

Quantum Set Theory Extending the Standard Probabilistic Interpretation of Quantum Theory*

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Abstract

The notion of equality between two observables will play many important roles in foundations of quantum theory. However, the standard probabilistic interpretation based on the conventional Born formula does not give the probability of equality between two arbitrary observables, since the Born formula gives the probability distribution only for a commuting family of observables. In this paper, quantum set theory developed by Takeuti and the present author is used to systematically extend the standard probabilistic interpretation of quantum theory to define the probability of equality between two arbitrary observables in an arbitrary state. We apply this new interpretation to quantum measurement theory, and establish a logical basis for the difference between simultaneous measurability and simultaneous determinateness.

Keywords: quantum logic, quantum set theory, quantum theory, quantum measurements, von Neumann algebras

1 Introduction

Set theory provides foundations of mathematics; all the mathematical notions like numbers, functions, relations, and structures are defined in the axiomatic set theory, ZFC (Zermelo-Fraenkel set theory with the axiom of choice), and all the mathematical theorems are required to be provable in ZFC. Quantum set theory, instituted by Takeuti [32] and developed by the present author [24], naturally extends the logical basis of set

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theory from classical logic to quantum logic to explore mathematics based on quantum logic.

Despite remarkable success in axiomatic foundations of quantum mechanics [35, 12], the quantum logic approach to quantum foundations has not been considered powerful enough to solve interpretational problems [29, 9]. However, this weakness is considered to be mainly due to the fact that the conventional study of quantum logic has been limited to propositional logic. Since quantum set theory extends the underlying logic from propositional logic to predicate logic, and provides set theoretical constructions of mathematical objects such as numbers, functions, relations, and structures based on quantum logic, we can expect that quantum set theory will provide much more systematic interpretation of quantum theory than the conventional quantum logic approach. This paper represents the first step towards establishing systematic interpretation of quantum theory based on quantum set theory, and naturally focusses on the most fundamental notion in mathematics, namely, equality.

The notion of equality between quantum observables will play many important roles in foundations of quantum theory, in particular, in the theory of measurement and disturbance [22, 23]. However, the standard probabilistic interpretation based on the conventional Born formula does not give the probability of equality between two arbitrary observables, since the Born formula gives the probability distribution only for a commuting family of observables [36]. In this paper, quantum set theory is used to systematically extend the probabilistic interpretation of quantum theory to define the probability of equality between two arbitrary observables in an arbitrary state based on the fact that real numbers defined in quantum set theory exactly corresponds to quantum observables [32, 24]. It is shown that every observational proposition on a quantum system corresponds to a statement in quantum set theory with the same projection-valued truth value and the same probability in any state. In particular, equality between real numbers in quantum set theory naturally provides a state-dependent notion of equality between quantum mechanical observables. It has been broadly accepted that we cannot speak of the values of quantum observables without assuming a hidden variable theory, which are severely constrained by Kochen-Specker type no-go theorems [14, 29]. However, quantum set theory enables us to do so without assuming hidden variables but alternatively with the consistent use of quantum logic. We apply this new interpretation to quantum measurement theory, and establish a logical basis for the difference between simultaneous measurability and simultaneous determinateness.

Section 2 provides preliminaries on complete orthomodular lattices, commutators of their subsets, quantum logic on Hilbert spaces, and the universe $V^{(\mathcal{Q})}$ of quantum

set theory over a logic \mathcal{Q} on a Hilbert space \mathcal{H} . We give a characterization of the commutator of a subset of a complete orthomodular lattice, improving Takeuti's characterization, and give a factorization of the double commutant of a subset of a complete orthomodular lattice into the maximal Boolean factor and a complete orthomodular lattice without non-trivial Boolean factor. Section 3 introduces a one-to-one correspondence obtained in Refs. [32, 24] between the reals $\mathbf{R}^{(\mathcal{Q})}$ in $V^{(\mathcal{Q})}$ and self-adjoint operators affiliated with the von Neumann algebra $\mathcal{M} = \mathcal{Q}''$ generated by \mathcal{Q} , determines commutators and equality in $\mathbf{R}^{(\mathcal{Q})}$, and gives the embedding of intervals in \mathbf{R} into $V^{(\mathcal{Q})}$. Section 4 formulates the standard probabilistic interpretation of quantum theory and also shows that the set of observational propositions for a quantum system can be embedded in a set of statements in quantum set theory without changing projection-valued truth value assignment. Section 5 extends the standard interpretation by introducing simultaneous determinateness, i.e., state-dependent commutativity of observables. We give several characterizations of simultaneous determinateness for finite number of quantum observables affiliated with an arbitrary von Neumann algebra in a given state, extending some previous results [23] on simultaneous determinateness for two observables. Section 6 extends the standard interpretation by introducing quantum equality, i.e., state-dependent equality for two arbitrary observables. We give several characterizations of quantum equality for two observables affiliated with an arbitrary von Neumann algebra in a given state, extending some previous results [23] on simultaneous determinateness for two observables. Sections 7 and 8 provide applications to quantum measurement theory. We discuss a state-dependent formulation of measurement of observables and simultaneous measurability, and establish a logical basis for the difference between simultaneous measurability and simultaneous determinateness. The conclusion is given in Section 9.

Whereas we will discuss the completely general case where \mathcal{M} is an arbitrary von Neumann algebra, some results for the case where $\dim(\mathcal{H}) < \infty$ and $\mathcal{M} = \mathcal{B}(\mathcal{H})$ have been previously reported in Ref. [25]. In this special case, we can avoid the use of quantum set theory to introduce simultaneous determinateness and quantum equality into the language of observational propositions, since simultaneous determinateness and quantum equality can be expressed, respectively, by observational propositions constructed by atomic formulas of the form $X = x$ with an observable X and a real number x . However, to prove a transfer theorem ensuring that all the classical tautologies have the truth value 1, mentioned without proof in Ref. [25], Theorem 3, it is necessary, even in this special case, to develop quantum set theory and to define the embedding of the language of observational propositions into the language of quantum

set theory. The required machinery will be, for the first time, fully constructed in this paper including the case with observables with continuous spectrum, though the full power of this machinery will be revealed when applied to mathematical theorems beyond tautologies after we have enriched the language of observational propositions, in the future research, with more sophisticated relations and functions than equality.

2 Quantum set theory

2.1 Quantum logic

A *complete orthomodular lattice* is a complete lattice \mathcal{Q} with an *orthocomplementation*, a unary operation \perp on \mathcal{Q} satisfying

$$(C1) \text{ if } P \leq Q \text{ then } Q^\perp \leq P^\perp,$$

$$(C2) \text{ } P^{\perp\perp} = P,$$

$$(C3) \text{ } P \vee P^\perp = 1 \text{ and } P \wedge P^\perp = 0, \text{ where } 0 = \bigwedge \mathcal{Q} \text{ and } 1 = \bigvee \mathcal{Q},$$

that satisfies the *orthomodular law*

$$(OM) \text{ if } P \leq Q \text{ then } P \vee (P^\perp \wedge Q) = Q.$$

In this paper, any complete orthomodular lattice is called a *logic*. A non-empty subset of a logic \mathcal{Q} is called a *subalgebra* iff it is closed under \wedge , \vee , and \perp . A subalgebra \mathcal{A} of \mathcal{Q} is said to be *complete* iff it has the supremum and the infimum in \mathcal{Q} of an arbitrary subset of \mathcal{A} . For any subset \mathcal{A} of \mathcal{Q} , the subalgebra generated by \mathcal{A} is denoted by $\Gamma_0 \mathcal{A}$. We refer the reader to Kalmbach [13] for a standard text on orthomodular lattices.

We say that P and Q in a logic \mathcal{Q} *commute*, in symbols $P \circ Q$, iff $P = (P \wedge Q) \vee (P \wedge Q^\perp)$. All the relations $P \circ Q$, $Q \circ P$, $P^\perp \circ Q$, $P \circ Q^\perp$, and $P^\perp \circ Q^\perp$ are equivalent. The distributive law does not hold in general, but the following useful propositions hold (Ref. [13], pp. 24–25).

Proposition 2.1. *If $P_1, P_2 \circ Q$, then the sublattice generated by P_1, P_2, Q is distributive.*

Proposition 2.2. *If $P_\alpha \circ Q$ for all α , then $\bigvee_\alpha P_\alpha \circ Q$, $\bigwedge_\alpha P_\alpha \circ Q$, $Q \wedge (\bigvee_\alpha P_\alpha) = \bigvee_\alpha (Q \wedge P_\alpha)$, and $Q \vee (\bigwedge_\alpha P_\alpha) = \bigwedge_\alpha (Q \vee P_\alpha)$.*

From Proposition 2.1, a logic \mathcal{Q} is a Boolean algebra if and only if $P \circ Q$ for all $P, Q \in \mathcal{Q}$ (Ref. [13], pp. 24–25).

For any subset $\mathcal{A} \subseteq \mathcal{Q}$, we denote by $\mathcal{A}^!$ the *commutant* of \mathcal{A} in \mathcal{Q} (Ref. [13], p. 23), i.e.,

$$\mathcal{A}^! = \{P \in \mathcal{Q} \mid P \circ Q \text{ for all } Q \in \mathcal{A}\}.$$

Then, $\mathcal{A}^!$ is a complete subalgebra of \mathcal{Q} . A *sublogic* of \mathcal{Q} is a subset \mathcal{A} of \mathcal{Q} satisfying $\mathcal{A} = \mathcal{A}^{!!}$. For any subset $\mathcal{A} \subseteq \mathcal{Q}$, the smallest logic including \mathcal{A} is $\mathcal{A}^{!!}$ called the *sublogic generated by \mathcal{A}* . Then, it is easy to see that a subset \mathcal{A} is a Boolean sublogic, or equivalently a distributive sublogic, if and only if $\mathcal{A} = \mathcal{A}^{!!} \subseteq \mathcal{A}^!$.

2.2 Commutators

Let \mathcal{Q} be a logic. Marsden [15] has introduced the commutator $\text{com}(P, Q)$ of two elements P and Q of \mathcal{Q} by

$$\text{com}(P, Q) = (P \wedge Q) \vee (P \wedge Q^\perp) \vee (P^\perp \wedge Q) \vee (P^\perp \wedge Q^\perp). \quad (1)$$

Bruns and Kalmbach [4] have generalized this notion to finite subsets of \mathcal{Q} by

$$\text{com}(\mathcal{F}) = \bigvee_{\alpha: \mathcal{F} \rightarrow \{\text{id}, \perp\}} \bigwedge_{P \in \mathcal{F}} P^{\alpha(P)} \quad (2)$$

for all $\mathcal{F} \in \mathcal{P}_\omega(\mathcal{Q})$, where $\mathcal{P}_\omega(\mathcal{Q})$ stands for the set of finite subsets of \mathcal{Q} , and $\{\text{id}, \perp\}$ stands for the set consisting of the identity operation id and the orthocomplementation \perp . Generalizing this notion to arbitrary subsets \mathcal{A} of \mathcal{Q} , Takeuti [32] defined $\text{com}(\mathcal{A})$ by

$$\text{com}(\mathcal{A}) = \bigvee T(\mathcal{A}), \quad (3)$$

$$T(\mathcal{A}) = \{E \in \mathcal{A}^! \mid P_1 \wedge E \circ P_2 \wedge E \text{ for all } P_1, P_2 \in \mathcal{A}\}, \quad (4)$$

of any $\mathcal{A} \in \mathcal{P}(\mathcal{Q})$, where $\mathcal{P}(\mathcal{Q})$ stands for the power set of \mathcal{Q} , and showed that $\text{com}(\mathcal{A}) \in T(\mathcal{A})$. Subsequently, Pulmannová [28] showed:

Theorem 2.3. *For any subset \mathcal{A} of a logic \mathcal{Q} , we have*

- (i) $\text{com}(\mathcal{A}) = \bigwedge \{\text{com}(\mathcal{F}) \mid \mathcal{F} \in \mathcal{P}_\omega(\mathcal{A})\}$,
- (ii) $\text{com}(\mathcal{A}) = \bigwedge \{\text{com}(P, Q) \mid P, Q \in \Gamma_0(\mathcal{A})\}$.

Here, we reformulate Takeuti's definition in a more convenient form. Let $\mathcal{A} \subseteq \mathcal{Q}$. Note that $\mathcal{A}^{!!}$ is the sublogic generated by \mathcal{A} , and $\mathcal{A}^! \cap \mathcal{A}^{!!}$ is the center of $\mathcal{A}^{!!}$, i.e., the set of elements of $\mathcal{A}^{!!}$ commuting with all elements of $\mathcal{A}^{!!}$. Denote by $L(\mathcal{A})$ the sublogic generated by \mathcal{A} , i.e., $L(\mathcal{A}) = \mathcal{A}^{!!}$, and by $Z(\mathcal{A})$ the center of $L(\mathcal{A})$, i.e.,

$Z(\mathcal{A}) = \mathcal{A}^! \cap \mathcal{A}^{!!}$. A *subcommutator* of \mathcal{A} is any $E \in Z(\mathcal{A})$ such that $P_1 \wedge E \downarrow P_2 \wedge E$ for all $P_1, P_2 \in \mathcal{A}$. Denote by $S(\mathcal{A})$ the set of subcommutators of \mathcal{A} , i.e.,

$$S(\mathcal{A}) = \{E \in Z(\mathcal{A}) \mid P_1 \wedge E \downarrow P_2 \wedge E \text{ for all } P_1, P_2 \in \mathcal{A}\}. \quad (5)$$

By the relation $Z(\mathcal{A}) \subseteq \mathcal{A}^!$, we immediately obtain the relation $\bigvee S(\mathcal{A}) \leq \text{com}(\mathcal{A})$. We shall show that the equality actually holds.

Lemma 2.4. *Let \mathcal{A} be any subset of a logic \mathcal{Q} . For any $P_1, P_2 \in \mathcal{A}$ and $E \in \mathcal{A}^!$, we have $P_1 \wedge E \downarrow P_2 \wedge E$ if and only if $P_1 \wedge E \downarrow P_2$.*

Proof. Let $E \in \mathcal{A}^!$ and $P_1, P_2 \in \mathcal{A}$. We have $(P_1 \wedge E) \wedge (P_2 \wedge E)^\perp = (P_1 \wedge E) \wedge P_2^\perp$, and hence

$$[(P_1 \wedge E) \wedge (P_2 \wedge E)] \vee [(P_1 \wedge E) \wedge (P_2 \wedge E)^\perp] = [(P_1 \wedge E) \wedge P_2] \vee [(P_1 \wedge E) \wedge P_2^\perp].$$

It follows that $P_1 \wedge E \downarrow P_2 \wedge E$ if and only if $P_1 \wedge E \downarrow P_2$. \square

For any $P, Q \in \mathcal{Q}$, the *interval* $[P, Q]$ is the set of all $X \in \mathcal{Q}$ such that $P \leq X \leq Q$. For any $\mathcal{A} \subseteq \mathcal{Q}$ and $P, Q \in \mathcal{A}$, we write $[P, Q]_{\mathcal{A}} = [P, Q] \cap \mathcal{A}$.

Theorem 2.5. *For any subset \mathcal{A} of a logic \mathcal{Q} , the following relations hold.*

- (i) $S(\mathcal{A}) = \{E \in Z(\mathcal{A}) \mid [0, E]_{\mathcal{A}} \subseteq Z(\mathcal{A})\}$.
- (ii) $\bigvee S(\mathcal{A})$ is the maximum subcommutator of \mathcal{A} , i.e., $\bigvee S(\mathcal{A}) \in S(\mathcal{A})$.
- (iii) $S(\mathcal{A}) = [0, \bigvee S(\mathcal{A})]_{L(\mathcal{A})}$.
- (iv) $\text{com}(\mathcal{A}) = \bigvee S(\mathcal{A})$.

Proof. (i) It is easy to see that $P_1 \wedge E \downarrow P_2$ for every $P_1, P_2 \in \mathcal{A}$ if and only if $[0, E] \cap \mathcal{A} \subseteq \mathcal{A}^!$, and hence the assertion follows from Lemma 2.4. (ii) Let $P_1, P_2 \in \mathcal{A}$. We have $P_1 \wedge E \downarrow P_2$ for every $E \in S(\mathcal{A})$ from Lemma 2.4, and $P_1 \wedge \bigvee S(\mathcal{A}) \downarrow P_2$ from Proposition 2.2. Since $S(\mathcal{A}) \subseteq Z(\mathcal{A})$, we have $\bigvee S(\mathcal{A}) \in Z(\mathcal{A})$. Thus, $\bigvee S(\mathcal{A}) \in S(\mathcal{A})$, and the assertion follows. (iii) If $P \in [0, \bigvee S(\mathcal{A})]_{L(\mathcal{A})}$ then $P = P \wedge \bigvee S(\mathcal{A})$ commutes with every element of $L(\mathcal{A})$. Thus, we have $[0, \bigvee S(\mathcal{A})]_{L(\mathcal{A})} = [0, \bigvee S(\mathcal{A})]_{Z(\mathcal{A})}$. Now, let $P \in [0, \bigvee S(\mathcal{A})]_{Z(\mathcal{A})}$. Then, $P_1 \downarrow P$ and $P_1 \downarrow P_2 \wedge \bigvee S(\mathcal{A})$, and hence $P_1 \downarrow P \wedge P_2 \wedge \bigvee S(\mathcal{A})$ and $P_1 \downarrow P_2 \wedge P$. Thus, we have $P \in S(\mathcal{A})$, and the assertion follows. (iv) Since $\text{com}(\mathcal{F}) \in Z(\mathcal{F})$ for every finite subset \mathcal{F} of \mathcal{A} , we have $\text{com}(\mathcal{A}) \in Z(\mathcal{A})$, and hence we have $\text{com}(\mathcal{A}) \in Z(\mathcal{A})$. Thus, relation (iv) follows. \square

The following proposition will be useful in later discussions.

Theorem 2.6. *Let \mathcal{B} be a maximal Boolean sublogic of a logic \mathcal{Q} and \mathcal{A} a subset of \mathcal{Q} including \mathcal{B} , i.e., $\mathcal{B} \subseteq \mathcal{A} \subseteq \mathcal{Q}$. Then, we have $\text{com}(\mathcal{A}) \in \mathcal{B}$ and $[0, \text{com}(\mathcal{A})]_{\mathcal{A}} \subseteq \mathcal{B}$.*

Proof. Since $\text{com}(\mathcal{A}) \in Z(\mathcal{A}) \subseteq \mathcal{B}^! = \mathcal{B}$, we have $\text{com}(\mathcal{A}) \in \mathcal{B}$. Let $P \in \mathcal{A}$. Then, $P \wedge \text{com}(\mathcal{A}) \downarrow Q$ for all $Q \in \mathcal{B}$, so that $P \wedge \text{com}(\mathcal{A}) \in \mathcal{B}^! = \mathcal{B}$, and hence $[0, \text{com}(\mathcal{A})]_{\mathcal{A}} \subseteq \mathcal{B}$. \square

The following theorem clarifies the significance of commutators.

Theorem 2.7. *Let \mathcal{A} be a subset of a logic \mathcal{Q} . Then, $L(\mathcal{A})$ is isomorphic to the direct product of the complete Boolean algebra $[0, \text{com}(\mathcal{A})]_{L(\mathcal{A})}$ and the complete orthomodular lattice $[0, \text{com}(\mathcal{A})^\perp]_{L(\mathcal{A})}$ without non-trivial Boolean factor.*

Proof. It follows from $\bigvee S(\mathcal{A}) \in Z(\mathcal{A})$ that $L(\mathcal{A}) \cong [0, \bigvee S(\mathcal{A})]_{L(\mathcal{A})} \times [0, \bigvee S(\mathcal{A})^\perp]_{L(\mathcal{A})}$. Then, $[0, \bigvee S(\mathcal{A})]_{L(\mathcal{A})}$ is a complete Boolean algebra, since $[0, \bigvee S(\mathcal{A})]_{L(\mathcal{A})} \subseteq Z(\mathcal{A})$. It follows easily from the maximality of $\bigvee S(\mathcal{A})$ that $[0, \bigvee S(\mathcal{A})^\perp]_{L(\mathcal{A})}$ has no non-trivial Boolean factor. Thus, the assertion follows from the relation $\bigvee S(\mathcal{A}) = \text{com}(\mathcal{A})$. \square

We refer the reader to Pulmannová [28] and Chevalier [5] for further results about commutators in orthomodular lattices.

2.3 Logic on Hilbert spaces

Let \mathcal{H} be a Hilbert space. For any subset $S \subseteq \mathcal{H}$, we denote by S^\perp the orthogonal complement of S . Then, $S^{\perp\perp}$ is the closed linear span of S . Let $\mathcal{C}(\mathcal{H})$ be the set of all closed linear subspaces in \mathcal{H} . With the set inclusion ordering, the set $\mathcal{C}(\mathcal{H})$ is a complete lattice. The operation $M \mapsto M^\perp$ is an orthocomplementation on the lattice $\mathcal{C}(\mathcal{H})$, with which $\mathcal{C}(\mathcal{H})$ is a logic.

Denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on \mathcal{H} and $\mathcal{Q}(\mathcal{H})$ the set of projections on \mathcal{H} . We define the *operator ordering* on $\mathcal{B}(\mathcal{H})$ by $A \leq B$ iff $(\psi, A\psi) \leq (\psi, B\psi)$ for all $\psi \in \mathcal{H}$. For any $A \in \mathcal{B}(\mathcal{H})$, denote by $\mathcal{R}(A) \in \mathcal{C}(\mathcal{H})$ the closure of the range of A , i.e., $\mathcal{R}(A) = (A\mathcal{H})^{\perp\perp}$. For any $M \in \mathcal{C}(\mathcal{H})$, denote by $\mathcal{P}(M) \in \mathcal{Q}(\mathcal{H})$ the projection operator of \mathcal{H} onto M . Then, $\mathcal{R}\mathcal{P}(M) = M$ for all $M \in \mathcal{C}(\mathcal{H})$ and $\mathcal{P}\mathcal{R}(P) = P$ for all $P \in \mathcal{Q}(\mathcal{H})$, and we have $P \leq Q$ if and only if $\mathcal{R}(P) \subseteq \mathcal{R}(Q)$ for all $P, Q \in \mathcal{Q}(\mathcal{H})$, so that $\mathcal{Q}(\mathcal{H})$ with the operator ordering is also a logic isomorphic to $\mathcal{C}(\mathcal{H})$. Any sublogic of $\mathcal{Q}(\mathcal{H})$ will be called a *logic on \mathcal{H}* . The lattice operations are characterized by $P \wedge Q = \text{weak-lim}_{n \rightarrow \infty} (PQ)^n$, $P^\perp = 1 - P$ for all $P, Q \in \mathcal{Q}(\mathcal{H})$.

Let $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$. We denote by \mathcal{A}' the *commutant* of \mathcal{A} in $\mathcal{B}(\mathcal{H})$. A self-adjoint subalgebra \mathcal{M} of $\mathcal{B}(\mathcal{H})$ is called a *von Neumann algebra* on \mathcal{H} iff $\mathcal{M}'' = \mathcal{M}$. For any self-adjoint subset $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, \mathcal{A}'' is the von Neumann algebra generated by \mathcal{A} . We denote by $\mathcal{P}(\mathcal{M})$ the set of projections in a von Neumann algebra \mathcal{M} . For any $P, Q \in \mathcal{Q}(\mathcal{H})$, we have $P \perp Q$ iff $[P, Q] = 0$, where $[P, Q] = PQ - QP$. For any subset $\mathcal{A} \subseteq \mathcal{Q}(\mathcal{H})$, we denote by \mathcal{A}' the *commutant* of \mathcal{A} in $\mathcal{Q}(\mathcal{H})$. For any subset $\mathcal{A} \subseteq \mathcal{Q}(\mathcal{H})$, the smallest logic including \mathcal{A} is the logic \mathcal{A}'' called the *logic generated* by \mathcal{A} . Then, a subset $\mathcal{Q} \subseteq \mathcal{Q}(\mathcal{H})$ is a logic on \mathcal{H} if and only if $\mathcal{Q} = \mathcal{P}(\mathcal{M})$ for some von Neumann algebra \mathcal{M} on \mathcal{H} (Ref. [24], Proposition 2.1).

We define the *implication* and the *logical equivalence* on \mathcal{Q} by $P \rightarrow Q = P^\perp \vee (P \wedge Q)$ and $P \leftrightarrow Q = (P \rightarrow Q) \wedge (Q \rightarrow P)$. We have the following characterization of commutators in logics on Hilbert spaces (Ref. [24], Theorems 2.5, 2.6).

Theorem 2.8. *Let \mathcal{Q} be a logic on \mathcal{H} and let $\mathcal{A} \subseteq \mathcal{Q}$. Then, we have the following relations.*

- (i) $\text{com}(\mathcal{A}) = \mathcal{P}\{\psi \in \mathcal{H} \mid [A, B]\psi = 0 \text{ for all } A, B \in \mathcal{A}'\}$.
- (ii) $\text{com}(\mathcal{A}) = \mathcal{P}\{\psi \in \mathcal{H} \mid [P_1, P_2]P_3\psi = 0 \text{ for all } P_1, P_2, P_3 \in \mathcal{A}'\}$.

2.4 Quantum set theory over logic on Hilbert spaces

We denote by V the universe of the Zermelo-Fraenkel set theory with the axiom of choice (ZFC). Let $\mathcal{L}(\in)$ be the first-order language with equality without constant symbols augmented by a binary relation symbol \in , bounded quantifier symbols $\forall x \in y$, $\exists x \in y$ (in addition to unbounded quantifier symbols $\forall x$, $\exists x$). For any class U , the language $\mathcal{L}(\in, U)$ is the one obtained by adding a name for each element of U .

Let \mathcal{Q} be a logic on \mathcal{H} . For each ordinal α , let

$$V_\alpha^{(\mathcal{Q})} = \{u \mid u : \text{dom}(u) \rightarrow \mathcal{Q} \text{ and } (\exists \beta < \alpha) \text{dom}(u) \subseteq V_\beta^{(\mathcal{Q})}\}. \quad (6)$$

The \mathcal{Q} -valued universe $V^{(\mathcal{Q})}$ is defined by

$$V^{(\mathcal{Q})} = \bigcup_{\alpha \in \text{On}} V_\alpha^{(\mathcal{Q})}, \quad (7)$$

where On is the class of all ordinals. For every $u \in V^{(\mathcal{Q})}$, the rank of u , denoted by $\text{rank}(u)$, is defined as the least α such that $u \in V_{\alpha+1}^{(\mathcal{Q})}$. It is easy to see that if $u \in \text{dom}(v)$ then $\text{rank}(u) < \text{rank}(v)$.

For any $u, v \in V^{(\mathcal{Q})}$, the \mathcal{Q} -valued truth values of atomic formulas $u = v$ and $u \in v$ are assigned by the following rules recursive in rank.

- (i) $[[u = v]]_{\mathcal{Q}} = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow [[u' \in v]]_{\mathcal{Q}}) \wedge \bigwedge_{v' \in \text{dom}(v)} (v(v') \rightarrow [[v' \in u]]_{\mathcal{Q}})$.
- (ii) $[[u \in v]]_{\mathcal{Q}} = \bigvee_{v' \in \text{dom}(v)} (v(v') \wedge [[u = v']]_{\mathcal{Q}})$.

To each statement ϕ of $\mathcal{L}(\in, V^{(\mathcal{Q})})$ we assign the \mathcal{Q} -valued truth value $[[\phi]]_{\mathcal{Q}}$ by the following rules.

- (iii) $[[\neg\phi]]_{\mathcal{Q}} = [[\phi]]_{\mathcal{Q}}^{\perp}$.
- (iv) $[[\phi_1 \wedge \phi_2]]_{\mathcal{Q}} = [[\phi_1]]_{\mathcal{Q}} \wedge [[\phi_2]]_{\mathcal{Q}}$.
- (v) $[[\phi_1 \vee \phi_2]]_{\mathcal{Q}} = [[\phi_1]]_{\mathcal{Q}} \vee [[\phi_2]]_{\mathcal{Q}}$.
- (vi) $[[\phi_1 \rightarrow \phi_2]]_{\mathcal{Q}} = [[\phi_1]]_{\mathcal{Q}} \rightarrow [[\phi_2]]_{\mathcal{Q}}$.
- (vii) $[[\phi_1 \leftrightarrow \phi_2]]_{\mathcal{Q}} = [[\phi_1]]_{\mathcal{Q}} \leftrightarrow [[\phi_2]]_{\mathcal{Q}}$.
- (viii) $[[\forall x \in u \phi(x)]]_{\mathcal{Q}} = \bigwedge_{u' \in \text{dom}(u)} (u(u') \rightarrow [[\phi(u')]]_{\mathcal{Q}})$.
- (ix) $[[\exists x \in u \phi(x)]]_{\mathcal{Q}} = \bigvee_{u' \in \text{dom}(u)} (u(u') \wedge [[\phi(u')]]_{\mathcal{Q}})$.
- (x) $[[\forall x \phi(x)]]_{\mathcal{Q}} = \bigwedge_{u \in V^{(\mathcal{Q})}} [[\phi(u)]]_{\mathcal{Q}}$.
- (xi) $[[\exists x \phi(x)]]_{\mathcal{Q}} = \bigvee_{u \in V^{(\mathcal{Q})}} [[\phi(u)]]_{\mathcal{Q}}$.

We say that a statement ϕ of $\mathcal{L}(\in, V^{(\mathcal{Q})})$ *holds* in $V^{(\mathcal{Q})}$ iff $[[\phi]]_{\mathcal{Q}} = 1$. A formula in $\mathcal{L}(\in)$ is called a Δ_0 -formula iff it has no unbounded quantifiers $\forall x$ or $\exists x$. The following theorem holds [24].

Theorem 2.9 (Δ_0 -Absoluteness Principle). *For any Δ_0 -formula $\phi(x_1, \dots, x_n)$ of $\mathcal{L}(\in)$ and $u_1, \dots, u_n \in V^{(\mathcal{Q})}$, we have*

$$[[\phi(u_1, \dots, u_n)]]_{\mathcal{Q}} = [[\phi(u_1, \dots, u_n)]]_{\mathcal{Q}(\mathcal{H})}.$$

Henceforth, for any Δ_0 -formula $\phi(x_1, \dots, x_n)$ and $u_1, \dots, u_n \in V^{(\mathcal{Q})}$, we abbreviate $[[\phi(u_1, \dots, u_n)]] = [[\phi(u_1, \dots, u_n)]]_{\mathcal{Q}}$, which is the common \mathcal{Q} -valued truth value in all $V^{(\mathcal{Q})}$ such that $u_1, \dots, u_n \in V^{(\mathcal{Q})}$.

The universe V can be embedded in $V^{(\mathcal{Q})}$ by the following operation $\vee : v \mapsto \check{v}$ defined by the \in -recursion: for each $v \in V$, $\check{v} = \{\check{u} \mid u \in v\} \times \{1\}$. Then we have the following [24].

Theorem 2.10 (Δ_0 -Elementary Equivalence Principle). *For any Δ_0 -formula $\phi(x_1, \dots, x_n)$ of $\mathcal{L}(\in)$ and $u_1, \dots, u_n \in V$, we have $\langle V, \in \rangle \models \phi(u_1, \dots, u_n)$ if and only if $[[\phi(\check{u}_1, \dots, \check{u}_n)]] = 1$.*

For $u \in V^{(\mathcal{Q})}$, we define the *support* of u , denoted by $\mathbb{L}(u)$, by transfinite recursion on the rank of u by the relation

$$\mathbb{L}(u) = \bigcup_{x \in \text{dom}(u)} \mathbb{L}(x) \cup \{u(x) \mid x \in \text{dom}(u)\}. \quad (8)$$

For $\mathcal{A} \subseteq V^{(\mathcal{Q})}$ we write $\mathbf{L}(\mathcal{A}) = \bigcup_{u \in \mathcal{A}} \mathbf{L}(u)$ and for $u_1, \dots, u_n \in V^{(\mathcal{Q})}$ we write $\mathbf{L}(u_1, \dots, u_n) = \mathbf{L}(\{u_1, \dots, u_n\})$. Let $\mathcal{A} \subseteq V^{(\mathcal{Q})}$. The *commutator* of \mathcal{A} , denoted by $\text{com}(\mathcal{A})$, is defined by

$$\text{com}(\mathcal{A}) = \text{com}(\mathbf{L}(\mathcal{A})). \quad (9)$$

For any $u_1, \dots, u_n \in V^{(\mathcal{Q})}$, we write $\text{com}(u_1, \dots, u_n) = \text{com}(\{u_1, \dots, u_n\})$. For bounded theorems, the following transfer principle holds [24].

Theorem 2.11 (ZFC Transfer Principle). *For any Δ_0 -formula $\phi(x_1, \dots, x_n)$ of $\mathcal{L}(\in)$ and $u_1, \dots, u_n \in V^{(\mathcal{Q})}$, if $\phi(x_1, \dots, x_n)$ is provable in ZFC, then we have*

$$\text{com}(u_1, \dots, u_n) \leq \llbracket \phi(u_1, \dots, u_n) \rrbracket.$$

3 Real numbers in quantum set theory

Let \mathbf{Q} be the set of rational numbers in V . We define the set of rational numbers in the model $V^{(\mathcal{Q})}$ to be $\check{\mathbf{Q}}$. We define a real number in the model by a Dedekind cut of the rational numbers. More precisely, we identify a real number with the upper segment of a Dedekind cut assuming that the lower segment has no end point. Therefore, the formal definition of the predicate $\mathbf{R}(x)$, “ x is a real number,” is expressed by

$$\begin{aligned} \mathbf{R}(x) \quad := \quad & \forall y \in x (y \in \check{\mathbf{Q}}) \wedge \exists y \in \check{\mathbf{Q}} (y \in x) \wedge \exists y \in \check{\mathbf{Q}} (y \notin x) \\ & \wedge \forall y \in \check{\mathbf{Q}} (y \in x \leftrightarrow \forall z \in \check{\mathbf{Q}} (y < z \rightarrow z \in x)). \end{aligned} \quad (10)$$

The symbol “ $:=$ ” is used to define a new formula, here and hereafter. We define $\mathbf{R}^{(\mathcal{Q})}$ to be the interpretation of the set \mathbf{R} of real numbers in $V^{(\mathcal{Q})}$ as follows.

$$\mathbf{R}^{(\mathcal{Q})} = \{u \in V^{(\mathcal{Q})} \mid \text{dom}(u) = \text{dom}(\check{\mathbf{Q}}) \text{ and } \llbracket \mathbf{R}(u) \rrbracket = 1\}. \quad (11)$$

The set $\mathbf{R}_{\mathcal{Q}}$ of real numbers in $V^{(\mathcal{Q})}$ is defined by

$$\mathbf{R}_{\mathcal{Q}} = \mathbf{R}^{(\mathcal{Q})} \times \{1\}. \quad (12)$$

Then, for any $u, v \in \mathbf{R}^{(\mathcal{Q})}$, the following relations hold in $V^{(\mathcal{Q})}$ [24].

- (i) $\llbracket (\forall u \in \mathbf{R}_{\mathcal{Q}}) u = u \rrbracket = 1$.
- (ii) $\llbracket (\forall u, v \in \mathbf{R}_{\mathcal{Q}}) u = v \rightarrow v = u \rrbracket = 1$.
- (iii) $\llbracket (\forall u, v, w \in \mathbf{R}_{\mathcal{Q}}) u = v \wedge v = w \rightarrow u = w \rrbracket = 1$.
- (iv) $\llbracket (\forall v \in \mathbf{R}_{\mathcal{Q}}) (\forall x, y \in v) x = y \wedge x \in v \rightarrow y \in v \rrbracket$.

$$(v) \quad [(\forall u, v \in \mathbf{R}_{\mathcal{Q}})(\forall x \in u)x \in u \wedge u = v \rightarrow x \in v].$$

From the above, the equality is an equivalence relation between real numbers in $V^{(\mathcal{Q})}$. For any $u_1, \dots, u_n \in \mathbf{R}^{(\mathcal{Q})}$, we have

$$[[u_1 = u_2 \wedge \dots \wedge u_{n-1} = u_n]] \leq \text{com}(u_1, \dots, u_n), \quad (13)$$

and hence commutativity follows from equality in $\mathbf{R}^{(\mathcal{Q})}$ [24].

Let \mathcal{M} be a von Neumann algebra on a Hilbert space \mathcal{H} and let $\mathcal{Q} = \mathcal{P}(\mathcal{M})$. A closed operator A (densely defined) on \mathcal{H} is said to be *affiliated* with \mathcal{M} , in symbols $A \eta \mathcal{M}$, iff $U^*AU = A$ for any unitary operator $U \in \mathcal{M}'$. Let A be a self-adjoint operator (densely defined) on \mathcal{H} and let $A = \int_{\mathbf{R}} \lambda dE^A(\lambda)$ be its spectral decomposition, where $\{E^A(\lambda)\}_{\lambda \in \mathbf{R}}$ is the resolution of identity belonging to A (Ref. [36], p. 119). It is well-known that $A \eta \mathcal{M}$ if and only if $E^A(\lambda) \in \mathcal{Q}$ for every $\lambda \in \mathbf{R}$. Denote by $\overline{\mathcal{M}}_{SA}$ the set of self-adjoint operators affiliated with \mathcal{M} . Two self-adjoint operators A and B are said to *commute*, in symbols $A \downarrow B$, iff $E^A(\lambda) \downarrow E^B(\lambda')$ for every pair λ, λ' of reals.

For any $u \in \mathbf{R}^{(\mathcal{Q})}$ and $\lambda \in \mathbf{R}$, we define $E^u(\lambda)$ by

$$E^u(\lambda) = \bigwedge_{\lambda < r \in \mathbf{Q}} u(\check{r}). \quad (14)$$

Then, it can be shown that $\{E^u(\lambda)\}_{\lambda \in \mathbf{R}}$ is a resolution of identity in \mathcal{Q} and hence by the spectral theorem there is a self-adjoint operator $\hat{u} \eta \mathcal{M}$ uniquely satisfying $\hat{u} = \int_{\mathbf{R}} \lambda dE^u(\lambda)$. On the other hand, let $A \eta \mathcal{M}$ be a self-adjoint operator. We define $\tilde{A} \in V^{(\mathcal{Q})}$ by

$$\tilde{A} = \{(\check{r}, E^A(r)) \mid r \in \mathbf{Q}\}. \quad (15)$$

Then, $\text{dom}(\tilde{A}) = \text{dom}(\mathbf{Q})$ and $\tilde{A}(\check{r}) = E^A(r)$ for all $r \in \mathbf{Q}$. It is easy to see that $\tilde{A} \in \mathbf{R}^{(\mathcal{Q})}$ and we have $(\hat{u})\check{r} = u$ for all $u \in \mathbf{R}^{(\mathcal{Q})}$ and $(\tilde{A})\check{r} = A$ for all $A \in \overline{\mathcal{M}}_{SA}$. Therefore, the correspondence between $\mathbf{R}^{(\mathcal{Q})}$ and $\overline{\mathcal{M}}_{SA}$ is a one-to-one correspondence. We call the above correspondence the *Takeuti correspondence*. Now, we have the following [24].

Theorem 3.1. *Let \mathcal{Q} be a logic on \mathcal{H} . The relations*

$$(i) \quad E^A(\lambda) = \bigwedge_{\lambda < r \in \mathbf{Q}} u(\check{r}) \text{ for all } \lambda \in \mathbf{Q},$$

$$(ii) \quad u(\check{r}) = E^A(r) \text{ for all } r \in \mathbf{Q},$$

for all $u = \tilde{A} \in \mathbf{R}^{(\mathcal{Q})}$ and $A = \hat{u} \in \overline{\mathcal{M}}_{SA}$ sets up a one-to-one correspondence between $\mathbf{R}^{(\mathcal{Q})}$ and $\overline{\mathcal{M}}_{SA}$.

For any $r \in \mathbf{R}$, we shall write $\tilde{r} = (r1)^\sim$, where $r1$ is the scalar operator on \mathcal{H} . Then, we have $\text{dom}(\tilde{r}) = \text{dom}(\check{\mathbf{Q}})$ and $\tilde{r}(\tilde{t}) = \llbracket \tilde{r} \leq \tilde{t} \rrbracket$, so that we have $\mathbf{L}(\tilde{r}) = \{0, 1\}$. Denote by $\mathcal{B}(\mathbf{R}^n)$ the σ -field of Borel subsets of \mathbf{R}^n and $B(\mathbf{R}^n)$ the space of bounded Borel functions on \mathbf{R}^n . A *spectral measure* [10] on \mathbf{R}^n in \mathcal{M} is a mapping E of $\mathcal{B}(\mathbf{R}^n)$ into $\mathcal{P}(\mathcal{M})$ satisfying $\sum_j E(\Delta_j) = 1$ for any disjoint sequence $\{\Delta_j\}$ in $\mathcal{B}(\mathbf{R}^n)$ such that $\bigcup_j \Delta_j = \mathbf{R}^n$. Let X be a self-adjoint operator affiliated with \mathcal{M} . For any $f \in B(\mathbf{R})$, the bounded self-adjoint operator $f(X) \in \mathcal{M}$ is defined by $f(X) = \int_{\mathbf{R}} f(\lambda) dE^X(\lambda)$. The *spectral measure of X* is a spectral measure E^A on \mathbf{R} in \mathcal{M} defined by $E^X(\Delta) = \chi_\Delta(X)$ for any $\Delta \in \mathcal{B}(\mathbf{R})$. Then, we have $E^X(\lambda) = E^X((-\infty, \lambda])$.

Proposition 3.2. *Let $r \in \mathbf{R}$, $s, t \in \mathbf{R}$, and $X \eta \mathcal{M}_{SA}$. We have the following relations.*

- (i) $\llbracket \check{r} \in \check{s} \rrbracket = \llbracket \check{s} \leq \check{r} \rrbracket = E^{s1}(t)$.
- (ii) $\llbracket \check{s} \leq \check{t} \rrbracket = \llbracket \check{s} \leq \check{t} \rrbracket = E^{s1}(t)$.
- (iii) $\llbracket \check{X} \leq \check{t} \rrbracket = E^X(t) = E^X((-\infty, t])$.
- (iv) $\llbracket \check{t} < \check{X} \rrbracket = 1 - E^X(t) = E^X((t, \infty))$.
- (v) $\llbracket \check{s} < \check{X} \leq \check{t} \rrbracket = E^X(t) - E^X(s) = E^X((s, t])$.
- (vi) $\llbracket \check{X} = \check{t} \rrbracket = E^X(t) - \bigvee_{r < t, r \in \mathbf{Q}} E^X(r) = E^X(\{t\})$.

Proof. Relations (i), (ii), and (iii) follows from [24, Proposition 5.11]. We have $\text{com}(\check{t}, \check{X}) = 1$, so that (iv) follows from the ZFC Transfer Principle (Theorem 2.11). Relation (v) follows from (iii) and (iv). We have

$$\begin{aligned}
\llbracket \check{X} = \check{t} \rrbracket &= \bigwedge_{r \in \mathbf{Q}} \check{X}(\check{r}) \rightarrow \llbracket \check{r} \in \check{t} \rrbracket \wedge \bigwedge_{r \in \mathbf{Q}} \check{t}(\check{r}) \rightarrow \llbracket \check{r} \in \check{X} \rrbracket \\
&= \bigwedge_{r \in \mathbf{Q}} E^X(r)^\perp \vee E^{t1}(r) \wedge \bigwedge_{r \in \mathbf{Q}} E^{t1}(r)^\perp \vee E^X(r) \\
&= \bigwedge_{r < t, r \in \mathbf{Q}} E^X(r)^\perp \wedge \bigwedge_{t \leq r \in \mathbf{Q}} E^X(r) \\
&= [1 - \bigvee_{r < t, r \in \mathbf{Q}} E^X(r)] \wedge E^X(t) \\
&= E^X(t) - \bigvee_{r < t, r \in \mathbf{Q}} E^X(r) \\
&= E^X(\{t\}).
\end{aligned}$$

Thus, relation (vi) follows. □

4 Standard probabilistic interpretation of quantum theory

Let \mathbf{S} be a quantum system described by a von Neumann algebra \mathcal{M} on a Hilbert space \mathcal{H} . According to the standard formulation of quantum theory, the *observables* of \mathbf{S} are defined as self-adjoint operators affiliated with \mathcal{M} , the *states* of \mathbf{S} are represented by density operators on \mathcal{H} , and a *vector state* ψ is identified with the state $|\psi\rangle\langle\psi|$. We denote by $\mathcal{O}(\mathcal{M})$ the set of observables, by $\mathcal{S}(\mathcal{H})$ the space of density operators. Observables $X_1, \dots, X_n \in \mathcal{O}(\mathcal{M})$ are said to be *mutually commuting* iff $X_j \downarrow X_k$ for all $j, k = 1, \dots, n$. If $X_1, \dots, X_n \in \mathcal{O}(\mathcal{M})$ are bounded, this condition is equivalent to $[X_j, X_k] = 0$ for all $j, k = 1, \dots, n$. The standard probabilistic interpretation of quantum theory defines the *joint probability distribution function* $F_\rho^{X_1, \dots, X_n}(x_1, \dots, x_n)$ for mutually commuting observables $X_1, \dots, X_n \in \mathcal{O}(\mathcal{M})$ in $\rho \in \mathcal{S}(\mathcal{H})$ by the *Born statistical formula*:

$$F_\rho^{X_1, \dots, X_n}(x_1, \dots, x_n) = \text{Tr}[E^{X_1}(x_1) \cdots E^{X_n}(x_n)\rho]. \quad (16)$$

To clarify the logical structure presupposed in the standard probabilistic interpretation, we define *observational propositions* for \mathbf{S} by the following rules.

- (R1) For any $X \in \mathcal{O}(\mathcal{M})$ and $x \in \mathbf{R}$, the expression $X \leq_o x$ is an observational proposition.
- (R2) If ϕ_1 and ϕ_2 are observational propositions, $\neg\phi_1$ and $\phi_1 \wedge \phi_2$ are also observational propositions.

Thus, every observational proposition is built up from “atomic” observational propositions $X \leq_o x$ by adding finite number of connectives \neg and \wedge . We denote by $\mathcal{L}_o(\mathcal{M})$ the set of observational propositions. We introduce the connective \vee by definition.

$$(D1) \quad \phi_1 \vee \phi_2 := \neg(\neg\phi_1 \wedge \neg\phi_2).$$

For each observational proposition ϕ , we assign its projection-valued truth value $[[\phi]]_o \in \mathcal{Q}(\mathcal{H})$ by the following rules [2].

- (T1) $[[X \leq_o x]]_o = E^X(x)$.
- (T2) $[[\neg\phi]]_o = [[\phi]]_o^\perp$.
- (T3) $[[\phi_1 \wedge \phi_2]]_o = [[\phi_1]]_o \wedge [[\phi_2]]_o$.

From (D1), (T2) and (T3), we have

$$(D2) \quad [[\phi_1 \vee \phi_2]]_o = [[\phi_1]]_o \vee [[\phi_2]]_o.$$

We define the *probability* $\Pr\{\phi\|\rho\}$ of an observational proposition ϕ in a state ρ by

$$(P1) \quad \Pr\{\phi\|\rho\} = \text{Tr}[[\phi]_o\rho].$$

We say that *an observational proposition ϕ holds in a state ρ* iff $\Pr\{\phi\|\rho\} = 1$.

The standard interpretation of quantum theory restricts observational propositions to be standard defined as follows.

(W1) An observational proposition including atomic formulas $X_1 \leq_o x_1, \dots, X_n \leq_o x_n$ is called *standard* iff X_1, \dots, X_n are mutually commuting.

All the standard observational propositions including only given mutually commuting observables X_1, \dots, X_n comprise a complete Boolean algebra under the logical order \leq defined by $\phi \leq \phi'$ iff $[[\phi]_o] \leq [[\phi']_o]$ and obey inference rules in classical logic. Suppose that $X_1, \dots, X_n \in \mathcal{O}(\mathcal{M})$ are mutually commuting. Let $x_1, \dots, x_n \in \mathbf{R}$. Then, $X_1 \leq_o x_1 \wedge \dots \wedge X_n \leq_o x_n$ is a standard observational proposition. We have

$$[[X_1 \leq_o x_1 \wedge \dots \wedge X_n \leq_o x_n]]_o = E^{X_1}(x_1) \wedge \dots \wedge E^{X_n}(x_n) = E^{X_1}(x_1) \cdots E^{X_n}(x_n). \quad (17)$$

Hence, we reproduce the Born statistical formula as

$$\Pr\{X_1 \leq_o x_1 \wedge \dots \wedge X_n \leq_o x_n\|\rho\} = \text{Tr}[E^{X_1}(x_1) \cdots E^{X_n}(x_n)\rho]. \quad (18)$$

From the above, our definition of the truth values of observational propositions are consistent with the standard probabilistic interpretation of quantum theory.

From Proposition 3.2 and (T1), we conclude

$$[[\tilde{X} \leq \tilde{x}]] = [[X \leq_o x]]_o \quad (19)$$

for all $X \in \mathcal{O}(\mathcal{M})$ and $x \in \mathbf{R}$. To every observational proposition ϕ the corresponding statement $\tilde{\phi}$ in $\mathcal{L}(\in, \mathbf{R}^{(\mathcal{Q})})$ is given by the following rules for any $X \in \mathcal{O}(\mathcal{M})$ and $x \in \mathbf{R}$, and observational propositions ϕ, ϕ_1, ϕ_2 .

$$(Q1) \quad \widetilde{X \leq_o x} := \tilde{X} \leq \tilde{x}.$$

$$(Q2) \quad \widetilde{\neg\phi} := \neg\tilde{\phi}.$$

$$(Q3) \quad \widetilde{\phi_1 \wedge \phi_2} := \tilde{\phi}_1 \wedge \tilde{\phi}_2.$$

Then, it is easy to see that the relation

$$[[\tilde{\phi}]] = [[\phi]_o] \quad (20)$$

holds for any observational proposition ϕ . Thus, all the observational propositions are embedded in the set of statements in $\mathcal{L}(\in, \mathbf{R}^{(2)})$ with the same projection-valued truth value.

We denote by $\text{Sp}(X)$ the spectrum of an observable $X \in \mathcal{O}(\mathcal{M})$, i.e., the set of all $\lambda \in \mathbf{R}$ such that $X - \lambda 1$ has a bounded inverse operator on \mathcal{H} . An observable $X \in \mathcal{O}(\mathcal{M})$ is called *finite* iff $\text{Sp}(X)$ is a finite set, and *infinite* otherwise. Denote by $\mathcal{O}_\omega(\mathcal{M})$ is the set of finite observables in $\mathcal{O}(\mathcal{M})$.

Let $X \in \mathcal{O}_\omega(\mathcal{M})$. Then, $\text{Sp}(X)$ coincides with the set of eigenvalues of X . Let

$$\delta(X) = \min_{x,y \in \text{Sp}(X), x \neq y} \{|x - y|/2, 1\}. \quad (21)$$

For any $x \in \mathbf{R}$, we define the observational proposition $X =_o x$ by

$$X =_o x := x - \delta(X) < X \leq_o x + \delta(X). \quad (22)$$

Then, it is easy to see that we have

$$[[X =_o x]]_o = E^X(\{x\}) \quad (23)$$

for all $x \in \mathbf{R}$.

In Ref. [25] we have introduced observational propositions for the case where $\dim(\mathcal{H}) < \infty$ and $\mathcal{M} = \mathcal{B}(\mathcal{H})$ by rules (R'1), (R'2) of well-formed formulas and rules (T'1)–(T'3) for projection-valued truth value assignment as follows.

- (R'1) For any $X \in \mathcal{O}(\mathcal{B}(\mathcal{H}))$ and $x \in \mathbf{R}$, the expression $X =_{o'} x$ is an observational proposition.
- (R'2) If ϕ_1 and ϕ_2 are observational propositions, $\neg\phi_1$ and $\phi_1 \wedge \phi_2$ are also observational propositions.
- (T'1) $[[X =_{o'} x]]_{o'} = E^X(x)$.
- (T'2) $[[\neg\phi]]_{o'} = [[\phi]]_{o'}^\perp$.
- (T'3) $[[\phi_1 \wedge \phi_2]]_{o'} = [[\phi_1]]_{o'} \wedge [[\phi_2]]_{o'}$.

Denote by $\mathcal{L}_{o'}(\mathcal{B}(\mathcal{H}))$ the set of observational propositions constructed by rules (R'1) and (R'2). In this language, for any observables $X \in \mathcal{O}(\mathcal{B}(\mathcal{H}))$ and any real number $x \in \mathbf{R}$, we can introduce the observational proposition $X \leq_{o'} x$ in $\mathcal{L}_{o'}(\mathcal{B}(\mathcal{H}))$ by

$$X \leq_{o'} x := \bigvee_{x_j \in \text{Sp}(X) \cap (-\infty, x]} X =_{o'} x_j, \quad (24)$$

where the observational proposition $\bigvee_j \phi_j$ is defined by $\bigvee_j \phi_j = \phi_1 \vee \cdots \vee \phi_n$ for any finite sequence of observational propositions ϕ_1, \dots, ϕ_n . Then, we have

$$\llbracket X \leq_{o'} x \rrbracket_{o'} = E^X(x). \quad (25)$$

Now, we can conclude that if $\dim(\mathcal{H}) < \infty$, the language $\mathcal{L}_o(\mathcal{B}(\mathcal{H}))$ and $\mathcal{L}_{o'}(\mathcal{B}(\mathcal{H}))$ are equivalent in the sense that there is a one-to-one correspondence Φ of $\mathcal{L}_{o'}(\mathcal{B}(\mathcal{H}))$ onto $\mathcal{L}_o(\mathcal{B}(\mathcal{H}))$ such that $\llbracket \Phi(\phi) \rrbracket_o = \llbracket \phi \rrbracket_{o'}$, $\Phi(X =_{o'} x) = (X =_o x)$, and $\Phi(X \leq_{o'} x) = (X \leq_o x)$ for all $\phi \in \mathcal{L}_{o'}(\mathcal{B}(\mathcal{H}))$, $X \in \mathcal{O}(\mathcal{B}(\mathcal{H}))$, and $x \in \mathbf{R}$. Thus, in what follows for the case where $\dim(\mathcal{H}) < \infty$ we shall identify the language $\mathcal{L}_{o'}(\mathcal{B}(\mathcal{H}))$ introduced in Ref. [25] with the language $\mathcal{L}_o(\mathcal{B}(\mathcal{H}))$; in this case we have $\mathcal{O}(\mathcal{B}(\mathcal{H})) = \mathcal{O}_\omega(\mathcal{B}(\mathcal{H}))$.

5 Simultaneous determinateness

In this section, we shall examine basic properties of the commutator $\text{com}(\tilde{X}_1, \dots, \tilde{X}_n)$ for observables $X_1, \dots, X_n \in \mathcal{O}(\mathcal{M})$. Let $X_1, \dots, X_n \in \mathcal{O}(\mathcal{M})$. We denote by $\{X_1, \dots, X_n\}''$ the von Neumann algebra generated by projections $E^{X_j}(\lambda)$ for all $j = 1, \dots, n$ and $\lambda \in \mathbf{R}$, and denote by $\mathcal{Z}(X_1, \dots, X_n)$ the center of $\{X_1, \dots, X_n\}''$, i.e., $\mathcal{Z}(X_1, \dots, X_n) = \{X_1, \dots, X_n\}'' \cap \{X_1, \dots, X_n\}'$. The cyclic subspace $\mathcal{C}(X_1, \dots, X_n; \rho)$ of \mathcal{H} generated by X_1, \dots, X_n , and ρ is defined by

$$\mathcal{C}(X_1, \dots, X_n; \rho) = \{X_1, \dots, X_n\}'' \overline{\text{ran}}(\rho),$$

where $\overline{\text{ran}}$ stands for the closure of the range. Then, $\mathcal{C}(X_1, \dots, X_n; \rho)$ is the least invariant subspace under $\{X_1, \dots, X_n\}''$ containing ρ . Denote by $C(X_1, \dots, X_n; \rho)$ the projection of \mathcal{H} onto $\mathcal{C}(X_1, \dots, X_n; \rho)$. Then, $C(X_1, \dots, X_n; \rho)$ is the smallest projection P in $\{X_1, \dots, X_n\}'$ such that $P\rho = \rho$.

Under the Takeuti correspondence, the commutator of observables are characterized as follows.

Theorem 5.1. *For any $X_1, \dots, X_n \in \mathcal{O}(\mathcal{M})$, the following relations hold.*

- (i) $\text{com}(\tilde{X}_1, \dots, \tilde{X}_n) = \mathcal{P}\{\psi \in \mathcal{H} \mid [A, B]\psi = 0 \text{ for all } A, B \in \{X_1, \dots, X_n\}''\}$.
- (ii) $\text{com}(\tilde{X}_1, \dots, \tilde{X}_n) = \mathcal{P}\{\psi \in \mathcal{H} \mid [E^{X_j}(r_1), E^{X_k}(r_2)]E^{X_l}(r_3)\psi = 0$
for all $r_1, r_2, r_3 \in \mathbf{Q}$ and $j, k, l = 1, \dots, n\}$.
- (iii) $\text{com}(\tilde{X}_1, \dots, \tilde{X}_n) = \max\{E \in \mathcal{P}(\mathcal{Z}(X_1, \dots, X_n)) \mid X_j E \perp X_k E$
for all $j, k = 1, \dots, n\}$.

Proof. Let $\mathcal{A} = \mathbf{L}(\tilde{X}_1, \dots, \tilde{X}_n)$. Then, $\text{com}(\tilde{X}_1, \dots, \tilde{X}_n) = \text{com}(\mathcal{A})$. We have

$$\mathbf{L}(\tilde{X}_1, \dots, \tilde{X}_n) = \{E^{X_j}(r_j) \mid r_j \in \mathbf{Q} \text{ and } j = 1, \dots, n\} \cup \{0, 1\},$$

and hence $\mathbf{L}(\tilde{X}_1, \dots, \tilde{X}_n)'' = \{X_1, \dots, X_n\}''$. Thus, relations (i) and (ii) follow from Theorem 2.8 (i) and (ii), respectively. From Theorem 2.5 we have

$$\begin{aligned} \text{com}(\tilde{X}_1, \dots, \tilde{X}_n) &= \max\{E \in \mathcal{Z}(\mathcal{A}) \mid P_1 \wedge E \perp P_2 \wedge E \text{ for all } P_1, P_2 \in \mathcal{A}\} \\ &= \max\{E \in \mathcal{P}(\mathcal{L}(X_1, \dots, X_n)) \mid X_j E \perp X_k E \text{ for all } j, k = 1, \dots, n\}, \end{aligned}$$

from which relation (iii) follows. \square

We say that observables $X_1, \dots, X_n \in \mathcal{O}(\mathcal{M})$ are *simultaneously determinate* in a state ρ iff $\text{Tr}[\text{com}(\tilde{X}_1, \dots, \tilde{X}_n)\rho] = 1$.

A probability measure μ on $\mathcal{B}(\mathbf{R}^n)$ is called a *joint probability distribution* of $X_1, \dots, X_n \in \mathcal{O}(\mathcal{M})$ in $\rho \in \mathcal{S}(\mathcal{H})$ iff for any polynomial $p(f_1(X_1), \dots, f_n(X_n))$ of observables $f_1(X_1), \dots, f_n(X_n)$, where $f_1, \dots, f_n \in B(\mathbf{R})$, we have

$$\text{Tr}[p(f_1(X_1), \dots, f_n(X_n))\rho] = \int \cdots \int_{\mathbf{R}^n} p(f_1(x_1), \dots, f_n(x_n)) d\mu(x_1, \dots, x_n). \quad (26)$$

A joint probability distribution of X_1, \dots, X_n in ρ is unique, if any. Since simultaneous determinateness is considered to be a state-dependent notion of commutativity, it is expected that simultaneous determinateness is equivalent to the state-dependent existence of the joint probability distribution. This is indeed shown below together with other useful characterizations of this notion.

Theorem 5.2. *For any observables $X_1, \dots, X_n \in \mathcal{O}(\mathcal{M})$ and a state $\rho \in \mathcal{S}(\mathcal{H})$, the following conditions are all equivalent.*

- (i) X_1, \dots, X_n are simultaneously determinate in ρ , i.e., $\text{Tr}[\text{com}(\tilde{X}_1, \dots, \tilde{X}_n)\rho] = 1$
- (ii) $\text{com}(\tilde{X}_1, \dots, \tilde{X}_n)\rho = \rho$.
- (iii) $C(X_1, \dots, X_n; \rho) \leq \text{com}(\tilde{X}_1, \dots, \tilde{X}_n)$.
- (iv) $[A, B]\rho = 0$ for all $A, B \in \{X_1, \dots, X_n\}''$.
- (v) There exists a joint probability distribution of X_1, \dots, X_n in ρ .
- (vi) $X_j C(X_1, \dots, X_n; \rho) \perp X_k C(X_1, \dots, X_n; \rho)$ for all $j, k = 1, \dots, n$.
- (vii) There exists a spectral measure E in \mathcal{M} on \mathbf{R}^n satisfying

$$E(\Delta_1 \times \cdots \times \Delta_n)\rho = E^{X_1}(\Delta_1) \wedge \cdots \wedge E^{X_n}(\Delta_n)\rho \quad (27)$$

for all $\Delta_1, \dots, \Delta_n \in \mathcal{B}(\mathbf{R})$.

(viii) *There exists a probability measure μ on \mathbf{R}^n satisfying*

$$\mu(\Delta_1 \times \cdots \times \Delta_n) = \text{Tr}[E^{X_1}(\Delta_1) \wedge \cdots \wedge E^{X_n}(\Delta_n)\rho] \quad (28)$$

for any $\Delta_1, \dots, \Delta_n \in \mathcal{B}(\mathbf{R})$.

Proof. Let $\mathcal{B} = \{X_1, \dots, X_n\}''$ and $C = C(X_1, \dots, X_n; \rho)$,

(i) \Rightarrow (ii): The assertion follows from the relation $\|P\sqrt{\rho} - \sqrt{\rho}\|_{HS}^2 = 1 - \text{Tr}[P\rho]$ for any projection P , where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm.

(ii) \Rightarrow (iii): Since $\text{com}(\tilde{X}_1, \dots, \tilde{X}_n) \in \mathcal{B}'$, (iii) follows from (ii) by minimality of $C(X_1, \dots, X_n; \rho)$.

(iii) \Rightarrow (iv): It follows from (iii) that $\text{ran}(\rho) \subseteq \text{ran}(\text{com}(\tilde{X}_1, \dots, \tilde{X}_n))$ so that (iv) follows from Theorem 5.1 (i).

(iv) \Rightarrow (v): It follows from assumption (iv) and Proposition 2.2 in Ref. [11] that the GNS representation $(\mathcal{H}, \pi, \Omega)$ of \mathcal{B} induced by ρ is abelian (i.e., $\pi(\mathcal{B})$ is abelian) and normal. Let $j = 1, \dots, n$. Let f_j be a bounded Borel function on \mathbf{R} . By normality of π , there is a self-adjoint operator $\pi(X_j)$ affiliated with $\pi(\mathcal{B})$ such that $E^{\pi(X_j)}(\Delta) = \pi(E^{X_j}(\Delta))$ for all $\Delta \in \mathcal{B}(\mathbf{R})$, and hence we have

$$\pi(f_j(X_j)) = f_j(\pi(X_j)).$$

Thus, the relation

$$\mu(\Delta_1 \times \cdots \times \Delta_n) = (\Omega, E^{\pi(X_1)}(\Delta_1) \cdots E^{\pi(X_n)}(\Delta_n)\Omega),$$

where $\Delta_1, \dots, \Delta_n \in \mathcal{B}(\mathbf{R})$, defines a probability measure μ on $\mathcal{B}(\mathbf{R}^n)$ satisfying

$$\int \cdots \int_{\mathbf{R}^n} p(f_1(x_1), \dots, f_n(x_n)) d\mu(x_1, \dots, x_n) = (\Omega, \pi(p(f_1(X_1), \dots, f_n(X_n)))\Omega)$$

for any polynomial $p(f_1(X_1), \dots, f_n(X_n))$ of $f_1(X_1), \dots, f_n(X_n)$. Thus, assertion (iv) follows from the relation

$$\text{Tr}[A\rho] = (\Omega, \pi(A)\Omega)$$

for any $A \in \mathcal{B}$ satisfied by the GNS representation $(\mathcal{H}, \pi, \Omega)$.

(v) \Rightarrow (i): Suppose that there exists a joint probability distribution μ of X_1, \dots, X_n in ρ . Then, for any $j, k, l = 1, \dots, n$ and $r_1, r_2, r_3 \in \mathbf{Q}$, we have

$$\text{Tr}[|E^{X_j}(r_1), E^{X_k}(r_2)|E^{X_l}(r_3)|^2\rho] = 0$$

and we have $[E^{X_j}(r_1), E^{X_k}(r_2)]E^{X_l}(r_3)\rho = 0$. From Theorem 5.1 (ii), it follows that $\text{com}(\tilde{X}_1, \dots, \tilde{X}_n)\rho\psi = \rho\psi$ for all $\psi \in \mathcal{H}$. Thus, we have $\text{com}(\tilde{X}_1, \dots, \tilde{X}_n)\rho = \rho$ and hence X_1, \dots, X_n are simultaneously determinate in a state ρ .

(iii) \Rightarrow (vi): Let $G = \text{com}(X_1, \dots, X_n)$ and $C = C(X_1, \dots, X_n; \rho)$. From Theorem 5.1 (iii), we have $X_j G \perp\!\!\!\perp X_k G$ for all $j, k = 1, \dots, n$. Since $X_j G \perp\!\!\!\perp C$ for all j , assertion (vi) follows from (iii).

(vi) \Rightarrow (vii): Obvious.

(vii) \Rightarrow (ii): Let μ be a probability measure on \mathbf{R}^n satisfying (28). Let $j, k, l \in \{1, \dots, n\}$. By taking an appropriate marginal measure of μ there exists a probability measure μ' on \mathbf{R}^3 such that

$$\mu'(\Delta_1 \times \Delta_2 \times \Delta_3) = \text{Tr}[E^{X_j}(\Delta_1) \wedge E^{X_k}(\Delta_2) \wedge E^{X_l}(\Delta_3)\rho]$$

for all $\Delta_1, \Delta_2, \Delta_3 \in \mathcal{B}(\mathbf{R})$. Let $\Delta_1, \Delta_2, \Delta_3 \in \mathcal{B}(\mathbf{R})$ and

$$\begin{aligned} P &= E^{X_l}(\Delta_3) - E^{X_k}(\Delta_2^c) \wedge E^{X_l}(\Delta_3) - E^{X_j}(\Delta_1^c) \wedge E^{X_k}(\Delta_2) \wedge E^{X_l}(\Delta_3) \\ &\quad - E^{X_j}(\Delta_1) \wedge E^{X_k}(\Delta_2) \wedge E^{X_l}(\Delta_3), \end{aligned}$$

where Δ^c stands for the complement of $\Delta \in \mathcal{B}(\mathbf{R})$. Then, by the additivity of μ' we have

$$\text{Tr}[P\rho] = \mu'(\mathbf{R} \times \mathbf{R} \times \Delta_3) - \mu'(\mathbf{R} \times \Delta_2^c \times \Delta_3) - \mu(\Delta_1^c \times \Delta_2 \times \Delta_3) - \mu(\Delta_1 \times \Delta_2 \times \Delta_3) = 0.$$

Since $\text{Tr}[(P\sqrt{\rho})^\dagger(P\sqrt{\rho})] = \text{Tr}[P\rho]$, we have $P\sqrt{\rho} = 0$, so that $E^{X_j}(\Delta_1)E^{X_k}(\Delta_2)P\rho = 0$, and hence $E^{X_j}(\Delta_1)E^{X_k}(\Delta_2)E^{X_l}(\Delta_3)\rho = E^{X_j}(\Delta_1) \wedge E^{X_k}(\Delta_2) \wedge E^{X_l}(\Delta_3)\rho$. By symmetry we also have $E^{X_k}(\Delta_2)E^{X_j}(\Delta_1)E^{X_l}(\Delta_3)\rho = E^{X_j}(\Delta_1) \wedge E^{X_k}(\Delta_2) \wedge E^{X_l}(\Delta_3)\rho$. Thus, we have $[E^{X_j}(\Delta_1), E^{X_k}(\Delta_2)]E^{X_l}(\Delta_3)\rho\psi = 0$ for all $\psi \in \mathcal{H}$. Since $\Delta_1, \Delta_2, \Delta_3$ were arbitrary, it follows from Theorem 5.1 that $\text{ran}(\rho) \subseteq \text{ran}(\text{com}(\tilde{X}_1, \dots, \tilde{X}_n))$, and (ii) follows. \square

The equivalence between (i) and (v) in the above theorem was previously reported in Theorem 2 of Ref. [25] for the case where $\mathcal{M} = \mathcal{B}(\mathcal{H})$ with $\mathcal{H} < \infty$. The equivalence of (ii), (vii), and (viii) was given in Theorem 5.1 of Ref. [23] for the case $n = 2$.

Note that for any $X_1, \dots, X_n \in \mathcal{O}(\mathcal{M})$ there exists a proposition ϕ in $\mathcal{L}_o(\mathcal{M})$ such that $[[\phi]]_o = \text{com}(\tilde{X}_1, \dots, \tilde{X}_n)$, since $\text{com}(\tilde{X}_1, \dots, \tilde{X}_n) \in \mathcal{O}_\omega(\mathcal{M})$ and $[[\text{com}(\tilde{X}_1, \dots, \tilde{X}_n) =_o 1]]_o = \text{com}(\tilde{X}_1, \dots, \tilde{X}_n)$. However, it is not in general possible to construct such ϕ from atomic propositions of the form $X_j \leq_o \lambda$ for $j = 1, \dots, n$ with $\lambda \in \mathbf{R}$. In what follows, we shall show that this is possible for finite observables.

For any finite observables $X_1, \dots, X_n \in \mathcal{O}_\omega(\mathcal{M})$ we define the observational proposition $\text{com}_o(X_1, \dots, X_n)$ by

$$\text{com}_o(X_1, \dots, X_n) := \bigvee_{x_1 \in \text{Sp}(X_1), \dots, x_n \in \text{Sp}(X_n)} X_1 =_o x_1 \wedge \dots \wedge X_n =_o x_n. \quad (29)$$

Then, we have the following theorem.

Theorem 5.3. *For any finite observables $X_1, \dots, X_n \in \mathcal{O}_\omega(\mathcal{M})$, we have*

$$[[\text{com}_o(X_1, \dots, X_n)]]_o = \text{com}(\tilde{X}_1, \dots, \tilde{X}_n). \quad (30)$$

Proof. Let $X_1, \dots, X_n \in \mathcal{O}_\omega(\mathcal{H})$. Let $x_j^{(1)} < \dots < x_j^{(n_j)} \in \mathbf{R}$ be the ascending sequence of eigenvalues of X_j . Then, we have

$$\mathbf{L}(\tilde{X}_1, \dots, \tilde{X}_n) = \{E^{X_j}(x) \mid x = x_j^{(1)}, \dots, x_j^{(n_j)}; j = 1, \dots, n\} \cup \{0\}.$$

Since $\mathbf{L}(X_1, \dots, X_n)$ is a finite set, it is easy to see that the relations

$$\begin{aligned} [[\text{com}_o(X_1, \dots, X_n)]]_o &= \text{com}(\{E^{X_j}(\{x\}) \mid x = x_j^{(1)}, \dots, x_j^{(n_j)}; j = 1, \dots, n\}) \\ &= \text{com}(\mathbf{L}(\tilde{X}_1, \dots, \tilde{X}_n)) \\ &= \text{com}(\tilde{X}_1, \dots, \tilde{X}_n) \end{aligned}$$

hold. □

The observational proposition $\text{com}_o(X_1, \dots, X_n)$ was previously introduced in Ref. [25] for the case where $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\dim(\mathcal{H}) < \infty$. The following theorem is a straightforward generalization of Theorem 1 in Ref. [25].

Theorem 5.4. *Finite observables $X_1, \dots, X_n \in \mathcal{O}_\omega(\mathcal{M})$ are simultaneously determinate in a vector state ψ if and only if the state ψ is a superposition of common eigenvectors of X_1, \dots, X_n .*

6 Quantum equality

In this section, we shall examine basic properties of the \mathcal{Q} -valued equality relation $[[\tilde{X} = \tilde{Y}]]$ defined through $V^{(\mathcal{Q})}$ for any two observables $X, Y \in \mathcal{O}(\mathcal{M})$, where $\mathcal{Q} = \mathcal{P}(\mathcal{M})$. From Theorem 6.3 of Ref. [24], we have the following characterizations.

Theorem 6.1. *For any $X, Y \in \mathcal{O}(\mathcal{M})$, the following relations hold.*

- (i) $[[\tilde{X} = \tilde{Y}]] = \mathcal{P}\{\psi \in \mathcal{H} \mid E^X(r)\psi = E^Y(r)\psi \text{ for all } r \in \mathbf{Q}\}$.
- (ii) $[[\tilde{X} = \tilde{Y}]] = \mathcal{P}\{\psi \in \mathcal{H} \mid f(X)\psi = f(Y)\psi \text{ for all } f \in B(\mathbf{R})\}$.
- (iii) $[[\tilde{X} = \tilde{Y}]] = \mathcal{P}\{\psi \in \mathcal{H} \mid (E^X(\Delta)\psi, E^Y(\Gamma)\psi) = 0$
for any $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$ with $\Delta \cap \Gamma = \emptyset\}$.

We introduce a new atomic observational proposition $X =_o Y$ in $\mathcal{L}_o(\mathcal{M})$ for all $X, Y \in \mathcal{O}(\mathcal{M})$ by the following additional rules for formation of observational propositions and for projection-valued truth values:

(R3) For any $X, Y \in \mathcal{O}(\mathcal{M})$, the expression $X =_o Y$ is an observational proposition.

(T4) $\llbracket X =_o Y \rrbracket_o = \llbracket \tilde{X} = \tilde{Y} \rrbracket$.

We extend the correspondence between observational propositions and formulas in $\mathcal{L}(\in, V^{(\mathcal{Q})})$ by the following rule for any $X, Y \in \mathcal{O}(\mathcal{M})$.

(Q4) $\widetilde{X =_o Y} := \tilde{X} = \tilde{Y}$.

Then, from (T4) it is easy to see that the relation

$$\llbracket \tilde{\phi} \rrbracket = \llbracket \phi \rrbracket_o \quad (31)$$

holds for any observational proposition ϕ . We denote by $\mathcal{L}_o(\mathcal{M}, =)$ the set of observational propositions constructed by rules (R1), (R2), (R3). Then, the language $\mathcal{L}_o(\mathcal{M}, =)$ is embedded in the set of statements in $\mathcal{L}(\in, \mathbf{R}^{(\mathcal{Q})})$ by rules (Q1), (Q2), (Q3), (Q4) with the same projection-valued truth value by rules (T1), (T2), (T3), (T4).

In general, the equality relation in $V^{(\mathcal{Q})}$ is not an equivalence relation in $V^{(\mathcal{Q})}$ [32]. From Theorem 6.3 of Ref. [24], however, we conclude that that \mathcal{Q} -valued equality between two observables is indeed a \mathcal{Q} -valued equivalence relation as follows.

Theorem 6.2. *For any observables $X, Y, Z \in \mathcal{O}(\mathcal{M})$, the following relations hold.*

- (i) $\llbracket X =_o X \rrbracket_o = 1$.
- (ii) $\llbracket X =_o Y \rrbracket_o = \llbracket Y =_o X \rrbracket_o$.
- (iii) $\llbracket X =_o Y \rrbracket_o \wedge \llbracket Y =_o Z \rrbracket_o \leq \llbracket X =_o Z \rrbracket_o$.

We say that observables X and Y are *equal in a state ρ* , in symbols $X =_\rho Y$, iff $\Pr\{X =_o Y \mid \rho\} = 1$, or equivalently iff $\llbracket X =_o Y \rrbracket_o \rho = \rho$. In general, we say that observables X and Y are *equal in a state ρ with probability $\Pr\{X =_o Y \mid \rho\}$* . On the other hand, we have explored another relation called quantum perfect correlation in Ref. [23] as follows. Two observables X and Y are called *perfectly correlated* in a state ρ iff $\text{Tr}[E^X(\Delta)E^Y(\Gamma)\rho] = 0$ for any disjoint Borel sets $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$. It is noted that the quantity $\text{Tr}[E^X(\Delta)E^Y(\Gamma)\rho] = 0$ for $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$ is called the *weak joint distribution* of X and Y in ρ , and known to be experimentally accessible by weak measurement and post-selection [26]. We shall show that the above two relations are equivalent together with other equivalent conditions to conclude that the relation $X =_\rho Y$ and the probability $\Pr\{X =_o Y \mid \rho\}$ are experimentally accessible.

Theorem 6.3. *For any observables $X, Y \in \mathcal{O}(\mathcal{M})$ and $\rho \in \mathcal{S}(\mathcal{H})$, the following conditions are all equivalent.*

- (i) $X =_{\rho} Y$, i.e., $\llbracket X =_o Y \rrbracket_o \rho = \rho$.
- (ii) X and Y are perfectly correlated in ρ , i.e., $\text{Tr}[E^X(\Delta)E^Y(\Gamma)\rho] = 0$ for all $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$ with $\Delta \cap \Gamma = \emptyset$.
- (iii) $X\psi = Y\psi$ for all $\psi \in \mathcal{C}(X, \rho)$.
- (iv) $\langle \psi, E^X(\Delta)\psi \rangle = \langle \psi, E^Y(\Delta)\psi \rangle$ for all $\psi \in \mathcal{C}(X, \rho)$.
- (v) $E^X(\Delta)\rho = E^Y(\Delta)\rho$ for all $\Delta \in \mathcal{B}(\mathbf{R})$.
- (vi) $f(X)C(X; \rho) = f(Y)C(X; \rho)$ for all $f \in B(\mathbf{R})$.
- (vii) $C(X; \rho) = C(Y; \rho)$ and $XC(X; \rho) = YC(X; \rho)$.
- (viii) There exists a joint probability distribution $\mu_{\rho}^{X,Y}(x,y)$ of X, Y in ρ that satisfies

$$\mu_{\rho}^{X,Y}(\{(x,y) \in \mathbf{R}^2 \mid x = y\}) = 1. \quad (32)$$

Proof. The assertions follow from Theorem 6.1 above and Theorems 3.2, Theorem 3.4, Theorem 4.3, and Theorem 5.3 in Ref. [23]. \square

The equivalence between (i) and (viii) was previously reported in Theorem 4 in Ref. [25] for the case where $\mathcal{H} < \infty$ and $\mathcal{M} = \mathcal{B}(\mathcal{H})$.

Let $\phi(X_1, \dots, X_n)$ be an observational proposition that is constructed by rules (R1), (R2), (R3) and includes symbols for observables only from the list X_1, \dots, X_n , i.e., $\phi(X_1, \dots, X_n)$ includes only atomic observational propositions of the form $X_j \leq_o x_j$ or $X_j = X_k$, where $j, k = 1, \dots, n$ and x_j is the symbol for an arbitrary real number. In this case, $\phi(X_1, \dots, X_n)$ is said to be an observational proposition in $\mathcal{L}_o(X_1, \dots, X_n)$. Then, $\phi(X_1, \dots, X_n)$ is said to be *contextually well-formed* in a state ρ iff X_1, \dots, X_n are simultaneously determinate in ρ . The following theorem answers the question as to in what state ρ the probability assignment satisfies rules for calculus of classical probability, and shows that for well-formed observational propositions $\phi(X_1, \dots, X_n)$ the projection-valued truth value assignment satisfies Boolean inference rules.

Theorem 6.4. *Let $\phi(X_1, \dots, X_n)$ be an observational proposition in $\mathcal{L}_o(X_1, \dots, X_n)$. If $\phi(X_1, \dots, X_n)$ is a tautology in classical logic, then we have*

$$\text{com}(\tilde{X}_1, \dots, \tilde{X}_n) \leq \llbracket \phi(X_1, \dots, X_n) \rrbracket_o.$$

Moreover, if $\phi(X_1, \dots, X_n)$ is contextually well-formed in a state ρ , then $\phi(X_1, \dots, X_n)$ holds in ρ .

Proof. Suppose that an observational proposition $\phi = \phi(X_1, \dots, X_n)$ is a tautology in classical logic. Let $\tilde{\phi}$ be the corresponding formula in $\mathcal{L}(\in, V^{(\mathcal{Q})})$. Then, it is easy

to see that there is a formula $\phi_0(u_1, \dots, u_n, v_1, \dots, v_m)$ in $\mathcal{L}(V^{(\mathcal{Q})})$ provable in ZFC satisfying $\phi_0(\tilde{X}_1, \dots, \tilde{X}_n, \tilde{r}_1, \dots, \tilde{r}_m) = \tilde{\phi}$ with some real numbers r_1, \dots, r_m . Then, by the ZFC Transfer Principle (Theorem 2.11), we have $\text{com}(\tilde{X}_1, \dots, \tilde{X}_n) \leq \llbracket \tilde{\phi} \rrbracket$. Thus, the assertion follows from relation (31). \square

The above theorem was previously announced as Theorem 3 in Ref. [25] for the case where $\mathcal{H} < \infty$ and $\mathcal{M} = \mathcal{B}(\mathcal{H})$, the proof of which needs quantum set theory and the embedding of the language of observational propositions into the language of quantum set theory developed in this paper.

Note that for any $X, Y \in \mathcal{O}(\mathcal{M})$ there exists a proposition ϕ in $\mathcal{L}_o(\mathcal{M})$ such that $\llbracket \phi \rrbracket_o = \llbracket \tilde{X} = \tilde{Y} \rrbracket$. In fact, we have $\llbracket \tilde{X} = \tilde{Y} \rrbracket \in \mathcal{O}_\omega(\mathcal{M})$ and $\llbracket \llbracket \tilde{X} = \tilde{Y} \rrbracket =_o 1 \rrbracket = \llbracket \tilde{X} = \tilde{Y} \rrbracket$. However, it is not in general possible to construct such ϕ from atomic propositions of the form $X_j \leq_o \lambda$ for $j = 1, \dots, n$ with $\lambda \in \mathbf{R}$. In what follows, we shall show that this is possible for finite observables.

For any finite observables X, Y , we define the observational proposition $X = Y$ by

$$X =_o Y := \bigvee_{x \in \text{Sp}(X)} X =_o x \wedge Y =_o x. \quad (33)$$

Then, we have the following.

Theorem 6.5. *For any finite observables $X, Y \in \mathcal{O}_\omega(\mathcal{H})$, we have*

$$\llbracket X =_o Y \rrbracket_o = \llbracket \tilde{X} = \tilde{Y} \rrbracket. \quad (34)$$

Proof. Let $\psi \in \mathcal{R}(\llbracket X = Y \rrbracket_o)$. Then, for any $x \in \text{Sp}(X)$, we have

$$E^X(\{x\})\psi = E^X(\{x\}) \cap E^Y(\{x\})\psi = E^Y(\{x\})\psi,$$

and for any $x \notin \text{Sp}(X)$, we have $E^X(\{x\})\psi = 0 = E^Y(\{x\})\psi$. Thus, $\psi \in \mathcal{R}(\llbracket \tilde{X} = \tilde{Y} \rrbracket)$ follows from Theorem 6.1 (i). Conversely, suppose $\psi \in \mathcal{R}(\llbracket \tilde{X} = \tilde{Y} \rrbracket)$. Then, for all $x \in \mathbf{R}$, we have $E^X(\{x\})\psi = E^Y(\{x\})\psi$ so that we have $E^X(\{x\}) \wedge E^Y(\{x\})\psi = E^X(\{x\})\psi$. Thus, we have $\llbracket X = Y \rrbracket_o \psi = \psi$. Therefore, the assertion follows. \square

As shown in Ref. [25] for the finite dimensional case, state-dependent equality between finite observables are generally characterized in terms of eigenvectors as follows.

Theorem 6.6. *Finite observables X and Y are equal in a vector state ψ if and only if the state ψ is a superposition of common eigenvectors of X and Y with common eigenvalues.*

7 Measurements of observables

In this and next sections, we shall discuss measurements for a quantum system described by a von Neumann algebra \mathcal{M} on a Hilbert space \mathcal{H} .

A *probability operator-valued measure (POVM)* for a von Neumann algebra \mathcal{M} on \mathbf{R}^n is a mapping $\Pi : \mathcal{B}(\mathbf{R}^n) \rightarrow \mathcal{M}$ satisfying the following conditions.

- (M1) $\Pi(\Delta) \geq 0$ for all $\Delta \in \mathcal{B}(\mathbf{R}^n)$.
- (M2) $\sum_j \Pi(\Delta_j) = 1$ for any disjoint sequence of Borel sets $\Delta_1, \Delta_2, \dots \in \mathcal{B}(\mathbf{R}^n)$ such that $\mathbf{R}^n = \bigcup_j \Delta_j$.

A *measuring process* for \mathcal{M} is defined to be a quadruple $(\mathcal{K}, \sigma, U, M)$ consisting of a Hilbert space \mathcal{K} , a state (density operator) σ on \mathcal{K} , a unitary operator U on $\mathcal{H} \otimes \mathcal{K}$, and an observable M on \mathcal{K} satisfying

$$\mathrm{Tr}_{\mathcal{K}} [U^\dagger (X \otimes E^M(\Delta)) U (1 \otimes \sigma)] \in \mathcal{M} \quad (35)$$

for every $X \in \mathcal{M}$ and $\Delta \in \mathcal{B}(\mathbf{R})$, where $\mathrm{Tr}_{\mathcal{K}}$ stands for the partial trace on \mathcal{K} [20, 16].

A measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{K}, \sigma, U, M)$ with *output variable* \mathbf{x} describes a measurement carried out by an interaction, called the *measuring interaction*, from time 0 to time Δt between the measured system \mathbf{S} described by \mathcal{M} and the *probe* system \mathbf{P} described by $\mathcal{B}(\mathcal{K})$ that is prepared in the state σ at time 0. The outcome of this measurement is obtained by measuring the observable M , called the *meter observable*, in the probe at time Δt . The unitary operator U describes the time evolution of $\mathbf{S} + \mathbf{P}$ from time 0 to Δt . We shall write $M(0) = 1 \otimes M$, $M(\Delta t) = U^\dagger M(0) U$, $X(0) = X \otimes 1$, and $X(\Delta t) = U^\dagger X(0) U$ for any observable $X \in \mathcal{O}(\mathcal{M})$. We can use the probabilistic interpretation for the system $\mathbf{S} + \mathbf{P}$. The *output distribution* $\Pr\{\mathbf{x} \in \Delta \mid \rho\}$, the probability distribution of the output variable \mathbf{x} of this measurement on input state $\rho \in \mathcal{S}(\mathcal{H})$, is naturally defined as

$$\Pr\{\mathbf{x} \in \Delta \mid \rho\} = \Pr\{M(\Delta t) \in \Delta \mid \rho \otimes \sigma\} = \mathrm{Tr}[E^{M(\Delta t)}(\Delta) \rho \otimes \sigma].$$

The *POVM of the measuring process* $\mathbf{M}(\mathbf{x})$ is defined by

$$\Pi(\Delta) = \mathrm{Tr}_{\mathcal{K}} [E^{M(\Delta t)}(\Delta) (I \otimes \sigma)].$$

Then, $\Pi(\Delta) \in \mathcal{M}$ for all $\Delta \in \mathcal{B}(\mathbf{R})$ by Eq. (35) and $\Pi : \mathcal{B}(\mathbf{R}) \rightarrow \mathcal{M}$ is a POVM for \mathcal{M} on \mathbf{R} satisfying

- (M3) $\Pr\{\mathbf{x} \in \Delta \mid \rho\} = \mathrm{Tr}[\Pi(\Delta) \rho]$.

Conversely, from a general result in Ref. [20] it can be easily seen that for every POVM Π for \mathcal{M} on \mathbf{R} , there is a measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{H}, \sigma, U, M)$ for \mathcal{M} satisfying (P3). In fact, for any fixed $\rho_0 \in \mathcal{S}(\mathcal{H})$ the relation $\mathcal{I}(\Delta)^* X = \text{Tr}[X\rho_0]\Pi(\Delta)$ for all $X \in \mathcal{M}$ and $\Delta \in \mathcal{B}(\mathbf{R})$ defines a completely positive instrument for $\mathcal{B}(\mathcal{H})$ on \mathbf{R} , and by Theorem 5.1 in Ref. [20] there exists a measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{H}, \sigma, U, M)$ for $\mathcal{B}(\mathcal{H})$ such that $\text{Tr}[X\rho_0]\Pi(\Delta) = \text{Tr}_{\mathcal{H}}[U^\dagger(X \otimes E^M(\Delta))U(1 \otimes \sigma)]$ for all $X \in \mathcal{M}$ and $\Delta \in \mathcal{B}(\mathbf{R})$. Then, it is easy to see that $\mathbf{M}(\mathbf{x})$ is a measuring process for \mathcal{M} and satisfies (P3). For further accounts of the universality of the class of measurement models described by measuring processes we refer the reader to Ref. [20] for quantum systems with finite degrees of freedom and to Ref. [16] for those with infinite degrees of freedom.

Let $A \in \mathcal{O}(\mathcal{M})$ and $\rho \in \mathcal{S}(\mathcal{H})$. A measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{H}, \sigma, U, M)$ for \mathcal{M} with the POVM Π is said to *measure* A in ρ if $A(0) =_{\rho \otimes \sigma} M(\Delta t)$, and *weakly measure* A in ρ iff $\text{Tr}[\Pi(\Delta)E^A(\Gamma)\rho] = \text{Tr}[E^A(\Delta \cap \Gamma)\rho]$ for any $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$. A measuring process $\mathbf{M}(\mathbf{x})$ is said to *satisfy the Born statistical formula* (BSF) for A in ρ iff it satisfies $\text{Pr}\{\mathbf{x} \in \Delta \mid \rho\} = \text{Tr}[E^A(\Delta)\rho]$ for all $x \in \mathbf{R}$. The following theorem characterizes measurements of an observable in a given state [23].

Theorem 7.1. *Let $\mathbf{M}(\mathbf{x}) = (\mathcal{H}, \sigma, U, M)$ be a measuring process for \mathcal{M} with the POVM $\Pi(\Delta)$. For any observable $A \in \mathcal{O}(\mathcal{M})$ and any state $\rho \in \mathcal{S}(\mathcal{H})$, the following conditions are all equivalent.*

- (i) $\mathbf{M}(\mathbf{x})$ *measures* A in ρ .
- (ii) $\mathbf{M}(\mathbf{x})$ *weakly measures* A in ρ .
- (iii) $\mathbf{M}(\mathbf{x})$ *satisfies the BSF for* A *in any vector state* $\psi \in \mathcal{C}(A, \rho)$.

In the conventional approach, a measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{H}, \sigma, U, M)$ with the POVM Π is considered to be a measurement of an observable A iff $\Pi = E^A$ [20], since in this case the probability distribution of A predicted by the Born formula is reproduced by the probability distribution of Π in any state. However, in this approach it is not clear whether a measurement of an observable A actually reproduces the value of the observable A just before the measurement. The following theorem, which is an immediate consequence of Theorem 7.1, ensures that this is indeed the case (cf. the remark after Theorem 8.2 in Ref. [23]).

Theorem 7.2. *Let $\mathbf{M}(\mathbf{x}) = (\mathcal{H}, \sigma, U, M)$ be a measuring process for \mathcal{M} with the POVM Π . Then, $\mathbf{M}(\mathbf{x})$ *measures* $A \in \mathcal{O}(\mathcal{M})$ *in any* $\rho \in \mathcal{S}(\mathcal{H})$ *if and only if* $\Pi = E^A$ *for all* $x \in \mathbf{R}$.*

8 Simultaneous measurability

For any measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{H}, \sigma, U, M)$ for \mathcal{M} and a real-valued Borel function f , the measuring process $\mathbf{M}(f(\mathbf{x}))$ with output variable $f(\mathbf{x})$ is defined by $\mathbf{M}(f(\mathbf{x})) = (\mathcal{H}, \sigma, U, f(M))$. Observables A, B are said to be *simultaneously measurable* in a state $\rho \in \mathcal{S}(\mathcal{H})$ by $\mathbf{M}(\mathbf{x})$ iff there are Borel functions f, g such that $\mathbf{M}(f(\mathbf{x}))$ and $\mathbf{M}(g(\mathbf{x}))$ measure A and B in ρ , respectively. Observables A, B are said to be *simultaneously measurable* in ρ iff there is a measuring process $\mathbf{M}(\mathbf{x})$ such that A and B are simultaneously measurable in ρ by $\mathbf{M}(\mathbf{x})$.

Simultaneous measurability and simultaneous determinateness are not equivalent notions under the state-dependent formulation, as the following theorem clarifies; the case where $\dim(\mathcal{H}) < \infty$ was previously reported in Ref. [25], Theorem10.

Theorem 8.1. (i) *Two observables $A, B \in \mathcal{O}(\mathcal{M})$ are simultaneously determinate in a state $\rho \in \mathcal{S}(\mathcal{H})$ if and only if there exists a POVM Π for \mathcal{M} on \mathbf{R}^2 satisfying*

$$\Pi(\Delta \times \mathbf{R}) = E^A(\Delta) \quad \text{on } \mathcal{C}(A, B, \rho) \text{ for all } \Delta \in \mathbf{R}, \quad (36)$$

$$\Pi(\mathbf{R} \times \Gamma) = E^B(\Gamma) \quad \text{on } \mathcal{C}(A, B, \rho) \text{ for all } \Gamma \in \mathbf{R}. \quad (37)$$

(ii) *Two observables $A, B \in \mathcal{O}(\mathcal{M})$ are simultaneously measurable in a state $\rho \in \mathcal{S}(\mathcal{H})$ if and only if there exists a POVM Π for \mathcal{M} on \mathbf{R}^2 satisfying*

$$\Pi(\Delta \times \mathbf{R}) = E^A(\Delta) \quad \text{on } \mathcal{C}(A, \rho) \text{ for all } \Delta \in \mathbf{R}, \quad (38)$$

$$\Pi(\mathbf{R} \times \Gamma) = E^B(\Gamma) \quad \text{on } \mathcal{C}(B, \rho) \text{ for all } \Gamma \in \mathbf{R}. \quad (39)$$

(iii) *Two observables $A, B \in \mathcal{O}(\mathcal{M})$ are simultaneously measurable in a state $\rho \in \mathcal{S}(\mathcal{H})$ if they are simultaneously determinate in ρ .*

Proof. Let $\mathcal{C} = \mathcal{C}(A, B, \rho)$ and $C = C(A, B; \rho)$.

(i) (only if part): Let $G = \text{com}(\tilde{A}, \tilde{B})$. Then, $G \in \mathcal{M}$ and $AG \perp BG$. Let Π be the joint spectral measure of AG and BG , i.e., $\Pi(\Delta \times \Gamma) = E^{AG}(\Delta)E^{BG}(\Gamma)$ for all $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$. Then, Π is a POVM for \mathcal{M} on \mathbf{R}^2 . Suppose that A and B are simultaneously determinate in a state ρ . Then, $\text{ran}(\rho) \subseteq \text{ran}(\text{com}(\tilde{A}, \tilde{B}))$. By the minimality of $C(A, B, \rho)$ among (A, B) -invariant subspaces, we have $C \leq G$ and $AG, BG \perp C$. Thus, we have $\Pi(\Delta \times \mathbf{R})C = E^{AG}(\Delta)C = E^{AC}(\Delta)C = E^A(\Delta)C$ and similarly $\Pi(\mathbf{R} \times \Gamma)C = E^B(\Gamma)C$ for all $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$. Thus, Π satisfies Eqs. (36) and (37).

(i) (if part): Let Π be a POVM for \mathcal{M} on \mathbf{R}^2 satisfying (36) and (37). Let Π' be a positive operator valued measure for $\mathcal{B}(\mathcal{H})$ on \mathbf{R}^2 defined by $\Pi'(\Delta \times \Gamma) = C\Pi(\Delta \times \Gamma)$.

$\Gamma)C$ for all $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$. Let Π'' be a POVM for $\mathcal{B}(\mathcal{C})$ on \mathbf{R}^2 obtained by restricting Π' to \mathcal{C} . Let $\Delta, \Gamma \in \mathcal{B}(\mathbf{R})$. By the definition of C , we have $E^A(\Delta)C = CE^A(\Delta) = CE^A(\Delta)C$ and $E^A(\Delta)C$ is a projection. Similarly, $E^B(\Gamma)C = CE^B(\Gamma) = CE^B(\Gamma)C$ and $E^B(\Gamma)C$ is a projection. Thus, we have $\Pi''(\Delta \times \mathbf{R}) = C\Pi(\Delta \times \mathbf{R})C = E^A(\Delta)C$, and similarly $\Pi''(\mathbf{R} \times \Gamma) = C\Pi(\mathbf{R} \times \Gamma)C = E^B(\Gamma)C$. Since Δ and Γ were arbitrary, the marginals of Π'' are projection-valued. By a well-know theorem (e.g., Ref. [8], Theorem 3.2.1), the marginals commute and Π'' is the product of their marginals. Thus, we have $AC \perp BC$, and hence by Theorem 5.2, A and B are simultaneously determinate.

(ii) (only if part): Suppose that $A, B \in \mathcal{O}(\mathcal{M})$ are simultaneously measurable in $\rho \in \mathcal{S}(\mathcal{H})$. Then, we have a measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{X}, \sigma, U, M)$ for \mathcal{M} and real-valued Borel functions f, g such that $\mathbf{M}(f(\mathbf{x}))$ measures A in ρ and $\mathbf{M}(g(\mathbf{x}))$ measures B in ρ . Let Π_0 be the POVM of $\mathbf{M}(\mathbf{x})$. Let Π be a POVM on \mathbf{R}^2 such that $\Pi(\Delta \times \Gamma) = \Pi_0(f^{-1}(\Delta) \cap g^{-1}(\Gamma))$. Then, it is easy to see that Π satisfies Eqs. (38) and (39).

(ii) (if part) Let Π be a POVM for \mathcal{H} on \mathbf{R}^2 satisfying Eqs. (38) and (39). Then, by the remark after condition (M3) in Section 7 there exists a measuring process $\mathbf{M}(\mathbf{x}) = (\mathcal{X}, \sigma, U, M)$ for \mathcal{M} and real-valued Borel functions f, g such that

$$\Pi(\Delta \times \Gamma) = \text{Tr}_{\mathcal{X}} [U^\dagger (I \otimes E^{f(M)}(\Delta) E^{g(M)}(\Gamma)) U (I \otimes \sigma)]. \quad (40)$$

Then, we have

$$\Pi(\Delta \times \mathbf{R}) = \text{Tr}_{\mathcal{X}} [U^\dagger (I \otimes E^{f(M)}(\Delta)) U (I \otimes \sigma)], \quad (41)$$

so that from Eq. (38) we have $\mathbf{M}(f(\mathbf{x}))$ measures A in ρ . Similarly, we can show that $\mathbf{M}(g(\mathbf{x}))$ measures B in ρ .

Assertion (iii) follows from (i) and (ii). \square

Discussions on physical significance of the state-dependent formulation of simultaneous measurability have been given in Ref. [25] for the finite dimensional case. Further discussions on the state-dependent formulation of quantum measurement theory will appear elsewhere.

9 Conclusion

Quantum set theory originated from the method of forcing introduced by Cohen [6, 7] for the independence proof of the continuum hypothesis and from quantum logic introduced by Birkhoff and von Neumann [3] for logical axiomatization of quantum mechanics. After Cohen's work, Scott and Solovay [30] reformulated the forcing method by Boolean-valued models of set theory [1], which have become a central method in

the field of axiomatic set theory. In 1978 Takeuti [31] started Boolean-valued analysis, which provides systematic applications of logical meta-theorems for Boolean-valued models to *not meta*-mathematical problems mainly in analysis. Among others, Boolean-valued analysis made a great successes in operator algebras [34, 33, 18] and especially in solving a long-standing open problem in the structure theory of type I algebras applying the forcing method for cardinal collapsing [17, 19, 21].

As a successor of those attempts, quantum set theory, a set theory based on the Birkhoff-von Neumann quantum logic, was introduced by Takeuti [32], who established the one-to-one correspondence between reals in the model (quantum reals) and quantum observables. Quantum set theory was recently developed by the present author [24, 27] to obtain the transfer principle to determine quantum truth values of theorems of the ZFC set theory, and to clarify the operational meaning of equality between quantum reals, which extends the probabilistic interpretation of quantum theory,

To formulate the standard probabilistic interpretation of quantum theory, we have introduced the language of observational propositions with rules (R1) and (R2) for well-formed formulas constructed from atomic formulas of the form $X \leq_o x$, rules (T1), (T2), and (T3) for projection-valued truth value assignment, and rule (P1) for probability assignment. Then, the standard probabilistic interpretation gives the statistical predictions for standard observational propositions formulated by (W1), which concern only a commuting family of observables. The Born statistical formula is naturally derived in this way. We have extended the standard interpretation by introducing the notion of simultaneous determinateness and atomic formulas of the form $X = Y$ for equality. To extended observational propositions formed through rules (R1), ..., (R4), the projection-valued truth values are assigned by rule (T1), ..., (T4), and the probabilities are assigned by rule (P1). Then, we can naturally extend the standard interpretation to a general and state-dependent interpretation for observational propositions including the relations of simultaneous determinateness and equality. Quantum set theory ensures that any contextually well-formed formula provable in ZFC has the probability assigned to be 1. This extends the classical inference for quantum theoretical predictions from commuting observables to simultaneously determinate observables. We apply this new interpretation to construct a theory of measurement of observables in the state-dependent approach, to which the standard interpretation cannot apply. We have reported only basic formulations here, but further development in this approach will be reported elsewhere.

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