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Emergent Lifshitz scaling from $\mathcal{N}=4$ SYM with supersymmetric heavy-quark density

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ABSTRACT: We consider supersymmetric configurations in Type IIB supergravity obtained by the beackreaction of fundamental strings ending on a stack of D3-branes and smeared uniformly in the three spatial directions along the D3-branes. These automatically include a distribution of D5-brane baryon vertices necessary to soak up string charge. The backgrounds are static, preserving eight supersymmetries, an SO(5) global symmetry and symmetry under spatial translations and rotations. We obtain the most general BPS configurations consistent with the symmetries. We show that the solutions to the Type IIB field equations are completely specified by a single function (the dilaton) satisfying a Poisson-like equation in two dimensions. We further find that the equation admits a class of solutions displaying Lifshitz-like scaling with dynamical critical exponent $z = 7$. The equations also admit an asymptotically $\mathrm{AdS}_5 \times \mathrm{S}^5$ solution deformed by the presence of backreacted string sources that yield a uniform density of heavy quarks in $\mathcal{N} = 4$ SYM.

Contents

1   Introduction and summary

Theories with spatially isotropic (nonrelativistic) scale invariance emerge as descriptions at quantum criticality [1–4] of various condensed matter systems. Such fixed point theories display a "dynamical" or Lifshitz scale invariance under the transformation,

$$ t \to \lambda^z t \,, \qquad \vec{x} \to \lambda \vec{x} \,, \qquad z \neq 1 \,, \tag{1.1} $$

which acts differently on the spatial ($\vec{x}$) and temporal (t) coordinates. Theories at Lifshitz points and at strong coupling are particularly interesting as they arise in the context of strongly correlated electron systems and models of high-T_c superconductors. The holographic duality between (large-N) quantum field theories (QFTs) and gravity/string theory [5, 6] has provided a natural setting for exploring properties of strongly interacting QFTs at Lifshitz points [7, 8], which can be further extended to theories with Lifshitz-like scaling accompanied by hyperscaling violation [9–11].

In this paper we will show that Lifshitz scaling arises in an interesting and unusual fashion in a family of supersymmetric ($\frac{1}{4}$-BPS) solutions within type IIB supergravity. The

backgrounds in question have a natural interpretation as long distance descriptions of a state in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory at large-N and strong coupling, with a spatially homogeneous distribution of static quark impurities. The configurations correspond to backreacted geometries of the intersections of D3-branes, F-strings (heavy quarks) and D5-branes (baryon vertices) [12]. The main result of this paper is the derivation of the most general conditions to be satisfied by static $\frac{1}{4}$-BPS configurations in IIB supergravity preserving ISO(3) × SO(5) symmetry,[1] which includes solutions discussed in [13], but also new ones as we describe below.

Our work is motivated by the goal of eventually obtaining holographic models suitable for understanding high density physics in QCD-like theories and unravelling the "condensed matter physics of QCD" [14–16] (albeit within holographic toy models). In order to make progress towards this goal, it is necessary to understand the dynamics of quark flavours and how they influence or backreact on the gluonic degrees of freedom at strong coupling. There are two reasons for this: the first reason is technical and a direct consequence of the large-N 't Hooft limit that accompanies any classical holographic dual description of gauge theories. Since QCD has "unquenched" quarks, it is necessary to address the backreaction of quark flavours in the large-N theory to model unquenched flavours. The second factor that necessitates inclusion of quark backreaction is intrinsic to physics at finite or high quark densities (in a deconfined phase) in the absence of temperature or any other comparable scale in the problem. In such a situation, when the quark density is not parametrically small, its backreaction on the large-N vacuum will determine the ground state of the system.

The necessity of incorporating flavour backreaction effects in holographic models at finite density (and low temperatures) has been emphasized in [17, 18]. While both issues above should be addressed simultaneously in principle, it is useful and interesting to first understand the possible manifestations of backreacting quark density within a holographic setting. If the quark flavours are (sufficiently) massive, at low enough energies we expect to be able to treat them as static objects. A state with a uniform density of these static quarks should, however, also be expected to backreact non-trivially on the glue degrees of freedom provided the quark density is $\sim \mathcal{O}(N^2)$ in the large-N limit.

In gauge theory, a heavy static quark corresponds to a straight timelike Wilson line (e.g. [19]). Therefore the state with a finite density of heavy-quarks can be viewed as the insertion of a distribution of Wilson line operators into the gauge theory. For densities scaling as N^2 in the large-N limit, we expect a non-trivial ground state (saddle-point) to emerge. This idea was implemented in a non-supersymmetric fashion in [20] in $\mathcal{N} = 4$ SYM theory at large-N and strong 't Hooft coupling. A heavy quark or straight Wilson line in $\mathcal{N} = 4$ SYM corresponds to a macroscopic, infinite string stretching radially from the conformal boundary of $\text{AdS}_5 \times \text{S}^5$ to the interior [21, 22]. As is well known, this (BPS-)Wilson line also carries an orientational SO(6) index associated to its location on the internal S^5. In [20], the distribution of static quarks was chosen to be both spatially uniform and

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SO(6)-symmetric, i.e. uniformly smeared around the S^5. The resulting non-supersymmetric background was shown to exhibit a flow from $AdS_5 \times S^5$ to an IR geometry $Lif_5 \times S^5$ displaying Lifshitz scaling with dynamical critical exponent $z = 7$. The scale invariance was found to be mildly broken by a logarithmically running dilaton.[2] Since the $\mathcal{N} = 4$ theory is scale invariant, there is no small parameter that controls the appearance of this IR scaling regime; any non-zero quark density leads to this (approximate) Lifshitz point.

The non-trivial picture above leaves several open questions. Firstly, the significance and the origin of the numerical value of $z = 7$ was *a priori* not understood. Second, given that the SO(6)-symmetric configuration is non-supersymmetric, the stability of the IR scaling solution was not established. Finally, although it is fairly clear that the picture should apply for dynamical massive quark flavours at low enough energy scales, its relevance for the large-N theory with massless quark flavours requires further clarification.[3]

In this paper, following on from an earlier publication [13], we derive the general BPS configurations describing supersymmetric backgrounds preserving eight supercharges in type IIB supergravity, generated by smeared strings intersecting with or ending on[4] a stack of D3-branes. In the latter case when semi-infinite strings end on branes, a non-vanishing density of baryon vertices or D5-branes is automatically induced. These appear and are necessary in order to soak up the string charge. The configurations described by our equations preserve an SO(5) subgroup of R-symmetry. The intersecting brane solutions with vanishing D5-charge were already explored in [13] and they are interpreted as quark-antiquark pairs with antipodal SO(6) orientation smeared uniformly in the gauge theory. The analysis in the present work allows us to further explore the situation where the strings end on the D3-branes and act as a source for non-zero quark density with all the quarks aligned with the same internal orientation (preserving an SO(5) internal symmetry).

The $\frac{1}{4}$-BPS configurations we find in this paper are determined by a single function, namely the dilaton, which satisfies a Poisson-like equation in two dimensions. It is somewhat remarkable then that this Poisson equation admits a class of solutions that exhibit Lifshitz scaling with $z = 7$ and a logarithmically running dilaton. Together with [20] and the F1-D3 intersection of [13], this provides the third distinct instance of backreacted quark impurities in $\mathcal{N} = 4$ SYM yielding identical scaling behaviour, independent of the global symmetries or supersymmetries of the configuration. This lends strong support to the physical picture found in [20] — that the static quark impurities trigger an RG flow at strong coupling in $\mathcal{N} = 4$ SYM to the long-wavelength Lifshitz scaling regime with $z = 7$. The F1-D3 intersections examined in [13] suggested that this scaling was a specific instance of a general dynamical critical exponent for F1-Dp intersections with $z = \frac{16-3p}{4-p}$. Indeed, a recent extensive and systematic study of smeared string configurations in Dp-brane theories with $p < 6$ has revealed this scaling accompanied by hyperscaling violating behaviour [24]. We will return to the potential significance of these results for addressing the larger questions that formed the motivation for this work, at the end of this paper.

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[2]The existence of this scaling solution has also been noted in [23].

[3]We would like to thank David Mateos and Javier Tarrio for enlightening discussions on this issue.

[4]Here we would like to make a distinction between the two situations: a string ending on a brane is semi-infinite and terminates at its endpoint on the brane, whilst a string intersecting a D-brane is of infinite extent and pierces through the D-brane.

In section 2 we present the ansätze and the broad categories of the BPS configurations for the setup described above. The detailed analysis of BPS conditions and the derivation of the equation to be satisfied, is presented in the appendices. In section 3 we obtain some solutions to the Poisson-like equation that determines the supersymmetric backgrounds of interest. We summarise our results and discuss future directions in section 4.

2   The setup and ansätze

The backgrounds we are interested in result from the backreaction of mutually BPS Wilson lines, represented in the brane picture by parallel strings ending on or intersecting a stack of N D3-branes. The worldine $\mathcal{C}$ of a static heavy quark associated to a string endpoint on the D3-branes is a straight timelike line, corresponding to the Maldacena-Wilson line which naturally incorporates a coupling with the six real scalars of the $\mathcal{N} = 4$ theory,

$$W_{\mathcal{R}}[\mathcal{C}] = \mathrm{Tr}_{\mathcal{R}} \mathcal{P} \exp \int_{\mathcal{C}} (i\, \dot{x}^{\mu} A_{\mu} + n^{i} \phi_{i})\, \mathrm{d}s\,. \tag{2.1}$$

Here ϕ_i, $i = 1, \cdots, 6$ are the scalars in the $\mathcal{N} = 4$ vector multiplet, and n is a constant unit six-vector. Note that one also needs to specify the representation $\mathcal{R}$ for the Wilson line. A single such Wilson line preserves the following set of symmetries: an SO(5) subgroup of the SO(6) R-symmetry group of $\mathcal{N} = 4$ SYM, 16 of the 32 supercharges, a spatial rotational SO(3) symmetry, and an SL$(2, \mathbb{R})$ subgroup of the full conformal group (generated by time translation, dilatation, and timelike special conformal transformations). Depending on the representation $\mathcal{R}$ and for small enough representations, the holographic dual involves either fundamental F-string probes or wrapped D3- and D5-brane probes carrying string charge [25–28]. For large representations with dim$[\mathcal{R}] \sim \mathcal{O}(N^2)$, the D-branes/Wilson loops can backreact to create a smooth supergravity geometry. Such backreacted "bubbling" geometries have been constructed in [29–31] for a single Wilson line.

For the configurations that we are interested in, the number density of mutually BPS heavy quarks scales as N^2 and we replace them by a smeared uniform spatial distribution so as to restore translation invariance. However, in doing so we give up dilatation invariance, since a scaling of the spatial coordinates would change the smearing density. Thus the symmetries of our static setup are $\mathbb{R}_t \times I$SO(3) $\times$ SO(5), where $\mathbb{R}_t$ represents time translations. Such configurations preserve one quarter of the original supersymmetry or eight real supercharges.

In order to find the appropriate supersymmetric backgrounds, our strategy is first to solve for the vanishing of the type IIB supergravity supersymmetry variations, and then to check which equations of motion remain to be solved. An elegant method for doing this, pioneered in papers such as [32–34], is to work with all possible *bilinears* of the Killing spinor, using the BPS equations to deduce the equivalent algebraic and differential equations governing the system. The details of our approach are presented in the appendix.

2.1   Supergravity ansatz

Based on the symmetries preserved by the putative backgrounds, the ansatz for the Einstein frame metric is,

$$\mathrm{d}s_{\mathrm{Einstein}}^2 = \mathrm{e}^{2A}\mathrm{d}x^i \mathrm{d}x^i + \mathrm{e}^{2B}\mathrm{d}\Omega_4^2 + g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}\,, \tag{2.2}$$

which describes an Euclidean three-plane and a four-sphere, with metric $\mathrm{d}\Omega_4^2$, both fibered over a base Lorentzian (2+1)-dimensional manifold $\mathcal{M}_3$ which itself has a metric $g_{\mu\nu}$. This ansatz is invariant under the rotational and translational symmetries $ISO(3)$ of the $\mathbb{R}^3$, as well as the $SO(5)$ isometry group of the S^4.

For the fluxes we choose the most general ansätze consistent with the same symmetries. Denoting the volume form of the manifold X as vol_X, our ansätze for the fluxes read

$$F_5 = \mathcal{F}_2 \wedge \mathrm{vol}_{\mathbb{R}^3} + \mathrm{d}f \wedge \mathrm{vol}_{S^4} \,, \tag{2.3}$$
$$G_3 = g \, \mathrm{vol}_{\mathbb{R}^3} + h \, \mathrm{vol}_{\mathcal{M}_3} \,,$$
$$P6 = P(x^\mu) \,, \qquad P_i = P_a = 0 \,,$$

where $f \in \mathbb{R}$ and $g, h \in \mathbb{C}$ are functions on $\mathcal{M}_3$, while $\mathcal{F}_2$ is a real two-form on $\mathcal{M}_3$. The complex one-form P contains the axio-dilaton (A.1) and has components only along $\mathcal{M}_3$. The complex three-form G_3 encodes both the RR and NS three-forms. For the specific case of a vanishing axion it is given by $G_3 = \mathrm{e}^{-\phi/2}H_3 + i\,\mathrm{e}^{\phi/2}F_3$. Furthermore, ten dimensional self-duality of F_5 implies that

$$\mathcal{F}_2 = \mathrm{e}^{(3A-4B)} *_3 \mathrm{d}f \tag{2.4}$$

where $*_3$ is the Hodge star on $\mathcal{M}_3$.

Upon substituting this ansatz into the fermionic variations of type IIB we find the most general supersymmetric configuration preserving (at least) 1/4 of the supercharges. The detailed analysis leading to this is presented in appendices A and B. In a particular S-duality frame in which the axion is vanishing, the supergravity fields take the following form

$$\mathrm{d}s^2 = -\mathrm{e}^{2(A+\phi)}\mathrm{d}t^2 + \mathrm{e}^{2A}\mathrm{d}x^i\mathrm{d}x^i + \mathrm{e}^{-2A}\left[\mathrm{e}^{-\phi}\left(\mathrm{d}y^2 + y^2\mathrm{d}\Omega_4^2\right) + \mathrm{e}^{\phi}\mathrm{d}x^2\right] \,, \tag{2.5}$$

$$F_5 = \frac{y^4}{4}(1+*)\left[-\partial_x(\mathrm{e}^{-4A-3\phi})\mathrm{d}y + \partial_y(\mathrm{e}^{-4A-\phi})\mathrm{d}x\right] \wedge \mathrm{vol}_{S^4} \,, \tag{2.6}$$

$$H_3 = \partial_y(\mathrm{e}^{2\phi})\,\mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}x \,, \qquad F_3 = -2\,\mathrm{e}^{4A}\partial_x(\mathrm{e}^{-\phi})\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \,, \tag{2.7}$$

where t, x and y are coordinates on $\mathcal{M}_3$. Given that we are only interested in static configurations, the warp factor A and the dilaton ϕ are only functions of x and y. Note that we have solved for the warp factor B appearing in (2.2) in terms of A and ϕ. The SUSY projection conditions on the ten-dimensional complex spinor ϵ are

$$\Gamma^{\hat{t}\hat{x}} D^{-1} \epsilon^* = \epsilon \qquad\qquad i\,\Gamma^{\hat{t}\hat{x}^1\hat{x}^2\hat{x}^3} \epsilon = \epsilon \,, \tag{2.8}$$

where D is the complex conjugation matrix and hatted coordinates denote flat indices.[5]

From the forms of the fluxes switched on, and the analysis presented in the appendix, it is clear that the supergravity fields are sourced by D3-, D5-branes and fundamental strings. We look for solutions which have at most localized (delta function) sources for D3 and D5 branes and fundamental strings. As explained in appendix C our equations can be easily extended to include smeared source distributions in the x-y plane, but we will

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[5]The spinor ϵ takes the form $\epsilon = \mathrm{e}^{(A+\phi)/2}\epsilon_0$, where ϵ_0 is a constant spinor satisfying the projections.
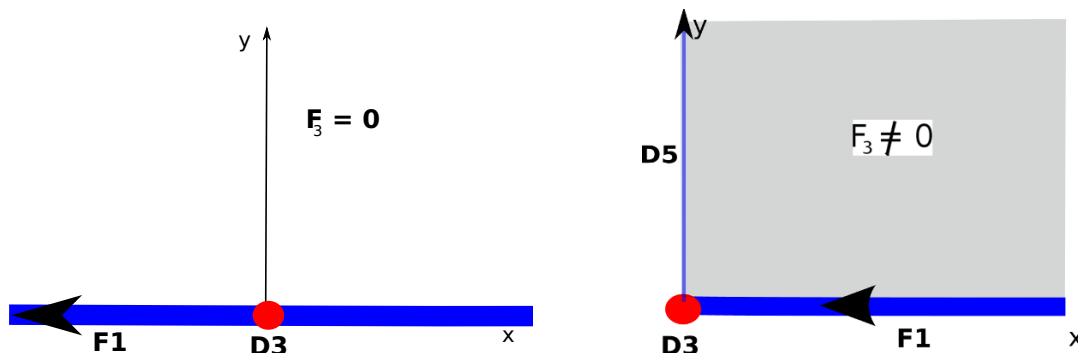
Figure 1. *Left*: solutions with $F_3 = 0$ or intersecting F1-D3 configurations depicted on the $x - y$ plane with D3-branes at the origin and F-strings on the x-axis piercing through them. *Right*: brane picture of configurations with $F_3 \neq 0$ correspond to semi-infinite strings ending on the branes. Such solutions are also endowed with D5-brane baryon vertices smeared along the gauge theory directions with the D5-branes wrapping S^4 and extending along the y-axis.

not explore these in this paper. Inspection of the equations of motion and the fluxes (2.7) shows that the strings and branes in the system must have the following orientations

$$
\begin{array}{c|cccccc}
 & x^1 & x^2 & x^3 & y & x & S^4 \\
\hline
\text{F1} & \cdot & \cdot & \cdot & \cdot & \times & \cdot \\
\text{D3} & \times & \times & \times & \cdot & \cdot & \cdot \\
\text{D5} & \cdot & \cdot & \cdot & \times & \cdot & \times
\end{array}
\tag{2.9}
$$

The kappa symmetry conditions for these brane orientations are consistent with the SUSY projectors.

The functions A and ϕ are not arbitrary, but are determined by the Einstein equations and equations of motion for the fluxes. We find that the solutions to the system of equations fall into two distinct categories:

- *Category I* ($F_3 = 0$): these solutions have a dilaton independent of x, i.e. $\partial_x \phi = 0$, which immediately implies $F_3 = 0$, as can be seen from eq. (2.7). Such backgrounds do not have D5-brane sources associated to them and are backreacted descriptions of supersymmetric D3-F1 intersections. The D-brane picture is indicated in figure 1 and corresponds to a bundle of parallel infinite strings piercing/intersecting the D3-branes. The entire family of such solutions was explored in earlier work [13] and includes the delocalised intersections studied in [36–39]. Such configurations were shown to yield both Lifshitz and hyper scaling violating IR geometries. These included a partially localised intersection with $z = 7$.

- *Category II* ($F_3 \neq 0$): for the second category of solutions the dilaton depends both on x and y coordinates. This automatically leads to a non-vanishing three-form flux. From the orientation of this flux we infer that it can be associated to D5-branes wrapping $S^4 \times \mathbb{R}_y$ and distributed uniformly along the spatial coordinates $x^{1,2,3}$. In particular,

$$
F_3 = Q_{\rm B}\, {\rm d}x^1 \wedge {\rm d}x^2 \wedge {\rm d}x^3 \,,
\tag{2.10}
$$

for some constant Q_B, proportional to the number of D5-branes. The precise normalization and its relation to the number density of strings can be inferred by comparison with [20]. Consistency with (2.7) requires the warp factor to be specified in terms of the dilaton, that is,

$$\mathrm{e}^{-4A} = -\frac{2}{Q_B}\, \partial_x \mathrm{e}^{-\phi} \tag{2.11}$$

which remains the only function to be determined by the equations of motion. Notice that once this replacement is executed the limit $Q_B \to 0$ is not smooth and for that case we refer again to [13]. Using the intermediate results quoted in the appendices, it is easy to compute the Maxwell charges associated to the different branes and strings in the system:

$$
\begin{aligned}
Q_{D5} &= \int_{\Sigma_3} F_3 = Q_B \operatorname{vol}\left(\mathbb{R}^3\right), \\
Q_{D3} &= \int_{\Sigma_5} 4\, F_5 = 4\, f \operatorname{vol}\left(S^4\right) = -\frac{1}{Q_B} y^4 \partial_y (\mathrm{e}^{-2\phi}) \operatorname{vol}\left(S^4\right) \\
Q_{F1} &= \int_{\Sigma_7} e^{-\phi} * H_3 = y^4 \partial_y (\mathrm{e}^{-2\phi}) \operatorname{vol}\left(\mathbb{R}^3\right) \operatorname{vol}\left(S^4\right).
\end{aligned}
\tag{2.12}
$$

As usual, the Maxwell charges are not gauge invariant (not topological). Note also that they are not constants and can depend on the radial coordinates x and y. Nevertheless, the expected relation between the number of D3-branes, the number of fundamental strings and the number of baryon vertices, which is of course gauge invariant, is verified

$$Q_{F1} = -Q_{D5}\, Q_{D3}. \tag{2.13}$$

3 Poisson-like equation and some solutions

The endpoint of the analysis of the BPS configurations and the ensuing type IIB field equations is that the whole system is governed by a non-linear Poisson-like equation

$$\frac{1}{y^4}\partial_y\left(y^4\partial_y \mathrm{e}^{-2\phi}\right) + \frac{1}{2}\,\partial_x^2 \mathrm{e}^{-4\phi} = \rho(x,y), \tag{3.1}$$

where we have allowed for a source term on the right hand side in keeping with the discussion in appendix C.1. In this paper we will not have explicit smeared sources on the x-y plane. All branes and strings will be taken to be localized in these coordinates (consistent with the depiction in figure 1), so that we can set $\rho(x,y) = 0$. Recall, for instance, that the strings are located on the x-axis, corresponding to a delta-function source for the Poisson equation.

In general the BPS configurations solve the equations of motion as well as the Einstein and dilaton equations with source terms for the strings when we identify the smearing form Ω_8 for the strings as $\Omega_8 = -y^4 \rho(x,y)\, \mathrm{d}y \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \operatorname{vol}_{S^4}$ (see appendix C.1).

The Poisson equation for the dilaton is similar in spirit to the Toda equation which appears in the analysis of [40, 41]. In that case there was an implicit variable change

which mapped the problem into a linear electrostatics problem. In the present case we are not aware of such a simplification and are confronted with a nonlinear partial differential equation.

The backreacted solutions contain two holographic "radial" coordinates x and y as would be expected of backgrounds generated by two sets of extended intersecting sources (e.g. [13] and references therein). In the UV limit (as discussed in detail below), one combination of these two directions will be identified with the standard radial direction in AdS_5 whilst a second independent combination measures the transverse distance from the F-string source distribution. From the field theoretic standpoint, it is not immediately obvious how to think about the two emergent holographic directions since the heavy quark impurities interact strongly with the degrees of freedom of $\mathcal{N} = 4$ SYM. However, intuitively, it is natural to associate one of the holographic directions to the scale dependence of the physics of observables of the $\mathcal{N} = 4$ theory, whilst the second holographic direction is probed by the scale dependence of observables associated to the impurity theory.

3.1 Scaling solution with $z = 7$

We now observe that the homogeneous version of eq. (3.1) possesses an interesting and physically relevant family of solutions. If we take

$$\mathrm{e}^{-2\phi} = \frac{Q_1}{y^3} \Psi(p), \qquad p \equiv \frac{x^2 y}{Q_1}, \tag{3.2}$$

where Q_1 is some constant, then the dependence on y simply factors out of eq. (3.1) and we obtain a non-linear ordinary differential equation for $\Psi(p)$:

$$p \left(4\Psi + p\right) \Psi'' + 4 p \, \Psi'^2 + 2(\Psi - p) \, \Psi' = 0. \tag{3.3}$$

For any $\Psi(p)$ it is easily seen (using (2.5) and (2.11)) that the resulting metric is

$$\mathrm{d}s^2 = \left(-\Psi' \Psi^{\frac{3}{2}} p^{\frac{1}{2}}\right)^{-\frac{1}{2}} \left[-y^{7/2} \, \mathrm{d}t^2 + y^{1/2} \Psi \, \mathrm{d}x^i \mathrm{d}x^i + \left(\frac{\mathrm{d}y^2}{y^2} + \mathrm{d}\Omega_4^2\right)(-\Psi') \Psi \, p^{\frac{1}{2}} \sqrt{\frac{2Q}{Q_\mathrm{B}}} \right.$$
$$\left. + y \, \mathrm{d}x^2 \left(-\Psi' p^{\frac{1}{4}}\right) \sqrt{\frac{2}{Q \, Q_\mathrm{B}}} \right], \tag{3.4}$$

where some numerical and other constants have been absorbed into rescalings of the coordinates t and x^i. We have also taken the constant charge densities Q_1 and Q_B to be positive so that it is necessary for $\Psi'(p)$ to be negative definite for a sensible solution. This metric is invariant under the transformations:

$$t \mapsto \lambda^7 t, \qquad x^i \mapsto \lambda x^i, \qquad x \mapsto \lambda^2 x, \qquad y \mapsto \lambda^{-4} y, \tag{3.5}$$

where y plays the role of the standard radial coordinate. It is quite remarkable that a similar realization of the $z = 7$ scaling was also found in the intersecting brane setup of [13] with vanishing F_3, through the dependence on the variables p and y as defined above. Note also that the equations of motion that determine the background for vanishing F_3 are effectively linear, so the actual solutions differ significantly from what we are discussing here. Since

the dilaton is also scale dependent as in (3.2), the background does not display exact scale invariance, precisely as found in the solutions of [13, 20]. The manner in which the scaling solution is realized is unusual because the background contains two radial coordinates x and y, or equivalently, the pair p and y, accompanied by an S^4 factor.

We can now be more specific about the function $\Psi(p)$. For small y, as the x-axis is approached (we can do this by taking the limit $y \to 0$ first followed by $p \to 0$) the dilaton (3.2) should be such that it yields the number density of fundamental strings as indicated by the Maxwell charge (2.12) when $y \to 0$. Note however that this number does not correspond to the number density of strings in the microsopic setup or to the number density of heavy-quarks in the boundary gauge theory. This is because of the presence of a non-zero F_3 and F_5 in the system.[6] Indeed, the number density of quarks in the gauge theory is determined by Q_B, the baryon number density. Requiring $\Psi(p)$ to approach a finite value, normalized to unity, in the limit $y \to 0$ we find

$$\Psi(p) = 1 + s_0 \, p^{1/2} - \frac{s_0^2}{2} \, p + \frac{s_0}{24} \left(5 + 12s_0^2\right) p^{3/2} + \mathcal{O}\left(p^2\right) \tag{3.6}$$

where all higher terms in the expansion are determined by a single integration constant s_0. Keeping the leading term we recover the usual Lifshitz metric with $z = 7$ associated with the string distribution, but the first correction is different from the one encountered in [13] as the expansion in the present context involves half-integral powers of p.

3.2 Asymptotically $\mathbf{AdS_5 \times S^5}$ solution

It is an important consistency check to verify that solutions to (3.1) yield a flow away from $\mathrm{AdS}_5 \times \mathrm{S}^5$ asymptotics in similar fashion to that encountered in [20] and [13]. In particular, the $\mathrm{AdS}_5 \times \mathrm{S}^5$ vacuum, which has $F_3 = 0$ and thus falls into the first category of solutions, is obtained when

$$\mathrm{e}^{-2\phi} = 1\,, \qquad \mathrm{e}^{-4A} = \frac{1}{(x^2 + y^2)^2}\,, \qquad x = r\cos\theta\,, \qquad y = r\sin\theta\,. \tag{3.7}$$

To extract the flow triggered by string sources, we linearize around the vanishing dilaton solution as

$$\mathrm{e}^{-2\phi} = 1 + \epsilon\, h_{(1)} + \mathcal{O}(\epsilon^2)\,. \tag{3.8}$$

The expansion parameter ϵ will eventually be related to the baryon vertex density Q_B. We find that the first correction $h_{(1)}$ satisfies the SO(5) symmetric Laplace equation on flat $\mathbb{R}^6$:

$$\frac{1}{y^4}\, \partial_y \left(y^4 \partial_y h_{(1)}\right) + \partial_x^2 h_{(1)} = 0\,. \tag{3.9}$$

This Laplace equation possesses a large family of solutions which takes the form of a sum over point charges,

$$h_{(1)} = \frac{1}{y^2} \sum_i q_i \left[\frac{(x - x_i)}{(x - x_i)^2 + y^2} + \frac{1}{y} \left(\arctan \frac{x - x_i}{y} + \frac{\pi}{2}\right) \right]\,. \tag{3.10}$$

---

[6]In contrast, when $F_3 = 0$ as in [13] the string charge has no such ambiguity.

Crucially, this family of solutions follows from the first order equation (2.11), upon using the linearized ansatz around $\mathrm{AdS}_5 \times \mathrm{S}^5$,

$$\epsilon \, \partial_x h_{(1)} = -Q_\mathrm{B} \sum_i q_i \frac{1}{\left[(x - x_i)^2 + y^2\right]^2} \,, \qquad (3.11)$$

where, on the right hand side we have introduced a particular multicentre distribution of D3-branes corresponding to a Coulomb branch configuration of $\mathcal{N} = 4$ SYM. The q_i represent the fraction of the total number D3-branes placed at the position x_i, so that $\sum_i q_i = 1$. This leads to the identification $Q_\mathrm{B} = -2\epsilon$, which implies that the ϵ-expansion is equivalent to an expansion in Q_B. For simplicity we focus attention on the origin of the Coulomb branch so that $x_i = 0$. Then we find the following asymptotic components of the spatial metric component in polar coordinates:

$$h_{ii} = r^2 - \frac{Q_\mathrm{B}}{16} \frac{1}{r} \frac{1}{\sin^3 \theta} \left(2(\pi - \theta) + \sin 2\theta\right) + \mathcal{O}\!\left(r^{-4}, Q_\mathrm{B}^2\right). \qquad (3.12)$$

This displays the $1/r$ potential term typical of backreacted string sources in AdS_5 [20, 42]. Similarly, the dilaton is

$$\mathrm{e}^{-2\phi} = 1 - \frac{Q_\mathrm{B}}{2} \frac{2(\pi - \theta) + \sin 2\theta}{\sin^3 \theta} \frac{1}{r^3} + \mathcal{O}(Q_\mathrm{B}^2) \qquad (3.13)$$

and to this order in Q_B the five-sphere is undeformed with respect to the vacuum $\mathrm{AdS}_5 \times \mathrm{S}^5$. The dependence of these corrections on the polar angle θ of the five-sphere is also consistent with the physical picture of the string sources being placed at one of the poles i.e at $\theta = 0$. This is reflected in a singularity at this point in the metric and dilaton corrections. There is no such singularity at $\theta = \pi$.

3.3 Smeared D5 solution

There is another simple solution to the homogeneous non-linear Poisson equation, obtained by forcing each term in the equation to vanish separately. It reads

$$\mathrm{e}^{-2\phi} = (1 + 2\,Q_\mathrm{B}\,x)^{\frac{1}{2}} \left(1 + \frac{Q_1}{y^3}\right) \equiv h_x^{\frac{1}{2}} \, h_y \,. \qquad (3.14)$$

The metric can be written in terms of these warp factors as

$$\mathrm{d}s^2 = h_x^{-\frac{1}{8}} h_y^{\frac{3}{4}} \left[h_y^{-2} \, \mathrm{d}t^2 + \left(\mathrm{d}y^2 + y^2 \mathrm{d}\Omega_4^2\right)\right] + h_x^{-\frac{5}{8}} h_y^{-\frac{1}{4}} \left[h_x \, \mathrm{d}x^i \mathrm{d}x^i + \mathrm{d}x^2\right] \,, \qquad (3.15)$$

supported by the fluxes

$$
\begin{aligned}
H_3 &= -h_x^{-\frac{1}{2}} \, \mathrm{d}h_y^{-1} \wedge \mathrm{d}t \wedge \mathrm{d}x \,, \\
F_3 &= Q_\mathrm{B} \, \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \,, \\
F_5 &= \frac{3}{4} \frac{Q_1}{Q_\mathrm{B}} \left(1 + *\right) \mathrm{d}h_x^{\frac{1}{2}} \wedge \mathrm{vol}_{S^4} \,.
\end{aligned}
\qquad (3.16)
$$

To interpret this solution it is useful to take the $Q_1 \to 0$ limit, i.e. $h_y \to 1$, in which case there are neither strings nor D3-branes. In this limit, the time coordinate combines with

y and the four-sphere to form a six-dimensional Minkowski space in which D5-branes are extended. As can be seen through an harmonic superposition analysis, the solution describes the backreacted geometry due to D5-branes homogeneously distributed along the x^i directions. Allowing for non-vanishing Q_1 includes smeared strings and the solution (3.15) describes the intersection of the strings with the D5-branes. Five-form flux is induced so that condition (2.13) is fulfilled.

3.4 A different scaling solution

We have also found an exact scaling solution of the type in eq. (3.2) with $\Psi(p) = \frac{1}{3}p$. Changing to the radial variables $y = \rho^4$ and $x = \sqrt{3}\,\sigma^{-2}$ this solution reads:

$$\mathrm{d}s^2 = L^2 \left(-\rho^{10}\,\sigma^4 \mathrm{d}t^2 + \rho^2\,\mathrm{d}x^i \mathrm{d}x^i + \frac{\mathrm{d}\rho^2}{\sigma^2} + \frac{3}{4}\frac{\rho^2}{\sigma^4}\,\mathrm{d}\sigma^2 + \frac{1}{16}\frac{\rho^2}{\sigma^2}\mathrm{d}\Omega_4^2 \right), \tag{3.17}$$

$$F_5 = \frac{L^6}{128\,Q_{\mathrm{B}}}\,(1+*)\,\mathrm{d}\left(\frac{\rho^4}{\sigma^4}\right) \wedge \mathrm{vol}_{S^4}, \tag{3.18}$$

$$H_3 = 4\sqrt{3}\,L^2\,\rho^7 \sigma\,\mathrm{d}t \wedge \mathrm{d}\rho \wedge \mathrm{d}\sigma, \tag{3.19}$$

$$F_3 = Q_{\mathrm{B}}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3, \tag{3.20}$$

$$\mathrm{e}^\phi = \rho^4 \sigma^2, \tag{3.21}$$

where $L = \frac{3^{1/4}}{2^{3/2}}Q_{\mathrm{B}}^{1/2}$. Surprisingly, the scaling properties of this metric are enhanced due to the interplay of both radii. Under a rescaling

$$t \mapsto \lambda^z\,t, \qquad x^i \mapsto \lambda\,x^i, \qquad \rho \mapsto \lambda^{\frac{3-z}{4}}\,\rho, \qquad \sigma \mapsto \lambda^{-1}\,\sigma, \tag{3.22}$$

the metric transforms as

$$\mathrm{d}s \mapsto \lambda^{\frac{7-z}{4}}\,\mathrm{d}s \tag{3.23}$$

signaling certain hyperscaling properties. Notice that the Lifshitz and hyperscaling coefficients are related but in principle arbitrary. We do not expect violations of energy conditions since the matter supporting the solution (smeared strings) does not have unphysical features. It is natural to ask whether the hyperscaling violation parameter seen in the boundary gauge theory satisfies the thermodynamic stability criterion $\theta \le d$ [11] which may in turn restrict the allowed values of z. However, it is unclear whether non-extremal counterparts of these solutions exist. Furthermore, although we are able to identify the hyperscaling violation in the 10D metric, the reduction to 5D (dual to a 4D boundary theory) is ambiguous due to the presence of the two radial directions which are both non-compact. Naively this would suggest that the IR theory is effectively five dimensional and while such a picture could be argued (via deconstruction) in earlier work [13], for the above class of solutions we do not have such compelling arguments.

It is possible to absorb the rescaling (3.23) of the metric into the radius through the parameter Q_{B}. This would mean that under a dilatation we are only changing the baryon density, or equivalently the string density. If we do so, the only fields that transform are the dilaton, H_3 and F_3, but in such a way that the relation between charges (2.13) is maintained.

Interestingly, F_5 only depends on the combination $\rho^4/\sigma^4 \sim x^2 y = p$. An analogous argument to that in [13] tells us that the D3-branes are not localized in these solutions. We further note that the curvature in string frame is

$$\mathcal{R}_{\text{String}} = -\frac{288}{L^2} \frac{\sigma}{\rho^4} \,. \tag{3.24}$$

Therefore there is a curvature singularity situated all along the x and y axes. The physical significance of these and other properties of this solution remain to be understood.

4 Conclusions

The main motivation behind this work was to understand whether the emergence of the IR $z = 7$ scaling found in [20] for the $\mathcal{N} = 4$ theory with heavy quarks could be reproduced within a supersymmetric setup. A significant outcome of our work is the derivation of the most general BPS configurations with eight supercharges in type IIB supergravity preserving $I\text{SO}(3) \times \text{SO}(5)$ global symmetry, showing that they are determined by the solutions to the two dimensional Poisson-like equation (3.1). We found solutions to this equation with $z = 7$ Lifshitz-like scaling, and also showed that the equation correctly captures the flow away from the asymptotically $\text{AdS}_5 \times \text{S}^5$ regime. Obtaining the flow interpolating between these two limits requires numerical integration of the PDE (3.1) which is interesting work for the future.

It would be extremely interesting to understand if there is a general structure under-lying the solutions of the Poisson equation (3.1) for this system, along the lines of the picture found in [29–31]. In particular, our equations, with suitable sources on the x-y plane, should also be able to describe distributions of Wilson lines/quarks in more general representations. Such information should be contained in a general linearized analysis of the UV asymptotics around $\text{AdS}_5 \times \text{S}^5$ which we have not explored completely in this paper. It would be interesting to know whether different choices of impurity representations have any effect on the long wavelength/IR description of the system or if they all flow to the same Lifshitz-like scaling solution.

The supersymmetric scaling solutions in this paper are singular due to the running of the dilaton. In order to make sense of such backgrounds it is important to have a non-extremal version of these solutions where the singular region is shielded behind a horizon and one may reliably speak about the scaling properties of physical quantities. This was easily achieved in the non-supersymmetric $\text{SO}(6)$ symmetric configuration of [20]. The corresponding generalization to the $\text{SO}(5)$-symmetric setup of this paper is not obvious due to the presence of effectively two radial directions in the bulk solutions.

The results of this paper also lend support to the general idea of applying the smearing technique to understand holographic backreaction of quark flavours at finite density [24, 43–47], in that long-distance properties of systems may not be sensitive to the details of the smearing procedure itself. It is important, however, to understand in detail the embedding of the scaling solutions found in [20, 24] within the setup of backreacted and smeared flavoured holographic duals [45]. This would open the way for understanding possible instabilities and their end-points that may lead to (colour) superconducting phases along the lines of the ideas presented in [17, 48, 49].

Acknowledgments

A BPS equations

We use the conventions of [31], which are stated explicitly in [35]. Note that the normalization of F_5 differs from that usually used in string theory by a factor of 4. Type IIB supergravity is written in terms of two field strengths P and Q, and string theory in terms of the dilaton ϕ and axion $C^{(0)}$. Following e.g. [34], we write the map between the supergravity and string theory variables:

$$P = \frac{1}{2}\mathrm{d}\phi + \frac{i}{2}\mathrm{e}^{\phi}\mathrm{d}C^{(0)}\,, \qquad Q = -\frac{1}{2}\mathrm{e}^{\phi}\mathrm{d}C^{(0)}\,. \tag{A.1}$$

The supergravity equations of motion have a local U(1) invariance with associated gauge field Q. Each field has a definite charge q under this U(1): ϵ has $q = 1/2$, P has $q = 2$ and G_3 has $q = 1$. The field strengths have corresponding Bianchi identities written in terms of the U(1)-covariant derivative $\mathrm{D} \equiv \nabla - iqQ$

$$\mathrm{D}P = 0 \tag{A.2}$$

$$\mathrm{d}Q = -iP \wedge P^* \tag{A.3}$$

which are automatically satisfied when we use the map to string theory variables. This formulation comes from a gauge fixing of the version of the theory with an extra auxiliary scalar field, and the remnant of this is that each SL$(2,\mathbb{R})$ action is accompanied by a local U(1) gauge transformation. This is the only way in which SL$(2,\mathbb{R})$ duality acts on the variables G, P, Q.

Type IIB supergravity has 32 real supercharges parametrized by a complex chiral ten-dimensional spinor $\Gamma\epsilon = -\epsilon$. We begin by writing down the SUSY variations in the Einstein frame:

$$\delta_\epsilon\lambda = i(\Gamma \cdot P)\mathcal{B}^{-1}\epsilon^* - \frac{i}{24}(\Gamma \cdot G)\,\epsilon \tag{A.4}$$

$$\delta_\epsilon\psi_M = \mathrm{D}_M\epsilon + \frac{i}{480}(\Gamma \cdot F)\Gamma_M\epsilon - \frac{1}{96}\left[\Gamma_M(\Gamma \cdot G) + 2(\Gamma \cdot G)\Gamma_M\right]\mathcal{B}^{-1}\epsilon^*\,. \tag{A.5}$$

The next step is to choose a basis of gamma matrices in ten dimensions. We choose:

$$\begin{aligned}
\Gamma_i &= \gamma_i \otimes \gamma_{S^4} \otimes \mathbb{1} \otimes \sigma^1 \\
\Gamma^a &= \mathbb{1} \otimes \gamma^a \otimes \mathbb{1} \otimes \sigma^1 \\
\Gamma_\mu &= \mathbb{1} \otimes \mathbb{1} \otimes \gamma^\mu \otimes \sigma^2
\end{aligned}$$

where $\gamma_{S^4} \equiv +\gamma^6\gamma^7\gamma^8\gamma^9$ is the chirality matrix on S^4. The ten dimensional chirality matrix is $\Gamma = \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma^3$, so that the IIB chirality condition reduces to

$$\sigma^3 \epsilon = -\epsilon \,. \tag{A.6}$$

We complete the basis by specifying gamma matrices within each factor space:

$$\begin{aligned}
i &: & \gamma^3 &= \sigma^1, & \gamma^4 &= \sigma^2, & \gamma^5 &= \sigma^3 \\
a &: & \gamma^6 &= \sigma^1 \otimes \mathbb{1}, \; \gamma^7 = \sigma^2 \otimes \mathbb{1}, \; \gamma^8 = \sigma^3 \otimes \sigma^1, \; \gamma^9 = \sigma^3 \otimes \sigma^2 \; \left(\Rightarrow \gamma_{S^4} = -\sigma^3 \otimes \sigma^3\right) \\
\mu &: & \gamma^0 &= i\sigma^2, & \gamma^1 &= \sigma^1, & \gamma^2 &= \sigma^3 \,.
\end{aligned}$$

Note that the basis for the gamma matrices on $\mathcal{M}_3$ is real (Majorana). The ten dimensional complex conjugation matrix $\mathcal{B}$, defined by $\{\mathcal{B}\,\Gamma^\mu \mathcal{B}^{-1} = (\Gamma^\mu)^*, \; \mathcal{B}^*\mathcal{B} = \mathbb{1}\}$, is now

$$\mathcal{B} = \sigma^2 \otimes \left(\sigma^2 \otimes \sigma^1\right) \otimes \mathbb{1} \otimes \sigma^3 \tag{A.7}$$

$$= b_3 \otimes b_4 \otimes \mathbb{1} \otimes \sigma^3 \tag{A.8}$$

where b_3 and b_4 are charge conjugation matrices in $\mathbb{R}^3$ and S^4 respectively:

$$b_3 \gamma^i b_3^{-1} = -(\gamma^i)^* \qquad b_4 \gamma^a b_4^{-1} = -(\gamma^a)^* \,.$$

We plug our ansatz and our Clifford algebra basis into the IIB SUSY variations (A.4) (A.5), and after a few pages of careful work we end up with the following set of BPS conditions on our ten dimensional complex spinor ϵ:

$$\not{P}\mathcal{B}^{-1}\epsilon^* - \frac{1}{4}\left(\mathrm{e}^{-3A}g\gamma_{S^4} - h\right)\epsilon = 0 \tag{A.9}$$

$$\frac{1}{3}\mathrm{e}^{-A}\gamma^i\tilde{\nabla}_i\epsilon + \frac{i}{2}\not{\partial}A\gamma_{S^4}\epsilon + \frac{1}{2}\mathrm{e}^{-4B}\not{\partial}f\epsilon - \frac{i}{16}\left(3\mathrm{e}^{-3A}g + h\gamma_{S^4}\right)\mathcal{B}^{-1}\epsilon^* = 0 \tag{A.10}$$

$$\mathrm{e}^{-B}\tilde{\nabla}_a\epsilon - \frac{i}{2}\gamma_a\not{\partial}B\epsilon + \frac{1}{2}\mathrm{e}^{-4B}\gamma_a\not{\partial}f\gamma_{S^4}\epsilon - \frac{i}{16}\gamma_a\left(\mathrm{e}^{-3A}g\gamma_{S^4} - h\right)\mathcal{B}^{-1}\epsilon^* = 0 \tag{A.11}$$

$$\mathrm{D}_\mu\epsilon + \frac{i}{2}\mathrm{e}^{-4B}\not{\partial}f\gamma_{S^4}\gamma_\mu\epsilon + \frac{1}{16}\left(\mathrm{e}^{-3A}g\gamma_{S^4} + 3h\right)\gamma_\mu\mathcal{B}^{-1}\epsilon^* = 0 \tag{A.12}$$

where here and in the following D_μ and ∇_μ denote derivatives on $\mathcal{M}_3$, and $\tilde{\nabla}_{i,a}$ are derivatives on $\mathbb{R}^3$ and S^4 respectively. Also note that when an operator appears which naturally acts within only one Clifford subspace, it should be taken as the tensor product with the identity matrix in the other tensor factors. For example, by $\gamma_{S^4}\epsilon$ we mean $(\mathbb{1}\otimes\gamma_{S^4}\otimes\mathbb{1}\otimes\mathbb{1})\epsilon$.

To proceed we must make an ansatz for the form of the ten dimensional spinor ϵ

$$\epsilon = \eta^\alpha \otimes \chi_a^\beta \otimes \epsilon_a^{\alpha\beta} \otimes \theta_a^{\alpha\beta} \,. \tag{A.13}$$

All repeated indices are to be summed over. The η_α are the linearly independent constant spinors on $\mathbb{R}^3$, $\alpha = 1, 2$, and the χ^β are the two sets of linearly independent Killing spinors on S^4, $\beta = 1, 2, 3, 4$, which can be taken to satisfy

$$\tilde{\nabla}_b \chi_a^\beta = \frac{a}{2} \gamma_{S^4} \gamma_b \chi_a^\beta \qquad\qquad \gamma_{S^4} \chi_a^\beta = \chi_{-a}^\beta \tag{A.14}$$

where we hope it is clear that the $a = \pm$ appearing here is not a spacetime index on S^4 but rather a label of the two different signs in the Killing spinor equation. The $\epsilon_a^{\alpha\beta}$ are commuting spinors on $\mathcal{M}_3$, and the $\theta_a^{\alpha\beta}$ are two-component spinors. The chirality condition (A.6) implies that $\sigma^3 \theta_a^{\alpha\beta} = -\theta_a^{\alpha\beta}$, so that without loss of generality we can set

$$\theta_a^{\alpha\beta} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \forall\, a, \alpha, \beta \,.$$

Following [31], we also note that, again without loss of generality, we can impose a reality condition on the basis Killing spinors.[7] Specifically, we impose[8]

$$(b_3 \otimes b_4)(\eta^* \otimes \chi_a^*) = \eta \otimes \chi_{-a} \,. \tag{A.15}$$

We can now reduce the BPS to three dimensions, by writing them in terms of the two complex two-component spinors $\epsilon_\pm$ on $\mathcal{M}_3$:

$$2\slashed{P}\,\epsilon_{-a}^* - \frac{1}{2}\mathrm{e}^{-3A} g\, \epsilon_{-a} + \frac{1}{2} h\, \epsilon_a = 0 \tag{A.16}$$

$$\frac{i}{2}\slashed{\partial}A\epsilon_{-a} + \frac{1}{2}\mathrm{e}^{-4B}\slashed{\partial}f\epsilon_a - \frac{i}{16}\left(3\mathrm{e}^{-3A} g\right)\epsilon_{-a}^* - \frac{i}{16} h\epsilon_a^* = 0 \tag{A.17}$$

$$-\frac{a}{2}\mathrm{e}^{-B}\epsilon_{-a} + \frac{i}{2}\slashed{\partial}B\epsilon_a - \frac{1}{2}\mathrm{e}^{-4B}\slashed{\partial}f\epsilon_{-a} + \frac{i}{16}\left(\mathrm{e}^{-3A} g\right)\epsilon_a^* - \frac{i}{16} h\epsilon_{-a}^* = 0 \tag{A.18}$$

$$\mathrm{D}_\mu \epsilon_a + \frac{i}{2}\mathrm{e}^{-4B}\slashed{\partial}f\gamma_\mu\epsilon_{-a} + \frac{1}{16}\left(\mathrm{e}^{-3A} g\right)\gamma_\mu\epsilon_a^* + \frac{1}{16}3h\gamma_\mu\epsilon_{-a}^* = 0 \,. \tag{A.19}$$

Since no operators which affect the $\alpha, \beta, \cdots$ indices appear in (A.12), these indices can be omitted, with the understanding that there is a $2 \times 4 = 8$-fold multiplicity in each set of solutions $\{\epsilon_+, \epsilon_-\}$ we shall find of (A.16)–(A.19).

It is convenient to introduce a 'tau-matrix' notation for these equations, as follows:

$$\left(\tau^I \epsilon\right)_a \equiv \tau_{ab}^I \epsilon_b \quad I = 0, 1, 2, 3 \tag{A.20}$$

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[7]We thank John Estes for pointing this out.

[8]It is impossible to impose $b_3 \eta^* = \eta$ on our basis, since $b_3^* b_3 = -1$ (there are no Majorana spinors in three Euclidean dimensions.) Furthermore we see that we cannot impose $\gamma_{S^4}\chi_a = \chi_{-a}$ and $b_4\chi_a = \chi_{-a}$ simultaneously, since $(\gamma_{S^4}b_4)^*(\gamma_{S^4}b_4) = -1$. However, we can impose a reality condition on the whole basis (rather than each factor indivually), since $(b_3\gamma_{S^4}b_4)^*(b_3\gamma_{S^4}b_4) = 1$.

where $\tau^{1,2,3}$ are the usual Pauli matrices acting on the a, b indices, and $\tau^0 \equiv \mathbb{1}_{ab}^{2\times2}$. Now the BPS equations reduced to $(2+1)$D read

$$\not{P}\epsilon^* - \frac{1}{4}\left(\mathrm{e}^{-3A}g - h\tau^1\right)\epsilon = 0 \tag{d}$$

$$\not{\partial}A\epsilon - i\mathrm{e}^{-4B}\not{\partial}f\tau^1\epsilon - \frac{1}{8}\left(3\mathrm{e}^{-3A}g + h\tau^1\right)\epsilon^* = 0 \tag{i}$$

$$-\mathrm{e}^{-B}\tau^2\epsilon + \not{\partial}B\epsilon + i\mathrm{e}^{-4B}\not{\partial}f\tau^1\epsilon + \frac{1}{8}\left(\mathrm{e}^{-3A}g - h\tau^1\right)\epsilon^* = 0 \tag{a}$$

$$\nabla_\mu\epsilon - \frac{i}{2}Q_\mu\epsilon + \frac{i}{2}\mathrm{e}^{-4B}\not{\partial}f\gamma_\mu\tau^1\epsilon + \frac{1}{16}\left(\mathrm{e}^{-3A}g + 3h\tau^1\right)\gamma_\mu\epsilon^* = 0\,. \tag{μ}$$

B Spinor bilinear analysis

We now solve the BPS system (d)–(μ) using the standard techniques of bilinear analysis. First we introduce the real bilinears

$$f^{(I)} \equiv \epsilon^\dagger\sigma^2\tau^I\epsilon \qquad V_\mu^{(I)} \equiv i\,\epsilon^\dagger\sigma^2\tau^I\gamma_\mu\epsilon\,. \tag{B.1}$$

Likewise we have the complex bilinears

$$\tilde{f}^{(I)} \equiv \epsilon^t\sigma^2\tau^I\epsilon \qquad \tilde{V}_\mu^{(I)} \equiv \epsilon^t\sigma^2\tau^I\gamma_\mu\epsilon\,. \tag{B.2}$$

Note that $\tilde{V}_{(2)}$ and $\tilde{f}^{(0,1,3)}$ all vanish identically since they are of the form $\epsilon^t M \epsilon$ where M is an antisymmetric matrix, and we have taken ϵ to be commuting. For typographical clarity, we use the (I) symbols as both subscripts and superscripts, but we intend no difference in meaning. The real bilinears have $q = 0$, and the complex ones have $q = 1$. We split the analysis into two, as is typical:

- On the one hand we have algebraic equations among the bilinears implied by the BPS equations. We will use these to define a preferred orthonormal basis for the tangent space of $\mathcal{M}_3$, namely an identity structure, and we express the fluxes in terms of this.

- On the other hand there are differential equations which give the 'torsion' of the identity structure, and which we use to define local coordinates and a metric.

At various points we will use the 3D Fierz identities, which express linear dependence between the bilinears.

B.1 Algebraic constraints

The first step is to the reduce the BPS equations to conditions on the minimum number of bilinears. We look at $\epsilon^\dagger\sigma^2[(\mathrm{i})+(\mathrm{a})]$ and take real and imaginary parts to find that $f^{(2)} = 0$ and $V^{(0)}\cdot\partial B = 0$. Now taking $\epsilon^\dagger\tau^{0,1}\sigma^2\{(\mathrm{i}), (\mathrm{d})^*\}$, we find

$$V^{(0)}\cdot X = 0 \qquad X = \mathrm{d}A,\,\mathrm{d}B,\,\mathrm{d}f,\,P\,. \tag{B.3}$$

Next $\epsilon^\dagger\sigma^2\tau^2\gamma_\mu[(\mathrm{i})-(\mathrm{a})]$+c.c. gives

$$-4\,\mathrm{e}^{-4B}\mathrm{d}f\,f^{(3)} = 0$$

which is only solved for $f^{(3)} = 0$, assuming $\mathrm{d}f \neq 0$. Since $\mathrm{d}f = 0$ would lead to solutions preserving more supersymmetry than the 8 SUSYs we are interested in (for example the D1-brane solution), we ignore this possibility. Finally, using Fierz identities we can now show that

$$f^{(0)} = 0 \qquad\qquad V^{(1)} = 0 \qquad\qquad (B.4)$$

$$V^2_{(2,3)} = -V^2_{(0)} = f^2_{(1)} \qquad V^{(I)} \cdot V^{(J)} = 0 \qquad \forall\, I \neq J\,. \qquad (B.5)$$

Now we have simplified things considerably. In particular we see that

$$\left\{ \frac{1}{f^{(1)}} V_{(0)}, \frac{1}{f^{(1)}} V_{(3)}, \frac{1}{f^{(1)}} V_{(2)} \right\} \equiv \left\{ e^0, e^1, e^2 \right\}$$

form an orthonormal basis for the cotangent space (and by raising indices, for the tangent space), and so we can take them as the vielbeine on $\mathcal{M}_3$. Thus the mininum set of bi-linears to consider consists of one real scalar, one complex scalar, and three real vectors: $\left\{ f_{(1)}, \tilde{f}_{(2)}, V^{(0)}, V^{(2)}, V^{(3)} \right\}$. We next find expressions for the fluxes in terms of them.

Complex 3-form. Since $\tilde{f}^{(2)}$ is the only complex bilinear, we expect its phase to control the phases of h, g and P. Taking $\epsilon^\dagger \tau^{3,2} \sigma^2 (\mathrm{d})^*$ and solving for g and h, we find that

$$h = \frac{4}{\tilde{f}^{(2)}} V^{(3)} \cdot P \qquad\qquad (B.6)$$

$$\mathrm{e}^{-3A} g = \frac{4i}{\tilde{f}^{(2)}} V^{(2)} \cdot P\,. \qquad\qquad (B.7)$$

Five-form. This can be obtained by taking $\epsilon^\dagger \sigma^2 \gamma_\mu(\mathrm{i}) + (\mathrm{i})^\dagger \sigma^2 \gamma_\mu \epsilon$:

$$- 2i\, f^{(1)}\, \mathrm{e}^{-4B} \partial_\mu f + 2\partial_\nu A \epsilon^\dagger \sigma^2 \gamma_\mu{}^\nu \epsilon + \frac{1}{8} \left(\left(3\mathrm{e}^{-3A} g \tilde{V}^{(0)*}_\mu + h \tilde{V}^{(1)*}_\mu \right) - \mathrm{h.c.} \right) = 0\,. \qquad (B.8)$$

Completeness of the tangent space implies that the $\tilde{V}^{(I)}$ are linear combinations of the other vector bilinears, and indeed a Fierzing gives $\tilde{V}^{(0)} = \frac{\tilde{f}^{(2)}}{f^{(1)}} V^{(3)}$, $\tilde{V}^{(1)} = i\frac{\tilde{f}^{(2)}}{f^{(1)}} V^{(2)}$. This leads to

$$f^2_{(1)}\, \mathrm{e}^{-4B} \mathrm{d}f = \left[V^{(2)} \cdot \left(\mathrm{d}A + \frac{3}{2} \mathrm{e}^{-2i\theta} P \right) \right] V^{(3)} - \left[V^{(3)} \cdot \left(\mathrm{d}A + \frac{1}{2} \mathrm{e}^{-2i\theta} P \right) \right] V^{(2)} \qquad (B.9)$$

where we have defined $\mathrm{e}^{i\theta}$ to be the phase of $\tilde{f}_{(2)}$.

Axiodilaton. We take the three combinations $\epsilon^\dagger \sigma^2 \tau^{2,3,0} ((\mathrm{i}) + (\mathrm{a}))$. Plugging in the expressions for the fluxes obtained above we get

$$V^{(0)} \cdot \left[\partial(A + B) + \mathrm{e}^{-2i\theta} P \right] = 0$$

$$V^{(2)} \cdot \left[\partial(A + B) + \mathrm{e}^{-2i\theta} P \right] = 0 \qquad\qquad (B.10)$$

$$V^{(3)} \cdot \left[\partial(A + B) + \mathrm{e}^{-2i\theta} P \right] = \mathrm{e}^{-B} f^{(1)}\,.$$

Using the orthonormality of our tangent space basis, the above equations imply that

$$\mathrm{e}^{-2i\theta}P = \frac{\mathrm{e}^{-B}}{f^{(1)}}V^{(3)} - \mathrm{d}(A+B)\,. \tag{B.11}$$

We can therefore see that $\mathrm{e}^{-2i\theta}P$ is real (this is discussed in section B.3). This implies that $P = \mathrm{e}^{2i\theta}\tilde{P}$, where $\tilde{P}$ is a real one-form.

As a last piece of information to take from the algebraic conditions, we take $\epsilon^t \sigma^2[(i) + (a)]$ and use the expressions for the fluxes to find that $|\tilde{f}_{(2)}|^2 = f_{(1)}^2$, so we can write

$$\tilde{f}^{(2)} = \mathrm{e}^{i\theta}\,f^{(1)}\,. \tag{B.12}$$

In summary, we have defined an identity structure, found the fluxes (B.6), (B.7), (B.9), (B.11) in terms of it, and obtained the relation (B.12).

B.2 Torsion

In section B.1 we reduced the problem of solving the BPS equations to finding two scalars $\{f_{(1)}, \tilde{f}_{(2)}\}$ and three vector bilinears $\{V^{(0)}, V^{(2)}, V^{(3)}\}$. These satisfy a system of differential equations which is implied by the BPS equations:

$$\mathrm{d}\left(\mathrm{e}^{A+2B}f^{(1)}\right) = 2\,\mathrm{e}^{A+B}V^{(3)} \tag{B.13}$$

$$\mathrm{D}\left(\mathrm{e}^{A+2B}\tilde{f}^{(2)}\right) = 2\,\mathrm{e}^{A+B}\tilde{V}^{(0)} \tag{B.14}$$

$$\mathrm{d}\left(\mathrm{e}^{2A+4B}V^{(0)}\right) = -4\,\mathrm{e}^{2A+3B} * V^{(2)} \tag{B.15}$$

$$\mathrm{d}\left(\mathrm{e}^{2A+4B}V^{(2)}\right) = -2\,\mathrm{e}^{2A}\mathrm{d}f \wedge V^{(3)} - 4\,\mathrm{e}^{2A+3B} * V^{(0)} \tag{B.16}$$

$$\mathrm{d}\left(\mathrm{e}^{2A+4B}V^{(3)}\right) = 2\,\mathrm{e}^{-4B}\mathrm{d}f \wedge V^{(2)}\,. \tag{B.17}$$

We begin by showing that $V^{(0)}$ is a timelike Killing vector. It is timelike since $V_{(0)}^2 = -f_{(1)}^2$ is negative, and it satisfies

$$\nabla_{(\mu}V_{\nu)}^{(0)} = -g_{\mu\nu}V^{(0)} \cdot \partial(A+2B) = 0\,, \tag{B.18}$$

where the last equality follows from (B.3). Therefore $V^{(0)}$ satisfies the Killing equation on $\mathcal{M}_3$, and together with (B.3) this implies it is a Killing vector of the whole 10D metric. We define the coordinate t such that $\partial/\partial t = V_{(0)}^{\#}$, where the notation denotes $V^{(0)}$ as a vector.

We would like to find out about the phase $\mathrm{e}^{i\theta}$, which as we have already seen governs the phases of the complex fields (g, h, P). First we Fierz $\tilde{V}^{(0)}$, and then equate the l.h.s.s of (B.14) and (B.13). Together with (B.12) this gives $Q = \mathrm{d}\theta$, so that the phase is just the U(1) holonomy.

Next we consider the $\mathrm{d}f^{(1)}$ equation (B.13). This implies that $\mathrm{e}^{A+B}V^{(3)}$ is a closed form, and defining a local coordinate $y^2 \equiv \mathrm{e}^{A+2B}f^{(1)}$ we have that

$$V^{(3)} = y\,\mathrm{e}^{-(A+B)}\mathrm{d}y\,. \tag{B.19}$$

Using (B.9) and (B.10), the $\mathrm{d}V^{(2)}$ equation (B.16) becomes

$$\mathrm{d}V^{(2)} = 3\left(\frac{\mathrm{d}y}{y} - \partial_y(A+B)\mathrm{d}y\right) \wedge V^{(2)} \tag{B.20}$$

so that we can write

$$V^{(2)} = y^3 \mathrm{e}^{-3(A+B)}\mathrm{d}x\,. \tag{B.21}$$

We take x to be the final coordinate. We have automatically $\partial/\partial x \cdot \partial/\partial y = 0$.

Lastly we turn to the equation for $\mathrm{d}V^{(0)}$, which reads

$$\mathrm{d}V^{(0)} = \left(-2\,\mathrm{d}(A+2B) + 4\frac{\mathrm{d}y}{y}\right) \wedge V^{(0)} \tag{B.22}$$

so that

$$V^{(0)} = y^4 \mathrm{e}^{-2A-4B}(\mathrm{d}t + \omega) \tag{B.23}$$

for some closed form $\mathrm{d}\omega = 0$.

In summary, we can now write down the full form of the metric in terms of the warp factors $\mathrm{e}^A, \mathrm{e}^B$:

$$\mathrm{d}s_3^2 = -y^4\mathrm{e}^{-2A-4B}(\mathrm{d}t + \omega)^2 + \frac{\mathrm{e}^{2B}}{y^2}\mathrm{d}y^2 + y^2\,\mathrm{e}^{-4A-2B}\mathrm{d}x^2 \tag{B.24}$$

and the fluxes are

$$P = \mathrm{e}^{2i\theta}\mathrm{d}(\log y - A - B) \tag{B.25a}$$

$$g = -4i\,\mathrm{e}^{i\theta}\mathrm{e}^{5A+B}y^{-1}\partial_x(A+B) \tag{B.25b}$$

$$h = 4\,\mathrm{e}^{i\theta}\mathrm{e}^{-B}y\,\partial_y(\log y - A - B) \tag{B.25c}$$

$$\mathrm{d}f = \frac{y^4}{4}\left[\partial_x\left(y^{-6}\mathrm{e}^{2A+6B}\right)\mathrm{d}y + \partial_y\left(y^{-2}\mathrm{e}^{-2A+2B}\right)\mathrm{d}x\right]. \tag{B.25d}$$

B.3 Reality condition and SL(2, $\mathbb{R}$)

We now address the reality condition implied by (B.11):

$$\mathfrak{Im}\left(\mathrm{e}^{-2i\theta}P\right) = 0\,. \tag{B.26}$$

This implies that $P = \mathrm{e}^{2i\theta}\tilde{P}$, where $\tilde{P}$ is a real one-form. Then (A.2) and (A.3) imply $\mathrm{d}Q = 0$ and $Q = \mathrm{d}\theta$, so that Q is pure gauge (these relations are actually implied by the BPS equations, as shown in (B.2)). Therefore by a local U(1) gauge transformation $\mathcal{U} = \mathrm{e}^{-i\theta}$, we can map to real P, i.e. vanishing axion. We know this must be the accompanying gauge transformation to an SL(2, $\mathbb{R}$) action, so the BPS equations give just the solution with $C^{(0)} = 0$ and its orbit under S-duality, parametrized by θ. This situation is familiar from e.g. [31].

C Equations of motion

In this appendix, we verify that our solution to the fermionic variations is also a solution to the equations of motion up to a unique Poisson type equation for the dilaton that has to be solved separately. Furthermore, our system is naturally equipped to include explicit sources in the form of fundamental strings and D3-branes smeared appropriately.

Since we have found that our configurations have $C^{(0)} = 0$ up to a duality transformation, we set $\theta = 0$ in this section, so that now $P = \frac{1}{2}\mathrm{d}\phi$. We also set $\omega = 0$. In this case, it is easy to write the solution to the BPS equations in terms of A and ϕ instead of A and B:

$$\mathrm{d}s^2 = -\mathrm{e}^{2(A+\phi)}\mathrm{d}t^2 + \mathrm{e}^{2A}\mathrm{d}x^i\mathrm{d}x^i + \mathrm{e}^{-2A}\left(\mathrm{e}^{-\phi}\left(\mathrm{d}y^2 + y^2\mathrm{d}\Omega_4^2\right) + \mathrm{e}^{\phi}\mathrm{d}x^2\right)$$

$$F_5 = \frac{y^4}{4}(1 + *)\left(-\partial_x\left(\mathrm{e}^{-4A-3\phi}\right)\mathrm{d}y + \partial_y\left(\mathrm{e}^{-4A-\phi}\right)\mathrm{d}x\right) \wedge \mathrm{vol}_{S^4} \tag{C.1}$$

$$H_3 = \partial_y\left(\mathrm{e}^{2\phi}\right)\mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}x \qquad F_3 = 2\,\mathrm{e}^{4A-\phi}\partial_x\phi\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\,.$$

We will now show in detail that all the equations of motion and Bianchi identities are satisfied provided we have $Q_B = -2\,\mathrm{e}^{4A}\partial_x\mathrm{e}^{-\phi}$, where Q_B is a (non-zero) real constant, and the following equation, which we will refer to as the 'Poisson equation', is satisfied:

$$\frac{1}{y^4}\partial_y\left(y^4\partial_y\mathrm{e}^{-2\phi}\right) + \frac{1}{2}\partial_x^2\mathrm{e}^{-4\phi} = 0\,. \tag{C.2}$$

The equations of motion deriving from Type IIB supergravity, with a vanishing axion, together with the Bianchi identities, read

$$\mathrm{d}(\mathrm{e}^{-\phi} * H_3) + 4F_5 \wedge F_3 = 0\,, \qquad \mathrm{d}F_5 - \frac{1}{4}H_3 \wedge F_3 = 0\,,$$

$$\mathrm{d}(\mathrm{e}^{\phi} * F_3) + 4H_3 \wedge F_5 = 0\,, \qquad\qquad\quad \mathrm{d}F_3 = 0\,, \tag{C.3}$$

$$\mathrm{d} * \mathrm{d}\phi + \frac{1}{2}G_3 \wedge *G_3 = 0\,.$$

On top of that we have the Einstein equations

$$R_{MN} = \frac{1}{2}\partial_M\phi\partial_N\phi + \frac{1}{6}(F_5)_{MP_1P_2P_3P_4}\,(F_5)_N{}^{P_1P_2P_3P_4}$$

$$+ \frac{1}{4}\mathfrak{Re}\left[(G_3)_{MP_1P_2}(G_3^*)_N{}^{P_1P_2}\right] - \frac{1}{48}g_{MN}(G_3)_{P_1P_2P_3}(G_3^*)^{P_1P_2P_3}\,. \tag{C.4}$$

F_3 Bianchi identity. This states that $F_3 = Q_B\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$ for some constant Q_B related to the number of D5-branes, and thus to the number of baryon vertices. Using the BPS expression for F_3, we obtain $Q_B = -2\,\mathrm{e}^{4A}\partial_x\mathrm{e}^{-\phi}$, which is the advertised relationship between A and ϕ.

F_5 Bianchi identity. Substituting in the solutions of the BPS equations we obtain explicit expressions for F_5

$$\mathrm{d}f = \frac{y^4}{4}\left[\partial_y\left(\mathrm{e}^{-4A-\phi}\right)\mathrm{d}x - \partial_x\left(\mathrm{e}^{-4A-3\phi}\right)\mathrm{d}y\right]$$

$$\Rightarrow \mathcal{F}_2 = \mathrm{e}^{3A-4B} *_3 \mathrm{d}f = -\frac{1}{4}\,\mathrm{e}^{8A+2\phi}\left(\partial_y\left(\mathrm{e}^{-4A-\phi}\right)\mathrm{d}y + \mathrm{e}^{2\phi}\partial_x\left(\mathrm{e}^{-4A-3\phi}\right)\mathrm{d}x\right) \wedge \mathrm{d}t \tag{C.5}$$

$$= \left(\frac{1}{4}\mathrm{d}\left(\mathrm{e}^{4A+\phi}\right) - \frac{1}{2}\mathrm{e}^{4A+2\phi}\partial_x(\mathrm{e}^{-\phi})\,\mathrm{d}x\right) \wedge \mathrm{d}t\,.$$

The Bianchi identity comes in two pieces: a piece proportional to vol_{S^4} and another proportional to $\mathrm{vol}_{\mathbb{R}^3}$. We deal with the second part first. Using the F_3 Bianchi, we can express the two sides as follows:

$$\mathcal{F}_2 = \frac{1}{4}\left(\mathrm{d}\left(\mathrm{e}^{4A+\phi}\right) + Q_B\,\mathrm{e}^{2\phi}\,\mathrm{d}x\right) \wedge \mathrm{d}t$$

$$\Rightarrow \mathrm{d}F_5 = \cdots + \frac{Q_B}{4}\partial_y\mathrm{e}^{2\phi}\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{d}t \wedge \mathrm{vol}_{\mathbb{R}^3} \tag{C.6}$$

$$\frac{i}{8}G \wedge G^* = \frac{1}{4}H_3 \wedge F_3 = \frac{1}{4}\partial_y\mathrm{e}^{2\phi}\mathrm{d}t \wedge \mathrm{d}y \wedge \mathrm{d}x \wedge (Q_B\,\mathrm{vol}_{\mathbb{R}^3}) = -\mathrm{d}F_5$$

so we see that this part is automatically satisfied. Now we turn to the first term. Setting this to zero amounts to saying we can locally find a function $f(x,y)$ whose derivative equals $\mathrm{d}f$. Again using the F_3 Bianchi we have

$$\mathrm{d}f = -\frac{y^4}{4\,Q_B}\left(\partial_y\partial_x\mathrm{e}^{-2\phi}\mathrm{d}x - \frac{1}{2}\partial_x^2\mathrm{e}^{-4\phi}\mathrm{d}y\right) \wedge \mathrm{vol}_{S^4}\,. \tag{C.7}$$

The integrability condition $\mathrm{d}^2 f = 0$ gives

$$\partial_x\left(\frac{1}{y^4}\partial_y\left(y^4\partial_y\mathrm{e}^{-2\phi}\right) + \frac{1}{2}\partial_x^2\mathrm{e}^{-4\phi}\right) = 0\,, \tag{C.8}$$

which is verified if the Poisson equation is satisfied. Moreover, we can integrate (C.7) to obtain f. This equation states that

$$\partial_x f = \partial_x\left(-\frac{y^4}{4\,Q_B}\partial_y\mathrm{e}^{-2\phi}\right) \quad \Rightarrow \quad f = -\frac{y^4}{4\,Q_B}\partial_y\mathrm{e}^{-2\phi} + g(y) \tag{C.9}$$

for some function $g(y)$. Using again (C.7) we have the compatibility condition

$$\partial_y f = -\frac{1}{4\,Q_B}\partial_y\left(y^4\partial_y\mathrm{e}^{-2\phi}\right) + g'(y) = \frac{1}{4Q_B}\frac{1}{2}\,y^4\partial_x^2\mathrm{e}^{-4\phi}\,. \tag{C.10}$$

Poisson's equation then states that $g(y)$ is at most a constant that we fix to zero, so we have

$$f = -\frac{y^4}{4\,Q_B}\partial_y\mathrm{e}^{-2\phi}\,, \tag{C.11}$$

which is the expression used to find the brane charges.

H_3 equation of motion. For the first term, we can write

$$\mathrm{e}^{-\phi} * H_3 = y^4\,\partial_y(\mathrm{e}^{-2\phi})\,\mathrm{vol}_{\mathbb{R}^3} \wedge \mathrm{vol}_{S^4}$$

$$\Rightarrow \mathrm{d}(\mathrm{e}^{-\phi} * H_3) = \left[\partial_y\left(y^4\partial_y(\mathrm{e}^{-2\phi})\right)\mathrm{d}y + y^4\partial_x\partial_y(\mathrm{e}^{-2\phi})\mathrm{d}x\right] \wedge \mathrm{vol}_{\mathbb{R}^3} \wedge \mathrm{vol}_{S^4}\,. \tag{C.12}$$

Analogously, for the the second term, using (C.7) we immediately have

$$-4F_5 \wedge F_3 = y^4\left(\partial_x\partial_y\mathrm{e}^{-2\phi}\mathrm{d}x - \frac{1}{2}\partial_x^2\mathrm{e}^{-4\phi}\mathrm{d}y\right) \wedge \mathrm{vol}_{\mathbb{R}^3} \wedge \mathrm{vol}_{S^4}\,. \tag{C.13}$$

Finally we obtain

$$\mathrm{d}(\mathrm{e}^{-\phi} * H_3) + 4F_5 \wedge F_3 = \left[\partial_y\left(y^4\partial_y\mathrm{e}^{-2\phi}\right) + \frac{y^4}{2}\partial_x^2\mathrm{e}^{-4\phi}\right]\mathrm{d}y \wedge \mathrm{vol}_{\mathbb{R}^3} \wedge \mathrm{vol}_{S^4}\,, \tag{C.14}$$

which is proportional to our Poisson equation. The equation of motion for F_3 works in a similar manner.

Dilaton equation of motion. This is

$$\mathrm{d} * \mathrm{d}\phi + \frac{1}{2}G_3 \wedge *G_3 = \frac{\mathrm{e}^{2A+3\phi}}{2}\left(\frac{1}{y^4}\partial_y\left(y^4\partial_y\mathrm{e}^{-2\phi}\right) + \frac{1}{2}\partial_x^2\mathrm{e}^{-4\phi}\right) * 1 \qquad (\text{C.15})$$

recognizable again as (C.2).

Einstein equations. The different components of the Einstein equations (C.4), in flat indices, read

$$\begin{aligned}
E_{\hat{i}\hat{i}} &= -\frac{\mathrm{e}^{2\phi+6A}}{Q_B y}\partial_x\left(\frac{1}{y^4}\partial_y\left(y^4\partial_y\mathrm{e}^{-2\phi}\right) + \frac{1}{2}\partial_x^2\mathrm{e}^{-4\phi}\right), \\
E_{\hat{a}\hat{a}} &= 0, \\
E_{\hat{0}\hat{0}} &= -\frac{\mathrm{e}^{3\phi+2A}}{2}\left(\frac{1}{y^4}\partial_y\left(y^4\partial_y\mathrm{e}^{-2\phi}\right) + \frac{1}{2}\partial_x^2\mathrm{e}^{-4\phi}\right).
\end{aligned} \qquad (\text{C.16})$$

We conclude that, once the Poisson equation (C.2) is verified, not only (C.1) is a supersymmetric configuration of Type IIB supergravity preserving 1/4 of the supercharges, but is additionally a solution to the equations of motion.

C.1 Sourcing

In this setup it is straightforward to include an explicit source on the r.h.s. of the Poisson equation

$$\frac{1}{y^4}\partial_y\left(y^4\partial_y\mathrm{e}^{-2\phi}\right) + \frac{1}{2}\partial_x^2\mathrm{e}^{-4\phi} = \rho(x,y). \qquad (\text{C.17})$$

In order to do so one needs to carefully smear distributions of both fundamental strings and D3-branes and let them backreact on the geometry. Of course, this changes the form of the equations of motion, including the appearance of new terms in the energy momentum tensor. The details as well as the conditions required to preserve supersymmetry in the process can be found in the appendix of [13]. In that case we studied distributions depending only on the coordinate y, but it is straightforward to incorporate the x-dependence.

In the notation of [13], we have now the smearing forms

$$\begin{aligned}
\Omega_8 &= -y^4\rho(x,y)\,\mathrm{d}y \wedge \mathrm{vol}_{\mathbb{R}^3} \wedge \mathrm{vol}_{S^4}, \\
\Omega_6 &= y^4\rho_{\mathrm{D3}}(x,y)\,\mathrm{d}y \wedge \mathrm{d}x \wedge \mathrm{vol}_{S^4}.
\end{aligned} \qquad (\text{C.18})$$

From these we can read the directions along which the strings and the branes are distributed. They are supplemented by the calibration forms

$$\begin{aligned}
\mathcal{K}_2 &= -\mathrm{e}^{\frac{3}{2}\phi}\,\mathrm{d}t \wedge \mathrm{d}x, \\
\mathcal{K}_4 &= \mathrm{e}^{4A+\phi}\,\mathrm{d}t \wedge \mathrm{vol}_{\mathbb{R}^3},
\end{aligned} \qquad (\text{C.19})$$

that verify the calibration conditions stated in [13], ensuring the supersymmetry of the configuration. It can be checked that all the equations of motion, modified by the presence of the sources, are verified given the sourced Poisson equation (C.17) and the condition

$$Q_B\,\rho_{\mathrm{D3}} = \partial_x\rho. \qquad (\text{C.20})$$

This connection between the distribution of D3-branes and strings is also responsible for the preservation of the relation among the charges, and is natural given that every string has to end on a D3-brane.

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