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On Davis–Putnam reductions for minimally unsatisfiable clause-sets

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ABSTRACT

DP-reduction $F \rightsquigarrow \mathrm{DP}_v(F)$, applied to a clause-set F and a variable v, replaces all clauses containing v by their resolvents (on v). A basic case, where the number of clauses is decreased (i.e., $c(\mathrm{DP}_v(F)) < c(F)$), is *singular DP-reduction* (sDP-reduction), where v must occur in one polarity only once. For minimally unsatisfiable $F \in \mathcal{MU}$, sDP-reduction produces another $F' := \mathrm{DP}_v(F) \in \mathcal{MU}$ with the same deficiency, that is, $\delta(F') = \delta(F)$; recall $\delta(F) = c(F) - n(F)$, using $n(F)$ for the number of variables. Let $\mathrm{sDP}(F)$ for $F \in \mathcal{MU}$ be the set of results of complete sDP-reduction for F; so $F' \in \mathrm{sDP}(F)$ fulfil $F' \in \mathcal{MU}$, are *nonsingular* (every literal occurs at least twice), and we have $\delta(F') = \delta(F)$. We show that for $F \in \mathcal{MU}$ all complete reductions by sDP must have the same length, establishing the *singularity index* of F. In other words, for $F', F'' \in \mathrm{sDP}(F)$ we have $n(F') = n(F'')$. In general the elements of $\mathrm{sDP}(F)$ are not even (pairwise) isomorphic. Using the fundamental characterisation by Kleine Büning, we obtain as application of the singularity index, that we have *confluence modulo isomorphism* (all elements of $\mathrm{sDP}(F)$ are pairwise isomorphic) in case $\delta(F) = 2$. In general we prove that we have confluence (i.e., $|\mathrm{sDP}(F)| = 1$) for saturated F (i.e., $F \in \mathcal{SMU}$). More generally, we show confluence modulo isomorphism for *eventually saturated* F, that is, where we have $\mathrm{sDP}(F) \subseteq \mathcal{SMU}$, yielding another proof for confluence modulo isomorphism in case of $\delta(F) = 2$.

1. Introduction

Minimally unsatisfiable clause-sets ("MUs") are a fundamental form of irredundant unsatisfiable clause-sets. Regarding the subset relation, they are the hardest examples for proof systems. A substantial amount of insight has been gained into their structure, as witnessed by the handbook article [12]. A related area of MU, which gained importance in recent industrial applications, is the study of "MUSs", that is minimally unsatisfiable *sub*-clause-sets $F' \in \mathcal{MU}$ with $F' \subseteq F$ as the "cores" of unsatisfiable clause-sets F; see [27] for a recent overview. For the investigations of this paper there are two main sources: The structure of MU (see Section 1.1), and the study of DP-reduction as started with [13,20,21]:

- A fundamental result shown there is that DP-reduction is commutative modulo subsumption (see Section 5.2 for the precise formulation).
- Singular DP-reduction is a special case of length-reducing DP-reduction (while in general one step of DP-reduction can yield a quadratic blow-up).

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– Confluence *modulo isomorphism* was shown in [13] (Theorem 13, Page 52) for a combination of subsumption elimination with special cases of length-reducing DP-reductions, namely DP-reduction in case no (non-tautological) resolvent is possible, and singular DP-reduction in case there is only one side clause, or the main clause is of length at most 2 (see Definition 6).

The basic questions for this paper are:

– When does singular DP-reduction, applied to MU, yield unique (non-singular) results (i.e., we have confluence)?
– And when are the results at least determined up to isomorphism (i.e., we have confluence modulo isomorphism)?

Different from the result from [13] mentioned above, we do not consider restricted versions of singular DP-reduction, but we restrict the class of clause-sets to which singular DP-reduction is applied (namely to subclasses of MU).

The conference-version of this article is [24] (two technical mistakes in [24] are corrected here; see Theorem 41 and remarks and Corollary 47 and remarks), while the underlying report is [25].

1.1. Investigations into the structure of $\mathcal{MU}(k)$

We give now a short overview on the problem of classifying $F \in \mathcal{MU}$ in terms of the deficiency $\delta(F) := c(F) - n(F)$, that is, the problem of characterising the levels $\mathcal{MU}_{\delta=k} := \{F \in \mathcal{MU} : \delta(F) = k\}$ (due to greater expressivity and generality, we prefer this notation over $\mathcal{MU}(k)$); see [12] for further information.

The field of the combinatorial study of minimally unsatisfiable clause-sets was opened by [1], showing the fundamental insight $\delta(F) \geq 1$ for $F \in \mathcal{MU}$ (see [16,12] for generalisations of the underlying method, based on autarky theory). Also $\mathcal{SMU}_{\delta=1}$ was characterised there, where $\mathcal{SMU} \subset \mathcal{MU}$ is the set of "saturated" minimally unsatisfiable clause-sets, which are minimal not only w.r.t. having no superfluous clauses, but also w.r.t. that no clause can be further weakened. The fundamental "saturation method" $F \in \mathcal{MU} \rightsquigarrow F' \in \mathcal{SMU}$ was introduced in [7] (see Definition 1). Basic for all studies of MU is detailed knowledge on minimal number of occurrences of a (suitable) variable (yielding a suitable splitting variable): see [23] for the current state-of-art. The levels $\mathcal{MU}_{\delta=k}$ are decidable in polynomial time by [6,15]; see [31,18] for further extensions.

"Singular" variables v in $F \in \mathcal{MU}$, that is, variables occurring in at least one polarity only once, play a fundamental role — they are degenerations which (usually) need to be eliminated by *singular DP-reduction*. Let $\mathcal{MU}' \subset \mathcal{MU}$ be the set of non-singular minimally unsatisfiable clause-sets (not having singular variables), that is, the results of applying singular DP-reduction to the elements of $\mathcal{MU}$ as long as possible. The fundamental problem is the characterisation of $\mathcal{MU}'_{\delta=k}$ for arbitrary $k \in \mathbb{N}$. Up to now only $k \leq 2$ has been solved: $\mathcal{MU}'_{\delta=1}$ has been determined in [4], while $\mathcal{MU}'_{\delta=2} = \mathcal{SMU}'_{\delta=2}$ has been determined in [11]. Regarding higher deficiencies, until now only (very) partial results in [32] exist. Regarding singular minimally unsatisfiable clause-sets, also $\mathcal{MU}_{\delta=1}$ is very well known (with further extensions and generalisations in [15], and generalised to non-boolean clause-sets in [19]), while for $\mathcal{MU}_{\delta=2}$ not much is known (Section 7 provides first insights).

For characterising $\mathcal{MU}'_{\delta=k}$, we need (very) detailed insights into (arbitrary) $\mathcal{MU}_{\delta<k}$, since the basic method to investigate $F \in \mathcal{MU}'_{\delta=k}$ is to split F into smaller parts from $\mathcal{MU}_{\delta<k}$ (usually containing singular variables). Assuming that we know $\mathcal{MU}'_{\delta<k}$, such insights can be based on some classification of $F \in \mathcal{MU}_{\delta<k}$ obtained from the set $\mathrm{sDP}(F) \subseteq \mathcal{MU}'_{\delta<k}$ of singular-DP-reduction results. The easiest case is when $|\mathrm{sDP}(F)| = 1$ holds (confluence), the second-easiest case is where all elements of $\mathrm{sDP}(F)$ are pairwise isomorphic. This is the basic motivation for the questions raised and partially solved in this article. For general k we have no conjecture yet how the classification of $\mathcal{MU}'_{\delta=k}$ could look like (besides the basic conjecture that enumeration of the isomorphism types can be done efficiently). However for unsatisfiable hitting clause-sets (two different clauses clash in at least one variable) we have the conjecture stated in [23], that for every $k \in \mathbb{N}$ there are only finitely many isomorphism types in $\mathcal{UHIT}'_{\delta=k}$ (unsatisfiable non-singular hitting clause-sets of deficiency k).

1.2. Overview of results

Section 3 introduces the basic notions regarding singularity, and the basic characterisations of singular DP-reduction on minimally unsatisfiable clause-sets are given in Section 3.2. In Section 4 we consider the question of confluence of singular DP-reduction, with the first main result Theorem 23, showing confluence for saturated clause-sets. Section 5 mainly considers the question of changing the order of DP-reductions without changing the result. The second main result of this article is Theorem 63, establishing the singularity index. Section 6 is devoted to show confluence modulo isomorphism on eventually saturated clause-sets (Theorem 68), the third main result. As an application we determine the "types" of (possibly singular) minimally unsatisfiable clause-sets of deficiency 2 via Theorem 74 (Section 7). We conclude with a collection of open problems in Section 8.

1.3. Applications

Our current main application, which motivated the questions tackled in this paper in the first place, is the project of classifying the structure of $\mathcal{MU}_{\delta=k}$ as discussed in Section 1.1: Knowing some form of invariance of singular DP-reduction enables one to classify also *singular* minimally unsatisfiable clause-sets, based on knowing the *non*-singular minimally unsatisfiable clause-sets of the same deficiency; see Section 7 for a first example.

For worst-case upper bounds of SAT decision (or related problems) we sometimes need to guarantee that certain reductions will yield a certain decrease in some parameter, for example the number of variables, independently of the special order of reductions — this is exactly established for singular DP-reduction by the singularity index (using Corollary 64).

Finally, singular DP-reduction is a very basic and efficient reduction, which should be helpful in the search for MUSs, using that a singular variable for F is also singular for $F' \subseteq F$ with $F' \in \mathcal{MU}$. The basic results of Section 3 make it possible to control the effects of singular DP-reduction, while our main results enable one to estimate the inherent non-determinism. We are aware of the following algorithms using sDP-reduction:

- A special case of singular DP-reduction, namely unit-clause propagation, has been exploited in [26] for searching for (some) MUSs; see Section 3.3 for further remarks. Note that in the general situation $F' \subseteq F$ with $F' \in \mathcal{MU}$, a singular variable for F' might not be singular for F (and thus might go unnoticed) — the problem is that we do not know F' in advance. However in the case of unit-clauses $\{x\} \in F$ we can discard all clauses $C \in F$ with $\{x\} \subset C$ (for a MUS involving $\{x\}$), and so the singular literal x will not be missed.
- DP-reduction in general has been used in theoretical as well as in practical SAT-algorithms:

1. [3] used DP-reductions for (complete) SAT solving, by unrestricted application of the reduction rule.
2. In [8] a simple case of DP-reduction, namely considering only variables occurring at most twice, has been analysed probabilistically.
3. DP-reductions has been used in the worst-case analysis of algorithms in [13,20,21]; especially in [20,21] it is shown that allowing reductions $F \rightsquigarrow \mathrm{DP}_v(F)$ with up to K new clauses for a fixed K, i.e., $c(\mathrm{DP}_v(F)) \le c(F) + K$, can improve worst-case performance.
4. In [9] this DP-reduction with bounded clause-number-increase has been used at each node of the search tree of a SAT solver, with $K \approx 200$.
5. In [30] another criterion analysed in [13,20,21], namely $\ell(\mathrm{DP}_v(F)) \le \ell(F)$ has been implemented, where $\ell(F) := \sum_{C \in F} |C|$ is the number of literal occurrences, this time as a free-standing preprocessor. Singular DP-reduction is not covered by this criterion (since the number of literal-occurrences can be increased by sDP-reduction).
6. This approach has been further developed in [5], but now using $K = 0$, i.e., $c(\mathrm{DP}_v(F)) \le c(F)$. Again a free-standing preprocessor has been provided, called "satELite". Now sDP-reduction is covered.

This preprocessor was incorporated into several recent SAT solvers, most notably into the `minisat` solvers from version 2.0 on. So a "minimal unsatisfiable core (or subset) extraction" algorithm like `Haifa-MUC` [28], the winner of the SAT 2011 competition regarding this task, applies sDP-reduction.

2. Preliminaries

We follow the general notations and definitions as outlined in [12]. We use $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Consider a relation $R \subseteq X^2$ on a set X; for us typically X is the set $\mathcal{CLS}$ of all clause-sets or the set $\mathcal{MU}$ of all minimally unsatisfiable clause-sets. We view R as a "reduction", and we write $x \rightsquigarrow x'$ for $(x, x') \in R$. Such a reduction is called *terminating* if there are no infinite chains $x_1 \rightsquigarrow x_2 \rightsquigarrow x_3 \rightsquigarrow \cdots$ of reductions. Using the reflexive-transitive closure $\rightsquigarrow^*$ (that is, zero, one or more reductions taking place), for a terminating reduction and every $x \in X$ there is at least one $x' \in X$ with $x \rightsquigarrow^* x'$ such that there is no $x'' \in X$ with $x' \rightsquigarrow x''$. A terminating reduction is called *confluent* if this x' is always unique.

An example for a terminating and confluent reduction-relation is unrestricted DP-reduction $F \rightsquigarrow \mathrm{DP}_v(F)$ for a clause-set $F \in \mathcal{CLS}$ and a variable $v \in \mathrm{var}(F)$, as defined below.

2.1. Clause-sets

The (infinite) set of all variables is $\mathcal{VA}$, while the set of all literals is $\mathcal{LIT}$, where we identify the positive literals with variables, that is, we assume $\mathcal{VA} \subset \mathcal{LIT}$. Complementation is an involution of $\mathcal{LIT}$, and is denoted for literals $x \in \mathcal{LIT}$ by $\overline{x} \in \mathcal{LIT}$. For a set L of literals we define $\overline{L} := \{\overline{x} : x \in L\}$ (so $\mathcal{LIT}$ is the disjoint union of $\mathcal{VA}$ and $\overline{\mathcal{VA}}$). A **clause** C is a finite and clash-free set of literals (i.e., $C \cap \overline{C} = \emptyset$), while a **clause-set** $F \in \mathcal{CLS}$ is a finite set of clauses. The empty clause is denoted by $\bot := \emptyset$, and the empty clause-set is denoted by $\top \in \mathcal{CLS}$. We denote by $\mathrm{var}(F)$ the set of (occurring) variables, by $n(F) := |\mathrm{var}(F)|$ the number of variables, by $c(F) := |F|$ the number of clauses, and finally by $\delta(F) := c(F) - n(F)$ the deficiency. For clause-sets F, G we denote by $\boldsymbol{F \cong G}$ that both clause-sets are **isomorphic**, that is, the variables of F can be renamed and potentially flipped so that F is turned into G; more precisely, an isomorphism α from F to G is a bijection α on literal-sets which preserves complementation and which maps the clauses of F precisely to the clauses of G. The *literal-degree* $\mathrm{ld}_F(x) \in \mathbb{N}_0$ of a literal x for a clause-set F is the number of clauses the literal appears in, i.e., $\mathrm{ld}_F(x) := |\{C \in F : x \in C\}|$. The *variable-degree* $\mathrm{vd}_F(v) \in \mathbb{N}_0$ for a variable v is the number of clauses the variable appears in, i.e., $\mathrm{vd}_F(v) := \mathrm{ld}_F(v) + \mathrm{ld}_F(\overline{v})$.

For a clause-set F and a variable v, by $\boldsymbol{\mathrm{DP}_v(F)}$ we denote the result of applying DP-reduction on v ("DP" stands for "Davis-Putnam", who introduced this operation in [3]), that is, removing all clauses containing v and adding all resolvents on v.

More formally

$$\mathrm{DP}_v(F) := \{C \in F : v \notin \mathrm{var}(C)\} \cup \{C \diamond D : C, D \in F,\ C \cap \overline{D} = \{v\}\},$$

where clauses C, D are resolvable iff they clash in exactly one literal, i.e., iff $|C \cap \overline{D}| = 1$, while for resolvable clauses C, D the resolvent $\boldsymbol{C \diamond D} := (C \cup D) \setminus \{x, \overline{x}\}$ for $C \cap \overline{D} = \{x\}$ is defined as the union minus the resolution literals (the two clashing literals). $\mathrm{DP}_v(F)$ is logically equivalent to the existential quantification of F by v, and thus F and $\mathrm{DP}_v(F)$ are satisfiability-equivalent, that is, $\mathrm{DP}_v(F)$ is satisfiable iff F is satisfiable.

We can define $\mathcal{SAT} \subset \mathcal{CLS}$, the set of all satisfiable clause-set, as the set of $F \in \mathcal{CLS}$ where reduction by DP will finally yield $\top$, the empty clause-set, while we can define $\mathcal{USAT} = \mathcal{CLS} \setminus \mathcal{SAT}$, the set of all unsatisfiable clause-set, as the set of $F \in \mathcal{CLS}$ where reduction by DP will finally yield $\{\bot\}$, the clause-set consisting of the empty clause.

Since DP-reduction on v removes at least variable v, every sequence of applications of DP until no variables are left must end up either in $\top$ or in $\{\bot\}$. The satisfiability-invariance of DP-reduction yields that the final result does not depend on the choices involved, but only on the satisfiability resp. unsatisfiability of the starting clause-set. So unrestricted DP-reduction is terminating and confluent; a proof of confluence from first principles (by combinatorial means) is achieved by Lemma 28.

2.2. Minimal unsatisfiability

The set of minimally unsatisfiable clause-sets is $\mathcal{MU} \subset \mathcal{USAT}$, the set of all clause-sets which are unsatisfiable, while removal of any clause makes them satisfiable. Furthermore the set of saturated minimally unsatisfiable clause-sets is $\mathcal{SMU} \subset \mathcal{MU}$, which is the set of minimally unsatisfiable clause-sets such that addition of any literal to any clause renders them satisfiable. Note that for $v \in \mathrm{var}(F)$ with $F \in \mathcal{MU}$ we have $\mathrm{vd}_F(v) \geq 2$. We recall the fact ([7] and Lemma 5.1 in [19]) that every minimally unsatisfiable clause-set $F \in \mathcal{MU}$ can be **saturated**, i.e., by adding literal occurrences to F we obtain $F' \in \mathcal{SMU}$ with $\mathrm{var}(F') = \mathrm{var}(F)$ such that there is a bijection $\alpha : F \to F'$ with $C \subseteq \alpha(C)$ for all $C \in F$. The details are as follows.

Definition 1. The operation $\mathbf{S}(\boldsymbol{F}, \boldsymbol{C}, \boldsymbol{x}) := (F \setminus \{C\}) \cup (C \cup \{x\}) \in \mathcal{CLS}$ (adding literal x to clause C in F) is defined if $F \in \mathcal{CLS}$, $C \in F$, and x is a literal with $\mathrm{var}(x) \in \mathrm{var}(F) \setminus \mathrm{var}(C)$. A **saturation** $F' \in \mathcal{SMU}$ of $F \in \mathcal{MU}$ is obtained by a sequence $F = F_0, \ldots, F_m = F'$, $m \in \mathbb{N}_0$,

- such that for $0 \leq i < m$ there are C_i, x_i with $F_{i+1} = \mathrm{S}(F_i, C_i, x_i)$,
- such that for all $1 \leq i \leq m$ we have $F_i \notin \mathcal{SAT}$,
- and such that the sequence cannot be extended.

Note that $n(F') = n(F)$ and $c(F') = c(F)$ holds (and thus $\delta(F') = \delta(F)$). More generally, a **partial saturation** of a clause-set $F \in \mathcal{MU}$ is a clause-set $F' \in \mathcal{MU}$ such that $\mathrm{var}(F') = \mathrm{var}(F)$ and there is a bijection $\alpha : F \to F'$ such that for all $C \in F$ we have $C \subseteq \alpha(C)$.

Please note that if for $F \in \mathcal{MU}$ and $F' := \mathrm{S}(F, C, x)$ we have $F' \notin \mathcal{SAT}$, then actually $F' \in \mathcal{MU}$ must hold. Thus if F' is a saturation of $F \in \mathcal{MU}$ in the sense of Definition 1, then actually F' is saturated (minimally unsatisfiable).

A clause-set F is **hitting** if every two different clauses clash in at least one literal. The set of hitting clause-sets is denoted by

$$\boldsymbol{\mathcal{HIT}} := \{F \in \mathcal{CLS} \mid \forall\, C, D \in F, C \neq D : C \cap \overline{D} \neq \emptyset\} \subset \mathcal{CLS},$$

the set of unsatisfiable hitting clause-sets by $\boldsymbol{\mathcal{UHIT}} := \mathcal{HIT} \cap \mathcal{USAT}$. When interpreting F as DNF, hitting clause-sets are known as "disjoint" or "orthogonal" DNF; see Chapter 7 in [2].

Lemma 2. *We have* $\mathcal{UHIT} \subset \mathcal{SMU}$.

Proof. For $F \in \mathcal{HIT}$ we have $F \in \mathcal{USAT}$ iff $\sum_{C \in F} 2^{-|C|} = 1$ (see [12]; the point is that two clashing clauses do not have a common falsifying assignment). Thus adding a literal to a clause of $F \in \mathcal{UHIT}$ makes F satisfiable. See Example 3 for an example showing that the inclusion is strict. □

Example 3. Two unsatisfiable hitting clause-sets used in various examples are:

$$\mathcal{F}_2 := \{\{v_1, v_2\}, \{\overline{v_1}, \overline{v_2}\}, \{\overline{v_1}, v_2\}, \{\overline{v_2}, v_1\}\}$$
$$\mathcal{F}_3 := \{\{v_1, v_2, v_3\}, \{\overline{v_1}, \overline{v_2}, \overline{v_3}\}, \{\overline{v_1}, v_2\}, \{\overline{v_2}, v_3\}, \{\overline{v_3}, v_1\}\}.$$

And an example for an element of $\mathcal{SMU} \setminus \mathcal{UHIT}$ is given by

$$\mathcal{F}_4 := \{\{v_1, v_2, v_3, v_4\}, \{\overline{v_1}, \overline{v_2}, \overline{v_3}, \overline{v_4}\}, \{\overline{v_1}, v_2\}, \{\overline{v_2}, v_3\}, \{\overline{v_3}, v_4\}, \{\overline{v_4}, v_1\}\}.$$

To see $\mathcal{F}_4 \in \mathcal{SMU}$ it is easiest to use Corollary 5.3 in [19], that is, we have to show that for all $v \in \mathrm{var}(\mathcal{F}_4)$ and $\varepsilon \in \{0, 1\}$ we have $\langle v \to \varepsilon \rangle * \mathcal{F}_4 \in \mathcal{MU}$. W.l.o.g. $v = v_1$ and $\varepsilon = 0$, and then $\langle v \to \varepsilon \rangle * \mathcal{F}_4 = \{\{v_2, v_3, v_4\}, \{\overline{v_2}, v_3\}, \{\overline{v_3}, v_4\}, \{\overline{v_4}\}\} \in \mathcal{MU}_{\delta=1}$. The clause-sets $\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ are elements of $\mathcal{MU}_{\delta=2}$; see Section 7 for more on this class.

The following (new) observation is fundamental for the study of hitting clause-sets:

Lemma 4. *For* $F \in \mathcal{HIT}$ *and a variable* v *we have* $\mathrm{DP}_v(F) \in \mathcal{HIT}$.

Proof. Consider clauses $E_1, E_2 \in \mathrm{DP}_v(F)$, $E_1 \neq E_2$. If $E_1, E_2 \in F$, then E_1, E_2 clash since F is hitting. The two remaining cases are (w.l.o.g.) $E_1 \in F, E_2 \notin F$ and $E_1, E_2 \notin F$. In the first case assume $E_2 = C_2 \diamond D_2$ for $C_2, D_2 \in F$ with $C_2 \cap \overline{D_2} = \{v\}$. Since $v \notin \mathrm{var}(E_1)$, it clashes E_1 with C_2 (as well as with D_2) and thus with E_2. For the second case also assume $E_1 = C_1 \diamond D_1$ for $C_1, D_1 \in F$ with $C_1 \cap \overline{D_1} = \{v\}$. We must have $C_1 \neq C_2$ or $D_1 \neq D_2$, yielding a clash between C_1, C_2 resp. D_1, D_2, and thus also E_1, E_2 clash. □

Since DP-reduction preserves unsatisfiability, we get:

Corollary 5. *For $F \in \mathcal{UHIT}$ and a variable v we have $\mathrm{DP}_v(F) \in \mathcal{UHIT}$.*

3. Singularity

In this section we present basic results on singular variables in minimally unsatisfiable clause-sets. Lemmas 9, 12 yield basic characterisations of singular DP-reduction for minimally unsatisfiable resp. saturated minimally unsatisfiable clause-sets (some of these results were discussed in [22]), while Lemma 14 shows that in the context of MU unit-clause propagation is a special case of singular DP-reduction. These results are straight-forward, but the choice of concepts is important, and the facts are somewhat subtle.

3.1. Singular variables

Definition 6. We call a variable v **singular** for a clause-set $F \in \mathcal{CLS}$ if we have $\min(\mathrm{ld}_F(v), \mathrm{ld}_F(\overline{v})) = 1$; the set of singular variables of F is denoted by $\boldsymbol{\mathrm{var}_s(F)} \subseteq \mathrm{var}(F)$. F is called **nonsingular** if F does not contain singular variables. Furthermore we use the following notations:

- $\boldsymbol{\mathcal{MU}'} := \{F \in \mathcal{MU} : \mathrm{var}_s(F) = \emptyset\}$ denotes the set of nonsingular MUs;
- $\boldsymbol{\mathcal{SMU}'} := \mathcal{SMU} \cap \mathcal{MU}'$ is the set of nonsingular saturated MUs;
- $\boldsymbol{\mathcal{UHIT}'} := \mathcal{UHIT} \cap \mathcal{SMU}' = \mathcal{HIT} \cap \mathcal{MU}'$ is the set of nonsingular unsatisfiable hitting clause-sets.

More precisely:

- We call variable v ***m*-singular** for F for some $m \in \mathbb{N}$, if v is singular for F with $m = \mathrm{vd}_F(v) - 1$. The set of 1-singular variables of F is denoted by $\boldsymbol{\mathrm{var}_{1s}(F)} := \{v \in \mathcal{VA} : \mathrm{ld}_F(v) = \mathrm{ld}_F(\overline{v}) = 1\} \subseteq \mathrm{var}_s(F)$.
- A **non-1-singular** variable is a variable which m-singular for some $m \geq 2$ (so "non-1-singular" variables are singular). The set of non-1-singular variables of F is denoted by $\boldsymbol{\mathrm{var}_{\neg 1s}(F)} := \mathrm{var}_s(F) \setminus \mathrm{var}_{1s}(F)$.

A **singular literal** for a singular variable v is a literal x with $\mathrm{var}(x) = v$ and $\mathrm{ld}_F(x) = 1$; if the underlying variable is 1-singular, then some choice is applied, so that we can speak of "the" singular literal of a singular variable. For the singular literal x for v we call the clause $C \in F$ with $x \in C$ the **main clause**, while the **side clauses** are the clauses $D_1, \dots, D_m \in F$ with $\overline{x} \in D_i$ (here v is m-singular).

Example 7. For $F := \{\{a\}, \{\overline{a}, b\}, \{\overline{a}, \overline{b}\}\}$, variable a is 2-singular, while variable b is 1-singular, and thus $\mathrm{var}_s(F) = \{a, b\}$, $\mathrm{var}_{1s}(F) = \{b\}$ and $\mathrm{var}_{\neg 1s}(F) = \{a\}$. The main clause of a is $\{a\}$, its side clauses are $\{\overline{a}, b\}$, $\{\overline{a}, \overline{b}\}$, while for the main clause of b there is the choice between $\{\overline{a}, b\}$ and $\{\overline{a}, \overline{b}\}$.

In general, if $F \in \mathcal{MU}$ contains a unit-clause $\{x\} \in F$, then $\mathrm{var}(x)$ is singular for F (see Lemma 14). Thus the clause-sets $\{\bot\}$ and $\mathcal{F}_2$ (recall Example 3) are the two smallest elements of $\mathcal{MU}'$, $\mathcal{SMU}'$ and $\mathcal{UHIT}'$ regarding the number of clauses.

3.2. Singular DP-reduction

The following special application of DP-reduction appears at many places in the literature (see [11], or Appendix B in [15] and subsequent [31,18]), and is fundamental for investigations of minimally unsatisfiable clause-sets:

Definition 8. A **singular DP-reduction** is a reduction $F \rightsquigarrow \mathrm{DP}_v(F)$, where v is singular for $F \in \mathcal{MU}$. For $F, F' \in \mathcal{MU}$ by $\boldsymbol{F \xrightarrow{\mathrm{sDP}} F'}$ we denote that F' is obtained from F by one step of singular DP-reduction; i.e., there is a singular variable v for F with $F' = \mathrm{DP}_v(F)$, where v is called the **reduction variable**. And we write $F \xrightarrow{\mathrm{sDP}}_* F'$ if F' is obtained from F by an arbitrary number of steps (possibly zero) of singular DP-reductions. The set of all nonsingular clause-sets obtainable from F by singular DP-reduction is denoted by $\boldsymbol{\mathrm{sDP}(F)}$:

$$\mathrm{sDP}(F) := \{F' \in \mathcal{MU}' : F \xrightarrow{\mathrm{sDP}}_* F'\}.$$

The following lemma is kind of "folklore", but apparently the only place where its assertions are (partially) stated in the literature (in a more general form) is [18], Lemma 6.1 (we add here various details):

Lemma 9. *Consider a clause-set F and a singular variable v for F. Then the following assertions are equivalent:*

1. *F is minimally unsatisfiable.*
2. *$\delta(\mathrm{DP}_v(F)) = \delta(F)$ and $\mathrm{DP}_v(F)$ is minimally unsatisfiable.*

3. $\mathrm{DP}_v(F)$ *is minimally unsatisfiable, and for the main clause C and the side clauses* $D_1, \ldots, D_m$ *for* v *(in F) we have:*
   (a) *Every* D_i *clashes with C in exactly one variable (namely in* v*).*
   (b) *For* $1 \le i \ne j \le m$ *we have* $C \diamond D_i \ne C \diamond D_j$.
   (c) *For* $E \in F$ *with* $v \notin \mathrm{var}(E)$ *and for all* $1 \le i \le m$ *we have* $C \diamond D_i \ne E$.

Proof. The equivalence of Part 1 and Part 2 is a special case of Lemma 6.1 in [18]. Part 2 implies Part 3, since if one of the conditions 3(a), 3(b) or 3(c) would not hold, then the deficiency of $\mathrm{DP}_v(F)$ would be (strictly) smaller than F, contradicting the assumption $\delta(\mathrm{DP}_v(F)) = \delta(F)$. Finally we show that Part 3 implies Part 1. Since $\mathrm{DP}_v(F)$ is minimally unsatisfiable, F is unsatisfiable. Now suppose that F is not minimally unsatisfiable. So for some clause $E \in F$ the clause-set $F' := F \setminus \{E\}$ is still unsatisfiable. By condition 3(a) we know that $C \diamond D_i$ must be in $\mathrm{DP}_v(F)$ for all $i \in \{1, \ldots, m\}$. Thus clause E cannot be the main clause C, and if $m = 1$, then E cannot be the side clause neither. So v is still a singular variable in F'. Since $\mathrm{DP}_v(F)$ is minimally unsatisfiable, while we have $\mathrm{DP}_v(F') \subseteq \mathrm{DP}_v(F)$, we obtain $\mathrm{DP}_v(F') = \mathrm{DP}_v(F)$, that is, either E is one of the side clauses and its resolvent with C was obtained by some other resolution or was already present, or E does not contain v, and thus E must be a resolvent. In any case we get a contradiction with one of 3(b) or 3(c). □

Corollary 10. *If* $F \in \mathcal{MU}$ *and* v *is a singular variable of F, then also* $\mathrm{DP}_v(F) \in \mathcal{MU}$*, where* $\delta(\mathrm{DP}_v(F)) = \delta(F)$*. So the classes* $\mathcal{MU}_{\delta=k}$ *for* $k \in \mathbb{N}$ *are stable under singular DP-reduction.*

Corollary 11. *Consider* $F \in \mathcal{MU}$ *and a singular variable* v *with singular literal x, with main clause C and side clauses* $D_1, \ldots, D_m$*. Then adding* $C \setminus \{x\}$ *to* D_i *for all* $i \in \{1, \ldots, m\}$ *is a partial saturation of F (recall Definition 1).*

Proof. Let F' be obtained from F by replacing the clauses D_i by the clauses $D_i \cup (C \setminus \{x\})$ for each $i \in \{1, \ldots, m\}$ (note that by Lemma 9, Part 3(a), the literal-sets $D_i \cup (C \setminus \{x\})$ are clash-free and thus indeed clauses). By Lemma 9 we know that $\mathrm{DP}_v(F) \in \mathcal{MU}$ holds. Now $\mathrm{DP}_v(F') = \mathrm{DP}_v(F)$, and so in order to show that $F' \in \mathcal{MU}$, we need to show that the three conditions of Part 3 of Lemma 9 hold. Condition 3(a) holds by definition. And conditions 3(b), 3(c) follow from the fact (which was already used for $\mathrm{DP}_v(F') = \mathrm{DP}_v(F)$), that the changed clauses D_i yield the same resolvents with clause C. □

Lemma 9 can be strengthened for saturated F by requiring special conditions for the occurrences of the singular variable.

Lemma 12. *Consider a clause-set F and a singular variable* v *for F. For the singular literal x for* v *consider the main clause C and the side clauses* $D_1, \ldots D_m \in F$*. Let* $C' := C \setminus \{x\}$ *and* $D'_i := D_i \setminus \{\overline{x}\}$*. The following assertions are equivalent:*

1. *F is saturated minimally unsatisfiable.*
2. *The following three conditions hold:*
   (a) $\mathrm{DP}_v(F)$ *is saturated minimally unsatisfiable;*
   (b) $C' = \bigcap_{i=1}^{m} D'_i$;
   (c) *for every* $E \in F$ *with* $v \notin \mathrm{var}(E)$ *we have* $C' \not\subseteq E$.
   Note that conditions 2(b), 2(c) together imply the condition that for $E \in F$ *we have* $C' \subseteq E$ *if and only if* $v \in \mathrm{var}(E)$ *holds.*

Proof. First assume that F is saturated minimally unsatisfiable. If there would be $E \in F$ with $v \notin \mathrm{var}(E)$ and $C' \subseteq E$, then for $F' := \mathrm{S}(F, E, \overline{v})$ we had $\mathrm{DP}_v(F') = \mathrm{DP}_v(F)$, and thus F' would be unsatisfiable, contradicting saturatedness of F. We have $C' \subseteq \bigcap_{i=1}^{m} D'_i$, since if there were a literal $y \in C'$ and $y \notin D'_i$ for some i, then $\mathrm{DP}_v(\mathrm{S}(F, D_i, y)) = \mathrm{DP}_v(F)$. And we have $C' \supseteq \bigcap_{i=1}^{m} D'_i$, since if there were a literal y contained in all D'_i, but not in C', then $\mathrm{DP}_v(\mathrm{S}(F, C, y)) = \mathrm{DP}_v(F)$.

By Lemma 9 we know that $\mathrm{DP}_v(F)$ is minimally unsatisfiable, and that all resolutions are carried out, with no contraction due to coinciding resolvents or coincidence of a resolvent with an existing clause. Assume that $\mathrm{DP}_v(F)$ is not saturated, that is, there is a clause E and a literal y with $G := \mathrm{S}(\mathrm{DP}_v(F), E, y) \in \mathcal{USAT}$. If $E \in F$ then $\mathrm{DP}_v(\mathrm{S}(F, E, y)) = G \in \mathcal{USAT}$, and so there is some $1 \le i \le m$ with $E = C \diamond D_i$. But now $\mathrm{DP}_v(\mathrm{S}(F, D_i, y)) = G$, yielding a contradiction.

Now we consider the opposite direction, that is, we assume that $C' = \bigcap_{i=1}^{m} D'_i$, that $\mathrm{DP}_v(F)$ is saturated minimally unsatisfiable, and that C' is contained in some clause of F iff this clause contains the variable v. First we establish the three conditions from Lemma 9, Part 3. Since clauses are clash-free, C' has no conflict with any D'_i, and thus the clash-freeness-condition is fulfilled. If we had $C \diamond D_i = C \diamond D_j$ for $i \ne j$, then w.l.o.g. there must be a literal $y \in C'$ with $y \in D'_i$ and $y \notin D'_j$, which is impossible since C' contains only literals which are common to all side clauses. Finally, since all resolvents $C \diamond D_i$ subsume the parent clause D_i, by the minimal unsatisfiability of F also Condition 3(c) is fulfilled. So we have established that F is minimally unsatisfiable.

Assume that F is not saturated, that is, there exists a clause $E \in F$ and a literal y with $G := \mathrm{S}(F, E, y) \in \mathcal{MU}$. Let $F' := \mathrm{DP}_v(F)$ and $G' := \mathrm{DP}_v(G)$ (note $G' \in \mathcal{USAT}$, and that $F' \in \mathcal{SMU}$ by assumption). Our strategy is to derive a contradiction by showing that literal occurrences can be added to F' in such a way that G' is obtained, contradicting that F' is saturated.

First consider $E \notin \{C\} \cup \{D_i\}_{1 \le i \le m}$. If $\mathrm{var}(y) \ne v$, then $G' = \mathrm{S}(F', E, y)$. If $y = \overline{v}$, then $G' = \mathrm{S}(F', \{E\}, C')$ (using Condition 2(c)). It remains the case $y = v$, but this case is impossible since then for all $1 \le i \le m$ we have $C \diamond D_i = D'_i \subseteq (E \cup \{v\}) \diamond D_i = E \cup D'_i$, and thus $\mathrm{DP}_v(G)$ would be satisfiability equivalent to $\mathrm{DP}_v(G \setminus \{E \cup \{v\}\})$, whence G would not be minimally unsatisfiable.

So we have $E \in \{C\} \cup \{D_i\}_{1\le i\le m}$, i.e., $v \in \mathrm{var}(C)$. If $E = C$, then $G' = \mathrm{S}(F', \{D'_i\}_{1\le i\le m}, y)$, using that C' is the intersection all the D'_i, and thus at least one D'_i does not contain y. And if $E = C_i$ for some i, then $G' = \mathrm{S}(F', D'_i, y)$. □

Corollary 13. *The class $\mathcal{SMU}$ is stable under singular DP-reduction.*

3.3. Unit-clauses

In this subsection we explore the observation that unit-clause propagation for minimally unsatisfiable clause-sets is a special case of singular DP-reduction. First we show that unit-clauses in minimally unsatisfiable clause-sets can be considered as special cases of singular variables in the following sense:

Lemma 14. *Consider $F \in \mathcal{MU}$.*

1. *If v is singular for F and occurs in every clause of F (positively or negatively), then we have $\{v\} \in F$ or $\{\overline{v}\} \in F$.*
2. *If $\{x\} \in F$ for some literal x, then $v := \mathrm{var}(x)$ is singular in F (with $\mathrm{ld}_F(x) = 1$). If here F is saturated, then v must occur in every clause of F.*

Proof. For Part 1 consider a main clause C for v, and assume w.l.o.g. $v \in C$. Since every other clause $D \in F \setminus \{C\}$ contains $\overline{v}$, while C has exactly one clash with D by Lemma 9, Part 3(a), literals in $C \setminus \{v\}$ are pure in F, and thus there cannot be any (that is, $C = \{v\}$ holds), since F is minimally unsatisfiable. For Part 2 we first observe that every other clause of F containing x would be subsumed by $\{x\}$, which is impossible since F is minimally unsatisfiable. If F is saturated, then every clause $D \in F \setminus \{\{x\}\}$ must contain $\overline{x}$ by Lemma 12, Part 2(c). □

So nonsingular minimally unsatisfiable clause-sets do not contain unit-clauses.

Example 15. Some examples illustrating the relation between unit-clauses and singular variables for $\mathcal{MU}$:

1. From $\mathcal{F}_2 = \{\{v_1, v_2\}, \{\overline{v_1}, \overline{v_2}\}, \{\overline{v_1}, v_2\}, \{\overline{v_2}, v_1\}\} \in \mathcal{UHIT}'_{\delta=2}$ (recall Example 3) we obtain, using "inverse unit-clause elimination":
   (a) $\{\{x\}, \{v_1, v_2, \overline{x}\}, \{\overline{v_1}, \overline{v_2}, \overline{x}\}, \{\overline{v_1}, v_2, \overline{x}\}, \{\overline{v_2}, v_1, \overline{x}\}\} \in \mathcal{UHIT}_{\delta=2}$
   (b) $\{\{x\}, \{v_1, v_2, \overline{x}\}, \{\overline{v_1}, \overline{v_2}, \overline{x}\}, \{\overline{v_1}, v_2, \overline{x}\}, \{\overline{v_2}, v_1\}\} \in \mathcal{MU}_{\delta=2} \setminus \mathcal{SMU}_{\delta=2}$.
2. $\{\{a, b\}, \{a, \overline{b}\}, \{\overline{a}, c\}, \{\overline{a}, \overline{c}\}\} \in \mathcal{UHIT}_{\delta=1}$ contains the two singular variables b, c, while not containing a unit-clause.

If $F \in \mathcal{MU}_{\delta=k}$ contains a unit-clause $\{x\} \in F$, then we can apply singular DP-reduction for the underlying variable of x, and the result $\mathrm{DP}_{\mathrm{var}(x)}(F) \in \mathcal{MU}_{\delta=k}$ is the same as the result of the usual unit-clause elimination for $\{x\}$ (setting x to true, and simplifying accordingly). We now consider the case where repeated unit-clause elimination, i.e., unit-clause propagation, yields the empty clause.

In [4] it has been shown that for minimally unsatisfiable clause-sets $F \in \mathcal{MU}$ the following properties are equivalent:

1. F can be reduced by sDP to $\{\perp\}$, i.e., $F \xrightarrow{\mathrm{sDP}}_* \{\perp\}$.
2. All sDP-reductions of F end with $\{\perp\}$, i.e., $\mathrm{sDP}(F) = \{\perp\}$.
3. $\delta(F) = 1$.

Let $\mathrm{r}_1 : \mathcal{CLS} \to \mathcal{CLS}$ denote unit-clause propagation, that is, $\mathrm{r}_1(F) := \{\perp\}$ if $\perp \in F$, $\mathrm{r}_1(F) := F$ if all clauses of F have length at least two, and otherwise $\mathrm{r}_1(F) := \mathrm{r}_1(\langle x \to 1\rangle * F)$ for $\{x\} \in F$, where $\langle x \to 1\rangle * F$ means setting literal x to true, i.e., removing clauses containing x, and removing literal $\overline{x}$ from the remaining clauses (see [14,17] for a proof of confluence, i.e., independence of the choice of the unit-clauses $\{x\}$, and for generalisations). So, if for $F \in \mathcal{MU}$ we have $\mathrm{r}_1(F) = \{\perp\}$, then we know $F \in \mathcal{MU}_{\delta=1}$. Now it is well-known (first shown in [10]) that for $F \in \mathcal{CLS}$ we have $\mathrm{r}_1(F) = \{\perp\}$ iff there is $F' \subseteq F$ with $F' \in \mathcal{MU} \cap \mathcal{RHO}$, where $\mathcal{RHO}$ is the class of renamable (or "hidden") Horn clause-sets, that is, we have $F' \in \mathcal{RHO}$ iff there is a Horn clause-set $F'' \in \mathcal{HO}$ with $F' \cong F''$, where $\mathcal{HO} := \{F \in \mathcal{CLS} \mid \forall C \in F : |C \cap \mathcal{VA}| \le 1\}$ (each clause contains at most one positive literal). Altogether follows the following well-known characterisation:

Lemma 16. *For $F \in \mathcal{MU}$ holds $\mathrm{r}_1(F) = \{\perp\}$ iff $F \in \mathcal{MU}_{\delta=1} \cap \mathcal{RHO}$.*

Proof. If for $F \in \mathcal{MU}$ holds $\mathrm{r}_1(F) = \{\perp\}$, then $F \in \mathcal{RHO}$ by [10], while $F \in \mathcal{MU}_{\delta=1}$ by [4] and Lemma 14. And if $F \in \mathcal{RHO}$ holds (for arbitrary $F \in \mathcal{USAT}$), then $\mathrm{r}_1(F) = \{\perp\}$ by [10]. □

If for a clause-set $F \in \mathcal{CLS}$ we have $\mathrm{r}_1(F) = \{\perp\}$, then by Lemma 16 there is $F' \subseteq F$ with $F' \in \mathcal{MU}_{\delta=1} \cap \mathcal{RHO}$. Construction of such F' is performed in [26]. In [26] also failed-literal elimination is discussed, i.e., the case $\mathrm{r}_2(F) = \{\perp\}$ (see [14,17]), where $\mathrm{r}_2 : \mathcal{CLS} \to \mathcal{CLS}$ is defined as $\mathrm{r}_2(F) := \mathrm{r}_2(\langle x \to 1\rangle * F)$ for a literal x with $\mathrm{r}_1(\langle x \to 0\rangle * F) = \{\perp\}$, while otherwise $\mathrm{r}_2(F) := F$.

Example 17. The following examples show that r_2 and sDP-reduction are incomparable regarding derivation of a contradiction:

1. $\mathcal{F}_2 \in \mathcal{UHIT}'_{\delta=2}$ (recall Example 3) has $\mathrm{r}_2(\mathcal{F}_2) = \{\perp\}$.
2. $F := \{\{a, b, c\}, \{a, b, \overline{c}\}, \{a, \overline{b}, d\}, \{a, \overline{b}, \overline{d}\}, \{\overline{a}, e, f\}, \{\overline{a}, e, \overline{f}\}, \{\overline{a}, \overline{e}, g\}, \{\overline{a}, \overline{e}, \overline{g}\}\}$ fulfils $F \in \mathcal{UHIT}_{\delta=1}$, while $\mathrm{r}_2(F) = F$ (all clauses of F have length 3).

4. Confluence of singular DP-reduction

In this section we introduce the question of confluence of singular DP-reduction. In Section 4.1 we define "confluence" and "confluence modulo isomorphism", and discuss basic examples. In Section 4.2 we obtain our first major result, namely confluence for $\mathcal{SMU}$ (Theorem 23).

4.1. The question of confluence

Definition 18. Let $\boldsymbol{\mathcal{CFMU}}$ be the set of $F \in \mathcal{MU}$ where singular DP-reduction is confluent, and let $\boldsymbol{\mathcal{CFIMU}}$ be the set of $F \in \mathcal{MU}$ where singular DP-reduction is confluent modulo isomorphism:

$$\mathcal{CFMU} := \{F \in \mathcal{MU} \mid |\mathrm{sDP}(F)| = 1\}$$
$$\mathcal{CFIMU} := \{F \in \mathcal{MU} \mid \forall F', F'' \in \mathrm{sDP}(F) : F' \cong F''\}.$$

Example 19. Examples illustrating $\mathcal{CFMU} \subset \mathcal{CFIMU} \subset \mathcal{MU}$:

1. In [4] it is shown that every $F \in \mathcal{MU}_{\delta=1}$ contains a 1-singular variable (see [15,23] for further generalisations). Thus by Corollary 10 we get that singular DP-reduction on $\mathcal{MU}_{\delta=1}$ must end in $\{\{\bot\}\}$, and we have $\mathcal{MU}'_{\delta=1} = \{\{\bot\}\}$. It follows $\mathcal{MU}_{\delta=1} \subseteq \mathcal{CFMU}$.
2. We now show $\mathcal{MU}_{\delta=2} \not\subseteq \mathcal{CFMU}$. Let $F \in \mathcal{MU}_{\delta=2}$ be obtained from $\mathcal{F}_2$ (recall Example 3) by "inverse singular DP-reduction", adding a new singular variable v and replacing the two clause $\{v_1, v_2\}, \{\overline{v_2}, v_1\} \in \mathcal{F}_2$ by the three clauses $\{v, v_1\}, \{\overline{v}, v_2\}, \{\overline{v}, \overline{v_2}\}$, obtaining F (the other two clauses in F are $\{\overline{v_1}, v_2\}, \{\overline{v_1}, \overline{v_2}\}$):

   $$F = \{\, \{v, v_1\}, \{\overline{v}, v_2\}, \{\overline{v}, \overline{v_2}\}, \{\overline{v_1}, v_2\}, \{\overline{v_1}, \overline{v_2}\} \,\}.$$

   Singular DP-reduction on v yields $\mathcal{F}_2$ (and thus by Lemma 9 we get indeed $F \in \mathcal{MU}_{\delta=2}$). The second singular variable of F is v_1, and sDP-reduction on v_1 yields $F' := \{\{v, v_2\}, \{v, \overline{v_2}\}, \{\overline{v}, v_2\}, \{\overline{v}, \overline{v_2}\}\}$, where $F' \neq \mathcal{F}_2$. Note however that we have $F' \cong \mathcal{F}_2$ (since F' consists of all binary clauses over the variables v, v_2), and in Theorem 74 we will indeed see that we have $\mathcal{MU}_{\delta=2} \subseteq \mathcal{CFIMU}$.
3. We show $\mathcal{MU}_{\delta=3} \not\subseteq \mathcal{CFIMU}$ by constructing $F \in \mathcal{MU}_{\delta=3}$ with $\mathrm{sDP}(F) = \{F_1, F_2\}$ where $F_1 \not\cong F_2$. Let $G_1 := \mathcal{F}_2$, and let G_2 be the variable-disjoint copy of G_1 obtained by replacing variables v_1, v_2 with v'_1, v'_2. Let w be a new variable, and obtain F_1 by "full gluing" of G_1, G_2 on w, that is, add literal w to all clauses of G_1, add literal $\overline{w}$ to all clauses of G_2, and let F_1 be the union of these two clause-sets:

   $$F_1 = \{\, \{w, v_1, v_2\}, \{w, v_1, \overline{v_2}\}, \{w, \overline{v_1}, v_2\}, \{w, \overline{v_1}, \overline{v_2}\}, \{\overline{w}, v'_1, v'_2\}, \{\overline{w}, v'_1, \overline{v'_2}\}, \{\overline{w}, \overline{v'_1}, v'_2\}, \{\overline{w}, \overline{v'_1}, \overline{v'_2}\} \,\}.$$

   We have $F_1 \in \mathcal{UHIT}'_{\delta=3}$. We obtain F from F_1 by inverse singular DP-reduction, adding a new (singular) variable v, and replacing the two clauses $\{w, v_1, v_2\}, \{w, v_1, \overline{v_2}\}$ by the three clauses $\{v, w, v_1\}, \{\overline{v}, v_2\}, \{\overline{v}, w, \overline{v_2}\}$:

   $$F = \{\, \{v, w, v_1\}, \{\overline{v}, v_2\}, \{\overline{v}, w, \overline{v_2}\},\ \{w, \overline{v_1}, v_2\}, \{w, \overline{v_1}, \overline{v_2}\}, \{\overline{w}, v'_1, v'_2\}, \{\overline{w}, v'_1, \overline{v'_2}\}, \{\overline{w}, \overline{v'_1}, v'_2\}, \{\overline{w}, \overline{v'_1}, \overline{v'_2}\} \,\}.$$

   Singular DP-reduction on v yields F_1, and thus $F \in \mathcal{MU}_{\delta=3}$. The second singular variable of F is v_1, and sDP-reduction on v_1 yields a clause-set F_2 containing one binary clause (since we left out w in the replacement-clause $\{\overline{v}, v_2\}$). Since all clauses in F_1 have length 3, we see $F_2 \not\cong F_1$.

4.2. Confluence on saturated $\mathcal{MU}$

Definition 20. For clause-sets F, G we write $\boldsymbol{F \subseteq^{\mapsto} G}$ if for all $C \in F$ there is $D \in G$ with $C \subseteq D$.

If $F \subseteq^{\mapsto} G$, then we say that "F is a subset of G mod(ulo) supersets". $\subseteq^{\mapsto}$ is a quasi-order on arbitrary clause-sets and a partial order on subsumption-free clause-sets, and thus $\subseteq^{\mapsto}$ is a partial order on $\mathcal{MU}$. The minimal element of $\subseteq^{\mapsto}$ on $\mathcal{CLS}$ is $\top$, the minimal element on $\mathcal{MU}$ is $\{\bot\}$. Now we show that "nonsingular saturated patterns" are not destroyed by singular DP-reduction:

Lemma 21. *Consider $F_0, F, F' \in \mathcal{MU}$ with $F \xrightarrow{sDP}_* F'$.*

1. *If F_0 is nonsingular, then $F_0 \subseteq^{\mapsto} F \Rightarrow F_0 \subseteq^{\mapsto} F'$.*
2. *If $F_0, F, F' \in \mathcal{SMU}$, then $F_0 \subseteq^{\mapsto} F' \Rightarrow F_0 \subseteq^{\mapsto} F$.*

Proof. W.l.o.g. we can assume for both parts that $F' = \mathrm{DP}_v(F)$ for a singular variable v of F. Part 1 follows from the facts that $v \notin \mathrm{var}(F_0)$ due to the nonsingularity of F_0, and that due to the minimal unsatisfiability of F no clause gets lost by an application of singular DP-reduction. For Part 2 assume $\mathrm{ld}_F(v) = 1$. Then the assertion follows from the fact, that due to the saturatedness of F we have for the clause $C \in F$ with $v \in C$ and for every clause $D \in F$ with $\overline{v} \in D$ that $C \setminus \{v\} \subseteq D \setminus \{\overline{v}\}$. □

Example 22. Illustrating the conditions of Lemma 21:

1. An example showing that in Part 1 nonsingularity of F_0 is needed, is given trivially by $F = F_0 = \{\{v\}, \{\overline{v}\}\}$.

2. While an example for Part 2 with $F \in \mathcal{MU} \setminus \mathcal{SMU}$ and $F_0 \not\subseteq^{\mapsto} F$ is given by $F_0 = F' = \mathcal{F}_3$ (recall Example 3) and

$$F = \{\{\overline{v_1}, \overline{v_2}, \overline{v_3}\}, \{v_1, v_2, v\}, \{\overline{v}, v_3\}, \{\overline{v_1}, v_2\}, \{\overline{v_2}, v_3\}, \{\overline{v_3}, v_1\}\}.$$

Theorem 23. *$\mathcal{SMU} \subset \mathcal{CFMU}$.*

Proof. Consider $F \in \mathcal{SMU}$ and two nonsingular $F', F'' \in \mathcal{SMU}$ with $F \xrightarrow{\text{sDP}}_* F'$ and $F \xrightarrow{\text{sDP}}_* F''$. From $F' \subseteq^{\mapsto} F'$ and $F \xrightarrow{\text{sDP}}_* F'$ by Lemma 21, Part 2 we get $F' \subseteq^{\mapsto} F$, and then by Part 1 we get $F' \subseteq^{\mapsto} F''$; in the same way we obtain $F'' \subseteq^{\mapsto} F'$ and thus $F' = F''$. □

5. Permutations of sequences of DP-reductions

This section contains central technical results on (iterated) singular DP-reduction. The basic observations are collected in Section 5.1, studying how literal degrees change under sDP-reductions. It follows an interlude on iterated general DP-reduction in Section 5.2, stating "commutativity modulo subsumption" and deriving the basic fact in Corollary 30, that in case a sequence of DP-reductions as well as some permutation both yield minimally unsatisfiable clause-sets, then actually these MUs are the same. In Section 5.3 then conclusions for singular DP-reductions are drawn, obtaining various conditions under which sDP-reductions can be permuted without changing the final result. A good overview on all possible sDP-reductions is obtained in Section 5.4 in case no 1-singular variables are present. In Section 5.5 we introduce the "singularity index", the minimal length of a maximal sDP-reduction sequence. Our second major result is Theorem 63, showing that in fact all maximal sDP-reduction-sequences have the same length.

5.1. Monitoring literal degrees under singular DP-reductions

First we analyse the changes for literal-degrees after one step of sDP-reduction.

Lemma 24. *Consider $F \in \mathcal{MU}$ and an m-singular variable v ($m \in \mathbb{N}$). Let C be the main clause, and let $D_1, \ldots, D_m$ be the side clauses; and let $F' := \mathrm{DP}_v(F)$. Consider a literal $x \in \mathcal{LIT}$; the task is to compare $\mathrm{ld}_F(x)$ and $\mathrm{ld}_{F'}(x)$.*

1. *If $\mathrm{var}(x) \neq v$ and $x \notin C$, then $\mathrm{ld}_{F'}(x) = \mathrm{ld}_F(x)$.*
2. *If $\mathrm{var}(x) = v$, then $\mathrm{ld}_F(x) + \mathrm{ld}_F(\overline{x}) = m + 1$, while $\mathrm{ld}_{F'}(x) = \mathrm{ld}_{F'}(\overline{x}) = 0$.*
   For the remaining items we assume $\mathrm{var}(x) \neq v$ and $x \in C$.
   Let $p := |\{i \in \{1, \ldots, m\} : x \notin D_i\}| \in \{0, \ldots, m\}$.
3. *$\mathrm{ld}_{F'}(x) = \mathrm{ld}_F(x) - 1 + p$.*
4. *$\max(m, \mathrm{ld}_F(x) - 1) \le \mathrm{ld}_{F'}(x) \le \mathrm{ld}_F(x) - 1 + m$.*
5. *If $m = 1$, then $\mathrm{ld}_{F'}(x) \le \mathrm{ld}_F(x)$.*
6. *We have $\mathrm{ld}_{F'}(x) > \mathrm{ld}_F(x)$ iff $p \ge 2$.*
7. *The following conditions are equivalent:*
   (a) *$\mathrm{ld}_{F'}(x) = \mathrm{ld}_F(x) - 1$.*
   (b) *$\mathrm{ld}_{F'}(x) < \mathrm{ld}_F(x)$.*
   (c) *$x \in C \cap D_1 \cap \cdots \cap D_m$.*
   (d) *$\mathrm{var}(x) \in \mathrm{var}(C) \cap \mathrm{var}(D_1) \cap \cdots \cap \mathrm{var}(D_m)$.*
8. *If $\mathrm{ld}_{F'}(x) < \mathrm{ld}_F(x)$, then $\mathrm{ld}_{F'}(x) = \mathrm{ld}_F(x) - 1 \ge m$.*

Proof. Parts 1–3 follow by definition, Parts 4, 6 follows by Part 3, Part 5 follows by Part 4. Part 7 follows by Parts 1–4 and the observation, that if $x \in C$, then $\overline{x} \notin D_1 \cup \cdots \cup D_m$ (due to $F \in \mathcal{MU}$). Part 8 follows by Part 7. □

We get that singular variables can only be created for 1-singular DP-reduction, while singular variables can only be destroyed for non-1-singular DP-reductions; the details are as follows:

Corollary 25. *Consider $F \in \mathcal{MU}$ and an m-singular variable v for F ($m \in \mathbb{N}$), and let $F' := \mathrm{DP}_v(F)$.*

1. (a) *If $m \ge 2$, then $\mathrm{var}_s(F') \subseteq \mathrm{var}_s(F)$ with $\mathrm{var}_{1s}(F') \subseteq \mathrm{var}_{1s}(F)$.*
   (b) *If $m = 1$ then $\mathrm{var}_{1s}(F') \setminus \mathrm{var}_{1s}(F) \subseteq \mathrm{var}_{\neg 1s}(F)$.*
2. (a) *If $m = 1$, then $\mathrm{var}_s(F) \setminus \{v\} \subseteq \mathrm{var}_s(F')$ with $\mathrm{var}_{1s}(F) \setminus \{v\} \subseteq \mathrm{var}_{1s}(F')$.*
   (b) *If $m \ge 2$ then $\mathrm{var}_s(F) \setminus \mathrm{var}_s(F') \subseteq \mathrm{var}_{\neg 1s}(F)$.*

Proof. We use Lemma 24 as follows: Part 1(a) follows by Part 8 (if a literal degree decreased, then it is at least $m \ge 2$). Part 1(b) follows by Part 7 (it cannot be that the degrees of a literal and its complement decrease at the same time). Part 2(a) follows by Part 5. And Part 2(b) follows by Part 6 (it cannot be that the degrees of a literal and its complement increase at the same time). □

For saturated clause-sets, singular DP-reduction cannot increase literal degrees:

Corollary 26. *Consider $F \in \mathcal{SMU}$ and a singular variable v; let $F' := \mathrm{DP}_v(F)$.*

1. *For all literals x holds $\mathrm{ld}_{F'}(x) \le \mathrm{ld}_F(x)$.*
2. *Thus if $w \neq v$ is a singular variable for F, then w is also singular for F'.*

Proof. By Lemma 12 we get in the situation of Lemma 24, Part 6, that $p = 0$ must hold. □

5.2. Iterated DP-reduction

Definition 27. Consider $F \in \mathcal{CLS}$ and a sequence $v_1, \ldots, v_n$ of variables for $n \in \mathbb{N}_0$. Then

$$\mathbf{DP}_{v_1,\ldots,v_n}(F) := \begin{cases} F & \text{if } n = 0 \\ \mathrm{DP}_{v_n}(\mathrm{DP}_{v_1,\ldots,v_{n-1}}(F)) & \text{if } n > 0. \end{cases}$$

Thus in "$\mathrm{DP}_{v_1,\ldots,v_n}$" DP-reduction is performed in order $v_1, \ldots, v_n$. We have $\mathrm{var}(\mathrm{DP}_{v_1,\ldots,v_n}(F)) \subseteq \mathrm{var}(F) \setminus \{v_1, \ldots, v_n\}$. In [20] (Lemma 7.4, page 33) as well as in [21] (Lemma 7.6, page 27) the following fundamental result on iterated DP-reduction is shown: If performing subsumption-elimination at the end, then iterated DP-reduction does not depend on the order of the variables, while additionally performing subsumption-elimination inbetween has no influence. More precisely:

Lemma 28. *Let $\mathrm{r_S} : \mathcal{CLS} \to \mathcal{CLS}$ be subsumption-elimination, that is, $\mathrm{r_S}(F)$ is the set of $C \in F$ which are minimal in F w.r.t. the subset-relation. And for $n \in \mathbb{N}_0$ let S_n be the set of permutations of $\{1, \ldots, n\}$. Then we have the following operator-equalities for all variable-sequences $v_1, \ldots, v_n \in \mathcal{VA}$ ($n \in \mathbb{N}_0$):*

1. $\mathrm{r_S} \circ \mathrm{DP}_{v_1,\ldots,v_n} = \mathrm{r_S} \circ \mathrm{DP}_{v_1,\ldots,v_n} \circ \mathrm{r_S}$.
2. *For all $\pi \in S_n$ we have* $\mathrm{r_S} \circ \mathrm{DP}_{v_1,\ldots,v_n} = \mathrm{r_S} \circ \mathrm{DP}_{v_{\pi(1)},\ldots,v_{\pi(n)}}$.

Definition 29. Consider $F \in \mathcal{CLS}$ and $v_1, \ldots, v_n \in \mathcal{VA}$ ($n \in \mathbb{N}_0$). Then a permutation $\pi \in S_n$ is called **equality-preserving** for F and $v_1, \ldots, v_n$ (for short: "eq-preserving"), if we have $\mathrm{DP}_{v_1,\ldots,v_n}(F) = \mathrm{DP}_{\pi(v_1),\ldots,\pi(v_n)}(F)$. The set of all eq-preserving $\pi \in S_n$ is denoted by $\mathbf{eqp(F, (v_1, \ldots, v_n))} \subseteq S_n$.

Note that if $\mathrm{var}(F) \subseteq \{v_1, \ldots, v_n\}$, i.e., all variables are included in the DP-reduction, then $\mathrm{eqp}(F, (v_1, \ldots, v_n)) = S_n$ (every DP-reduction for these variables will end in either $\top$, if $F \in \mathcal{SAT}$, or else $\{\bot\}$). We can now show the fundamental characterisation of equality-preserving permutations in case of minimal unsatisfiability:

Corollary 30. *Consider $F \in \mathcal{CLS}$ and variables $v_1, \ldots, v_n$ ($n \in \mathbb{N}_0$) such that $\mathrm{DP}_{v_1,\ldots,v_n}(F) \in \mathcal{MU}$. Then we have for $\pi \in S_n$:*

$$\pi \in \mathrm{eqp}(F, (v_1, \ldots, v_n)) \Leftrightarrow \mathrm{DP}_{v_{\pi(1)},\ldots,v_{\pi(n)}}(F) \in \mathcal{MU}.$$

Proof. The direction from left to right follows by assumption. For the direction from right to left let $F' := \mathrm{DP}_{v_1,\ldots,v_n}(F)$ and $F'' := \mathrm{DP}_{v_{\pi(1)},\ldots,v_{\pi(n)}}(F)$. Assume that $F'' \in \mathcal{MU}$ holds. By Lemma 28 we have $\mathrm{r_S}(F') = \mathrm{r_S}(F'')$, and thus $F' = F''$, since minimally unsatisfiable clause-sets do not contain subsumptions. □

Since hitting clause-sets do not contain subsumptions, by Lemma 4 we obtain:

Corollary 31. *For clause-sets $F \in \mathcal{HIT}$ and variables $v_1, \ldots, v_n$ ($n \in \mathbb{N}_0$) we have* $\mathrm{eqp}(F, (v_1, \ldots, v_n)) = S_n$.

5.3. Iterated sDP-reduction via singular tuples

Generalising Definition 6 we consider "singular tuples":

Definition 32. Consider $F \in \mathcal{MU}$. A tuple $\boldsymbol{v} = (v_1, \ldots, v_n)$ of variables ($n \in \mathbb{N}_0$) is called **singular** for F if for all $i \in \{1, \ldots, n\}$ we have that v_i is singular for $\mathrm{DP}_{v_1,\ldots,v_{i-1}}(F)$. Note that for a singular tuple $(v_1, \ldots, v_n)$ all variables must be different. We call variable v_i ($i \in \{1, \ldots, n\}$) m**-singular** ($m \in \mathbb{N}$) for $\boldsymbol{v}$ and F, if v_i is m-singular for $\mathrm{DP}_{v_1,\ldots,v_{i-1}}(F)$. And the **singularity-degree tuple** of $\boldsymbol{v}$ w.r.t. F is the tuple $(m_1, \ldots, m_n)$ of natural numbers such that v_i is m_i-singular for $\boldsymbol{v}$ and F.

Example 33. Consider $F := \{\{a\}, \{\overline{a}, b\}, \{\overline{a}, \overline{b}\}\}$ (recall Example 7). There are 5 singular tuples for F, namely $()$, (a), (b), (a, b), (b, a). Considering $\boldsymbol{v} := (a, b)$, variable a is 2-singular for $\boldsymbol{v}$ and F, and b is 1-singular for $\boldsymbol{v}$ and F, and thus its singularity-degree sequence is $(2, 1)$, while considering $\boldsymbol{v}' := (b, a)$, both a and b are 1-singular for $\boldsymbol{v}'$ and F, and thus the singularity-degree sequence is $(1, 1)$.

For the understanding of sDP-reduction of $F \in \mathcal{MU}$, understanding the set of singular tuples for F is an important task. Two basic properties are:

1. F has only the empty singular tuple iff F is nonsingular.
2. If $(v_1, \ldots, v_n)$ is a singular tuple for F, then for all $i \in \{0, \ldots, n\}$ the tuple $(v_1, \ldots, v_i)$ is also singular for F.

Definition 34. Consider $F \in \mathcal{MU}$ and a singular tuple $(v_1, \ldots, v_n)$ for F. A permutation $\pi \in S_n$ is called **singularity-preserving** for F and $(v_1, \ldots, v_n)$ (for short: "s-preserving"), if also $(v_{\pi(1)}, \ldots, v_{\pi(n)})$ is singular for F. The set of all s-preserving $\pi \in S_n$ is denoted by $\mathbf{sp(F, (v_1, \ldots, v_n))} \subseteq S_n$.

By Corollary 30 we obtain the fundamental lemma, showing that singularity-preservation implies equality-preservation:

Lemma 35. *For $F \in \mathcal{MU}$ and a singular tuple $\boldsymbol{v}$ we have* $\mathrm{sp}(F, \boldsymbol{v}) \subseteq \mathrm{eqp}(F, \boldsymbol{v})$.

Thus singular tuples with the same variables yield the same reduction-result:

Corollary 36. *Consider two singular tuples $(v_1, \dots, v_n), (v'_1, \dots, v'_n)$ for $F \in \mathcal{MU}$. If $\{v_1, \dots, v_n\} = \{v'_1, \dots, v'_n\}$, then $\mathrm{DP}_{v_1,\dots,v_n}(F) = \mathrm{DP}_{v'_1,\dots,v'_n}(F)$.*

Preparing our results on singularity-preserving permutations, we consider first "homogeneous" singular pairs in the following two (easy) lemmas.

Lemma 37. *Consider $F \in \mathcal{MU}$ and two different non-1-singular variables v, w for F. Let C be the main clause for v, and let D be the main clause for w. There are precisely two cases now:*

1. *If $C = D$, then $w \notin \mathrm{var}_s(\mathrm{DP}_v(F))$ and $v \notin \mathrm{var}_s(\mathrm{DP}_w(F))$.*
2. *If $C \neq D$, then $w \in \mathrm{var}_{\neg 1s}(\mathrm{DP}_v(F))$ and $v \in \mathrm{var}_{\neg 1s}(\mathrm{DP}_w(F))$.*

Proof. Part 1 follows by Lemma 24, Part 4, and Part 2 follows by Part 8 of that lemma (to see that the complements occur at least twice after the DP-reductions). □

Example 38. We illustrate the two cases of Lemma 37:

1. Let $F := \{\{v, w\}, \{\overline{v}, a\}, \{\overline{v}, \overline{a}\}, \{\overline{w}, b\}, \{\overline{w}, \overline{b}\}\} \in \mathcal{MU}_{\delta=1}$. Then $C = D = \{v, w\}$, and w is not singular in $\mathrm{DP}_v(F) = \{\{w, a\}, \{w, \overline{a}\}, \{\overline{w}, b\}, \{\overline{w}, \overline{b}\}\}$, and v is not singular in $\mathrm{DP}_w(F) = \{\{\overline{v}, a\}, \{\overline{v}, \overline{a}\}, \{v, b\}, \{v, \overline{b}\}\}$.
2. Let $F := \{\{v, a\}, \{w, \overline{a}\}, \{\overline{v}, a, b\}, \{\overline{v}, a, \overline{b}\}, \{\overline{w}, \overline{a}, b\}, \{\overline{w}, \overline{a}, \overline{b}\}\} \in \mathcal{SMU}_{\delta=2}$. Then $C = \{v, a\} \neq D = \{w, \overline{a}\}$, and now w is singular in $\mathrm{DP}_v(F) = \{\{w, \overline{a}\}, \{a, b\}, \{a, \overline{b}\}, \{\overline{w}, \overline{a}, b\}, \{\overline{w}, \overline{a}, \overline{b}\}\}$, and v is singular in $\mathrm{DP}_w(F) = \{\{v, a\}, \{\overline{v}, a, b\}, \{\overline{v}, a, \overline{b}\}, \{\overline{a}, b\}, \{\overline{a}, \overline{b}\}\}$.

Lemma 39. *Consider $F \in \mathcal{MU}$ and a singular tuple (v, w) for F with singularity-degree tuple $(1, 1)$. Let $C, D \in F$ be the two occurrences of v.*

1. *Assume w is not 1-singular in F:*
   (a) *Then w is 2-singular in F. Let $E_0 \in F$ be the main-clause of w, and let $E_1, E_2 \in F$ be the two side-clauses.*
   (b) *We have $\{E_1, E_2\} = \{C, D\}$.*
   (c) *So v is 1-singular in $\mathrm{DP}_w(F)$.*
   (d) *Thus (w, v) is a singular tuple with singularity-degree tuple $(2, 1)$.*
2. *Otherwise w is 1-singular in F.*
   (a) *v is 1-singular in $\mathrm{DP}_w(F)$.*
   (b) *Thus (w, v) is a singular tuple with singularity-degree tuple $(1, 1)$.*
   (c) *Let E_1, E_2 be the two occurrences of w in F: $|\{C, D\} \cap \{E_1, E_2\}| \leq 1$.*

Proof. For Part 1 we use Lemma 24, Part 7, and we see that w is 2-singular for F (sDP-reduction can only reduce literal-degrees by one), and the complement of the singular literal of w must occur in all occurrences of variable v; we also see that DP-reduction on w does not change the degree of v. For Part 2 we use Corollary 25, Part 2(a), together we the fact that the occurrences of two 1-singular variables cannot completely coincide, since then we had more than one clash between the main clause and the side clause (see Lemma 9, Part 3(a)). □

Example 40. We illustrate the two cases of Lemma 39:

1. Let $F := \{\{v, w\}, \{\overline{v}, w\}, \{\overline{w}\}\} \in \mathcal{SMU}_{\delta=1}$, with $\mathrm{DP}_v(F) = \{\{w\}, \{\overline{w}\}\}$. Then $\{C, D\} = \{\{v, w\}, \{\overline{v}, w\}\}$, and $E_0 = \{\overline{w}\}$ and $\{E_1, E_2\} = \{C, D\}$, where $\mathrm{DP}_w(F) = \{\{v\}, \{\overline{v}\}\}$.
2. We give examples for both cases of $|\{C, D\} \cap \{E_1, E_2\}| \in \{0, 1\}$:
   (a) Let $F := \{\{v, a\}, \{\overline{v}, a\}, \{w, \overline{a}\}, \{\overline{w}, \overline{a}\}\} \in \mathcal{SMU}_{\delta=1}$. Then $\mathrm{DP}_v(F) = \{\{a\}, \{w, \overline{a}\}, \{\overline{w}, \overline{a}\}\}$ and $\mathrm{DP}_w(F) = \{\{v, a\}, \{\overline{v}, a\}, \{\overline{a}\}\}$, where $\{C, D\} = \{\{v, a\}, \{\overline{v}, a\}\}$ and $\{E_1, E_2\} = \{\{w, \overline{a}\}, \{\overline{w}, \overline{a}\}\}$.
   (b) Let $F := \{\{v\}, \{\overline{v}, w\}, \{\overline{w}\}\} \in \mathcal{MU}_{\delta=1}$. Then $\mathrm{DP}_v(F) = \{\{w\}, \{\overline{w}\}\}$ and $\mathrm{DP}_w(F) = \{\{v\}, \{\overline{v}\}\}$, where $\{C, D\} = \{\{v\}, \{\overline{v}, w\}\}$ and $\{E_1, E_2\} = \{\{\overline{v}, w\}, \{\overline{w}\}\}$.

Now we are ready to show the central "exchange theorem", characterising s-preserving neighbour-exchanges (recall that every permutation is a composition of neighbour-exchanges): The gist of Theorem 41 is that in most cases neighbours in a singular tuple can be exchanged safely (i.e., s-preserving), except of the cases where a 1-singular DP-reduction is followed by a non-1-singular DP-reduction (Case 3(b)).

Theorem 41. *Consider $F \in \mathcal{MU}$ and a singular tuple $\boldsymbol{v} = (v_1, \dots, v_n)$ with $n \geq 2$, and let $(m_1, \dots, m_n)$ be the singularity-degree tuple of $\boldsymbol{v}$ w.r.t. F. Consider $i \in \{1, \dots, n-1\}$, and let $\pi \in S_n$ be the neighbour-exchange $i \leftrightarrow i+1$ (i.e., $\pi(j) = j$ for $j \in \{1, \dots, n\} \setminus \{i, i+1\}$, while $\pi(i) = i+1$ and $\pi(i+1) = i$). Let $(m'_1, \dots, m'_n)$ be the singularity-degree tuple of $\boldsymbol{v}'$ w.r.t. F, where $\boldsymbol{v}' := (v_{\pi(1)}, \dots, v_{\pi(n)})$, in case of $\pi \in \mathrm{sp}(F, \boldsymbol{v})$. The task is to characterise when $\pi \in \mathrm{sp}(F, \boldsymbol{v})$ holds; we also need to be able to apply such s-preserving neighbour-exchanges consecutively, by controlling the changes in the singularity-degrees.*

1. *If $\pi \in \mathrm{sp}(F, \boldsymbol{v})$, then for $j \in \{1, \dots, n\} \setminus \{i, i+1\}$ we have $m'_j = m_j$.*
2. *Assume $m_i \geq 2$.*
   (a) *$\pi \in \mathrm{sp}(F, \boldsymbol{v})$.*
   (b) *$m'_i \leq m_{i+1} + 1$.*
   (c) *$m'_{i+1} \geq m_i - 1$.*

   (d) *If* $m_{i+1} = 1$, *then* $m'_i = 1$.
   (e) *If* $m_{i+1} \geq 2$, *then* $m'_{i+1} \geq 2$.
3. *Assume* $m_i = 1$.
   (a) *Assume* $m_{i+1} = 1$.
      i. $\pi \in \mathrm{sp}(F, \boldsymbol{v})$.
      ii. $m'_{i+1} = 1$ *and* $m'_i \in \{1, 2\}$.
   (b) *Assume* $m_{i+1} \geq 2$.
      i. $\pi \in \mathrm{sp}(F, \boldsymbol{v})$ *if and only if* v_{i+1} *is singular in* $\mathrm{DP}_{v_1,\ldots,v_{i-1}}(F)$.
      ii. *If* $\pi \in \mathrm{sp}(F, \boldsymbol{v})$, *then* $m'_i \geq 2$.

Proof. Part 1 follows by Lemma 35. For the remainder let $F_0 := F$, and $F_i := \mathrm{DP}_{v_i}(F_{i-1})$ for $i \in \{1, \ldots, n\}$.

Now consider Part 2; so we assume $m_i \geq 2$ here. For Part 2(a) we need to show that v_{i+1} is singular for F_i and v_i is singular for $\mathrm{DP}_{v_{i+1}}(F_i)$: The former follows by Corollary 25, Part 1(a), while the latter follows by Part 2(a) of that Corollary (if v_{i+1} is 1-singular for F_i) and by both parts of Lemma 37 (if v_{i+1} is non-1-singular for F_i; the main clauses for v_i, v_{i+1} in F_i cannot be the same).

Part 2(b), 2(c) follow by Part 7 of Lemma 24, while Part 2(d) follows by Part 8 of that Lemma. Now consider Part 2(e), and so we assume $m_{i+1} \geq 2$. If $m'_i \geq 2$ then $m'_{i+1} \geq 2$ follows from Part 1(a) of Corollary 25; it remains the case $m'_i = 1$. Let x be the singular literal of v_i in F_i, and let y be the singular literal of v_{i+1} in F_{i+1}. Since sDP-reduction by v_i in F_i increased the number of occurrences of $\overline{y}$, for the main clause C of v_i in F_i (thus $x \in C$) we must have $\overline{y} \in C$. Let D be the main clause of v_{i+1} in F_i, that is, $y \in D$ (note that C, D are the only occurrences of variable v_{i+1} in F_i). If $m'_{i+1} = 1$ would be the case, then we would have $\overline{x} \in C, D$ contradicting $x \in C$.

Finally consider Part 3, assuming $m_i = 1$. Part 3(a) follows with Lemma 39. For Part 3(b) assume $m_{i+1} \geq 2$. For Part 3(b)i the direction from left to right follows by definition, while the direction from right to left follows by Part 2(b) of Lemma 25. And Part 3(b)ii by Part 5 of Lemma 24. □

We remark that for Part 2(e) of Theorem 41, in the conference version we also asserted that $m'_i \geq 2$ would be the case (Lemma 26, Part 2, in [24]), which is false as shown in Example 43.

Corollary 42. *Consider* $F \in \mathcal{MU}$ *and a singular tuple* $\boldsymbol{v} = (v_1, \ldots, v_n)$ $(n \geq 2)$ *with* $1 \leq i < n$. *Then a sufficient condition for the neighbour exchange* $i \leftrightarrow i+1$ *to be s-preserving is:*

> v_i *is non-1-singular for* $\boldsymbol{v}$, *or* v_{i+1} *is 1-singular for* $\boldsymbol{v}$, *or* v_{i+1} *is singular for* $\mathrm{DP}_{v_1,\ldots,v_{i-1}}(F)$.

Proof. To show the sufficiency of the three conditions, use Parts 2(a) resp. 3(a)i resp. 3(b)i of Theorem 41 (in a cascading form). □

5.3.1. Examples

We now give various examples showing that the bounds from Theorem 41 are sharp in general. First we show that a swap of two non-1-singular variables can create a 1-singular variables.

Example 43. Consider $k \in \mathbb{N}$. The following $F \in \mathcal{MU}$ and $v, w \in \mathrm{var}(F)$ have the properties that (v, w) is a singular tuple with singularity-degree tuple (k, k) while (w, v) is a singular tuple with singularity-degree tuple $(1, k)$.

1. Let $F := \{ \{v, w\}, \{\overline{v}, x_1\}, \ldots, \{\overline{v}, x_k\}, \{\overline{w}\}, \{\overline{x_1}, \ldots, \overline{x_k}\} \} \in \mathcal{MU}_{\delta=1}$.
2. v is k-singular for F, while w is 1-singular for F.
3. We have $\mathrm{var}_s(F) = \mathrm{var}(F) = \{v, w, x_1, \ldots, x_k\}$ and $\mathrm{var}_{\neg 1s}(F) = \{v\}$.
4. Let $F' := \mathrm{DP}_v(F) = \{\{w, x_1\}, \ldots, \{w, x_k\}, \{\overline{w}\}, \{\overline{x_1}, \ldots, \overline{x_k}\}\}$.
5. Now w is k-singular for F', and thus the associated singularity-degree tuple for (v, w) and F is (k, k).
6. While the singular tuple (w, v) has singularity-degree tuple $(1, k)$.

Next we give examples showing that the bounds from Part 2 of Theorem 41 are sharp in general.

Example 44. All examples (again) are in $\mathcal{MU}_{\delta=1}$.

1. First we consider Part 2(b), showing that the two extreme cases $m'_i = 1$ and $m'_i = m_{i+1} + 1$ are possible.
   (a) Example 43 yields $m_i = m_{i+1} = k \geq 2$ and $m'_i = 1, m'_{i+1} = k$.
   (b) That is, the original pair (v_i, v_{i+1}) has singularity-degree tuple (k, k), while after swap we have $(1, k)$. In the sequel we will describe the examples in this manner.
   (c) For $k \in \mathbb{N}$ let $F_1 := \{\{v, \overline{w}\}, \{\overline{v}, \overline{w}, x_1\}, \ldots, \{\overline{v}, \overline{w}, x_k\}, \{\overline{x_1}, \ldots, \overline{x_k}\}, \{w\}\}$. Then for (v, w) we have (k, k), while for (w, v) we have $(k+1, k)$.
2. Now we consider Part 2(c), showing that $m'_{i+1} = m_i - 1 + p$ for all $p \in \mathbb{N}_0$ is possible.
   (a) For $p = 0$ we just re-use F_1, but in the other direction, from (w, v) with $(k+1, k)$ to (v, w) with (k, k).
   (b) Let $F_2 := \{\{v\}, \{\overline{v}, w, y\}, \{\overline{v}, \overline{y}\}, \{\overline{w}, x_1\}, \ldots, \{\overline{w}, x_p\}, \{\overline{x_1}, \ldots, \overline{x_p}\}\}$ for $p \geq 1$. For (v, w) we have $(2, p)$, while for (w, v) for have $(p, p+1)$.

3. Finally we consider Part 2(d), showing that $m'_{i+1} = k$ for all $k \in \mathbb{N}$ is possible.
   (a) For $k = 1$ consider $F_3 := \{\{v\}, \{\overline{v}, w\}, \{\overline{v}, \overline{w}\}\}$. For (v, w) we have $(2, 1)$, and for (w, v) we have $(1, 1)$.
   (b) Let $F_4 := \{\{v\}, \{\overline{v}, x_1\}, \dots, \{\overline{v}, x_k\}, \{w, \overline{x_1}, \dots, \overline{x_k}\}, \{\overline{w}, \overline{x_1}, \dots, \overline{x_k}\}\}$ for $k \geq 2$. For (v, w) we have $(k, 1)$, and for (w, v) we have $(1, k)$.

Finally we give examples showing that the bounds from Part 3 (the case $m_i = 1$) of Theorem 41 are sharp in general.

Example 45. All examples (again) are in $\mathcal{MU}_{\delta=1}$.

1. For Part 3(a) ($m_{i+1} = 1$), that is, the singularity-degree tuple $(1, 1)$, it is trivial that after swap we can have $(1, 1)$ again, while to obtain $(2, 1)$ consider F_3 from Example 44 in the other direction.
2. Consider Part 3(b) ($m_{i+1} \geq 2$).
   (a) An example showing that the swap can be impossible is given by $F := \{\{v, w\}, \{\overline{v}, w\}, \{\overline{w}, x_1\}, \dots, \{\overline{w}, x_k\}, \{\overline{x_1}, \dots, \overline{x_k}\}\}$ for $k \geq 2$: For (v, w) we have $(1, k)$, while (w, v) is not singular.
   (b) And to obtain swap-results $(1, k) \rightsquigarrow (k, k)$ we use Example 43, but in the other direction.

5.3.2. Applications

We first consider singular tuples where all permutations are also singular:

Definition 46. Consider $F \in \mathcal{MU}$ and a tuple $\boldsymbol{v} = (v_1, \dots, v_n)$ ($n \in \mathbb{N}_0$). $\boldsymbol{v}$ is called **totally singular** for F if $\boldsymbol{v}$ is singular for F with $\mathrm{sp}(F, (v_1, \dots, v_n)) = S_n$.

Corollary 47. *Consider $F \in \mathcal{MU}$ and a singular tuple $\boldsymbol{v} = (v_1, \dots, v_n)$ ($n \in \mathbb{N}_0$) such that each v_i is non-1-singular in F (i.e., $\{v_1, \dots, v_n\} \subseteq \mathrm{var}_{\neg 1s}(F)$). Then $\boldsymbol{v}$ is totally singular for F, and for each permutation $\boldsymbol{v}'$ every variable is non-1-singular for $\boldsymbol{v}$.*

Proof. With Part 2(a) of Theorem 41 and Part 1(a) of Corollary 25. □

We remark that in the conference version, that is Corollary 27 in [24], a more general version is stated, only assuming for $\boldsymbol{v}$ that every variable is not-1-singular for it (not, as in Corollary 47, already for F). We believe this more general statement is true, but the proof there is false. The more general version is not needed for any of the other results of [24] or the article at hand. Furthermore a false additional assertion is given in Corollary 27 in [24], namely that all permutation of $\boldsymbol{v}$ would also be non-1-singular, which is refuted by the following example.

Example 48. Consider $F := \{\{v, a\}, \{\overline{a}\}, \{\overline{v}, b\}, \{\overline{v}, \overline{b}\}\} \in \mathcal{MU}_{\delta=1}$. Then (v, a) has the property that all variables are non-1-singular for it, while (a) is 1-singular for F.

We mention another (simpler) case of total singularity (which already follows by Corollary 25, Part 2(a)):

Corollary 49. *Consider $F \in \mathcal{MU}$ and a singular tuple $\boldsymbol{v} = (v_1, \dots, v_n)$ such that $\{v_1, \dots, v_n\} \subseteq \mathrm{var}_{1s}(F)$. Then $\boldsymbol{v}$ is totally singular, and for each permutation $\boldsymbol{v}'$ of $\boldsymbol{v}$ each variable is 1-singular (for $\boldsymbol{v}'$).*

Proof. With Part 3(a)i of Theorem 41 and Part 2(a) of Corollary 25. □

Finally we get some normal form of a singular tuple $\boldsymbol{v}$ for $F \in \mathcal{MU}$ by moving the singular variables from F to the front, followed by further 1-singular DP-reductions, and concluded by non-1-singular DP-reductions:

Corollary 50. *Consider $F \in \mathcal{MU}$ and a singular tuple $\boldsymbol{v} = (v_1, \dots, v_n)$. Let $V := \{v_1, \dots, v_n\} \cap \mathrm{var}_{1s}(F)$ and $p := |V|$. Consider any $\pi_0 : \{1, \dots, p\} \to \{1, \dots, n\}$ such that $\{v_{\pi_0(i)} : i \in \{1, \dots, p\}\} = V$. Then there exists $q \in \{p, \dots, n\}$ and an s-preserving permutation π for $\boldsymbol{v}$ such that π extends π_0, and $v_{\pi(i)}$ is 1-singular for $(v_{\pi(1)}, \dots, v_{\pi(n)})$ and $i \in \{1, \dots, n\}$ if and only if $i \leq q$.*

Proof. The sorting of $\boldsymbol{v}$ is computed via singularity-preserving neighbour swaps, in four steps ("processes"). Process I establishes that in the associated singularity-degree tuple all entries equal to 1 appear in the front-part (the first q elements). This is achieved by noting that a neighbouring degree-pair $(\geq 2, 1)$ can be swapped and becomes $(1, \geq 1)$. Thus we can grow the 1-singular front part by every value 1 occurring not in it, and we obtain a permutation where all singularity-degrees of value 1 appear in the (consecutive) front-part (while the back-part has all singularity-degrees of values ≥ 2).

Process II now additionally moves variables in V occurring in the back-part to the front-part as follows: If there is still such a variable, then this cannot be the first place in the back-part, and so the variable can be moved one place to the left. Possibly process I has to applied after this step (if it does, then the front-part grows at least by one element). This process can be repeated and terminates once all of V is in the front part. Now the variables in the front part and especially q have been determined. In the remainder the front part is put into a suitable order.

Process III only considers the front part, and the task is to move all variables in V to its front. This is unproblematic, since 1-singular DP-reduction does not increase literal degrees. Finally process IV commutes the variables in V into the given order. □

Comparing two different singular tuples, they do not need to overlap, however they need to have a "commutable beginning" via appropriate permutations, given they contain at least two variables:

Lemma 51. *Consider $F \in \mathcal{MU}$ and singular tuples $(v_1, \ldots, v_p)$, $(w_1, \ldots, w_q)$ for F with $p, q \geq 2$. Then there is an s-preserving permutation π for $(v_1, \ldots, v_p)$ and an s-preserving permutation π' for $(w_1, \ldots, w_q)$, such that both $(v_{\pi(1)}, w_{\pi'(1)})$ and $(w_{\pi'(1)}, v_{\pi(1)})$ are singular for F.*

Proof. If one of the two tuples contains a 1-singular variable $v_i \in \mathrm{var}_{1s}(F)$ resp. $w_i \in \mathrm{var}_{1s}(F)$, then the assertion follows by Corollary 50 and Part 2 of Corollary 25. So assume that neither contains a 1-singular variable from F. Note that if none of the variables of a singular tuple is 1-singular for F, then all the variables in it must be singular for F, since new singular variables are only created by 1-singular DP-reduction according to Corollary 25, Part 1(a). Thus the assertion follows by Corollary 47 and Lemma 37. □

5.4. Without 1-singular variables

If $F \in \mathcal{MU}$ has no 1-singular variables, then we know its maximal singular tuples (singular tuples which cannot be extended), as we will show in Lemma 57, namely they are given by choosing exactly one singular literal from each clause which contains singular literals. In this context the concept of "singularity hypergraph" is useful, so that we can recognise such maximal singular tuples as minimal "transversals". Recall that a *hypergraph* G is a pair $G = (V, E)$, where V is a set, the elements called "vertices", while E is a set of subsets of V, the elements called "hyperedges"; the notations $V(G) := V$ and $E(G) := E$ are used.

Definition 52. For $F \in \mathcal{MU}$ we define the **singularity hypergraph $S(F)$** as follows:

- The vertex set is $\mathrm{var}(F)$ (the variables of F).
- For every $v \in \mathrm{var}_s(F)$ let x_v be the singular literal (which depends on the given choice in case v is 1-singular), and let $L := \{x_v : v \in \mathrm{var}_s(F)\}$.
- Now the hyperedges are given by $\mathrm{var}(C \cap L)$ for $C \in F$ with $C \cap L \neq \emptyset$.

I.e.,

$$S(F) := (\mathrm{var}(F), \{\mathrm{var}(C \cap L) : C \in F \wedge C \cap L \neq \emptyset\}).$$

Note that the hyperedges of $S(F)$ are non-empty and pairwise disjoint.

Example 53. Continuing Example 19:

1. For F as in Part 2 we have $S(F) = (\{v, v_1, v_2\}, \{\{v, v_1\}\})$.
2. For F as in Part 3 we have $S(F) = (\{v, w, v_1, v_2, v_1', v_2'\}, \{\{v, v_1\}\})$.

Example 54. With another inverse sDP-reduction, applied to F from Part 2 of Example 19 and introducing variable v', we obtain

$$F = \{\{v, v_1\}, \{\overline{v}, v_2\}, \{\overline{v}, \overline{v_2}\}, \{v', \overline{v_1}\}, \{\overline{v'}, v_2\}, \{\overline{v'}, \overline{v_2}\}\}.$$

We have $\mathrm{var}_s(F) = \{v_1, v, v'\}$ and $\mathrm{var}_{1s}(F) = \{v_1\}$. Choosing v_1 resp. $\overline{v_1}$ as the singular literal for v_1, we have $S(F) = (\{v, v', v_1, v_2\}, \{\{v, v_1\}, \{v'\}\})$ resp. $= (\{v, v', v_1, v_2\}, \{\{v\}, \{v', v_1\}\})$.

Example 55. Consider

$$F := \{\, \{a, b\}, \{\overline{a}, x, v\}, \{\overline{a}, y, v'\}, \{\overline{b}, x, v\}, \{\overline{b}, y, v'\}, \{\overline{x}, v\}, \{\overline{y}, v'\}, \{\overline{v}, \overline{v'}\} \,\}.$$

We have $S(F) = (\{a, b, x, y, v, v'\}, \{\{a, b\}, \{x\}, \{y\}, \{v, v'\}\})$. We have furthermore the properties $F \in \mathcal{MU}_{\delta=2} \setminus \mathcal{SMU}_{\delta=2}$ and $\mathrm{var}(F) = \mathrm{var}_{\neg 1s}(F)$.

Definition 56. Consider $F \in \mathcal{MU}$. A singular tuple $(v_1, \ldots, v_n)$ for F is called **maximal**, if there is no singular tuple extending it, i.e., $\mathrm{DP}_{v_1,\ldots,v_n}(F)$ is nonsingular.

Lemma 57. *Consider $F \in \mathcal{MU}$ with $\mathrm{var}_{1s}(F) = \emptyset$. The variable-sets of maximal singular tuples for F are precisely the minimal transversals of $S(F)$ (minimal sets of vertices intersecting every hyperedge). And the maximal singular tuples of F are precisely obtained as (arbitrary) linear orderings of these variable-sets.*

Proof. By Corollary 25, Part 1(a), for each singular tuple $(v_1, \ldots, v_n)$ of F we have $\{v_1, \ldots, v_n\} \subseteq \mathrm{var}_{\neg 1s}(F) = \mathrm{var}_s(F)$. So by Corollary 47 all permutations are singular. Finally, for $v \in \mathrm{var}_s(F)$ let $F_v := \mathrm{DP}_v(F)$, let $C_v \in F$ be the main clause of v, and let $H_v := \mathrm{var}(C_v) \cap \mathrm{var}_s(F)$. Then we have $S(F_v) = (V(S(F)) \setminus \{v\}, E(S(F)) \setminus \{H_v\})$. The assertion of the lemma follows now easily by induction. □

Example 58. Continuing Example 19 (and Example 53): For F as in Part 2 as well as in Part 3 the two maximal singular tuples are (v) and (v_1).

Example 59. Continuing Example 55: We have $2 \cdot 2 = 4$ minimal transversals, namely $\{a, x, y, v\}$, $\{b, x, y, v\}$, $\{a, x, y, v'\}$, $\{b, x, y, v'\}$. There are thus 4 elements in $\mathrm{sDP}(F)$; Theorem 74 will show that they are necessarily all isomorphic to $\mathcal{F}_2$ (since after reduction 2 variables remain; recall Example 3). Finally we remark that F has precisely $4 \cdot 4! = 96$ maximal singular tuples.

Since two different minimal transversals of $S(F)$ remove different variables, they result in different sDP-reduction results. So the elements of $\mathrm{sDP}(F)$ are here in bijective correspondence to the minimal transversals of F, and we get:

Corollary 60. *For* $F \in \mathcal{MU}$ *with* $\mathrm{var}_{1s}(F) = \emptyset$ *we have that* $|\mathrm{sDP}(F)|$ *is the number of minimal transversals of* $S(F)$.

5.5. The singularity index

Definition 61. Consider $F \in \mathcal{MU}$. The **singularity index** of F, denoted by $\mathbf{si}(F) \in \mathbb{N}_0$, is the minimal $n \in \mathbb{N}_0$ such that a maximal singular tuple of length n exists for F.

So $\mathrm{si}(F) = 0 \Leftrightarrow F \in \mathcal{MU}'$. See Corollary 69, Part 1, for a characterisation of $F \in \mathcal{MU}$ with $\mathrm{si}(F) = 1$. In Theorem 63 we will see that all maximal singular tuples are of the same length (given by the singularity index). As a first step towards this result we show a special case:

Lemma 62. *Consider* $F \in \mathcal{MU}$ *not having 1-singular variables (i.e.,* $\mathrm{var}_{1s}(F) = \emptyset$*). Then every maximal singular tuple has length* $\mathrm{si}(F)$*, which is the number of different clauses of* F *containing at least one singular literal.*

Proof. The assertion follows by Lemma 57 and the fact, that the hyperedges of the singularity hypergraph are pairwise disjoint. □

More general than Lemma 62 (but with less details), we show next that for all minimally unsatisfiable clause-sets all maximal singular tuples (i.e., maximal sDP-reduction sequences) have the same length. The basic idea is to utilise the good commutativity properties of 1-singular variables, so that induction on the singularity index can be used.

Theorem 63. *For* $F \in \mathcal{MU}$ *and every maximal singular tuple* $(v_1, \ldots, v_m)$ *for* F *we have* $m = \mathrm{si}(F)$.

Proof. We prove the assertion by induction on $\mathrm{si}(F)$. For $\mathrm{si}(F) = 0$ the assertion is trivial, so assume $\mathrm{si}(F) > 0$. If F has no 1-singular variables, then the assertion follows by Lemma 62, and so we assume that F has a 1-singular variable v. First we show that we can choose v such that $\mathrm{si}(\mathrm{DP}_v(F)) = n - 1$.

Consider a maximal singular tuple $(v_1, \ldots, v_n)$ of length $n = \mathrm{si}(F)$. Note that $\mathrm{si}(\mathrm{DP}_{v_1}(F)) = n - 1$. If v_1 is 1-singular, then we can use $v := v_1$ and we are done, and so assume v_1 is not 1-singular. The induction hypothesis, applied to $\mathrm{DP}_{v_1}(F)$, yields $\mathrm{si}(\mathrm{DP}_{v_1,v}(F)) = n - 2$. Now by Corollary 25, Part 2, both tuples (v_1, v) and (v, v_1) are singular for F, whence $\mathrm{DP}_{v_1,v}(F) = \mathrm{DP}_{v,v_1}(F)$ holds (Corollary 36), and so $\mathrm{si}(\mathrm{DP}_{v,v_1}(F)) = n - 2$. We obtain $\mathrm{si}(\mathrm{DP}_v(F)) \le n - 1$, and thus $\mathrm{si}(\mathrm{DP}_v(F)) = n - 1$ as claimed.

Now consider an arbitrary maximal singular tuple $(w_1, \ldots, w_m)$. It suffices to show that $\mathrm{si}(\mathrm{DP}_{w_1}(F)) \le n-1$, from which by induction hypothesis the assertion follows. The argument is now similar to above. The claim holds for $w_1 = v$, and so assume $w_1 \neq v$. By induction hypothesis we have $\mathrm{si}(\mathrm{DP}_{v,w_1}(F)) = n - 2$. By Corollary 25, Part 2, both tuples (v, w_1) and (w_1, v) are singular for F. Thus $\mathrm{si}(\mathrm{DP}_{w_1,v}(F)) = n - 2$. We obtain $\mathrm{si}(\mathrm{DP}_{w_1}(F)) \le n - 1$ as claimed. □

Corollary 64. *For* $F \in \mathcal{MU}$ *and* $F', F'' \in \mathrm{sDP}(F)$ *we have* $n(F') = n(F'')$.

6. Confluence modulo isomorphism on eventually $\mathcal{SMU}$

Finally we are able to show our third major result, confluence modulo isomorphism of singular DP-reduction in case all maximal sDP-reductions yield saturated clause-sets.

Definition 65. A minimally unsatisfiable clause-set F is called **eventually saturated**, if all nonsingular F' with $F \xrightarrow{\mathrm{sDP}}_* F'$ are saturated; the set of all eventually saturated clause-sets is $\boldsymbol{\mathcal{ESMU}} := \{F \in \mathcal{MU} : \mathrm{sDP}(F) \subseteq \mathcal{SMU}\}$.

By Corollary 13 we have $\mathcal{SMU} \subseteq \mathcal{ESMU}$. If $\mathcal{C} \subseteq \mathcal{MU}$ is stable under sDP-reduction, then we have $\mathcal{C} \subseteq \mathcal{ESMU}$ iff $\mathcal{C} \cap \mathcal{MU}' \subseteq \mathcal{SMU}$. In order to show $\mathcal{ESMU} \subseteq \mathcal{CFIMU}$ (recall Definition 18), we show first that "divergence in one step" is enough, that is, if we have a clause-set $F \in \mathcal{MU}$ such that sDP-reduction is not confluent modulo isomorphism, then we can obtain from F by sDP-reduction the clause-set $F' \in \mathcal{MU}$ with singularity index 1 (thus using $\mathrm{si}(F) - 1$ reduction steps) such that also for F' sDP-reduction is not confluent modulo isomorphism:

Lemma 66. *Consider* $F \in \mathcal{MU} \setminus \mathcal{CFIMU}$*. So* $\mathrm{si}(F) \ge 1$*. Then there is a singular tuple* $(v_1, \ldots, v_{\mathrm{si}(F)-1})$ *for* F*, such that for* $F' := \mathrm{DP}_{v_1,\ldots,v_{\mathrm{si}(F)-1}}(F)$ *we still have* $\mathrm{sDP}(F') \in \mathcal{MU} \setminus \mathcal{CFIMU}$ *(note* $\mathrm{si}(F') = 1$*).*

Proof. We prove the assertion by induction on $\mathrm{si}(F) \ge 1$. The assertion is trivial for $\mathrm{si}(F) = 1$, and so consider $n := \mathrm{si}(F) \ge 2$. If there is a singular variable $v \in \mathrm{var}_s(F)$ with $\mathrm{DP}_v(F) \in \mathcal{MU} \setminus \mathcal{CFIMU}$, then the assertion follows by induction hypothesis. So assume for the sake of contradiction, that for all singular variables v we have $\mathrm{DP}_v(F) \in \mathcal{CFIMU}$. Consider (maximal) singular tuples $(v_1, \ldots, v_n)$, $(w_1, \ldots, w_n)$ for F such that $\mathrm{DP}_{\boldsymbol{v}}(F)$ and $\mathrm{DP}_{\boldsymbol{w}}(F)$ are not isomorphic. By Lemma 51 w.l.o.g. we can assume that (v_1, w_1) and (w_1, v_1) are both singular for F, whence $\mathrm{DP}_{v_1,w_1}(F) = \mathrm{DP}_{w_1,v_1}(F)$ by Corollary 36. We have $\mathrm{DP}_{v_1}(F), \mathrm{DP}_{w_1}(F) \in \mathcal{CFIMU}$ by assumption, and we obtain the contradiction that $\mathrm{DP}_{\boldsymbol{v}}(F)$ and $\mathrm{DP}_{\boldsymbol{w}}(F)$ are isomorphic, since $\mathrm{DP}_{\boldsymbol{v}}(F)$ is isomorphic to the result obtained by reducing F via a (maximal) singular tuple $\boldsymbol{v}' = (v_1, w_1, \ldots)$ of length n, where permuting the first two elements in $\boldsymbol{v}'$ yields the singular tuple $\boldsymbol{w}' = (w_1, v_1, \ldots)$ with the same result, which in turn is isomorphic to $\mathrm{DP}_{\boldsymbol{w}}(F)$. □

Corollary 67. *Consider a class $\mathcal{C} \subseteq \mathcal{MU}$ which is stable under application of singular DP-reduction. Then we have $\mathcal{C} \subseteq \mathcal{CFIMU}$ if and only if $\{F \in \mathcal{C} : \mathrm{si}(F) = 1\} \subseteq \mathcal{CFIMU}$.*

Proof. Obviously the stated condition for $\mathcal{C} \subseteq \mathcal{CFIMU}$ is necessary. Now assume that $\mathcal{C} \not\subseteq \mathcal{CFIMU}$, and consider some $F \in \mathcal{C} \setminus \mathcal{CFIMU}$. Then for the F' as defined in Lemma 66 we have $F' \in \mathcal{C} \setminus \mathcal{CFIMU}$ with $\mathrm{si}(F') = 1$. □

Now we analyse the main case where all sDP-reductions give saturated results:

Lemma 68. *Consider $F \in \mathcal{MU}$ and a clause $C \in F$. Let $C' := \{x \in C : \mathrm{ld}_F(x) = 1\}$ be the set of singular literals in C, establishing C as the main clause for the underlying singular variables $\mathrm{var}(x)$ (for $x \in C'$), and let $F_x := \{D \in F : \overline{x} \in D\}$ be the set of side clauses of $\mathrm{var}(x)$ for $x \in C'$. Due to $F \in \mathcal{MU}$ the sets F_x are non-empty and pairwise disjoint (note that $\mathrm{var}(x)$ is $|F_x|$-singular in F for $x \in C'$). Now assume $|C'| \ge 2$, and that for all $x \in C'$ we have $\mathrm{DP}_{\mathrm{var}(x)}(F) \in \mathcal{SMU}$. Then:*

1. $|C'| = 2$.
2. $\forall x \in C' \, \forall D \in F_x : (C \setminus C') \subseteq D$.
3. *For $x, y \in C'$ we have that $\mathrm{DP}_{\mathrm{var}(x)}(F)$ and $\mathrm{DP}_{\mathrm{var}(y)}(F)$ are isomorphic.*

Proof. Consider (any) literals $x, y \in C'$ with $x \neq y$. Then for $D \in F_x$ we have $(C \setminus \{x, y\}) \subseteq D$ by Corollary 11, since otherwise the corollary can be applied to $\mathrm{var}(x)$, replacing D by $D \cup (C \setminus \{x, y\})$, which yields the partial saturation $F' \in \mathcal{MU}$ of F with singular variable $\mathrm{var}(y)$, and where then $\mathrm{DP}_{\mathrm{var}(y)}(F')$ would yield a proper partial saturation G of $\mathrm{DP}_{\mathrm{var}(y)}(F)$, contradicting that the latter is saturated. It follows that actually $C' = \{x, y\}$ must be the case, since if there would be $z \in C' \setminus \{x, y\}$, then $\mathrm{ld}_F(z) \ge 2$ contradicting the definition of C'. It follows Part 2. Finally for Part 3 we note that now $F \rightsquigarrow \mathrm{DP}_x(F)$ just replaces $\overline{x}$ in the clauses of F_x by y, while $F \rightsquigarrow \mathrm{DP}_y(F)$ just replaces $\overline{y}$ in the clauses of F_y by x, and thus renaming y in $\mathrm{DP}_x(F)$ to $\overline{x}$ yields $\mathrm{DP}_y(F)$. □

Corollary 69. *For $F \in \mathcal{MU}$ with $\mathrm{si}(F) = 1$ we have:*

1. *If $|\mathrm{var}_s(F)| \ge 2$:*
   (a) $\mathrm{var}_s(F) = \mathrm{var}_{\neg 1s}(F)$*, that is, all singular variables are non-1-singular.*
   (b) *The main clauses of the singular variables coincide (that is, there is $C \in F$ such that for all singular literals x for F we have $x \in C$).*
   (c) *If $F \in \mathcal{ESMU}$ then $|\mathrm{var}_s(F)| = 2$.*
2. *If $F \in \mathcal{ESMU}$ then $F \in \mathcal{CFIMU}$.*

Proof. Part 1(a) follows by Part 2(a) of Corollary 25, and Part 1(b) follows by Lemma 37. Now Parts 1(c), 2 follow from Lemma 68. □

Example 70. The two clause-sets F from Example 19 (recall Example 58) fulfil $\mathrm{si}(F) = 1$ and $|\mathrm{var}_s(F)| = 2$. For F from in Part 2 there we have $F \in \mathcal{ESMU}$, for F from Part 3 we have $F \notin \mathcal{CFIMU}$.

By Corollary 67 we obtain from Part 2 of Corollary 69:

Theorem 71. $\mathcal{ESMU} \subset \mathcal{CFIMU}$*.*

7. Applications to $\mathcal{MU}_{\delta=2}$

If $F \in \mathcal{CFIMU}$, then we can speak of *the non-singularity type* of F as the (unique) isomorphism type of the elements of $\mathrm{sDP}(F)$. In this section we show that for $F \in \mathcal{MU}_{\delta=2}$ these assumptions are fulfilled. First we recall the fundamental classification:

Definition 72. Consider $n \ge 2$, let addition for the indices of variables $v_1, \ldots, v_n$ be understood modulo n (so $n + 1 \rightsquigarrow 1$), and define $P_n := \{v_1, \ldots, v_n\}$, $N_n := \{\overline{v_1}, \ldots, \overline{v_n}\}$, $C_i := \{\overline{v_i}, v_{i+1}\}$ for $i \in \{1, \ldots, n\}$, and finally $\boldsymbol{\mathcal{F}_n} := \{P_n, N_n\} \cup \{C_i : i \in \{1, \ldots, n\}\} \in \mathcal{MU}'_{\delta=2}$.

So $n(\mathcal{F}_n) = n$ and $c(\mathcal{F}_n) = n + 2$. Recall Example 3, where $\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4$ were already given. The clause-sets $\mathcal{F}_n$ are precisely (up to isomorphism) the non-singular elements of $\mathcal{MU}_{\delta=2}$:

Theorem 73 ([11]). *For $F \in \mathcal{MU}'_{\delta=2}$ we have $F \cong \mathcal{F}_{n(F)}$.*

We show now that for $F \in \mathcal{MU}_{\delta=2}$ we have the non-singularity type of F, which can be encoded as the number of variables left after complete sDP-reduction, using that the isomorphism types in $\mathcal{MU}'_{\delta=2}$ are determined by their number of variables:

Theorem 74. $\mathcal{MU}_{\delta=2} \subseteq \mathcal{CFIMU}$*.*

Proof. The first proof is obtained by applying Corollary 64 and the observation that non-isomorphic elements of $\mathcal{MU}'_{\delta=2}$ have different numbers of variables. The second proof is obtained by applying Theorem 71 and the fact that $\mathcal{MU}'_{\delta=2} \subseteq \mathcal{SMU}$, whence $\mathcal{MU}_{\delta=2} \subseteq \mathcal{ESMU}$. □

Definition 75. By Theorem 74 to every $F \in \mathcal{MU}_{\delta=2}$ we can associate its **non-singularity type $\mathbf{nst_2}(F)$** $\in \mathbb{N}_{\ge 2}$, the unique n such that F by singular DP-reduction can be reduced to a clause-set isomorphic to $\mathcal{F}_n$.

So, considering the structure of $\mathcal{F}_n$ as a "contradictory cycle", we can say that every $F \in \mathcal{MU}_{\delta=2}$ contains a contradictory cycle, where the length of that cycle is $\mathrm{nst}_2(F)$ (and thus uniquely determined), while, as Example 19 shows, the variables constituting such a cycle are not uniquely determined.

8. Conclusion and open problems

We have discussed questions regarding confluence of singular DP-reduction on minimally unsatisfiable clause-sets. Besides various detailed characterisations, we obtained the invariance of the length of maximal sDP-reduction-sequences, confluence for saturated and confluence modulo isomorphism for eventually saturated clause-sets. The main open questions regarding these aspects are:

1. Can we obtain a better overview on singular tuples for $F \in \mathcal{MU}$?
   (a) What are the structural properties of the set of all singular tuples, for $F \in \mathcal{MU}$, $\mathcal{SMU}$, $\mathcal{UHIT}$?
   (b) Especially for $F \in \mathcal{UHIT}$ it should hold that if $\mathrm{si}(F)$ is "large", then $|\mathrm{var}_s(F)|$ must be "large". More precisely:
   Conjecture 76. *For every $k \in \mathbb{N}$ there are $a \in \mathbb{N}$ and $\alpha \in \mathbb{R}_{>0}$ such that for all $F \in \mathcal{UHIT}_{\delta=k}$ with $\mathrm{si}(F) \geq a$ we have $|\mathrm{var}_s(F)| \geq \alpha \cdot \mathrm{si}(F)$.*
2. Can we characterise $\mathcal{CFMU}$ and/or $\mathcal{CFIMU}$? Especially, what is the decision complexity of these classes?
3. Are there other interesting classes for which we can show confluence resp. confluence mod isomorphism of singular DP-reduction?

As a first application of our results, in Section 7 we considered the types of (arbitrary) elements of $\mathcal{MU}_{\delta=2}$. This detailed knowledge is a stepping stone for the determination of the isomorphism types of the elements of $\mathcal{MU}'_{\delta=3}$, which we have obtained meanwhile (to be published; based on a mixture of general insights into the structure of $\mathcal{MU}$ and detailed investigations into $\mathcal{MU}_{\delta\leq 2}$).

The major open problem of the field is the classification (of isomorphism types) of $\mathcal{MU}'_{\delta=k}$ for arbitrary k. The point of departure is the conjecture stated in [23] that for $F \in \mathcal{UHIT}'_{\delta=k}$ the number $n(F)$ of variables is bounded.

Regarding the potential applications from Section 1.3, applying singular DP-reductions in algorithms searching for MUSs is a natural next step.

Finally, a promising direction is the generalisation of the results of this paper beyond minimal unsatisfiability, possibly to arbitrary clause-sets: The analysis of sDP-reduction is much simplified by the fact that for $F \in \mathcal{MU}$ all possible resolutions must actually occur, without producing tautologies and without producing any contractions. To handle arbitrary $F \in \mathcal{CLS}$, these complications have to be taken into account.

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