Swansea University
Prifysgol Abertawe

Cronfa
Setting Research Free

Cronfa - Swansea University Open Access Repository

_____

This is an author produced version of a paper published in :
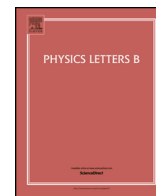*Physics Letters B*

_____

Cronfa URL for this paper:
http://cronfa.swan.ac.uk/Record/cronfa20694

_____

_____

CrossMark

The quark condensate in multi-flavour QCD – planar equivalence confronting lattice simulations

Adi Armoni [a], Mikhail Shifman [b], Graham Shore [a,*], Gabriele Veneziano [c,d]

[a] *Department of Physics, Swansea University, Singleton Park, Swansea, SA2 8PP, UK*
[b] *William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455, USA*
[c] *PH Department, CERN, CH-1211 Geneva 23, Switzerland*
[d] *Collège de France, 11 place M. Berthelot, 75005 Paris, France*

A B S T R A C T

Planar equivalence between the large N limits of $\mathcal{N} = 1$ Super Yang–Mills (SYM) theory and a variant of QCD with fermions in the antisymmetric representation is a powerful tool to obtain analytic non-perturbative results in QCD itself. In particular, it allows the quark condensate for $N = 3$ QCD with quarks in the fundamental representation to be inferred from exact calculations of the gluino condensate in $\mathcal{N} = 1$ SYM. In this paper, we review and refine our earlier predictions for the quark condensate in QCD with a general number n_f of flavours and confront these with lattice results.

© 2014 The Authors. Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP³.

1. The QCD condensate and planar equivalence

One of the challenges in theoretical studies of QCD is to find analytic, non-perturbative methods for calculations of strong-coupling quantities such as the quark or gluon condensates. Such methods are an important complement to direct evaluations using lattice gauge theory and give extra physical insight into the underlying dynamical mechanisms. One proposal is to use special properties of supersymmetric theories to perform exact non-perturbative calculations in $\mathcal{N} = 1$ gauge theories, then relate these in suitable limits to infer results in QCD itself. This approach has been pioneered in Refs. [1,2].

The key idea is to exploit a remarkable property of $SU(N)$ QCD with a single flavour of Dirac fermions in the antisymmetric representation,[1] QCD_{AS}, *viz.* that as N is varied, it interpolates between three theories of special importance – pure Yang–Mills theory, QCD with one flavour of fundamental fermions and $\mathcal{N} = 1$ super-Yang–Mills theory.

Specifically, for $N = 2$, the antisymmetric representation becomes trivial and QCD_{AS} becomes simply $SU(2)$ Yang–Mills. For $N = 3$, the antisymmetric representation (which has dimension $\frac{1}{2}N(N-1)$) coincides with the fundamental representation (dimension N) and so $\text{QCD}_{\text{AS}}(N = 3)$ is identical to one-flavour

$SU(3)$ QCD.[2] In the large N limit, $\text{QCD}_{\text{AS}}(N \to \infty)$ becomes equivalent to a theory with $SU(N)$ gauge group and a single real fermion in the adjoint representation (dimension $N^2 - 1$). Crucially, this theory is supersymmetric, *viz.* $\mathcal{N} = 1$ super Yang–Mills (SYM), and this is the key to being able to perform the exact non-perturbative calculations we exploit.

The relation of QCD_{AS} at large N with $\mathcal{N} = 1$ SYM has been extensively described in a series of earlier papers on "planar equivalence" [1–4]. It has been shown that in the 't Hooft large-N limit the two theories become equivalent in the common bosonic C-parity even sector. A necessary and sufficient condition for planar equivalence to hold is that charge conjugation symmetry is not broken spontaneously [5]. This was verified by a dedicated lattice simulation [6] (see also [7]) where it was shown that charge conjugation symmetry is broken if one dimension is compactified on a small-enough circle, but is restored at large (in particular infinite) compactification radius.

In this paper, we focus on a single issue – the prediction of the value of the quark condensate in QCD, its N and n_f dependence, and its confrontation with lattice data. The gluino condensate [8,9] has been evaluated exactly in $\mathcal{N} = 1$ SYM [10,11] and the idea here is to use QCD_{AS} with varying N to infer the value of the quark condensate for one-flavour $N = 3$ QCD by interpolating between

---

* Corresponding author.

[1] This is the theory referred to in [1–3] as "QCD-OR" or "Orientifold QCD". This name highlights its origin in string theory [12], though this will play no role in the analysis here.

[2] Here we use n_f for the *total* number of quark flavours. Note this is in contrast to [4] where it was used as the *additional* number of fundamental flavours in a "QCD-OR'" theory comprising one antisymmetric flavour plus fundamentals, which we call $\text{QCD}_{\text{AS-F}}$ here.

Table 1

Renormalisation group coefficients for general N and n_f for the theories considered here. Gamma is the anomalous dimension for the condensate, *i.e.* the running mass anomalous dimension. In general, for a (complex) representation R with n_f flavours, $\beta_0 = \frac{11}{3}C_2(A) - \frac{4}{3}n_f T(R)$, $\beta_1 = \frac{17}{3}C_2(A)^2 - \frac{1}{3}n_f T(R)(10C_2(A) + 6C_2(R))$ and $\gamma = 3C_2(R)$, where $C_2(A)$ is the quadratic Casimir for the gauge group.

	$\text{QCD}_{\text{AS-F}}(N, n_f)$	Yang–Mills	$\text{QCD}_F(N, n_f)$	$\mathcal{N} = 1$ SYM
β_0	$3N + 2 - \frac{2}{3}n_f$	$\frac{11}{3}N$	$\frac{11}{3}N - \frac{2}{3}n_f$	$3N$
β_1	$3N^2 + \frac{17}{2}N - \frac{9}{2N} - \frac{1}{6}n_f(13N - \frac{3}{N})$	$\frac{17}{3}N^2$	$\frac{17}{3}N^2 - \frac{1}{6}n_f(13N - \frac{3}{N})$	$3N^2$
γ	$\frac{3}{N}(N-2)(N+1)$ [AS]	–	$\frac{3}{2N}(N^2 - 1)$	$3N$
	$\frac{3}{2N}(N^2 - 1)$ [F]			

$\mathcal{N} = 1$ SYM at large N and pure Yang–Mills at $N = 2$, where of course the condensate disappears. For many flavours, we consider a generalisation to a theory, $\text{QCD}_{\text{AS-F}}$, with one AS representation fermion and $(n_f - 1)$ fundamentals.

The calculation of the gluino condensate $\langle \lambda\lambda \rangle \equiv \langle \lambda^{a\alpha} \lambda^a_\alpha \rangle$ in $\mathcal{N} = 1$ SYM relies on the holomorphy of F-terms in supersymmetric theories to analytically continue a weak-coupling, semi-classical evaluation of the condensate in a deformed version of the theory to the strong-coupling regime of $\mathcal{N} = 1$ SYM itself. Specifically, in Ref. [10], additional matter fields with mass m are added allowing the condensate to be calculated from the one-instanton contribution in a weak-coupling regime at small non-zero m before taking $m \to \infty$ to decouple the extra fields and recover the original theory. In Ref. [11], $\mathcal{N} = 1$ SYM itself is considered on a compactified space $R^3 \times S$ (with β the radius of the compactified dimension) and the condensate is evaluated initially in the limit of small β where the theory is weakly-coupled and the condensate is dominated by contributions from monopole configurations, both conventional BPS type and additional Kaluza–Klein monopoles. Both approaches agree and, quoting the result for the condensate for an $SU(N)$ gauge group in terms of the scale $\Lambda_{\overline{MS}}$ appropriate to SYM (see below), we have

$$\langle \lambda\lambda \rangle_{\overline{MS}} = -\frac{N^2}{2\pi^2} \frac{3}{2\lambda(\mu)} \Lambda^3_{\overline{MS}}\big|_{\text{SYM}}. \tag{1.1}$$

Before proceeding, we need to carefully specify our conventions and the definitions of the key quantities used below.[3] First, for ease of reference, in Tables 1 and 2 we collect the N and n_f dependence of the main group theoretical parameters and the renormalisation group coefficients for the theories considered here.

Our results are presented first in terms of renormalisation group invariant quantities, written in terms of the 't Hooft coupling. We define the RG invariant scale parameter

$$\Lambda_c = \mu \big(c\lambda(\mu)\big)^{-\beta_1/\beta_0^2} e^{-N/(\beta_0\lambda)}, \tag{1.2}$$

and the RG invariant condensate for a Dirac fermion ψ as

$$\langle \bar\psi\psi \rangle_{\tilde c} = \big(\tilde c\lambda(\mu)\big)^{\gamma/\beta_0} \langle \bar\psi\psi \rangle_{\overline{MS}}, \tag{1.3}$$

where $\langle \bar\psi\psi \rangle_{\overline{MS}}$ denotes the renormalised condensate in the $\overline{MS}$ scheme at scale μ. The normalisation parameters c and $\tilde c$ are essentially arbitrary, but should admit an expansion in $1/N$ around a finite $O(1)$ large-N limit. This ensures the condensate matching condition (1.5) below is consistent with planar equivalence at large N [3]. The conventional $\overline{MS}$ definition of the scale parameter $\Lambda_{\overline{MS}}$

Table 2

The Dynkin index $T(R)$ and quadratic Casimir $C_2(R)$ for various representations of $SU(N)$. For a representation R of $SU(N)$ with generators t^a they are defined as $\text{tr}\, t^a t^b = T(R)\delta^{ab}$ and $(t^a t^a)_{ij} = C_2(R)\delta_{ij}$ and satisfy $T(R)\dim(A) = C_2(R)\dim(R)$.

Representation (R)	$T(R)$	$C_2(R)$	$\dim(R)$
Antisymmetric (AS)	$\frac{1}{2}(N-2)$	$\frac{1}{N}(N-2)(N+1)$	$\frac{1}{2}N(N-1)$
Fundamental (F)	$\frac{1}{2}$	$\frac{1}{2N}(N^2 - 1)$	N
Adjoint (A)	N	N	$N^2 - 1$

is simply Λ_c with $c = \beta_0/2N$. Later, we will choose $\tilde c = \beta_0/N$ to facilitate easy comparison with lattice results.

Expressed entirely in terms of these RG invariant quantities, the SYM gluino condensate is therefore

$$\langle \lambda\lambda \rangle_{\tilde c}/\Lambda^3_c\big|_{\text{SYM}} = -\frac{N^2}{2\pi^2} c\tilde c, \tag{1.4}$$

noting that for $\mathcal{N} = 1$ SYM both γ/β_0 and $3\beta_1/\beta_0^2$ are simply 1.

One flavour, QCD$_{\text{AS}}$:

To determine the condensate in one-flavour QCD, we start from the QCD_{AS} theory, where planar equivalence has been firmly established. Our basic ansatz for the QCD_{AS} condensate is

$$\langle \bar\Psi\Psi \rangle_{\tilde c}/\Lambda^3_c\big|_{\text{AS}}$$
$$= -\frac{N^2}{2\pi^2}\left(1 - \frac{2}{N}\right) c^{3\beta_1/\beta_0^2} \tilde c^{\gamma/\beta_0} K_{\text{AS}}(1/N; n_f = 1), \tag{1.5}$$

where Ψ denotes a fermion in the AS representation of $SU(N)$ and the appropriate RG coefficients can be read off from Table 1. The content of (1.5) is that the most significant $1/N$ correction to the leading large N behaviour of $\langle \bar\Psi\Psi \rangle_{\tilde c}$ as determined by planar equivalence with the exact SYM result (1.4) is given by the relative $(1 - 2/N)$ factor. Assuming a smooth dependence of $\langle \bar\Psi\Psi \rangle_{\tilde c}$ on N in the QCD_{AS} theory, this is the simplest interpolating factor between the large N SYM result and the vanishing of the condensate for $N = 2$, where the antisymmetric representation is trivial and QCD_{AS} degenerates to pure $SU(2)$ Yang–Mills. Notice that this factor is simply the ratio of the Dynkin indices for the AS and adjoint representations, a feature we may conjecture to be more generally valid. The remaining sub-leading corrections are encoded in the factor $K_{\text{AS}} = 1 + O(1/N)$, which we initially assume to be relatively small.

Given the arbitrariness in the normalisation of the RG invariant condensates and scale parameters, it is natural to separate the dependence on the c, $\tilde c$ factors explicitly on the rhs of (1.5). Notice[4] that in the *ratio of ratios* between (1.5) and (1.4) for the

---

[3] Our conventions follow those of the Particle Data Group, QCD review, 2008 [13]. Since we work here with the 't Hooft coupling $\lambda = g^2 N/8\pi^2$ rather than $\alpha_s = g^2/4\pi$, it is more convenient to use the RG coefficients $\beta_0, \beta_1, \dots$ rather than the more recent PDG 2014 [14] definitions $b_0, b_1, \dots$ to absorb convention-dependent factors of 4π. The gluino field in (1.1) is normalised so that its kinetic term in the SYM Lagrangian is $\mathcal{L} = i\bar\lambda\slashed{\partial}\lambda$.

[4] Explicitly, for a single AS representation, $\gamma/\beta_0 = 1 - \frac{13}{9}\frac{1}{N}$ and $3\beta_1/\beta_0^2 = 1 + \frac{11}{9}\frac{1}{N}$, so $\tilde c^{-1+\gamma/\beta_0} = 1 - \frac{13}{9}\log\tilde c + O(1/N^2)$ and $c^{-1+3\beta_1/\beta_0^2} = 1 + \frac{11}{9}\log c + O(1/N^2)$.

AS and adjoint representation condensates, both these factors are $1 + O(1/N)$ since the RG factors in the exponents are both $O(1/N)$ (see Table 1), so could in principle be absorbed into the K_{AS} factor. However, this would not be appropriate since they are clearly convention dependent whereas K_{AS} should reflect the basic $O(1/N)$ physics of the theory.

Our prediction for the condensate can be expressed in several ways, which will be useful for comparing with lattice data. In particular, we may write

$$\langle \bar{\Psi}\Psi \rangle_{\tilde{c}} / \Lambda^3_{\overline{MS}} \big|_{AS} = -\frac{N^2}{2\pi^2}\left(1 - \frac{2}{N}\right)\left(\frac{\beta_0}{2N}\right)^{3\beta_1/\beta_0^2}$$
$$\times \tilde{c}^{\gamma/\beta_0} K_{AS}(1/N; n_f = 1), \qquad (1.6)$$

and

$$\langle \bar{\Psi}\Psi \rangle_{\overline{MS}} / \Lambda^3_{\overline{MS}} \big|_{AS} = -\frac{N^2}{2\pi^2}\left(1 - \frac{2}{N}\right)\left(\frac{\beta_0}{2N}\right)^{3\beta_1/\beta_0^2}$$
$$\times \lambda(\mu)^{-\gamma/\beta_0} K_{AS}(1/N; n_f = 1), \qquad (1.7)$$

where in the latter form, $\langle \bar{\Psi}\Psi \rangle_{\overline{MS}}$ is μ-dependent and we need to find the 't Hooft coupling $\lambda(\mu)$ by inverting the relation (1.2) for $\Lambda_{\overline{MS}}$. Finally, as used in Ref. [4], we could express the condensate entirely in terms of the 't Hooft coupling at scale μ, viz.

$$\langle \bar{\Psi}\Psi \rangle_{\overline{MS}} = -\mu^3 \frac{N^2}{2\pi^2}\left(1 - \frac{2}{N}\right)\lambda(\mu)^{-3\beta_1/\beta_0^2 - \gamma/\beta_0}$$
$$\times e^{-3N/(\beta_0 \lambda(\mu))} K_{AS}(1/N; n_f = 1). \qquad (1.8)$$

n_f flavours, QCD$_{AS-F}$:

So far, we have discussed the condensate in theories with only a single flavour, where planar equivalence with $\mathcal{N} = 1$ SYM has been demonstrated for QCD$_{AS}$. In Ref. [4], we explored to what extent planar equivalence could be shown directly in a multi-flavour theory. Since we need the additional flavours to decouple in the large-N limit, and since we ultimately wish to discuss $N = 3$ QCD with quarks in the fundamental representation, we considered the hybrid theory QCD$_{AS-F}$, viz. QCD with one AS and $(n_f - 1)$ fundamental fermions (see footnote 2).

The demonstration of planar equivalence and matching of condensates with $\mathcal{N} = 1$ SYM in this case involved comparison of Wilson loops and the construction from anomalous chiral Ward identities of a 'decoupling' current, which defines a sector in which the Goldstone bosons of spontaneously broken chiral symmetry do not affect the relevant correlation functions. These theoretical considerations are described at length in [4]. Here we just quote our conclusions for the RG-invariant condensates:

$$\langle \bar{\Psi}\Psi \rangle_{\tilde{c}} / (\Lambda^{(n_f)}_c)^3 \big|_{AS-F} = -\frac{N^2}{2\pi^2}\left(1 - \frac{2}{N}\right)$$
$$\times c^{3\beta_1/\beta_0^2} \tilde{c}^{\gamma/\beta_0} K_{AS}(1/N; n_f), \qquad (1.9)$$

for the AS fermion Ψ, while for the fundamental fermions q,

$$\langle \bar{q}q \rangle_{\tilde{c}} / (\Lambda^{(n_f)}_c)^3 \big|_{AS-F} = -\frac{N}{2\pi^2} c^{3\beta_1/\beta_0^2} \tilde{c}^{\gamma/\beta_0} K_F(1/N; n_f). \qquad (1.10)$$

Once again, the K factor for the AS representation is $K_{AS} = 1 + O(1/N)$, with the n_f dependence contained in the $O(1/N)$ terms. For the fundamental representation fermions, however, we do not necessarily need to impose this. All that is actually required is self-consistency for $N = 3$ when the two representations coincide, i.e. $K_F(1/3; n_f) = K_{AS}(1/3; n_f)$.
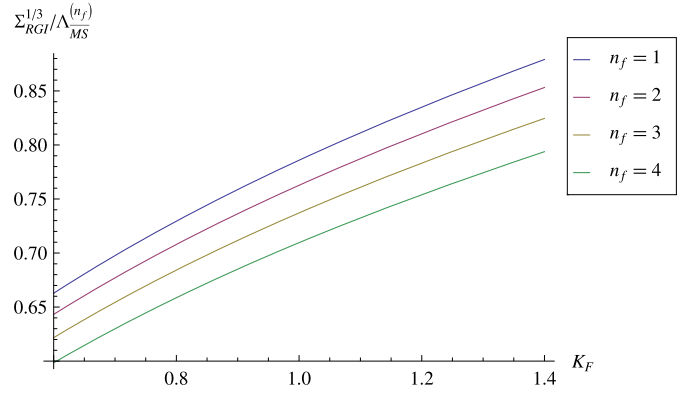

Fig. 1. Plot showing the dependence of the RG-invariant condensate $\Sigma^{1/3}_{RGI} / \Lambda^{(n_f)}_{\overline{MS}}$ on the $K_F(1/3; n_f)$ parameter for different numbers n_f of flavours.

These K factors encode the sub-dominant $1/N$ corrections, which we conjecture to be relatively small. Our initial condensate predictions for QCD are therefore based on taking the relevant $K \simeq 1$ and confronting these with lattice data. Further dynamical insight and assumptions may subsequently be used to refine the prediction. For example, in Ref. [4] we used the argument that QCD with n_f flavours, the K factors should go *smoothly* to zero as the conformal window is approached to estimate their flavour dependence, finding a rather mild dependence. Ideally, lattice simulations with sufficient precision to pin down the variation of the K factors for different numbers of flavours could ultimately give information on the behaviour of the condensate near the conformal window and the nature of the transition.

2. Numerical predictions and lattice data

We now specialise to QCD with $N = 3$ and n_f fundamental flavours and present numerical predictions for the $\langle \bar{q}q \rangle$ condensate based on the formulae above. These predictions will then be critically compared with available results from lattice gauge theory.

First we need to emphasise that the result of any calculation, analytic or lattice, is a dimensionless ratio, since the overall QCD scale is the free parameter of the theory. The cleanest way to present our results is therefore in terms of the ratio $\langle \bar{q}q \rangle^{1/3}_{\tilde{c}} / \Lambda^{(n_f)}_{\overline{MS}}$ of the RG-invariant condensate to the QCD scale parameter in the $\overline{MS}$ scheme for the relevant number of flavours. For ease of comparison with the lattice, we adopt here the convention of Ref. [15] for the RG-invariant condensate, viz. take $\tilde{c} = \beta_0/N$. We denote this condensate by $\Sigma_{RGI} \equiv -\langle \bar{q}q \rangle_{\tilde{c}=\beta_0/N}$. The results of the previous section imply:

$$\Sigma^{1/3}_{RGI} / \Lambda^{(n_f)}_{\overline{MS}} = \left(\frac{3}{2\pi^2}\right)^{1/3}\left(\frac{\beta_0}{6}\right)^{\beta_1/\beta_0^2}\left(\frac{\beta_0}{3}\right)^{\gamma/3\beta_0} K^{1/3}_F(1/3; n_f). \qquad (2.1)$$

Our fundamental prediction (2.1) is shown in Fig. 1, where we plot the ratio of the RG-invariant condensate to $\Lambda_{\overline{MS}}$ for different numbers of flavours as a function of the K_F parameter. As we see below, lattice data supports the view that K_F is close to 1, so for orientation we list here our predictions taking $K_F = 1$:

$$\Sigma^{1/3}_{RGI} / \Lambda^{(n_f)}_{\overline{MS}} = 0.786\,(n_f = 1), \quad 0.763\,(n_f = 2),$$
$$0.737\,(n_f = 3), \quad 0.710\,(n_f = 4). \qquad (2.2)$$

It is also useful to express our results in terms of the $\overline{MS}$ condensate $\Sigma_{\overline{MS}} = \langle \bar{q}q \rangle |_{\mu=2\ \text{GeV}}$. This requires the relation between $\Lambda_{\overline{MS}}$ and the 't Hooft coupling $\lambda(\mu)$. In fact, for accuracy in the
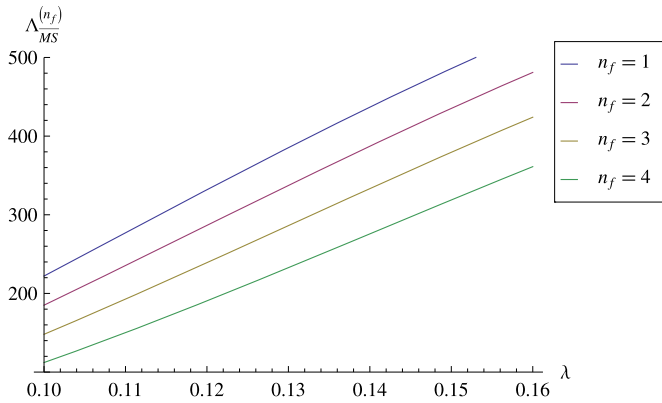
Fig. 2. Plot showing the relation between the scale parameter $\Lambda_{\overline{MS}}^{(n_f)}$ and the 't Hooft coupling $\lambda(\mu)$ evaluated at $\mu = 2$ GeV for different numbers n_f of flavours.
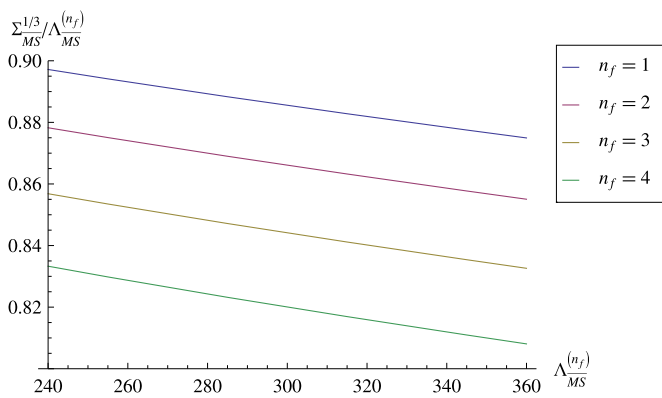

Fig. 3. Plot showing the condensate ratio $\Sigma_{\overline{MS}}^{1/3}/\Lambda_{\overline{MS}}^{(n_f)}$ in the $\overline{MS}$ scheme in terms of the scale parameter $\Lambda_{\overline{MS}}^{(n_f)}$ for different numbers n_f of flavours.

numerical predictions, we do this using the three-loop RG formula, rather than two-loop expression given above, *viz.*

$$\Lambda_{\overline{MS}}^{(n_f)} = \mu \left(\frac{\beta_0}{2N} \lambda \right)^{-\beta_1/\beta_0^2} e^{-N/\beta_0\lambda}$$
$$\times \left[1 - \frac{\lambda}{8N} \frac{1}{\beta_0^2} \left(\beta_2 - \frac{8\beta_1^2}{\beta_0} \right) + \ldots \right]. \tag{2.3}$$

This is shown, for $N = 3$, in Fig. 2.

From the previous section, we have the following formula for the $\overline{MS}$ condensate in $N = 3$ QCD:

$$\Sigma_{\overline{MS}}^{1/3}/\Lambda_{\overline{MS}}^{(n_f)} = \left(\frac{3}{2\pi^2} \right)^{1/3} \left(\frac{\beta_0}{6} \right)^{\beta_1/\beta_0^2}$$
$$\times \lambda(\mu)^{-\gamma/3\beta_0} K_F^{1/3}(1/3; n_f). \tag{2.4}$$

This is plotted, taking $K_F = 1$ and evaluating at the standard scale $\mu = 2$ GeV, in Fig. 3 for the ratio $\Sigma_{\overline{MS}}^{1/3}/\Lambda_{\overline{MS}}^{(n_f)}$ and in Fig. 4 for the condensate $\Sigma_{\overline{MS}}^{1/3}$ itself expressed in MeV units inherited from the n_f-dependent scale parameter.

To confront these predictions with lattice results, we need to be careful about interpreting scale-dependent data in variants of 'real-world' QCD in which the number of flavours is varied. These are in principle distinct theories with their own independent free scale parameter $\Lambda_{\overline{MS}}$. Only for real-world QCD (which we consider as $n_f = 3$ light flavours with quark masses taken into account) can predictions be unambiguously linked to experimental data, al-

lowing results to be expressed in genuine MeV units.[5] This means that the only strictly meaningful comparisons to be made are between predictions of dimensionless ratios. For our purposes, this requires comparing our predictions to a lattice calculation that *self-consistently* determines the ratio of the condensate to $\Lambda_{\overline{MS}}$.

While, as we discuss in Appendix A, there are several evaluations in the literature of the condensate for various n_f, these are usually expressed in some definition of MeV units and are not linked to a self-consistent determination of $\Lambda_{\overline{MS}}$. This makes a precision confrontation of lattice data with our planar equivalence predictions difficult.

An exception is the recent work of Engel et al. [15] and the ALPHA lattice collaboration [16] in which they quote self-consistent evaluations of both the RG-invariant condensate $\Sigma_{RGI}^{1/3}$ and the scale parameter $\Lambda_{\overline{MS}}^{(2)}$ for $n_f = 2$. The condensate is determined by studying the rate of condensation of the low eigenvalues of the Dirac operator near the limit of vanishing quark mass. For the ratio, they quote[6]

$$\Sigma_{RGI}^{1/3}/\Lambda_{\overline{MS}}^{(2)} = 0.77 \ (4). \tag{2.5}$$

Comparing with Eq. (2.2) for $n_f = 2$, this is in quite remarkable agreement with our $K_F = 1$ prediction of 0.763.

To illustrate this further, in Fig. 5 we restrict the plot of $\Sigma_{RGI}^{1/3}/\Lambda_{\overline{MS}}^{(2)}$ (see Fig. 1) to $n_f = 2$ and superimpose our prediction for the condensate as a function of K_F with the one-sigma error band of the lattice result (2.5). The lattice constraint on K_F is therefore

$$K_F(1/3; n_f = 2) = 1.03 \ (16) \tag{2.6}$$

in excellent agreement with the planar equivalence prediction and our understanding that the corrections to $K_F \simeq 1$ are relatively small.

Nonetheless, despite this success, it is clear that if we are to rely on the lattice to determine K_F with the precision to gain insight into the flavour-dependence of the quark condensate and the transition to the conformal window, the accuracy of lattice calculations needs to be increased, along with the extension to self-consistent determinations of *both* the condensate and $\Lambda_{\overline{MS}}^{(n_f)}$ parameters for other values of n_f.

The challenge to the lattice is therefore to extend determinations of the quark condensate in QCD to different numbers of flavours with the accuracy required to find a real discrimination amongst different n_f. Comparison with the planar equivalence predictions may also be stringently tested by simulations for different numbers of colours, $N \neq 3$, or different fermion representations. For example, in Ref. [17], a lattice study of the condensate was carried out in the quenched approximation with fermions in the AS, symmetric and adjoint representations of $SU(N)$ for various values of N. This broadly confirms the planar equivalence expectations

---

[5] In practice, a compromise is usually made whereby the MeV scale for QCD with $n_f \neq 3$ is set by fixing some quantity which is considered to be only relatively weakly dependent on n_f to its experimental, real-world QCD, value. This is of course potentially dangerous if we are to use lattice results to determine the n_f-dependence of the condensate and constrain the K_F parameter.

[6] In Ref. [15], the results are given in terms of an auxiliary scale F as

$$\Sigma_{RGI}^{1/3}/F = 2.77 \ (2) \ (4), \qquad \Lambda_{\overline{MS}}^{(2)}/F = 3.6 \ (2).$$

Setting MeV units by supplementing the theory with a quenched strange quark and fixing the scale through a fit to the physical decay constant F_K, they quote

$$\Sigma_{\overline{MS}}^{1/3}\big|_{2 \text{ GeV}} = 263 \ (3) \ (4) \text{ MeV}, \qquad \Lambda_{\overline{MS}}^{(2)} = 311 \ (19) \text{ MeV}.$$
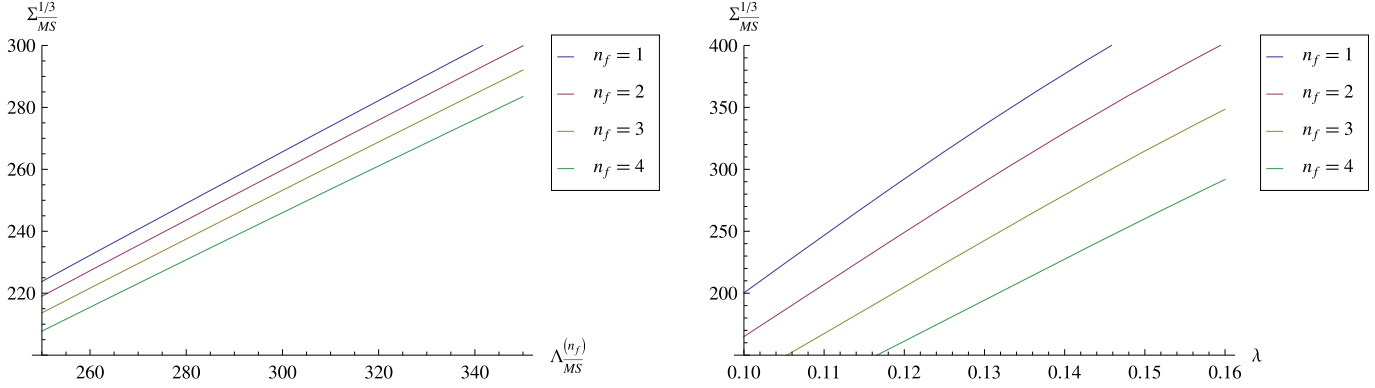
Fig. 4. Plots showing the condensate ratio $\Sigma_{\overline{MS}}^{1/3}$ in the $\overline{MS}$ scheme at $\mu = 2$ GeV in terms of the scale parameter $\Lambda_{\overline{MS}}^{(n_f)}$ (left) and 't Hooft coupling $\lambda(\mu)$ (right) for different numbers n_f of flavours.
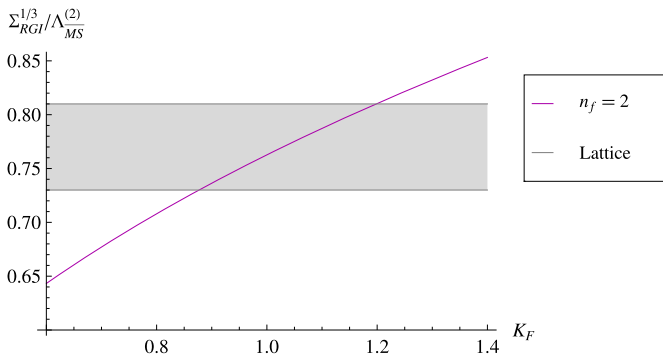

Fig. 5. Plot showing the dependence of the RG-invariant condensate $\Sigma_{RGI}^{1/3}/\Lambda_{\overline{MS}}^{(n_f)}$ on the $K_F(1/3, n_f)$ parameter for $n_f = 2$ superimposed with the one-sigma error band of the lattice determination of Engel et al.

and in particular the result (1.5) that to leading order, the ratio of condensates for different representations is given by the ratio of their Dynkin indices. In particular, we anticipate the following expression for the quark condensate in a theory with fermions in the symmetric (S) representation:

$$\langle \bar{\Psi}\Psi \rangle_{\tilde{c}}/\Lambda_{\overline{MS}}^3 \big|_S = -\frac{N^2}{2\pi^2} \left(1 + \frac{2}{N}\right) \left(\frac{\beta_0}{2N}\right)^{3\beta_1/\beta_0^2}$$
$$\times \tilde{c}^{\gamma/\beta_0} K_S(1/N; n_f = 1). \qquad (2.7)$$

Meanwhile, it would be interesting to extend the planar equivalence programme further by attempting analytic calculations of further quantities beyond the gluino and quark condensates, identifying other scale-setting quantities more suited to comparison with the lattice than $\Lambda_{\overline{MS}}$, and looking for further relations between $\mathcal{N} = 1$ SYM and QCD.

Appendix A. Lattice calculations of the condensate for $n_f = 1, 2, 3$

We present here a brief review and update of earlier exploratory determinations of the quark condensate in $N = 3$ QCD

with $n_f = 1, 2$ and 3 fundamental flavours on the lattice, and their comparison with our planar equivalence predictions. Unfortunately, this lattice data is not sufficiently accurate to give a reliable discrimination between different n_f, although as we show it agrees within its uncertainties with our predictions.

The first comparison of the planar equivalence result with lattice simulations was made in Ref. [3] with the work of DeGrand et al. [18] (see also [19]) for $n_f = 1$. Scale-setting in the $n_f = 1$ theory was performed in [3] by equating $\Lambda_{\overline{MS}}^{(1)}$ with the value of $\Lambda_{\overline{MS}}^{(3)}$ inferred from the experimental value of $\lambda(\mu = 2$ GeV) in physical $n_f = 3$ QCD to obtain a prediction in MeV units. However, this does not correspond with the scale-setting used in the lattice calculation. Here, we improve on this comparison and update the result of [18] using more recent lattice data for the scales involved.

The essential result of [18] is a value for the $\overline{MS}$ condensate at $\mu = 2$ GeV in units of the Sommer parameter r_0, viz. $r_0 \Sigma_{\overline{MS}}^{1/3} = 0.68$ (2). The scale $\Lambda_{\overline{MS}}$ was introduced using the then current values of the ALPHA collaboration [20], viz. $r_0 \Lambda_{\overline{MS}}^{(0)} = 0.60$ (8) and $r_0 \Lambda_{\overline{MS}}^{(2)} = 0.62$ (6) with $r_0 \simeq 0.5$ fm $\simeq (400$ MeV$)^{-1}$, corresponding within errors to an approximately n_f-independent value taken as $\Lambda_{\overline{MS}}^{(2)} = 245$ (20) MeV. We can, however, improve on this if we take the most recent ALPHA determination of $\Lambda_{\overline{MS}}^{(2)}$ from [16] and, still assuming $r_0 \Lambda_{\overline{MS}}^{(n_f)}$ is not too sensitive to $n_f = 1$ or 2, use this to set the scale for the DeGrand et al. calculation. We therefore take $r_0 = 0.503$ (10) fm and $r_0 \Lambda_{\overline{MS}}^{(2)} = 0.78$ (6), corresponding to $\Lambda_{\overline{MS}}^{(2)} = 310$ (20) MeV [16], and combining this with the value of $r_0 \Sigma_{\overline{MS}}^{1/3}$ given above, we now deduce

$$\Sigma_{\overline{MS}}^{1/3}/\Lambda_{\overline{MS}}^{(1)} = 0.87 \ (7), \qquad (A.1)$$

and $\Sigma_{\overline{MS}}^{1/3} = 270$ (20) MeV. This is to be compared with the $K_F = 1$ planar equivalence prediction for $n_f = 1$ (see Fig. 3)

$$\Sigma_{\overline{MS}}^{1/3}/\Lambda_{\overline{MS}}^{(1)} = 0.884, \qquad (A.2)$$

corresponding to $\Sigma_{\overline{MS}}^{1/3} = 274$ MeV. With this improved scale-setting, we see that the $n_f = 1$ lattice result is indeed now in good agreement, within its significant uncertainty, with the planar equivalence prediction.

A similar improvement can be applied to the original $n_f = 2$ condensate prediction by DeGrand et al. in Ref. [21]. Taking the

result[7] given there as $r_0 \Sigma_{\overline{MS}}^{1/3} = 0.69$ (2), we find $\Sigma_{\overline{MS}}^{1/3}/\Lambda_{\overline{MS}}^{(2)} = 0.88$ (7). This is to be compared with the $K_F = 1$ planar equivalence prediction 0.864 (see Fig. 3) which, with $\Lambda_{\overline{MS}}^{(2)} = 311$ MeV [15], corresponds to $\Sigma_{\overline{MS}}^{1/3} = 269$ MeV. Again we recover reasonable agreement, bringing the result of Ref. [21] into line with the precision calculation of Engel et al. [15], for which this ratio is 0.85 (5).

For $n_f = 3$, our planar equivalence prediction is

$$\Sigma_{\overline{MS}}^{1/3}/\Lambda_{\overline{MS}}^{(3)} = 0.839. \tag{A.3}$$

If we set the scale by using the PDG [14] value for the 't Hooft coupling $\lambda(\mu = 2\ \text{GeV}) = 0.143$, corresponding to $\Lambda_{\overline{MS}}^{(3)} = 339$ (10) MeV our $K_F = 1$ prediction is $\Sigma_{\overline{MS}}^{1/3} = 284$ MeV. This is again supported by recent lattice results, taking *e.g.* $\Sigma_{\overline{MS}}^{1/3} = 283$ (2) MeV [22] as a representative figure.

References

[1] A. Armoni, M. Shifman, G. Veneziano, Exact results in non-supersymmetric large N orientifold field theories, Nucl. Phys. B 667 (2003) 170, arXiv:hep-th/0302163.

[2] A. Armoni, M. Shifman, G. Veneziano, Refining the proof of planar equivalence, Phys. Rev. D 71 (2005) 045015, arXiv:hep-th/0412203.

[3] A. Armoni, M. Shifman, G. Veneziano, QCD quark condensate from SUSY and the orientifold large-N expansion, Phys. Lett. B 579 (2004) 384, arXiv:hep-th/0309013.

[4] A. Armoni, G. Shore, G. Veneziano, Quark condensate in massless QCD from planar equivalence, Nucl. Phys. B 740 (2006) 23, arXiv:hep-ph/0511143.

[5] M. Unsal, L.G. Yaffe, (In)validity of large N orientifold equivalence, Phys. Rev. D 74 (2006) 105019, arXiv:hep-th/0608180.

[6] T. DeGrand, R. Hoffmann, QCD with one compact spatial dimension, J. High Energy Phys. 0702 (2007) 022, arXiv:hep-lat/0612012.

[7] B. Lucini, A. Patella, C. Pica, Baryon currents in QCD with compact dimensions, Phys. Rev. D 75 (2007) 121701, arXiv:hep-th/0702167.

[8] E. Witten, Constraints on supersymmetry breaking, Nucl. Phys. B 202 (1982) 253.

[9] G. Veneziano, S. Yankielowicz, An effective Lagrangian for the pure $N = 1$ supersymmetric Yang–Mills theory, Phys. Lett. B 113 (1982) 231.

[10] M.A. Shifman, A.I. Vainshtein, On gluino condensation in supersymmetric gauge theories SU(N) and O(N) groups, Nucl. Phys. B 296 (1988) 445, Sov. Phys. JETP 66 (1987) 1100.

[11] N.M. Davies, T.J. Hollowood, V.V. Khoze, M.P. Mattis, Gluino condensate and magnetic monopoles in supersymmetric gluodynamics, Nucl. Phys. B 559 (1999) 123, arXiv:hep-th/9905015.

[12] A. Sagnotti, Some properties of open string theories, in: SUSY 95, Palaiseau, 1995, pp. 473–484, arXiv:hep-th/9509080;
A. Sagnotti, Surprises in open string perturbation theory, Nucl. Phys. B, Proc. Suppl. 56 (1997) 332, arXiv:hep-th/9702093.

[13] C. Amsler, et al., Particle Data Group Collaboration, Review of particle physics, Phys. Lett. B 667 (2008) 1.

[14] K.A. Olive, et al., Particle Data Group Collaboration, Review of particle physics, Chin. Phys. C 38 (2014) 090001.

[15] G.P. Engel, L. Giusti, S. Lottini, R. Sommer, Chiral symmetry breaking in QCD Lite, arXiv:1406.4987 [hep-ph].

[16] P. Fritzsch, F. Knechtli, B. Leder, M. Marinkovic, S. Schaefer, R. Sommer, F. Virotta, The strange quark mass and Lambda parameter of two flavor QCD, Nucl. Phys. B 865 (2012) 397, arXiv:1205.5380 [hep-lat].

[17] A. Armoni, B. Lucini, A. Patella, C. Pica, Lattice study of planar equivalence: the quark condensate, Phys. Rev. D 78 (2008) 045019, arXiv:0804.4501 [hep-th].

[18] T. DeGrand, R. Hoffmann, S. Schaefer, Z. Liu, Quark condensate in one-flavor QCD, Phys. Rev. D 74 (2006) 054501, arXiv:hep-th/0605147.

[19] T. DeGrand, The one-flavor quark condensate and related problems, PoS LAT2006 (2006) 005, arXiv:hep-lat/0609031.

[20] M. Della Morte, et al., ALPHA Collaboration, Computation of the strong coupling in QCD with two dynamical flavors, Nucl. Phys. B 713 (2005) 378, arXiv:hep-lat/0411025.

[21] T. DeGrand, Z. Liu, S. Schaefer, Quark condensate in two-flavor QCD, Phys. Rev. D 74 (2006) 094504;
T. DeGrand, Z. Liu, S. Schaefer, Phys. Rev. D 74 (2006) 099904, arXiv:hep-lat/0608019 (Erratum).

[22] C. McNeile, A. Bazavov, C.T.H. Davies, R.J. Dowdall, K. Hornbostel, G.P. Lepage, H.D. Trottier, Direct determination of the strange and light quark condensates from full lattice QCD, Phys. Rev. D 87 (2013) 034503, arXiv:1211.6577 [hep-lat].

---

[7] In fact, Ref. [21] quotes three values for the condensate corresponding to simulations with different quark masses. Two of these are in close agreement, while the third, for the lightest quark mass, is substantially higher and is disregarded in the average quoted above.