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LOGIC OF DESCRIPTIONS.
A NEW APPROACH TO THE FOUNDATIONS
OF MATHEMATICS AND SCIENCE

Abstract. We study a new formal logic LD introduced by Prof. Grzegorczyk. The logic is based on so-called *descriptive equivalence*, corresponding to the idea of shared meaning rather than shared truth value. We construct a semantics for LD based on a new type of algebras and prove its soundness and completeness. We further show several examples of classical laws that hold for LD as well as laws that fail. Finally, we list a number of open problems.

Keywords: non-classical logic, logic of descriptions, equivalence connective, paradoxes of implication

1. Introduction

Logic arose from philosophical and linguistic reflections that began in ancient Greece and later spread throughout Europe. In the 20$^{\text{th}}$ century, *formal logical systems*, especially for classical two-valued logic, achieved perfection and became the gold standard in the foundations of mathematics (and of science in general). However, every now and then a philosopher, a logician or a mathematician has expressed doubts and objections concerning this standard. These objections have been made on various grounds, and many so-called "non-classical" logics have been proposed to rectify the perceived faults, such as modal, intuitionistic, conditional, relevant, paraconsistent, free, quantum, fuzzy, independence-friendly, and so on. Nevertheless, none of these logics has been generally accepted as the right one, and a resolution to the arguments about their practical and philosophical merits and drawbacks is nowhere in sight.

Historically, logic was born out of attempts to explain the structure of human reasoning. It should be emphasized that ancient logicians did not aspire to create an abstract model of human thought, akin to modern attempts at passing the Turing test. The lofty goal of their reflections on

logical principles was to find their way into the very essence of reality. This goal, however utopian it may have been, was consistently pursued by the philosophers who brought forth the logic revolution. However, the invention of formal methods in mathematics – and hence a means to achieve unprecedented rigour – led to logic being "taken over" by engineers and computer scientists. This observation is not meant as a criticism. The results obtained in the field of mathematical logic, as well as its fruitful applications in information technology are impressive indeed. Nevertheless, despite the great success of logic in these areas, we may still ask whether the formal systems commonly used in mathematical logic can serve as adequate tools for understanding human reasoning.

Prof. Grzegorczyk treats this question in his recent article [2011], which can be described as a manifesto calling for the creation of new logical principles suitable for *scientific description of reality* and for the revision of the current standard; that is, various versions of classical two-valued logic.

One of Prof. Grzegorczyk's objections to classical logic is the fact that it "restricts itself to considering only one, admittedly the most important, parameter of the content of a claim, namely its *truth value*" [2011, p. 446], which – as the author points out – is the source of the paradoxical nature of certain tautologies involving implication and equivalence, such as *false implies everything, anything implies the truth, any true sentences are equivalent regardless of their content.* As the author explains, such tautologies are useful in formal deductions in the technical sense, but do not otherwise contribute to understanding. Of course, the paradoxes of material implication have been widely discussed elsewhere, and several non-classical logics have been created in order to solve them. However, the main point of the paper does not involve material implication as such, but rather the problematic nature of material equivalence, which is clearly seen in the following example. In mathematics, one may say that the equations $x + 2 = 3$ and $1 - x = 0$ "mean the same" or "say the same thing in different ways" because they are logically equivalent; that is, their truth values are the same for any given value of x. On the other hand, any two true propositions are equivalent to each other. So, if we consistently speak of equivalent propositions as "meaning the same", we end up claiming that "$2 + 2 = 4$" means the same as "Warsaw lies on the Vistula river." According to Prof. Grzegorczyk, this shows that

> Such combined propositions, established as equivalences according to classical formal logic, do not lead to anything interesting. The paradoxical nature of the classical concept of equivalence arises because we may expect equivalence

to open the possibility of linking the content of one claim with that of the other. We would like equivalent sentences not only to be equally true, but also TO SPEAK ABOUT THE SAME SUBJECT. It seems (from a philosophical point of view) that claims that are not connected by a common subject cannot be treated as fully equivalent. [2011, p. 447]

To remedy the ills of classical equivalence and to avoid the above-mentioned paradox, we should carefully distinguish between two kinds of equivalence, which are:

1. "truth-functional equivalence" – the condition that the truth values of two propositions are the same; this is classical equivalence $\leftrightarrow$ ("coarse, even cynically paradoxical").
2. "descriptive equivalence"[1] – the condition that the meanings of two propositions are the same ("more subtle, but not totally determined, allowing for an intuitive interpretation of being connected by a shared subject").

Introducing a new connective involves describing its usage, which naturally leads one to consider a new logical formalism in which the classical equivalence connective $\leftrightarrow$ has been replaced with a descriptive equivalence connective, which, according to Prof. Grzegorczyk, better reflects human ways of thinking.

We will use the symbol $\equiv$ to denote the new connective. The new logic, denoted here by LD, is defined by rules of inference and a set of axioms. In the article [Grzegorczyk, 2011], a number of important questions concerning the new logic are raised. Firstly, do the new equivalence and the corresponding implication coincide with their respective classical counterparts? Secondly, can we define a semantics for which the given syntactic proof system is sound and complete?

In the present article, we analyze the logic LD as presented in [Grzegorczyk, 2011]. We will show that the descriptive equivalence connective is indeed different from the classical one. In fact, our further results show that the new logic is substantially different from many other known ones, representing various kinds of non-classical logics. One of our central results is the construction of a semantics for LD, which, in turn, allows us to prove further, rather peculiar, properties of LD, shedding some light on the obscure secrets of descriptive equivalence.

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[1] In a draft version of the article [Grzegorczyk, 2011], the new connective is called the *equimeaning* connective, while in its final, published version, it is named the *perceptive equivalence* connective. Later on, Prof. Grzegorczyk came to the conclusion that the most fitting name for the new connective is the *descriptive equivalence* connective, and this is the terminological convention we will follow.

It is worth noting that the distinction between classical and descriptive equivalence is essentially the same as the distinction between the truth value and the meaning of a sentence. Roman Suszko, among others, argued for the need, and even necessity, to consider the latter distinction when building the semantical basis for a logical system, and he introduced so-called *non-Fregean logic* as a formalization of this idea (see [Suszko, 1968]). The sentential version of non-Fregean logic (called *Sentential Calculus with Identity*, SCI) is obtained from classical sentential logic by adding a new *identity connective*. Suszko's philosophical motivations for creating SCI were similar to those of Prof. Grzegorczyk for creating LD. By coincidence, both of them chose the symbol $\equiv$ for essentially the same purpose: to denote descriptive equivalence in LD and identity of sentences in SCI. However, their intuitions and underlying philosophical assumptions have led to two very different formalisms. The logic SCI is based on classical logic, which is simply extended by adding new axioms expressing the properties of sentential identity. On the other hand, LD is built from the ground up to reflect the interactions between descriptive equivalence and the basic connectives of negation, disjunction and conjunction, introducing counterparts of many classical laws involving equivalence and omitting others. Nevertheless, the logics of Suszko and Grzegorczyk have many common elements, which can be seen especially in our construction of a semantics for LD.

The paper is organized as follows. In Section 2, we present the language and the Hilbert-style axiomatization of LD with examples of LD-provable formulas. We present a semantics and prove its soundness and completeness in Section 3. In Section 4, we discuss some interesting properties of LD, in particular classical results that fail for LD and the independence of the axioms. In Section 5, we study two proposed alternative formulations of LD, showing how they fail to fulfill the philosophical motivations behind LD. Conclusions and open problems are presented in Section 6. To avoid cluttering the main text with excessive tables, we present most example models in the Appendix.

Our results rely heavily on computer software for semi-automatic proof generation and model checking, written by the second author. The software and associated files are available upon request.

2. Logic LD: axiomatization

Logic LD belongs to the family of propositional logics. The vocabulary of the logic LD consists of the following pairwise disjoint sets of symbols:

- $\mathbb{V} = \{p_0, p_1, p_2, \ldots\}$ – an infinite countable set of propositional variables,
- $\{\neg, \vee, \wedge, \equiv\}$ – propositional operations of negation $\neg$, disjunction $\vee$, conjunction $\wedge$, and descriptive equivalence $\equiv$.

In practice, we will use the symbols p, q, r instead of the "official" subscripted ones.

As usual in propositional logics, we define the set of LD-formulas as the smallest set that contains all the propositional variables and is closed under all the propositional operations. The logic LD is given by the Hilbert-style axiomatization. Below we list the axioms and rules of inference of LD in their original forms from [Grzegorczyk, 2011].

Axioms:[2]

Ax1	$p \equiv p$
Ax2	$(p \equiv q) \equiv (q \equiv p)$
Ax3	$(p \equiv q) \equiv [(p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r))]$
Ax4	$(p \equiv q) \equiv (\neg p \equiv \neg q)$
Ax5	$(p \equiv q) \equiv [(p \equiv q) \wedge ((p \vee r) \equiv (q \vee r))]$
Ax6	$(p \equiv q) \equiv [(p \equiv q) \wedge ((p \wedge r) \equiv (q \wedge r))]$
Ax7	$(p \vee q) \equiv (q \vee p)$
Ax8	$(p \vee (q \vee r)) \equiv ((p \vee q) \vee r)$
Ax9	$p \equiv (p \vee p)$
Ax10	$(p \wedge q) \equiv (q \wedge p)$
Ax11	$(p \wedge (q \wedge r)) \equiv ((p \wedge q) \wedge r)$
Ax12	$p \equiv (p \wedge p)$
Ax13	$(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$
Ax14	$(p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r))$
Ax15	$\neg(p \vee q) \equiv (\neg p \wedge \neg q)$
Ax16	$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$
Ax17	$\neg\neg p \equiv p$
Ax18	$\neg(p \wedge \neg p)$

Observe that among the eighteen axioms of LD, only one axiom, namely Ax18, does not involve the descriptive equivalence connective. Moreover, the rest of the axioms can be divided into three groups, with Ax3 playing a double role. First, the axioms Ax1–Ax3 express the basic properties of

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[2] Here we adopt a revised form of the axiom Ax3, presented in Errata 2012 to [Grzegorczyk, 2011]. We will discuss the original form in Section 5.

descriptive equivalence, namely its reflexivity, symmetry, and transitivity. Axioms Ax3–Ax6 formulate the idea that equals can be substituted for each other. Axioms Ax7–Ax17 state some basic properties of equivalence of compound formulas built with the classical connectives of negation, conjunction, and disjunction, which are: associativity, commutativity, and idempotency of conjunction and disjunction (axioms Ax7–Ax12), distributivity of conjunction (resp. disjunction) over disjunction (resp. conjunction) (axioms Ax13 and Ax14), involution of negation that additionally satisfies de Morgan laws (axioms Ax15–Ax17).

Rules of inference:

$$(\mathsf{MP_{LD}}) \quad \frac{\varphi \equiv \psi, \varphi}{\psi} \qquad\qquad (\mathsf{Sub}) \quad \frac{\varphi(p_0, \ldots, p_n)}{\varphi(p_0/\psi_0, \ldots, p_n//\psi_n)}$$

$$(\wedge_1) \quad \frac{\varphi, \psi}{\varphi \wedge \psi} \qquad\qquad (\wedge_2) \quad \frac{\varphi \wedge \psi}{\varphi, \psi}$$

For technical reasons, we impose the additional restriction that the rule (Sub) may be applied only to axioms. As in the classical case, this restriction is not essential when no additional assumptions are used in the proof.

The rules (Sub), ($\wedge_1$), and ($\wedge_2$) are standard in classical logic. However, the crucial feature of LD is that it contains a *modus ponens*-type rule only with respect to descriptive equivalence, while the classical modus ponens rule is not present. Furthermore, LD does not have any rule for introduction or elimination of disjunction or negation. As we will show later, only a disjunction introduction rule is derivable in LD.

An LD-formula φ is said to be LD-*provable* ($\vdash \varphi$ for short) whenever there exists a finite sequence $\varphi_1, \ldots, \varphi_n$ of LD-formulas, $n \geq 1$, such that $\varphi_n = \varphi$ and each φ_i, $i \in \{1, \ldots, n\}$, is an axiom or follows from earlier formulas in the sequence by one of the rules of inference. If X is any set of LD-formulas, then φ is said to be LD-*provable from* X ($X \vdash \varphi$ for short) whenever there exists a finite sequence $\varphi_1, \ldots, \varphi_n$ of LD-formulas, $n \geq 1$, such that $\varphi_n = \varphi$ and for each $i \in \{1, \ldots, n\}$, φ_i is an axiom or $\varphi_i \in X$ or φ_i follows from earlier formulas in the sequence by one of the rules of inference.

Now, it is worth noting that the logic LD is consistent, as the interpretation of $\equiv$ as the usual classical equivalence yields that all the axioms are classical tautologies and all the rules preserve classical validity. Hence, one of all possible models of LD is the two-element Boolean algebra of the classical propositional logic. This implies also that the formula $p \equiv \neg p$ is not LD-provable.

The axiomatization of LD enables us to prove many of the classical laws, but not all. In particular, the formula $\varphi \vee \neg\varphi$ is LD-provable.

$\vdash (\varphi \vee \neg\varphi)$, for any LD-formula φ.

(1) $\quad \neg(\varphi \wedge \neg\varphi)$ $\hfill$ (Sub) to Ax18 for p/φ

(2) $\quad \neg(\varphi \wedge \neg\varphi) \equiv (\neg\varphi \vee \neg\neg\varphi)$ $\hfill$ (Sub) to Ax16 for p/φ, $q/\neg\varphi$

(3) $\quad \neg\varphi \vee \neg\neg\varphi$ $\hfill$ (MP_{LD}) to (1) and (2)

(4) $\quad (\neg\varphi \vee \neg\neg\varphi) \equiv (\neg\neg\varphi \vee \neg\varphi)$ $\hfill$ (Sub) to Ax7 for $p/\neg\varphi$, $q/\neg\neg\varphi$

(5) $\quad \neg\neg\varphi \vee \neg\varphi$ $\hfill$ (MP_{LD}) to (3) and (4)

(6) $\quad \neg\neg\varphi \equiv \varphi$ $\hfill$ (Sub) to Ax17 for p/φ

(7) $\quad (\neg\neg\varphi \equiv \varphi) \equiv [(\neg\neg\varphi \equiv \varphi) \wedge ((\neg\neg\varphi \vee \neg\varphi) \equiv (\varphi \vee \neg\varphi))]$
$\hfill$ (Sub) to Ax5 for $p/\neg\neg\varphi$, q/φ, r/φ

(8) $\quad (\neg\neg\varphi \equiv \varphi) \wedge ((\neg\neg\varphi \vee \neg\varphi) \equiv (\varphi \vee \neg\varphi))$ $\hfill$ (MP_{LD}) to (6) and (7)

(9) $\quad (\neg\neg\varphi \vee \neg\varphi) \equiv (\varphi \vee \neg\varphi)$ $\hfill$ $(\wedge_2)$ to (8)

(10) $\quad \varphi \vee \neg\varphi$ $\hfill$ (MP_{LD}) to (5) and (9)

Also the following formula is provable in LD:

$$\neg(\varphi \wedge (\neg\varphi \vee \psi)) \vee \psi^3.$$

Below we sketch its proof.

$\vdash \neg(\varphi \wedge (\neg\varphi \vee \psi)) \vee \psi$, for all LD-formulas φ and ψ.

(1) $\quad (\neg\varphi \vee \psi) \vee \neg(\neg\varphi \vee \psi)$ $\hfill$ (Sub) to Ax18 for $p/(\neg\varphi \vee \psi)$

(2) $\quad (\neg\varphi \vee \neg(\neg\varphi \vee \psi)) \vee \psi$ $\hfill$ from (1) by Ax7 and Ax8

(3) $\quad \neg(\varphi \wedge (\neg\varphi \vee \psi)) \vee \psi$ $\hfill$ from (2) by Ax16

It is also easy to see that the following rule is a *derived* LD-rule:

$$(\text{tran}) \qquad \frac{\varphi \equiv \psi, \psi \equiv \vartheta}{\varphi \equiv \vartheta},$$

which can be proved as follows:

(1) $\quad \varphi \equiv \psi$

(2) $\quad \psi \equiv \vartheta$

(3) $\quad (\varphi \equiv \psi) \equiv [(\varphi \equiv \psi) \wedge ((\varphi \equiv \vartheta) \equiv (\psi \equiv \vartheta))]$
$\hfill$ (Sub) to Ax3 for p/φ, q/ψ, r/ϑ

(4) $\quad (\varphi \equiv \psi) \wedge ((\varphi \equiv \vartheta) \equiv (\psi \equiv \vartheta))$ $\hfill$ (MP_{LD}) to (1) and (3)

(5) $\quad (\varphi \equiv \vartheta) \equiv (\psi \equiv \vartheta)$ $\hfill$ $(\wedge_2)$ to (4)

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^{3} If the usual definition of the classical implication is assumed, that is, $(p \rightarrow q) \equiv (\neg p \vee q)$, then the formula $\neg(\varphi \wedge (\neg\varphi \vee \psi)) \vee \psi$ can be abbreviated as $(\varphi \wedge (\varphi \rightarrow \psi)) \rightarrow \psi$.

(6) $[(\varphi \equiv \vartheta) \equiv (\psi \equiv \vartheta)] \equiv [(\psi \equiv \vartheta) \equiv (\varphi \equiv \vartheta)]$

$\qquad\qquad\qquad\qquad$ (Sub) to Ax2 for $p/(\varphi \equiv \vartheta),\ q/(\psi \equiv \vartheta)$

(7) $(\psi \equiv \vartheta) \equiv (\varphi \equiv \vartheta)$ $\qquad\qquad\qquad\qquad$ (MP$_{\text{LD}}$) to (5) and (6)

(8) $(\varphi \equiv \vartheta)$ $\qquad\qquad\qquad\qquad\qquad\qquad$ (MP$_{\text{LD}}$) to (2) and (7)

However, a distinguishing feature of LD is that the algebra of its formulas does not form a Boolean algebra, since neither the absorption laws

$$p \vee (p \wedge q) \equiv p \quad \text{and} \quad p \wedge (p \vee q) \equiv p,$$

nor the boundness laws

$$(p \vee \neg p) \equiv (q \vee \neg q) \quad \text{and} \quad \neg(p \wedge \neg q) \equiv \neg(q \wedge \neg q),$$

are adopted as axioms, nor – as we will show later – are they LD-provable. One of the motivations for not allowing these laws is that we do not wish to treat two tautologies as identical in meaning if their contents are completely different. For instance, the sentences, "Professor Grzegorczyk is in the next room or he is not there" and "President Obama is in the next room or he is not there" are both tautologies and hence logically equivalent, but they are derived from claims concerning different persons, so their contents are different. To quote Prof. Grzegorczyk: "Of course, mathematicians do not concern themselves with anything outside imagined reality, where everything consistent is acceptable. They may thus consider all tautologies to have the same meaning. However, a philosopher ought to be more careful."[4]

A further interesting feature of the logic LD is concerned with possible derived rules allowed in LD. As we showed, the formula $\neg(\varphi \wedge (\neg\varphi \vee \psi)) \vee \psi$ is provable in LD, but the corresponding rule, that is, the classical rule of modus ponens:

$$\frac{\varphi, (\neg\varphi \vee \psi)}{\psi},$$

is not derivable in LD in the sense that the premises may be satisfied by a valuation that fails to satisfy the conclusion, as we will show in Section 4.

We end this section with a short discussion of the redundancy of the LD-axiomatization presented above. It can be easily seen that some LD-axioms are redundant, since they follow from the others. For example, the reflexivity of $\equiv$ follows from its symmetry and transitivity and the fact that every formula is equivalent to some other formula. The proof can be formalized easily enough.

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[4] Personal communication, 2012-01-21

(1)	$\neg\neg p \equiv p$	Ax17
(2)	$(\neg\neg p \equiv p) \equiv ((\neg\neg p \equiv p) \wedge ((\neg\neg p \equiv p) \equiv (p \equiv p)))$	
		(Sub) to Ax3 $p/\neg\neg p, q/p, r/p$
(3)	$(\neg\neg p \equiv p) \wedge ((\neg\neg p \equiv p) \equiv (p \equiv p))$	MP_{LD} to (1) and (2)
(4)	$(\neg\neg p \equiv p) \equiv (p \equiv p)$	$(\wedge_2)$ to (3)
(5)	$p \equiv p$	MP_{LD} to (1) and (4)

Moreover, it seems intuitively plausible that Ax5 and Ax6 should be provable from each other using DeMorgan laws. This is indeed the case, even though the formal proofs turn out to be quite lengthy. In the same way, we can eliminate half of the axioms Ax7 through Ax16. More precisely, let LD_{red} be obtained from LD by removing Ax1, Ax5, Ax7, Ax8, Ax9, Ax14, and Ax16. Then, the following holds:

Proposition 1

The axioms Ax1, Ax5, Ax7, Ax8, Ax9, Ax14, and Ax16 are provable in LD_{red}.

Proof.

Ax1: Already proved above.
Ax5: We have

$$
\begin{aligned}
(p \equiv q) \ &\equiv \ (\neg p \equiv \neg q) \\
&\equiv \ [(\neg p \equiv \neg q) \wedge ((\neg p \wedge \neg r) \equiv (\neg q \wedge \neg r))] \\
&\equiv \ [(\neg p \equiv \neg q) \wedge (\neg(p \vee r) \equiv \neg(q \vee r))] \\
&\equiv \ [(p \equiv q) \wedge ((p \vee r) \equiv (q \vee r))].
\end{aligned}
$$

Even though this outline seems simple and convincing enough, as it is easy to check that each step involves only substitutions that are directly justified by axioms and do not occur in the scope of a disjunction, expanding it to a full formal proof is surprisingly tedious. Our computer-generated proof consists of 8 axioms and 75 applications of rules. One could plausibly find a significantly shorter proof, but we assume it would still consist of several dozen lines, as intuitively obvious substitutions sometimes require rather complicated formal manipulations.

We will prove Ax16 next to be able to use it in the remaining proofs.

$$
\begin{aligned}
\text{Ax16:} \qquad \neg(p \wedge q) \ &\equiv \ \neg(\neg\neg p \wedge \neg\neg q) \\
&\equiv \ \neg\neg(\neg p \vee \neg q) \\
&\equiv \ (\neg p \vee \neg q).
\end{aligned}
$$

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Ax7:
$$(p \vee q) \equiv \neg\neg(p \vee q)$$
$$\equiv \neg(\neg p \wedge \neg q)$$
$$\equiv \neg(\neg q \wedge \neg p)$$
$$\equiv \neg\neg(q \vee p)$$
$$\equiv (q \vee p).$$

Ax8:
$$(p \vee (q \vee r)) \equiv \neg\neg(p \vee (q \vee r))$$
$$\equiv \neg(\neg p \wedge \neg(q \vee r))$$
$$\equiv \neg(\neg p \wedge (\neg q \wedge \neg r))$$
$$\equiv \neg((\neg p \wedge \neg q) \wedge \neg r)$$
$$\equiv \neg(\neg(p \vee q) \wedge \neg r)$$
$$\equiv \neg\neg((p \vee q) \vee r)$$
$$\equiv ((p \vee q) \vee r).$$

Ax9:
$$(p \equiv (p \vee p)) \equiv (\neg p \equiv \neg(p \vee p))$$
$$\equiv (\neg p \equiv (\neg p \wedge \neg p)),$$

and the last equivalence is obtained from Ax12 by substitution.

Ax14:
$$((p \vee (q \wedge r)) \equiv ((p \vee q) \wedge (p \vee r)))$$
$$\equiv (\neg(p \vee (q \wedge r)) \equiv \neg((p \vee q) \wedge (p \vee r)))$$
$$\equiv ((\neg p \wedge \neg(q \wedge r)) \equiv (\neg(p \vee q) \vee \neg(p \vee r)))$$
$$\equiv ((\neg p \wedge (\neg q \vee \neg r)) \equiv ((\neg p \wedge \neg q) \vee (\neg p \wedge \neg r))),$$

and again, the last element of the equivalence chain is obtained by substitution, this time from Ax13. ∎

The mostly computer-generated full formal proofs and program sources are available from the authors upon request. In Section 4 we will show that all axioms of LD_{red} are independent from each other.

3. Logic LD: semantics

The logic LD is originally given by Hilbert-style axiomatization. So, a natural problem to be solved is to provide a sound as well as *complete* semantics for LD. In this section, on the basis of some modifications of non-Fregean models for the logic SCI, we define a suitable class of structures for LD, and then we prove its soundness and completeness. First, we introduce some useful notions.

A structure $(U, \oplus, \otimes)$ is said to be a *distributive bisemilattice* whenever the following hold, for all $a, b, c \in U$ and for any $\odot \in \{\otimes, \oplus\}$:

- $a \odot b = b \odot a,$ (commutativity of $\otimes, \oplus$)
- $a \odot (b \odot c) = (a \odot b) \odot c,$ (associativity of $\otimes, \oplus$)
- $a \odot a\ = a,$ (idempotency of $\otimes, \oplus$)
- $a \oplus (b \otimes c) = (a \oplus b) \otimes (a \oplus c),$ (distributivity of $\otimes$ over $\oplus$)
- $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c).$ (distributivity of $\oplus$ over $\otimes$)

    A *de Morgan bisemilattice* is a structure $(U, \sim, \oplus, \otimes)$ such that $(U, \oplus, \otimes)$ is a distributive bisemilattice and for all $a, b \in U$, the following hold:

- $\sim\sim a\ = a,$
- $\sim(a \oplus b) = \sim a\ \otimes \sim b.$

    A *Grzegorczyk algebra* is a structure $(U, \sim, \oplus, \otimes, \circ)$ such that $(U, \sim, \oplus, \otimes)$ is a de Morgan bisemilattice and for all $a, b, c \in U$, the following hold:

- $a \circ b = b \circ a,$
- $a \circ b = \sim a\ \circ \sim b,$
- $a \circ b = (a \circ b) \otimes ((a \circ c) \circ (b \circ c)),$
- $a \circ b = (a \circ b) \otimes ((a \oplus c) \circ (b \oplus c)),$
- $a \circ b = (a \circ b) \otimes ((a \otimes c) \circ (b \otimes c)).$

Fact 2

    A structure $(U, \sim, \oplus, \otimes, \circ)$ is a Grzegorczyk algebra if and only if the following conditions hold, for all $a, b, c \in U$:

(LD1) $a \circ b = b \circ a,$

(LD2) $a \circ b = (a \circ b) \otimes ((a \circ c) \circ (b \circ c)),$

(LD3) $a \circ b = \sim a\ \circ \sim b,$

(LD4) $a \circ b = (a \circ b) \otimes ((a \oplus c) \circ (b \oplus c)),$

(LD5) $a \circ b = (a \circ b) \otimes ((a \otimes c) \circ (b \otimes c)),$

(LD6) $a \oplus b = b \oplus a,$

(LD7) $a \oplus (b \oplus c) = (a \oplus b) \oplus c,$

(LD8) $a \oplus a\ = a,$

(LD9) $a \otimes b = b \otimes a,$

(LD10) $a \otimes (b \otimes c) = (a \otimes b) \otimes c,$

(LD11) $a \otimes a\ = a,$

(LD12) $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c),$

(LD13) $a \oplus (b \otimes c) = (a \oplus b) \otimes (a \oplus c),$

(LD14) $\sim(a \oplus b) = \sim a\ \otimes \sim b,$

(LD15) $\sim(a \otimes b) = \sim a\ \oplus \sim b,$

(LD16) $\sim\sim a\ = a.$

It is worth emphasizing the following fact:

Fact 3

Boolean algebras, Kleene algebras, and de Morgan algebras are Grzegorczyk algebras.

However, the converse of the above does not hold. The class of Grzegorczyk algebras is quite extensive and contains subclasses that form bases for semantics of various non-classical logics of different types. Grzegorczyk algebras will be a basis for structures of LD.

An LD-*structure* is of the form $(U, \sim, \oplus, \otimes, \circ, D)$, where:

- U, D are non-empty sets such that $D \subseteq U$,

- $(U, \sim, \oplus, \otimes, \circ)$ is a Grzegorczyk algebra,

- For all $a, b \in U$, the following hold:
    $(a \otimes b) \in D$ if and only if $a \in D$ and $b \in D$,
    $(a \circ b) \in D$ if and only if $a = b$,
    $\sim(a \otimes \sim a) \in D$ and $(a \otimes \sim a) \notin D$.

Let $\mathcal{M} = (U, \sim, \oplus, \otimes, \circ, D)$ be an LD-structure. A *valuation* on $\mathcal{M}$ is any mapping $v \colon \mathbb{V} \to U$ such that for all LD-formulas φ and ψ:

- $v(\neg\varphi) = \sim v(\varphi)$,

- $v(\varphi \wedge \psi) = v(\varphi) \otimes v(\psi)$,

- $v(\varphi \vee \psi) = v(\varphi) \oplus v(\psi)$,

- $v(\varphi \equiv \psi) = v(\varphi) \circ v(\psi)$.

A formula φ is said to be *satisfied in an* LD-*structure by a valuation v* if and only if $v(\varphi) \in D$. It is *true in $\mathcal{M}$* whenever it is satisfied in $\mathcal{M}$ by all the valuations on $\mathcal{M}$, and it is LD-*valid* if it is true in all LD-structures.

We may think of LD-structures as variants of non-Fregean structures, introduced by Suszko in [Suszko, 1971]. The universe of a non-Fregean structure consists of the correlates of sentences in a given language, and its elements are known as *situations* or *states of affairs*. The correlates of true sentences are *factual situations* and the correlates of false sentences (ones whose negations are true) are *counterfactual situations*. Unlike Suszko, we do not insist on logical two-valuedness but allow for sentences that are neither true nor false; their correlates are *undetermined situations*. The set D is the set of factual situations.

Example 4

Let $U = \{0, 1, 2, 3\}$, $D = \{2, 3\}$, and define the operations as follows:

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	3	0	0	0
1	0	3	0	0
2	0	0	3	0
3	0	0	0	3

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	0
2	0	1	2	3
3	0	0	3	3

$\oplus$	0	1	2	3
0	0	0	3	3
1	0	1	2	3
2	3	2	2	3
3	3	3	3	3

It can be verified that the above tables indeed define a Grzegorczyk algebra, but the absorption laws do not hold, as $1 \oplus (1 \otimes 0) = 1 \otimes (1 \oplus 0) = 0 \neq 1$, for instance.

First, we will prove that LD is sound with respect to the class of all LD-structures as defined above. Thus, we need to show that all LD-axioms are LD-valid and all LD-rules preserve LD-validity. To be more precise, a rule of the form $\frac{\varphi_1,\dots,\varphi_n}{\psi_1,\dots,\psi_m}$, for $n, m \leq 2$, is called *weakly* LD-*correct* whenever the LD-validity of $\varphi_1, \dots, \varphi_n$ implies the LD-validity of $\psi_1, \dots, \psi_m$, and *strongly* LD-*correct* whenever for every LD-structure $\mathcal{M}$ and every valuation v on $\mathcal{M}$ such that $\mathcal{M}, v \models \varphi_1, \dots, \varphi_n$, it holds that $\mathcal{M}, v \models \psi_1, \dots, \psi_m$.

Proposition 5

All the LD-rules except (Sub) are strongly LD-correct. Moreover, (Sub) is weakly LD-correct.

Proof.

The proofs of the weak correctness of (Sub) and the strong correctness of $(\wedge_1)$ and $(\wedge_2)$ are easy to carry out. So by way of example, we will show the strong correctness of the rule $(\mathsf{MP}_{\mathrm{LD}})$.

Let $\mathcal{M} = (U, \sim, \oplus, \otimes, \circ, D)$ be an LD-structure and v a valuation on $\mathcal{M}$ such that $\mathcal{M}, v \models \varphi \equiv \psi$ and $\mathcal{M}, v \models \varphi$. By the assumption, $v(\varphi \equiv \psi) \in D$ and $v(\varphi) \in D$. Since $v(\varphi \equiv \psi) \in D$, we have also $v(\varphi) \circ v(\psi) \in D$, so $v(\varphi) = v(\psi)$. Therefore $v(\psi) \in D$. Hence, the rule $\mathsf{MP}_{\mathrm{LD}}$ is strongly LD-correct. ∎

Proposition 6

All the LD-axioms are LD-valid.

Proof.

Let $\mathcal{M} = (U, \sim, \oplus, \otimes, \circ, D)$ be an LD-structure and let v be a valuation on $\mathcal{M}$. We need to show that if ϕ is an LD-axiom, then $v(\phi) \in D$.

First, by the definition of an LD-structure, we have the following: $v(p) = v(p)$ iff $v(p) \circ v(p) \in D$ iff $v(p \equiv p) \in D$. Hence, the axiom Ax2 is LD-valid.

By Fact 2(LD1), for all $a, b \in U$, $(a \circ b) = (b \circ a)$. Therefore, $(v(p) \circ v(q)) = (v(q) \circ v(p))$. On the other hand, by the definition of an LD-structure, we have also: $(v(p) \circ v(q)) = (v(q) \circ v(p))$ iff $(v(p) \circ v(q)) \circ (v(q) \circ v(p)) \in D$ iff $v((p \equiv q) \equiv (q \equiv p)) \in D$. Thus, axiom Ax2 is LD-valid. In a similar way, we can prove LD-validity of axioms Ax3–Ax16. Generally, for $i \in \{3, \dots, 17\}$, LD-validity of axiom Axi follows from condition (LDi−1) of Fact 2. Furthermore, by the definition of an LD-structure, axiom Ax18 is obviously LD-valid. ∎

Propositions 5 and 6 yield soundness of the logic LD with respect to the class of all LD-structures:

Proposition 7 (Soundness of LD)

Every LD-provable formula is LD-valid.

Now, we will proceed to completeness. Let R be the following binary relation on the set of all LD-formulas:

$$\varphi \, R \, \psi \text{ if and only if } \varphi \equiv \psi \text{ is provable in LD.}$$

Fact 8

The relation R is an equivalence relation on the set of all LD-formulas. Moreover, R is compatible with all LD-connectives.

Proof.

Let φ, ψ, ϑ be any LD-formulas. Clearly, $\varphi \equiv \varphi$ is provable in LD, so R is reflexive. Assume $\varphi \, R \, \psi$, that is, $\varphi \equiv \psi$ is provable in LD. By axiom Ax2, $((\varphi \equiv \psi) \equiv (\psi \equiv \varphi))$ is provable in LD. Thus, by the assumption, so is $\psi \equiv \varphi$, which implies $\psi \, R \, \varphi$. Hence, R is symmetric. Now, assume that $\varphi \, R \, \psi$ and $\psi \, R \, \vartheta$, which means that $\varphi \equiv \psi$ and $\psi \equiv \vartheta$ are provable in LD. Since R is symmetric, $\psi \, R \, \varphi$, so $\psi \equiv \varphi$ is provable in LD. By axiom Ax3, LD proves:

$$(\psi \equiv \varphi) \equiv [(\psi \equiv \varphi) \wedge ((\psi \equiv \vartheta) \equiv (\varphi \equiv \vartheta))].$$

Applying the rule $(\mathsf{MP_{LD}})$, and then $(\wedge_2)$, we obtain that LD proves:

$$(\psi \equiv \vartheta) \equiv (\varphi \equiv \vartheta).$$

Applying again the rule $(\mathsf{MP_{LD}})$ and the assumption $\psi \equiv \vartheta$, we have that $\varphi \equiv \vartheta$ is provable in LD, which implies $\varphi \, R \, \vartheta$. Hence, R is transitive.

Assume then that $\varphi_1 \, R \, \varphi_2$ and $\psi_1 \, R \, \psi_2$. Then $\neg\varphi_1 \, R \, \neg\varphi_2$ by Ax4, so R is compatible with $\neg$. Moreover,

$$\varphi_1 \equiv \psi_1 \, R \, \varphi_2 \equiv \psi_1 \, R \, \psi_1 \equiv \varphi_2 \, R \, \psi_2 \equiv \varphi_2 \, R \, \varphi_2 \equiv \psi_2,$$

so by the transitivity of R, it follows that R is also compatible with $\equiv$. The proofs of compatibility with $\wedge$ and $\vee$ are similar. ∎

By the above fact, we can define a structure $\mathcal{M}^{\mathrm{LD}} = (U, \sim, \oplus, \otimes, \circ, D)$ as follows:

- $U = \{ |\varphi|_R : \varphi$ is an LD-formula $\}$, that is, U is the set of equivalence classes of R on the set of all LD-formulas,
- $D = \{ |\varphi|_R : \varphi$ is provable in LD $\}$, that is, D is the set of equivalence classes of R on the set of all provable formulas in LD,
- For all $|\varphi|_R, |\psi|_R \in U$:

$$\sim|\varphi|_R \overset{\mathrm{df}}{=} |\neg\varphi|_R, \qquad\qquad |\varphi|_R \circ |\psi|_R \overset{\mathrm{df}}{=} |\varphi \equiv \psi|_R,$$
$$|\varphi|_R \otimes |\psi|_R \overset{\mathrm{df}}{=} |\varphi \wedge \psi|_R, \qquad\qquad |\varphi|_R \oplus |\psi|_R \overset{\mathrm{df}}{=} |\varphi \vee \psi|_R.$$

Proposition 9

The structure $\mathcal{M}^{\mathrm{LD}}$ is an LD-structure.

Proof.

First, we will show that $\mathcal{M}^{\mathrm{LD}} = (U, \sim, \oplus, \otimes, \circ)$ is a Grzegorczyk algebra. By Fact 2, it suffices to show that $\mathcal{M}^{\mathrm{LD}}$ satisfies all the conditions (LD1), (LD2), ..., (LD16). Let φ, ψ, ϑ be any LD-formulas.

*Proof of (*LD1*)*
$$|\varphi|_R \circ |\psi|_R = |\varphi \equiv \psi|_R = |\psi \equiv \varphi|_R = |\psi|_R \circ |\varphi|_R.$$

Proof of (LD2)
$$\begin{aligned}
(|\varphi|_R \circ |\psi|_R) &= |\varphi \equiv \psi|_R \\
&= |(\varphi \equiv \psi) \wedge ((\varphi \equiv \vartheta) \equiv (\psi \equiv \vartheta))|_R \\
&= |\varphi \equiv \psi|_R \otimes |(\varphi \equiv vartheta) \equiv (\psi \equiv \vartheta)|_R \\
&= (|\varphi|_R \circ |\psi|_R) \otimes ((|\varphi|_R \circ |\vartheta|_R) \circ (|\psi|_R \circ |\vartheta|_R)).
\end{aligned}$$

Proof of (LD3)
$$\begin{aligned}
|\varphi|_R \circ |\psi|_R &= |\varphi \equiv \psi|_R = |\neg\varphi \equiv \neg\psi|_R \\
&= |\neg\varphi|_R \circ |\neg\psi|_R = \sim|\varphi|_R \circ \sim|\psi|_R.
\end{aligned}$$

Proof of (LD4)
$$\begin{aligned}
|\varphi|_R \circ |\psi|_R &= |\varphi \equiv \psi|_R \\
&= |(\varphi \equiv \psi) \wedge ((\varphi \vee \vartheta) \equiv (\psi \vee \vartheta))|_R \\
&= |\varphi \equiv \psi|_R \otimes |(\varphi \vee \vartheta) \equiv (\psi \vee \vartheta)|_R \\
&= (|\varphi|_R \circ |\psi|_R) \otimes ((|\varphi|_R \oplus |\vartheta|_R) \circ (|\psi|_R \oplus |\vartheta|_R)).
\end{aligned}$$

Proof of (LD5)

$$|\varphi|_R \circ |\psi|_R = |\varphi \equiv \psi|_R$$
$$= |(\varphi \equiv \psi) \wedge ((\varphi \wedge \vartheta) \equiv (\psi \wedge \vartheta))|_R$$
$$= |\varphi \equiv \psi|_R \otimes |(\varphi \wedge \vartheta) \equiv (\psi \wedge \vartheta)|_R$$
$$= (|\varphi|_R \circ |\psi|_R) \otimes ((|\varphi|_R \otimes |\vartheta|_R) \circ (|\psi|_R \otimes |\vartheta|_R)).$$

Proof of (LD6)

$$|\varphi|_R \oplus |\psi|_R = |\varphi \vee \psi|_R = |\psi \vee \varphi|_R = |\psi|_R \oplus |\varphi|_R.$$

Proof of (LD7)

$$|\varphi|_R \oplus (|\psi|_R \oplus |\vartheta|_R) = |\varphi \vee (\psi \vee \vartheta)|_R = |(\varphi \vee \psi) \vee \vartheta|_R$$
$$= (|\varphi|_R \oplus |\psi|_R) \oplus |\vartheta|_R.$$

Proof of (LD8)

$$|\varphi|_R \oplus |\varphi|_R = |\varphi \vee \varphi|_R = |\varphi|_R.$$

Proof of (LD9)

$$|\varphi|_R \otimes |\psi|_R = |\varphi \wedge \psi|_R = |\psi \wedge \varphi|_R = |\psi|_R \otimes |\varphi|_R.$$

Proof of (LD10)

$$|\varphi|_R \otimes (|\psi|_R \otimes |\vartheta|_R) = |\varphi \wedge (\psi \wedge \vartheta)|_R = |(\varphi \wedge \psi) \wedge \vartheta|_R$$
$$= (|\varphi|_R \otimes |\psi|_R) \otimes |\vartheta|_R.$$

Proof of (LD11)

$$|\varphi|_R \otimes |\varphi|_R = |\varphi \wedge \varphi|_R = |\varphi|_R.$$

Proof of (LD12)

$$|\varphi|_R \otimes (|\psi|_R \oplus |\vartheta|_R) = |\varphi \wedge (\psi \vee \vartheta)|_R = |(\varphi \wedge \psi) \vee (\varphi \wedge \vartheta)|_R$$
$$= (|\varphi|_R \otimes |\psi|_R) \oplus (|\varphi|_R \otimes |\vartheta|_R).$$

Proof of (LD13)

$$|\varphi|_R \oplus (|\psi|_R \otimes |\vartheta|_R) = |\varphi \vee (\psi \wedge \vartheta)|_R = |(\varphi \vee \psi) \wedge (\varphi \vee \vartheta)|_R$$
$$= (|\varphi|_R \oplus |\psi|_R) \otimes (|\varphi|_R \oplus |\vartheta|_R).$$

Proof of (LD14)

$$\sim(|\varphi|_R \oplus |\psi|_R) = \sim|\varphi \vee \psi|_R = |\neg(\varphi \vee \psi)|_R = |\neg\varphi \wedge \neg\psi|_R$$
$$= |\neg\varphi|_R \otimes |\neg\psi|_R = \sim|\varphi|_R \otimes \sim|\psi|_R.$$

Proof of (LD15)

$$\sim(|\varphi|_R \otimes |\psi|_R) = \sim|\varphi \wedge \psi|_R = |\neg(\varphi \wedge \psi)|_R = |\neg\varphi \vee \neg\psi|_R$$
$$= |\neg\varphi|_R \oplus |\neg\psi|_R = \sim|\varphi|_R \oplus \sim|\psi|_R.$$

Proof of (LD16)

$$\sim(\sim|\varphi|_R) = \sim|\neg\varphi|_R = |\neg\neg\varphi|_R = |\varphi|_R.$$

Hence, we have shown that $(U, \sim, \oplus, \otimes, \circ)$ is a Grzegorczyk algebra. Now, we will prove that $\mathcal{M}^{LD}$ satisfies all other conditions required in the definition of LD-structures. Clearly, U and D are non-empty sets such that $D \subseteq U$. By the definition of $\mathcal{M}^{LD}$, for any formula φ: $|\varphi|_R \in D$ if and only if φ is provable in LD. Let $|\varphi|_R, |\psi|_R \in U$. Then, $|\varphi|_R \otimes |\psi|_R = |\varphi \wedge \psi|_R \in D$ iff $\varphi \wedge \psi$ is provable in LD iff φ and ψ are provable in LD iff $|\varphi|_R \in D$ and $|\psi|_R \in D$. Therefore, $|\varphi|_R \otimes |\psi|_R \in D$ if and only if $|\varphi|_R \in D$ and $|\psi|_R \in D$. Furthermore, we have also: $|\varphi|_R \circ |\psi|_R = |\varphi \equiv \psi|_R \in D$ iff $\varphi \equiv \psi$ is provable in LD iff $\varphi R \psi$ iff $|\varphi|_R = |\psi|_R$. Hence, $|\varphi|_R \circ |\psi|_R \in D$ if and only if $|\varphi|_R = |\psi|_R$. By axiom Ax18, for any formula φ, $\neg(\varphi \wedge \neg\varphi)$ is provable in LD, so $\sim(|\varphi|_R \otimes \sim|\varphi|_R) \in D$. On the other hand, for any formula φ, $\varphi \wedge \neg\varphi$ is not provable in LD, since otherwise by Proposition 7, it would be true in all LD-structures, which is impossible. Therefore, we obtain: $(|\varphi|_R \otimes \sim|\varphi|_R) \notin D$. Hence, $\mathcal{M}^{LD}$ is an LD-structure. ∎

From now on, the structure $\mathcal{M}^{LD}$ is referred to as *canonical*.

Proposition 10 (Completeness of LD)

For every LD-formula φ, if φ is LD-valid, then it is LD-provable.

Proof.

Let φ be an LD-valid formula, and let v be the valuation on $\mathcal{M}^{LD}$ such that $v(\psi) = |\psi|_R$ for every ψ. It is easy to check that v is indeed a valuation. Now, by the assumption, $v(\varphi) \in D$, and hence φ is LD-provable. ∎

Finally, by Propositions 7 and 10, we obtain:

Theorem 11 (Soundness and Completeness of LD)

For every LD-formula φ, the following conditions are equivalent:
1. φ is LD-provable.
2. φ is LD-valid.

By the completeness theorem, LD-structures will be referred to as LD-*models*.

Next, we consider LD-consistency.

Definition 12

Let S be a set of LD-formulas.
1. S is LD-*satisfiable* if there are an LD-model $\mathcal{M}$ and a valuation v on M such that for every $\varphi \in S$, it holds that $\mathcal{M}, v \models \varphi$.

2. S is LD-*inconsistent* if there is some formula φ such that $S \vdash \varphi \wedge \neg\varphi$. Otherwise, S is LD-*consistent*.

Proposition 13

A set S of LD-formulas is LD-satisfiable if and only if S is LD-consistent.

Proof.

Assume first that S is LD-satisfiable. Let $\mathcal{M}$ be an LD-model and let v be a valuation on $\mathcal{M}$ such that $\mathcal{M}, v \models S$. Let φ be an LD-formula. Suppose $S \vdash \varphi \wedge \neg\varphi$. Then, $\mathcal{M}, v \models \varphi \wedge \neg\varphi$, which means that $v(\varphi \wedge \neg\varphi) \in D$, so $v(\varphi) \in D$ and $v(\neg\varphi) \in D$, which contradicts the definition of an LD-structure. Hence, S is LD-consistent.

Assume then that S is LD-consistent. We build a model in the same way as the canonical model above. So, let R be the binary relation on the set of all LD-formulas defined as: $\varphi R \psi$ if and only if $S \vdash \varphi \equiv \psi$. As before, R is an equivalence relation compatible with all connectives, and hence we can define a Grzegorczyk algebra $(U, \sim, \oplus, \otimes, \circ)$ from R exactly as in the definition of the canonical model. The earlier proof works almost verbatim. Let further $D = \{ |\varphi|_R : S \vdash \varphi \}$. Again, the proof of the required properties of D is otherwise essentially the same as before, but showing that there is no φ such that both $|\varphi|_R \in D$ and $\sim|\varphi|_R \in D$ require some extra care. Assume towards a contradiction that φ is a counterexample. Then it follows from the definitions that $S \vdash \varphi$ and $S \vdash \neg\varphi$, which contradicts the assumption that S is LD-consistent. ∎

4. Some interesting properties

In [Grzegorczyk, 2011], Prof. Grzegorczyk raises important questions about the relationship between equality of descriptions and material equivalence, as well as the corresponding implications. Let us introduce the following definitions:

- $(p \to q) \stackrel{\mathrm{df}}{=\!=} (\neg p \vee q)$ (classical implication)
- $(p \leftrightarrow q) \stackrel{\mathrm{df}}{=\!=} (p \to q) \wedge (q \to p)$ (classical equivalence)
- $(p \Rightarrow q) \stackrel{\mathrm{df}}{=\!=} (p \equiv (p \wedge q))$ (descriptive implication)

Now, the questions about relationships between descriptive and classical equivalences as well as between descriptive and classical implications can be formalized as follows:

(Q1)   Is the formula $(p \equiv q) \equiv (p \leftrightarrow q)$ provable in LD?

(Q2)   Is the formula $(p \Rightarrow q) \equiv (p \rightarrow q)$ provable in LD?

Both questions have negative answers, as the following proposition shows.

Proposition 14

 1. The formula $(p \equiv q) \equiv (p \leftrightarrow q)$ is not provable in LD.

 2. The formula $(p \Rightarrow q) \equiv (p \rightarrow q)$ is not provable in LD.

Proof.
    Let $(U, \sim, \oplus, \otimes, \circ, D)$ be as in Example 4 above, and let $v(p) = v(q) = 2$. Then

$$v(p \rightarrow q) = \sim v(p) \oplus v(q) = \sim 2 \oplus 2 = 1 \oplus 2 = 2,$$

but

$$v(p \Rightarrow q) = v(p) \circ (v(p) \otimes v(q)) = 2 \circ (2 \otimes 2) = 2 \circ 2 = 3.$$

In the same way, we see that $v(p \leftrightarrow q) = 2$ but $v(p \equiv q) = 3$.   ■

Next, we present some LD-provable formulas and derived rules as well as classical results that fail in LD. Due to the excessive lengths of the formal proofs, we omit the details of most of them, showing only outlines. The models we use as counterexamples are listed in the Appendix.

Even though there are no explicit rules concerning disjunction, there is a derived disjunction introduction rule.

Proposition 15

    The following rule is strongly LD-correct:

$$\frac{\varphi}{\psi \vee \varphi}$$

Proof.
    Assume $\mathcal{M}, v \models \varphi$. The formula $\psi \vee \neg\psi$ is LD-valid, so we get $\mathcal{M}, v \models \varphi \wedge (\psi \vee \neg\psi)$. Further,

$$\varphi \wedge (\psi \vee \neg\psi) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi)$$
$$\equiv ((\varphi \wedge \psi) \vee \varphi) \wedge ((\varphi \wedge \psi) \vee \neg\psi)$$
$$\equiv ((\varphi \vee \varphi) \wedge (\psi \vee \varphi)) \wedge ((\varphi \wedge \psi) \vee \neg\psi),$$

whence $\mathcal{M}, v \models \psi \vee \varphi$.   ■

Proposition 16

The formula $(p \equiv p) \equiv (q \equiv q)$ is not LD-provable.

Proof.

See Example 36 in the Appendix. ∎

Proposition 17

The formula $\neg(p \equiv \neg p)$ is not LD-provable.

Proof.

See Example 38 in the Appendix. Now, actually for any $a \in U$, it holds that $\sim(a \circ \sim a) = 2 \notin D$, so there is no valuation v such that

$$\mathcal{M}, v \models \neg(p \equiv \neg p).$$ ∎

Note that there cannot be any LD-model $\mathcal{M}$ and valuation v such that $\mathcal{M}, v \models p \equiv \neg p$, as $v(p \wedge \neg p) \notin D$ but $v(p \vee \neg p) \in D$.

Fact 3 shows that many familiar types of algebras are also Grzegorczyk algebras. However, there are Grzegorczyk algebras that do not belong to any of those types, and they often seem to be complicated and difficult to understand intuitively. However, they do illustrate various unexpected aspects of LD. For instance, we can show the failure of the classical modus ponens rule, despite the provability of the corresponding formula, by constructing a suitable model and choosing a valuation that satisfies the premises but not the supposed conclusion. Moreover, it follows directly from the definition of an LD-model that if a formula φ is satisfied in a model $\mathcal{M}$ by a valuation v, then $\neg\varphi$ is not satisfied by v in $\mathcal{M}$, but the converse does not hold, as we saw above. This trait of LD contrasts strongly with most other well-known logics, that is, ones that follow the negation clause of Tarski's truth definition. On the other hand, LD is also quite unlike intuitionistic logic, as there is no negation introduction rule, but the axioms include DeMorgan's laws and double negation is treated classically. It follows from this combination of DeMorgan's laws, a classical conjunction, and a non-standard negation, that also disjunction behaves in an unexpected way. Indeed, the formula $p \equiv \neg p$ is unsatisfiable and $(p \equiv \neg p) \vee \neg(p \equiv \neg p)$ is a tautology, but $\neg(p \equiv \neg p)$ is not provable. Hence, the connection between the truth values of a disjunction and the disjuncts is less definite than in classical logic. This property is, at least to a degree, in accordance with the philosophical motivations that LD is based on, as the LD-provability of a formula gives us information not only of its necessary truth, but also of its connection to the axioms.

The soundness of LD with respect to the class of LD-models allows us to derive further unprovability results.

Proposition 18

The following formulas are not provable in LD:

1. $(\varphi \vee (\varphi \wedge \psi)) \equiv \varphi$.
2. $(\varphi \wedge (\varphi \vee \psi)) \equiv \varphi$.
3. $(\varphi \vee \neg\varphi) \equiv (\psi \vee \neg\psi)$.
4. $(\varphi \wedge \neg\varphi) \Rightarrow \psi$.
5. $\varphi \Rightarrow (\psi \vee \neg\psi)$.

Moreover, the following rule is not strongly LD-correct:

$$(\vee_1) \quad \frac{\varphi, \neg\varphi \vee \psi}{\psi}.$$

Proof.

For (1) and (2), see Example 34 in the Appendix. For (3), (4), and (5), see Example 35. For $(\vee_1)$, see Example 37. ∎

Note that the formulas (3), (4), and (5) are instances of the classical paradoxes of equivalence and implication: "any true statements are equivalent to each other", "false implies everything", and "the truth is implied by anything", respectively. Hence their failure indicates that LD indeed avoids these paradoxes.

Now, we can prove that all the axioms of $\mathrm{LD_{red}}$ are independent of each other. By a *quasi* LD-*structure* we will mean a structure $\mathcal{M} = (U, \sim, \otimes, \oplus, \circ, D)$ such that U, D are nonempty sets, $D \subseteq U$, $\sim$ is a unary operation on U and $\otimes, \oplus, \circ$ are binary operations on U. The notions of valuation, satisfaction, and the truth in a quasi LD-structure are defined in the same way as for LD-models.

Proposition 19

Let $S = \{\mathsf{Ax2}, \mathsf{Ax3}, \mathsf{Ax4}, \mathsf{Ax6}, \mathsf{Ax10}, \mathsf{Ax11}, \mathsf{Ax12}, \mathsf{Ax13}, \mathsf{Ax15}, \mathsf{Ax17}, \mathsf{Ax18}\}$. Then, for each $\varphi \in S$, there is a quasi LD-structure $\mathcal{M}_\varphi$ such that $\mathcal{M}_\varphi \not\models \varphi$ but for each $\psi \in S \setminus \varphi$, it holds that $\mathcal{M}_\varphi \models \psi$.

Proof.

We will list the structures in the Appendix. ∎

So, the set S forms an independent set of axioms for LD.

5. Alternative versions of LD

During discussions about LD in seminar meetings, some alternative forms of Ax3 were suggested. One of the motivations for adopting Ax3 was to express the transitivity of descriptive equivalence, but at least superficially,

Ax3 appears to say something stronger. Therefore, it was natural to consider alternative versions and their respective consequences. As we mentioned above in Section 2, the original version of Ax3 as given in [Grzegorczyk, 2011] was different, but our version is the intended one, as published in Errata [2012]. We will now discuss the logics obtained by replacing Ax3 with two alternative forms: Ax3* and Ax3'. The axiom Ax3* is the original version presented in [Grzegorczyk, 2011], whereas Ax3' was proposed by Stanisław Krajewski, and it is already mentioned in the Errata. These axioms have the following forms:

Ax3* $(p \equiv q) \equiv [(p \equiv r) \equiv (q \equiv r)]$,

Ax3' $[(p \equiv q) \wedge (q \equiv r)] \Rightarrow (p \equiv r)$.

Recall that $\Rightarrow$ is an LD-implication defined as:

$$\varphi \Rightarrow \psi \overset{\text{df}}{=} \varphi \equiv (\varphi \wedge \psi).$$

Thus, the explicit form of axiom Ax3' is:

$$[(p \equiv q) \wedge (q \equiv r)] \equiv [(p \equiv q) \wedge (q \equiv r) \wedge (p \equiv r)].$$

By LD* (resp. LD') we will denote the logic obtained from LD by replacing the axiom Ax3 with Ax3* (resp. Ax3'). It is again easy to see that both logics are consistent, since under the interpretation of $\equiv$ as the usual classical equivalence, both axioms Ax3* and Ax3' are classical tautologies.

First, we will discuss the logic LD*. We can actually prove Ax3 in LD*, as the following proof shows:

(1) $[(p \equiv q) \equiv ((p \equiv r) \equiv (q \equiv r))] \equiv$
    $\equiv [((p \equiv q) \equiv ((p \equiv r) \equiv (q \equiv r))) \wedge$
        $\wedge (((p \equiv q) \wedge (p \equiv q)) \equiv (((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)))]$
            (Sub) to Ax6 for $p/(p \equiv q)$, $q/((p \equiv r) \equiv (q \equiv r))$, $r/(p \equiv q)$

(2) $(p \equiv q) \equiv ((p \equiv r) \equiv (q \equiv r)]$ Ax3*

(3) $((p \equiv q) \equiv ((p \equiv r) \equiv (q \equiv r))) \wedge$
    $\wedge (((p \equiv q) \wedge (p \equiv q)) \equiv (((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)))$
                                            (MP$_{\text{LD}}$) to (1) and (2)

(4) $((p \equiv q) \wedge (p \equiv q)) \equiv (((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q))$ ($\wedge_2$) to (3)

(5) $[((p \equiv q) \wedge (p \equiv q)) \equiv (((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q))] \equiv$
    $\equiv \{(((p \equiv q) \wedge (p \equiv q)) \equiv (p \equiv q)) \equiv$
        $\equiv ((((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv (p \equiv q))\}$
                                (Sub) to Ax3* for $p/((p \equiv q) \wedge (p \equiv q))$,
                                $q/(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q))$, $r/(p \equiv q)$

(6) $(((p \equiv q) \wedge (p \equiv q)) \equiv (p \equiv q)) \equiv$
$$\equiv ((((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv (p \equiv q))$$
$$(\mathsf{MP}_{\mathrm{LD}}) \text{ to (4) and (5)}$$

(7) $(p \equiv q) \equiv ((p \equiv q) \wedge (p \equiv q))$ $\qquad$ (Sub) to Ax12 for $p/(p \equiv q)$

(8) $[(p \equiv q) \equiv ((p \equiv q) \wedge (p \equiv q))] \equiv [((p \equiv q) \wedge (p \equiv q)) \equiv (p \equiv q)]$
$$(\mathsf{Sub}) \text{ to Ax2 for } p/(p \equiv q), \ q/((p \equiv q) \wedge (p \equiv q))$$

(9) $((p \equiv q) \wedge (p \equiv q)) \equiv (p \equiv q)$ $\qquad$ $(\mathsf{MP}_{\mathrm{LD}})$ to (7) and (8)

(10) $(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv (p \equiv q)$ $\qquad$ $(\mathsf{MP}_{\mathrm{LD}})$ to (6) and (9)

(11) $(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))$
$$(\mathsf{Sub}) \text{ to Ax10}$$
$$\text{for } p/((p \equiv r) \equiv (q \equiv r)), \ q/(p \equiv q)$$

(12) $[(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv (p \equiv q)] \equiv$
$$\equiv \{[(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))] \equiv$$
$$\equiv [(p \equiv q) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))]\}$$
$$(\mathsf{Sub}) \text{ to Ax3}^* \text{ for } p/(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)),$$
$$q/(p \equiv q), \ r/((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))$$

(13) $[(((p \equiv r) \equiv (q \equiv r)) \wedge (p \equiv q)) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))] \equiv$
$$\equiv [(p \equiv q) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))]$$
$$(\mathsf{MP}_{\mathrm{LD}}) \text{ to (10) and (12)}$$

(14) $(p \equiv q) \equiv ((p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r)))$ $\quad$ $(\mathsf{MP}_{\mathrm{LD}})$ to (11) and (13).

Thus, we get the following proposition.

Proposition 20

Every LD-formula φ that is provable in LD is also provable in LD*.

It turns out that replacing Ax3 with Ax3* defeats the purpose of introducing a new connective, as the following proposition shows:

Proposition 21

The following rules are strongly correct in LD*:

$$\frac{p \leftrightarrow q}{p \equiv q} \qquad\qquad \frac{p \equiv q}{p \leftrightarrow q}$$

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Proof.

First, it is easy to see that $(p \equiv p) \equiv (q \equiv q)$ holds for any p, q, as both sides are equal to $(p \equiv q) \equiv (p \equiv q)$, by Ax3* and symmetry. Let us denote this common value by 1. Then,

$$
\begin{aligned}
(p \equiv 1) &\equiv ((p \equiv 1) \equiv (1 \equiv 1)) \\
&\equiv ((p \equiv 1) \equiv 1),
\end{aligned}
$$

whence $p \equiv (p \equiv 1)$. In particular, if φ is provable in LD*, then so is $\varphi \equiv 1$. Hence,

$$
\begin{aligned}
p \vee 1 &\equiv p \vee (p \vee \neg p) \\
&\equiv p \vee \neg p \\
&\equiv 1.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
p &\equiv (p \equiv 1) \\
&\equiv (p \equiv 1) \wedge ((p \wedge 1) \equiv (1 \wedge 1)) \\
&\equiv p \wedge (p \wedge 1),
\end{aligned}
$$

so $(p \wedge 1) \equiv p$ for all p.

If we substitute 1 for q in axioms Ax5 and Ax6 and simplify, we get, respectively,

$$
\begin{aligned}
p &\equiv p \wedge ((p \wedge r) \equiv r), \\
p &\equiv p \wedge (p \vee r).
\end{aligned}
$$

So, it follows that

$$
(p \wedge q) \equiv [(p \wedge q) \wedge ((p \wedge q) \equiv p) \wedge ((p \wedge q) \equiv q)].
$$

By applying DeMorgan's laws and some further manipulations, we also get

$$
(\neg p \wedge \neg q) \equiv [(\neg p \wedge \neg q) \wedge ((p \wedge q) \equiv p) \wedge ((p \wedge q) \equiv q)].
$$

So,

$$
(p \leftrightarrow q) \equiv [(p \leftrightarrow q) \wedge ((p \wedge q) \equiv p) \wedge ((p \wedge q) \equiv q)].
$$

From this and the transitivity of $\equiv$, the claim follows.

On the other hand, we can prove the tautology $q \leftrightarrow q$ as in the classical case, and hence $(q \leftrightarrow q) \equiv 1$ and further $r \equiv (r \equiv (q \leftrightarrow q))$, for any r. Moreover,

$$
(p \equiv q) \equiv [(p \equiv q) \wedge ((p \leftrightarrow q) \equiv (q \leftrightarrow q))],
$$

and therefore $(p \equiv q) \equiv ((p \equiv q) \wedge (p \leftrightarrow q))$. ∎

So, LD^* is effectively just an unnecessarily complex reformulation of classical propositional logic. It also follows that the converse of Proposition 20 does not hold.

Let us now consider $\mathsf{Ax3'}$. We can define an $\mathsf{LD'}$-model by changing the definition of an LD-model appropriately, that is, by replacing the condition $a \circ b = (a \circ b) \otimes ((a \circ c) \circ (b \circ c))$ with $(a \circ b) \otimes (b \circ c) = (a \circ b) \otimes (b \circ c) \otimes (a \circ c)$. Now, every $\mathsf{LD'}$-provable formula is true in every $\mathsf{LD'}$-model, which can be proved essentially in the same way as in the case of LD. However, the converse implication does not hold. Let φ be the formula $(p \equiv q) \equiv ((p \wedge p) \equiv q)$. The structure presented in Example 39 satisfies all $\mathsf{LD'}$-axioms and rules but does not satisfy φ, which means that φ is not $\mathsf{LD'}$-provable. On the other hand, φ is clearly true in every $\mathsf{LD'}$-model, as $\equiv$ is interpreted as equality. Moreover, the philosophical motivations for LD suggest that φ should be true. Therefore, we will not study $\mathsf{LD'}$ any further.

6. Conclusions

We have defined the logic LD in terms of syntactic deduction rules and axioms, defined a corresponding semantics, and proved a soundness and completeness theorem. We have given several examples of classical laws that hold in LD as well as laws that fail in LD. We have shown that the original axiomatization is redundant and found an independent set of axioms. We have considered two proposed alternative forms of $\mathsf{Ax3}$ and found both of them unsatisfying as replacements for $\mathsf{Ax3}$.

In studying the properties of LD, we have found computer-assisted methods indispensable. We developed the following tools for LD:

- Two proof checkers, accepting two different input syntaxes and using different methods for checking, generally being as independent as possible apart from the fact that both were written by the second author.
- An automatic proof builder, which accepts a set of targets and some intermediate steps and attempts to output a full formal proof of the targets, in a format suitable for either checker or LaTeX.
- A model checker, which inputs a finite structure in the signature of LD as well as some deduction rules and formulas, and checks whether the structure obeys the rules and satisfies the formulas.

As we mentioned before, the programs and sample input and output files are available on request.

There are several interesting questions about LD that we have not answered here.

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1. Is LD decidable?

   We conjecture LD has the finite model property, formulated in the following way due to the non-classical negation:

   For every formula φ such that there are an LD-model $\mathcal{M}$ and a valuation v on $\mathcal{M}$ such that $\mathcal{M}, v \not\models \varphi$, there are a finite LD-model $\mathcal{M}'$ and a valuation v on $\mathcal{M}'$ such that $\mathcal{M}', v' \not\models \varphi$.

   For logics with a Tarskian negation, this formulation is, of course, equivalent with the usual one. In the case of LD, however, this is the form that we need. Indeed, if the conjecture is true, we can prove the decidability of LD in the usual way.

2. If LD is decidable, what is the complexity of deciding whether an LD-formula is provable?

   Our conjecture about the finite model property, mentioned above, is based on a construction of a finite model whose size is doubly exponential in the size of the formula. If the construction is correct, there is an obvious decision algorithm that runs in doubly exponential space and hence triply exponential time: simply search for a small enough counterexample.

3. Is the classical modus ponens rule weakly correct for LD?

   We have a counterexample showing that MP is not strongly correct. However, we do not know whether it is weakly correct.

4. Are there other interesting variants of LD?

   We showed that LD* is too strong and LD$'$ too weak to formalize the motivating philosophical ideas. However, Ax3$'$ appears plausible in its own right, and instead of replacing Ax3 with Ax3$'$, one could simply add Ax3$'$ to LD. So far, our preliminary results suggest such an extension would be similar to LD, with only minor technical differences.

5. Can one generate LD-proofs fully automatically in practice?

   Our prover needs a human-generated outline consisting of intermediate steps, which it then attempts to expand to a full proof by applying some derived rules. If the outline is not sufficiently detailed, the prover fails. The non-classical nature of negation and disjunction prevents a straightforward implementation of a tableau-based prover. So far, we have not found a practical proof strategy for LD. Of course, a brute-force search is possible in principle.

6. Is there a normal form for LD-formulas?

   A suitable normal form may simplify the task of finding an automatic proof system, among other things.

Acknowledgments

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Appendix

In this section, we will list the models that show the unprovability claims made in the main text.

Example 22

Here is the simplest possible LD-model, unique up to isomorphism.

$U = \{0, 1\}$, $D = \{1\}$

$\sim$	0	1
	1	0

$\circ$	0	1
0	1	0
1	0	1

$\otimes$	0	1
0	0	0
1	0	1

$\oplus$	0	1
0	0	1
1	1	1

Example 23

This model shows that Ax2 is independent of $\mathsf{LD_{red}}$. That is, the formula $(p \equiv q) \equiv (q \equiv p)$ is not true in it, but all other axioms of $\mathsf{LD_{red}}$ are.

$U = \{0, 1, 2, 3, 4, 5\}$, $D = \{3, 4, 5\}$; $v(p) = 0$, $v(q) = 2$.

$\sim$	0	1	2	3	4	5
	5	4	3	2	1	0

$\circ$	0	1	2	3	4	5
0	3	1	1	1	1	1
1	1	4	1	2	1	2
2	2	1	4	1	2	1
3	1	2	1	4	1	2
4	2	1	2	1	4	1
5	1	1	1	1	1	3

$\otimes$	0	1	2	3	4	5
0	0	1	2	1	2	1
1	1	1	1	1	1	1
2	2	1	2	1	2	1
3	1	1	1	3	3	3
4	2	1	2	3	4	3
5	1	1	1	3	3	5

$\oplus$	0	1	2	3	4	5
0	0	2	2	4	4	4
1	2	1	2	3	4	3
2	2	2	2	4	4	4
3	4	3	4	3	4	3
4	4	4	4	4	4	4
5	4	3	4	3	4	5

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Example 24

The axiom $(p \equiv q) \equiv [(p \equiv q) \wedge ((p \equiv r) \equiv (q \equiv r))]$ (Ax3) is independent.

$U = \{0,1,2,3\}$, $D = \{2,3\}$; $v(p) = 0$, $v(q) = 0$, $v(r) = 0$.

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	2	0	0	0
1	0	3	0	0
2	0	0	3	0
3	0	0	0	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	3
3	0	0	3	3

$\oplus$	0	1	2	3
0	0	0	3	3
1	0	1	3	3
2	3	3	2	3
3	3	3	3	3

Example 25

The axiom $(p \equiv q) \equiv (\neg p \equiv \neg q)$ (Ax4) is independent.

$U = \{0,1,2,3\}$, $D = \{2,3\}$; $v(p) = 1$, $v(q) = 1$.

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	2	0	0	0
1	0	3	0	0
2	0	0	2	0
3	0	0	0	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	3
3	0	0	3	3

$\oplus$	0	1	2	3
0	0	0	3	3
1	0	1	3	3
2	3	3	2	3
3	3	3	3	3

Example 26

The axiom $(p \equiv q) \equiv [(p \equiv q) \wedge ((p \wedge r) \equiv (q \wedge r))]$ (Ax6) is independent.

$U = \{0,1,2,3\}$, $D = \{2,3\}$; $v(p) = 3$, $v(q) = 3$, $v(r) = 1$.

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	3	0	0	0
1	0	2	0	0
2	0	0	2	0
3	0	0	0	3

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\oplus$	0	1	2	3
0	0	0	1	2
1	1	1	1	3
2	2	2	3	2
3	3	3	3	3

Example 27

The axiom $(p \wedge q) \equiv (q \wedge p)$ (Ax10) is independent.

$U = \{0,1,2,3\}$, $D = \{2,3\}$; $v(p) = 0$, $v(q) = 1$.

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	2	0	0	0
1	0	2	0	0
2	0	0	2	0
3	0	0	0	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	1	1	1	1
2	0	0	2	2
3	1	1	3	3

$\oplus$	0	1	2	3
0	0	0	2	2
1	1	1	3	3
2	2	2	2	2
3	3	3	3	3

Example 28

The axiom $(p \wedge (q \wedge r)) \equiv ((p \wedge q) \wedge r)$ (Ax11) is independent.

$U = \{0, 1, 2, 3, 4, 5\}$, $D = \{3, 4, 5\}$; $v(p) = 0$, $v(q) = 0$, $v(r) = 4$.

$\sim$	0	1	2	3	4	5
	5	4	3	2	1	0

$\circ$	0	1	2	3	4	5
0	3	2	2	2	2	2
1	2	3	2	2	2	2
2	2	2	3	2	2	2
3	2	2	2	3	2	2
4	2	2	2	2	3	2
5	2	2	2	2	2	3

$\otimes$	0	1	2	3	4	5
0	0	0	2	2	1	1
1	0	1	2	2	1	1
2	2	2	2	2	2	2
3	2	2	2	3	3	3
4	1	1	2	3	4	4
5	1	1	2	3	4	5

$\oplus$	0	1	2	3	4	5
0	0	1	2	3	4	4
1	1	1	2	3	4	4
2	2	2	2	3	3	3
3	3	3	3	3	3	3
4	4	4	3	3	4	5
5	4	4	3	3	5	5

Example 29

The axiom $p \equiv (p \wedge p)$ (Ax12) is independent.

$U = \{0, 1, 2, 3\}$, $D = \{2, 3\}$; $v(p) = 0$.

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	2	1	1	1
1	1	2	1	1
2	1	1	2	1
3	1	1	1	2

$\otimes$	0	1	2	3
0	1	1	1	1
1	1	1	1	1
2	1	1	2	2
3	1	1	2	3

$\oplus$	0	1	2	3
0	0	1	2	2
1	1	1	2	2
2	2	2	2	2
3	2	2	2	2

Example 30

The axiom $(p \wedge (q \vee r)) \equiv ((p \wedge q) \vee (p \wedge r))$ (Ax13) is independent.

$U = \{0, 1, 2, 3\}$, $D = \{2, 3\}$; $v(p) = 1$, $v(q) = 1$, $v(r) = 2$.

$\sim$	0	1	2	3
	3	2	1	0

$\circ$	0	1	2	3
0	2	0	0	0
1	0	2	0	0
2	0	0	2	0
3	0	0	0	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	0	0
2	0	0	2	2
3	0	0	2	3

$\oplus$	0	1	2	3
0	0	1	3	3
1	1	1	3	3
2	3	3	2	3
3	3	3	3	3

Example 31

The axiom $\neg(p \vee q) \equiv (\neg p \wedge \neg q)$ (Ax15) is independent.

$U = \{0, 1, 2, 3\}$, $D = \{2, 3\}$; $v(p) = 0$, $v(q) = 1$.

$\sim$	0	1	2	3
	2	3	0	1

$\circ$	0	1	2	3
0	2	0	0	0
1	0	2	0	0
2	0	0	2	0
3	0	0	0	2

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

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Example 32

The axiom $\neg\neg p \equiv p$ (Ax17) is independent.

$U = \{0, 1, 2, 3, 4, 5, 6, 7\}$, $D = \{4, 5, 6, 7\}$; $v(p) = 1$.

~	0	1	2	3	4	5	6	7
	7	6	5	4	3	1	2	0

∘	0	1	2	3	4	5	6	7
0	4	0	0	0	0	0	0	0
1	0	4	0	0	0	0	0	0
2	0	0	4	0	0	0	0	0
3	0	0	0	4	0	0	0	0
4	0	0	0	0	4	0	0	0
5	0	0	0	0	0	4	0	0
6	0	0	0	0	0	0	4	0
7	0	0	0	0	0	0	0	4

⊗	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	0	1	1	1	1	1
2	0	0	2	2	2	2	2	2
3	0	1	2	3	3	3	3	3
4	0	1	2	3	4	4	4	4
5	0	1	2	3	4	5	4	5
6	0	1	2	3	4	4	6	6
7	0	1	2	3	4	5	6	7

⊕	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	1	3	3	4	5	6	7
2	2	3	2	3	4	5	6	7
3	3	3	3	3	4	5	6	7
4	4	4	4	4	4	5	6	7
5	5	5	5	5	5	5	7	7
6	6	6	6	6	6	7	6	7
7	7	7	7	7	7	7	7	7

Example 33

The axiom $\neg(p \wedge \neg p)$ (Ax18) is independent.

$U = \{0, 1, 2, 3\}$, $D = \{3\}$; $v(p) = 1$.

~	0	1	2	3
	3	2	1	0

∘	0	1	2	3
0	3	0	0	0
1	0	3	0	0
2	0	0	3	0
3	0	0	0	3

⊗	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

⊕	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

Example 34

In this model, the absorption law $(p \wedge (p \vee q)) \equiv p$ does not hold.

$U = \{0, 1, 2, 3\}$, $D = \{2, 3\}$; $v(p) = 1$, $v(q) = 0$.

~	0	1	2	3
	3	2	1	0

∘	0	1	2	3
0	3	0	0	0
1	0	3	0	0
2	0	0	3	0
3	0	0	0	3

⊗	0	1	2	3
0	0	0	0	0
1	0	1	1	0
2	0	1	2	3
3	0	0	3	3

⊕	0	1	2	3
0	0	0	3	3
1	0	1	2	3
2	3	2	2	3
3	3	3	3	3

Example 35

Here the formulas $(p \vee \neg p) \equiv (q \vee \neg q)$, $(p \wedge \neg p) \Rightarrow q$, and $\neg q \Rightarrow (p \vee \neg p)$ are not true.

$U = \{0, 1, 2, 3\}$, $D = \{2, 3\}$; $v(p) = 1$, $v(q) = 0$.

~	0	1	2	3
	3	2	1	0

∘	0	1	2	3
0	3	0	0	0
1	0	3	0	0
2	0	0	3	0
3	0	0	0	3

⊗	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

⊕	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

Example 36

In this model, the formula $(p \equiv p) \equiv (q \equiv q)$ is not true.

$U = \{0, 1, 2, 3, 4, 5\}$, $D = \{3, 4, 5\}$; $v(p) = 0$, $v(q) = 1$.

~	0	1	2	3	4	5
	5	4	3	2	1	0

∘	0	1	2	3	4	5
0	5	0	0	0	0	0
1	0	3	0	0	0	0
2	0	0	5	0	0	0
3	0	0	0	5	0	0
4	0	0	0	0	3	0
5	0	0	0	0	0	5

⊗	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	0	0	2
2	0	2	2	0	0	2
3	0	0	0	3	3	3
4	0	0	0	3	4	3
5	0	2	2	3	3	5

⊕	0	1	2	3	4	5
0	0	2	2	3	3	5
1	2	1	2	5	5	5
2	2	2	2	5	5	5
3	3	5	5	3	3	5
4	3	5	5	3	4	5
5	5	5	5	5	5	5

Example 37

In this model, the modus ponens rule is not correct. That is, the formula $p \wedge (p \rightarrow q)$ is satisfied by a valuation that does not satisfy q.

$U = \{0, 1, 2, 3, 4, 5\}$, $D = \{4, 5\}$; $v(p) = 4$, $v(q) = 3$.

~	0	1	2	3	4	5
	5	4	3	2	1	0

∘	0	1	2	3	4	5
0	5	0	0	0	0	0
1	0	5	0	0	0	0
2	0	0	5	0	0	0
3	0	0	0	5	0	0
4	0	0	0	0	5	0
5	0	0	0	0	0	5

⊗	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	0	1	1
2	0	1	2	0	1	2
3	0	0	0	3	3	3
4	0	1	1	3	4	4
5	0	1	2	3	4	5

⊕	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	1	2	4	4	5
2	2	2	2	5	5	5
3	3	4	5	3	4	5
4	4	4	5	4	4	5
5	5	5	5	5	5	5

Example 38

In this example, the formula $\neg(p \equiv \neg p)$ is not true, even though one can prove a contradiction from $p \equiv \neg p$. Thus, there cannot be a negation introduction rule.

$U = \{0, 1, 2, 3\}$, $D = \{3\}$; $v(p) = 0$.

~	0	1	2	3
	3	2	1	0

∘	0	1	2	3
0	3	2	2	2
1	2	3	2	2
2	2	2	3	2
3	2	2	2	3

⊗	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

⊕	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

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Example 39

This example shows that the formula $(p \equiv q) \equiv ((p \wedge p) \equiv q)$ is not LD$'$-provable.

$U = \{0, 1, 2, 3, 4, 5\}$, $D = \{3, 4, 5\}$; $v(p) = 2$, $v(q) = 0$.

$\sim$	0	1	2	3	4	5
	5	4	3	2	1	0

$\circ$	0	1	2	3	4	5
0	5	1	0	0	1	1
1	1	5	4	0	1	1
2	0	4	5	1	0	0
3	0	0	1	5	4	0
4	1	1	0	4	5	1
5	1	1	0	0	1	5

$\otimes$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	1	0	0	1
2	0	1	1	0	0	1
3	0	0	0	4	4	4
4	0	0	0	4	4	4
5	0	1	1	4	4	5

$\oplus$	0	1	2	3	4	5
0	0	1	1	4	4	5
1	1	1	1	5	5	5
2	1	1	1	5	5	5
3	4	5	5	4	4	5
4	4	5	5	4	4	5
5	5	5	5	5	5	5

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