsymbol{F})(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})\, \mathrm{d}S'\right\}\right)\hat{\boldsymbol{r}}$$
$$= \int_{\partial B} (\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times \boldsymbol{F})(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})^2 \hat{\boldsymbol{r}}\, \mathrm{d}S'. \tag{A.4}$$

The result

$$\int_{\partial B} (\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{n}}' \times \boldsymbol{F})\boldsymbol{r}'\, \mathrm{d}S' = \hat{\boldsymbol{r}} \times \int_{\partial B} \boldsymbol{r}' \times \hat{\boldsymbol{n}}' \times \boldsymbol{F}\, \mathrm{d}S' + \int_{\partial B} (\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{n}}' \times \boldsymbol{F}\, \mathrm{d}S'$$
$$= \frac{1}{2}\hat{\boldsymbol{r}} \times \int_{\partial B} \boldsymbol{r}' \times \hat{\boldsymbol{n}}' \times \boldsymbol{F}\, \mathrm{d}S' + \int_{\partial B} \boldsymbol{r}'(\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times \boldsymbol{F})(\boldsymbol{r}' \cdot \hat{\boldsymbol{r}})\, \mathrm{d}S', \tag{A.5}$$

follows from application of the vector triple product identity with $\boldsymbol{A} = \hat{\boldsymbol{r}}$, $\boldsymbol{B} = \boldsymbol{r}'$ and $\boldsymbol{C} = \hat{\boldsymbol{n}}' \times \boldsymbol{F}$ and (A.2).

Unlike the previous identifies, which follow from (A.1) and/or the use of vector identities, the derivation of the final identity (A.9), which follows below, does not. We first prove the following lemma.

Lemma A.1 If $\boldsymbol{F}$ is a differentiable vector function defined everywhere in the neighbourhood of a closed surface ∂B, then

$$\int_{\partial B} \boldsymbol{r}' \cdot \hat{\boldsymbol{n}}' \times \boldsymbol{F}\, \mathrm{d}S' = \frac{1}{2}\int_{\partial B} (\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times \boldsymbol{F})\boldsymbol{r}' \cdot \boldsymbol{r}'\, \mathrm{d}S'. \tag{A.6}$$

Proof. The vector field $\boldsymbol{r}'$ can be expressed as $\boldsymbol{r}' = r_1'\hat{\boldsymbol{i}}_1 + r_2'\hat{\boldsymbol{i}}_2 + r_3'\hat{\boldsymbol{i}}_3$. Noting that $\hat{\boldsymbol{i}}_1 = \nabla_r r_1 = \nabla_{r'} r_1'$ and that $r_1'\nabla_{r'} r_1' = \frac{1}{2}\nabla_{r'}(r_1')^2$, then, by using similar results for other components, we see that $\boldsymbol{r}' = \frac{1}{2}(\nabla_{r'}(r_1')^2 + \nabla_{r'}(r_2')^2 + \nabla_{r'}(r_3')^2)$. Thus,

$$\int_{\partial B} \boldsymbol{r}' \cdot \hat{\boldsymbol{n}}' \times \boldsymbol{F}\, \mathrm{d}S' = \frac{1}{2}\left(\sum_{i=1}^{3}\int_{\partial B} (\nabla_{r'}(r_i')^2 \cdot \hat{\boldsymbol{n}}' \times \boldsymbol{F})\, \mathrm{d}S'\right). \tag{A.7}$$

Application of the properties of the scalar triple product and cross product enable us to write

$$\int_{\partial B} (\nabla_{r'}(r_i')^2 \cdot \hat{\boldsymbol{n}}' \times \boldsymbol{F}\, \mathrm{d}S' = -\int_{\partial B} (\hat{\boldsymbol{n}}' \cdot \nabla_{r'}(r_i')^2 \times \boldsymbol{F})\, \mathrm{d}S'$$
$$= -\int_{\partial B} (\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times (\boldsymbol{F}(r_i')^2))\, \mathrm{d}S' + \int_{\partial B} (\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times \boldsymbol{F})(r_i')^2\, \mathrm{d}S', \tag{A.8}$$

where the first term on the right-hand side of (A.8) is zero for each i since $\int_{\partial B} \hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times \boldsymbol{G})\,\mathrm{d}S' = 0$, for a differentiable vector field $\boldsymbol{G}$, when B is closed. The lemma then immediately follows from (A.7) and (A.8). □

This lemma then enables us to write the final integral identity as

$$\int_{\partial B} (\boldsymbol{r}' \cdot \hat{\boldsymbol{n}}' \times \boldsymbol{F})\hat{\boldsymbol{r}}\,\mathrm{d}S' = \frac{1}{2}\int_{\partial B} (\hat{\boldsymbol{n}}' \cdot \nabla_{r'} \times \boldsymbol{F})(\boldsymbol{r}' \cdot \boldsymbol{r}')\hat{\boldsymbol{r}}\,\mathrm{d}S', \tag{A.9}$$

which follows since $\hat{\boldsymbol{r}}$ can be taken outside the integral sign.

Appendix B. General and generalized polarization tensors

We recall that the fundamental solution to the Laplace equation in $\mathbb{R}^3$ corresponds to u_0 (i.e. setting $k = 0$ in (3.3)) and that, for a bounded Lipschitz domain B with boundary ∂B, the associated single-layer potential applied to a function $f \in L^2(B)$ is defined as

$$\mathcal{S}_B f(\boldsymbol{r}) = \int_{\partial B} u_0(\boldsymbol{r} - \boldsymbol{r}') f(\boldsymbol{r}')\,\mathrm{d}S', \tag{B.1}$$

with $\boldsymbol{r} \in \mathbb{R}^3$. The single-layer potential also satisfies

$$\mathcal{S}_B f|_+(\boldsymbol{r}) = \mathcal{S}_B f|_-(\boldsymbol{r}), \tag{B.2}$$

$$\frac{\partial}{\partial \hat{\boldsymbol{n}}}\mathcal{S}_B f|_\pm(\boldsymbol{r}) = \left(\pm\frac{1}{2}\mathbb{I} + \mathcal{K}_B^*\right) f(\boldsymbol{r}) \tag{B.3}$$

for $\boldsymbol{r} \in \partial B$ almost everywhere (Ammari & Kang, 2007). In the above $\mathbb{I}$ is the identity operator,

$$\mathcal{K}_B f(\boldsymbol{r}) = p.v. \int_{\partial B} \frac{\hat{\boldsymbol{n}}' \cdot (\boldsymbol{r}' - \boldsymbol{r})}{4\pi |\boldsymbol{r}' - \boldsymbol{r}|^3} f(\boldsymbol{r}')\,\mathrm{d}S', \tag{B.4}$$

and $\mathcal{K}_B^*$ is the L^2 adjoint of $\mathcal{K}_B$.

To simplify the notation in the remainder of this appendix, we drop the prime where no confusion arises. In Ammari & Kang (2007, Definition 4.1), the authors define their symmetric generalized polarization tensor in terms of multi-indices α, β as

$$(\mathscr{M}_B(c))_{\alpha\beta} = \int_{\partial B} \boldsymbol{r}^\beta \hat{\vartheta}_\alpha(c, \boldsymbol{r})\,\mathrm{d}S, \tag{B.5}$$

where $\hat{\vartheta}_\alpha(c, \boldsymbol{r}) = (\lambda\mathcal{I} - \mathcal{K}_B^*)^{-1}(\hat{\boldsymbol{n}}' \cdot \nabla_{r'}(\boldsymbol{r}'^\alpha))(\boldsymbol{r})$, c is the material contrast and $\lambda(c) = (1 + c)/(2(c - 1))$. In Lemma 4.3, they show that an alternative expression for $(\mathscr{M}_B(c))_{\alpha\beta}$ is

$$(\mathscr{M}_B(c))_{\alpha\beta} = (c - 1)\int_{\partial B} \boldsymbol{r}^\beta \frac{\partial \boldsymbol{r}^\alpha}{\partial \hat{\boldsymbol{n}}}\,\mathrm{d}S + (c - 1)^2 \int_{\partial B} \boldsymbol{r}^\beta \left.\frac{\partial \vartheta_\alpha(c)}{\partial \hat{\boldsymbol{n}}}\right|_- \mathrm{d}S, \tag{B.6}$$

where $\vartheta_\alpha(c)$ satisfies the transmission problem

$$\begin{aligned} \nabla \cdot c\nabla \vartheta_\alpha &= 0 \quad \text{in } B, \\ \nabla^2 \vartheta_\alpha &= 0 \quad \text{in } \mathbb{R}^3 \setminus B, \\ \vartheta_\alpha|_+ - \vartheta_\alpha|_- &= 0 \quad \text{on } \partial B, \\ \left.\frac{\partial \vartheta_\alpha}{\partial \hat{\boldsymbol{n}}}\right|_+ - c\left.\frac{\partial \vartheta_\alpha}{\partial \hat{\boldsymbol{n}}}\right|_- &= \hat{\boldsymbol{n}} \cdot \nabla(\boldsymbol{r}^\alpha) \quad \text{on } \partial B, \\ \vartheta_\alpha &\to 0 \quad \text{as } |\boldsymbol{r}| \to \infty. \end{aligned} \tag{B.7}$$

By restricting consideration to $|\alpha| = |\beta| = 1$, the generalized polarization tensor of Ammari and Kang becomes a symmetric matrix $[\![\mathscr{M}_B(c)]\!]_{ij}$, $i, j = 1, 2, 3$, which is identical to (4.3).

Kleinman & Senior (1982) and Dassios & Kleinman (2000, p. 168) considers the following scalar transmission problem for $\theta_j(c)$, $j = 1, 2, 3$:

$$\begin{aligned} \nabla \cdot c\nabla \theta_j &= 0 \quad \text{in } B, \\ \nabla^2 \theta_j &= 0 \quad \text{in } \mathbb{R}^3 \setminus B, \\ \theta_j|_+ - \theta_j|_- &= 0 \quad \text{on } \partial B, \\ \left.\frac{\partial \theta_j}{\partial \hat{\boldsymbol{n}}}\right|_+ - c\left.\frac{\partial \theta_j}{\partial \hat{\boldsymbol{n}}}\right|_- &= 0 \quad \text{on } \partial B, \\ \theta_j - r_j &\to 0 \quad \text{as } |\boldsymbol{r}| \to \infty, \end{aligned} \tag{B.8}$$

and present a number of alternative expressions for their general polarization tensor $[\![M_B(c)]\!]$ including

$$[\![M_B(c)]\!]_{ij} = \int_{\partial B} \hat{n}_i \theta_j(c)|_+ - r_i \left.\frac{\partial \theta_j(c)}{\partial \hat{\boldsymbol{n}}}\right|_+ \mathrm{d}S, \tag{B.9}$$

$$= (1-c) \int_{\partial B} \hat{n}_i \theta_j(c)|_+ \,\mathrm{d}S, \tag{B.10}$$

$$= (1-c) \int_{\partial B} r_i \left.\frac{\partial \theta_j(c)}{\partial \hat{\boldsymbol{n}}}\right|_- \mathrm{d}S, \tag{B.11}$$

where the equivalence between the different expressions follows from the manipulation of the transmission conditions and application of the divergence theorem (Kleinman & Senior, 1982).

Lemma B.1 In the case of $|\alpha| = |\beta| = 1$ the generalized polarization tensor of Ammari and Kang is identical (apart from a minus sign) to the general polarization tensor as presented by Kleinman & Senior (1982), which, in turn, is a symmetric matrix $[\![M_B(c)]\!]_{ij}$, $i, j = 1, 2, 3$ that also satisfies $[\![\mathscr{M}_B(c)]\!] = -[\![M_B(c)]\!]$.

Proof. Transforming the transmission problem (B.8), using $\chi_j(c)=\theta_j(c)-r_j$, we see that χ_j satisfies

$$\begin{aligned} \nabla\cdot c\nabla\chi_j &= 0 \quad \text{in } B,\\ \nabla^2\chi_j &= 0 \quad \text{in } \mathbb{R}^3\setminus B,\\ \chi_j|_+ - \chi_j|_- &= 0 \quad \text{on } \partial B,\\ \left.\frac{\partial\chi_j}{\partial\hat{\boldsymbol{n}}}\right|_+ - c\left.\frac{\partial\chi_j}{\partial\hat{\boldsymbol{n}}}\right|_- &= (c-1)\frac{\partial r_j}{\partial\hat{\boldsymbol{n}}} \quad \text{on } \partial B,\\ \chi_j &\to 0 \quad \text{as } |\boldsymbol{r}|\to\infty. \end{aligned} \tag{B.12}$$

The solution to (B.12), for each j, can be expressed in terms of a single-layer potential as $\chi_j(\boldsymbol{r})=\mathcal{S}_B\hat{\chi}_j(\boldsymbol{r})$, $\boldsymbol{r}\in\mathbb{R}^3$. This representation automatically satisfies the first transmission condition on ∂B and the second condition becomes

$$\left.\frac{\partial\mathcal{S}_B\hat{\chi}_j}{\partial\hat{\boldsymbol{n}}}\right|_+ - c\left.\frac{\partial\mathcal{S}_B\hat{\chi}_j}{\partial\hat{\boldsymbol{n}}}\right|_- = (c-1)\frac{\partial r_j}{\partial\hat{\boldsymbol{n}}} \quad \text{on } \partial B.$$

Then, by using (B.3), we can deduce, in a similar way to Ammari & Kang (2007, p. 77), that

$$\hat{\chi}_j(\boldsymbol{r}) = (\lambda\mathcal{I}-K_B^*)^{-1}(\hat{\boldsymbol{n}}'\cdot\nabla_{r'}r_j')(\boldsymbol{r}'), \quad \boldsymbol{r}\in\partial B. \tag{B.13}$$

Clearly, $\hat{\vartheta}_\alpha(\boldsymbol{r})$ has similarities to $\hat{\chi}_j(\boldsymbol{r})$. In fact, by limiting consideration to $|\alpha|=|\beta|=1$, the generalized polarization tensor of Ammari and Kang can instead be written in terms of $\hat{\chi}$ as the symmetric matrix

$$[\![\mathscr{M}_B(c)]\!]_{ij} = \int_{\partial B} r_j\hat{\chi}_i(\boldsymbol{r})\,\mathrm{d}S = \int_{\partial B} r_i\hat{\chi}_j(\boldsymbol{r})\,\mathrm{d}S. \tag{B.14}$$

Finally, we can relate $[\![\mathscr{M}_B(c)]\!]$ to $[\![M_B(c)]\!]$ by using a similar approach to Lemma 4.3 in Ammari & Kang (2007): Recalling that $\theta_j=r_j+\chi_j=r_j+\mathcal{S}_B\hat{\chi}_j$, then

$$\left.\frac{\partial\theta_j}{\partial\hat{\boldsymbol{n}}}\right|_- = \hat{\boldsymbol{n}}\cdot\nabla r_j + \left(-\frac{1}{2}+K_B^*\right)\hat{\chi}_j(\boldsymbol{r}), \quad \boldsymbol{r}\in\partial B, \tag{B.15}$$

and noting that $-\frac{1}{2}\mathbb{I}+K_B^* = -(\lambda\mathbb{I}-K_B^*)+(\lambda-\frac{1}{2})\mathbb{I}$, it can be shown that

$$\begin{aligned} \int_{\partial B} r_i\left.\frac{\partial\theta_j}{\partial\hat{\boldsymbol{n}}}\right|_- \mathrm{d}S &= \left(\lambda-\frac{1}{2}\right)\int_{\partial B} r_i(\lambda\mathbb{I}-\mathcal{K}_B^*)^{-1}(\hat{\boldsymbol{n}}'\cdot\nabla_{r'}r_j')(\boldsymbol{r})\,\mathrm{d}S\\ &= \left(\lambda-\frac{1}{2}\right)[\![\mathscr{M}_B(c)]\!]_{ij}. \end{aligned} \tag{B.16}$$

Thus, since $\lambda-\frac{1}{2}=1/(c-1)$, and by considering (B.11), we have $[\![\mathscr{M}_B(c)]\!]_{ij}=-[\![M_B(c)]\!]_{ij}$. In Dassios & Kleinman (2000, p. 168), the authors present an alternative form of their general polarization tensor where the equivalence with Ammari and Kang's tensor is almost immediate. □