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Truthlikeness for Theories on Countable Languages*

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1 Introduction

With only slight oversimplification one may assert that the various theories of truthlikeness are governed by a single guiding metaphor: Truth, or better, the true theory W, is conceptualized as being located in a logical space of theories in such a way that the difference between W and a false theory F is a sort of distance between W and F: the larger the distance $d(W, F)$ the more F differs from the whole truth. Indeed Popper, who is considered as one of the founding fathers of a theory of truthlikeness or verisimilitude, seems to have been captivated by the above mentioned spatial metaphor when in *Conjectures and Refutations* he asked the question (1963, Chapter 10, § X):

> Is it not dangerously misleading to talk as if Tarskian truth were located somewhere in a kind of metrical or at least topological space so that we can sensibly say of two theories — say an earlier theory t_1 and a later theory t_2, that t_2 has superseded t_1, or progressed beyond t_1, by approaching more closely to the truth than t_1?
>
> I do not think that this kind of talk is at all misleading. On the contrary, I believe that we simply cannot do without something like this idea of a better or worse approximation to truth.

In order to overcome a merely metaphorical conception of 'distance from the truth', one has to show that for a given logical space of theories one

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can define a non-trivial metric structure. Or, at least, one has to show that for general reasons such a metrical structure exists. Popper addressed this problem properly only for finite logical spaces for propositional theories (cp. Popper 1976, § 2). In this contribution I should like to argue that Popper was more right than he might have imagined. Theories on countable languages may indeed be conceptualized as being located in a metrical space. This thesis is to be understood literally, not metaphorically. More precisely, I propose to construct for theories that can be formulated in a countable language L a logical space $T(L)$ endowed with an appropriate metric that satisfies most of the structural requirements that have been proposed for such a metric in the literature. This metric is objective in the sense that it does not depend on 'intuitive' choices but is determined by structural facts, at least to a large extent. The point at which a conventional choice has to be made, can be determined quite precisely, namely when it comes to the choice of a metric that is compatible with the 'objective' ordered topological structure.

The outline of the paper is as follows: In the next section we recall the rudiments of topology, order theory, and metric spaces we need in the following sections. In particular we state a fundamental theorem of Urysohn-Carruth, which asserts that under mild restrictions every ordered metrical space can be assumed to have a metric that is compatible with the order structure. This theorem will be crucial for the main result of this paper. In § 3 we characterize the theories $T(L)$ of a language L as the class of deductively closed sets of the Lindenbaum algebra LINDA(L) of logically equivalent propositions. This sets the stage for the topological representation of LINDA(L) carried out in § 4. In § 5 the logical space $T(L)$ of theories, conceived of as a Vietoris space of closed subsets of the Stone space $ST(L)$ is constructed. Thereby the Urysohn-Carruth metrization theorem can be used to show that this space can be endowed with a metric that is adequate for the purposes of truth approximation. In § 6 some relations between the account of truthlikeness advocated in this paper and Poincaré's conventionalist stance in geometry are pointed out.

2 Topology, metric, and order

In this section the basic notions of topology, theory of metric spaces and order structures are recalled that will be needed in the following. This section can offer only a very brief and incomplete survey. For a more thorough introduction to the topics mentioned in this section the reader is recommended to consult the literature (Davey & Priestley 1990, James 1999, Nachbin 1965, Nagata 1985, Niiniluoto 1987, §§ 1.1 and 1.2). Let us start with topology.

Definition 2.1 Let X be a set. A topological structure $\mathcal{O}(X)$ on X is given by a subset $\mathcal{O}(X) \subseteq \mathcal{P}(X)$ of the power set $\mathcal{P}(X)$ of X that is assumed to satisfy the following properties:

(i) $\emptyset$ and X belong to $\mathcal{O}(X)$.

(ii) If A_i is an arbitrary collection of subsets of X belonging to $\mathcal{O}(X)$, then the set-theoretical union $\bigcup A_i$ also is an element of $\mathcal{O}(X)$.

(iii) If A and B belong to $\mathcal{O}(X)$, then the intersection $A \cap B$ belongs to $\mathcal{O}(X)$.

The elements of $\mathcal{O}(X)$ are called *open sets*. An open set U that contains some element α is called an (open) neighbourhood of α. The set-theoretical complements of open sets are called *closed* sets.[1] The set of closed sets of a topological space $\langle X, \mathcal{O}(X)\rangle$ is denoted by $\mathcal{C}(X)$. Closed sets obey the dual conditions to those of Definition 2.1. Obviously, a topological structure on X is defined by $\mathcal{C}(X)$ as well as by $\mathcal{O}(X)$.

Definition 2.2 Let $\langle X, \mathcal{O}(X)\rangle$ be a topological space. A *base* of the topology $\mathcal{O}(X)$ is a subset B of $\mathcal{O}(X)$ closed under finite intersections, such that any open set is a union of elements of B. A *subbase* of $\mathcal{O}(X)$ is a subset S of $\mathcal{O}(X)$ such that any element of $\mathcal{O}(X)$ is the union of finite intersections of elements of S. Dually, a *closed base* is a subset $D \subseteq \mathcal{C}(X)$ that is closed under finite unions such that any element of $\mathcal{C}(X)$ is an intersection of elements of D. A topological space $\langle X, \mathcal{O}(X)\rangle$ is *second countable* if & only if it has a *countable* base. If $\langle X, \mathcal{O}(X)\rangle$ is a topological space the Cartesian product $X \times X$ is endowed with the topology that has $\{U \times V \mid U, V \in \mathcal{O}(X)\}$ as a base. This topology is called the *product topology* on $X \times X$.

Denote the set of real numbers by $\mathbf{R}$. The standard topology of $\mathbf{R}$ is the topology having as a base the set of open rational intervals $(x, z) := \{y \mid x < y < z\}$. Correspondingly, the class of closed intervals provides a closed base for this topology. With respect to this topology, $\mathbf{R}$ is second countable. More generally, let $\langle X, d\rangle$ be a metrical space, that is, a set X endowed with a real valued function $d : X \times X \longmapsto \mathbf{R}$ satisfying the following axioms:

$$\begin{array}{lrcl} (1) & d(x,y) & \geq & 0 \\ (2) & d(x,y) = 0 & \iff & x = y \\ (3) & d(x,y) & = & d(y,x) \\ (4) & d(x,z) & \leq & d(x,y) + d(y,z). \end{array}$$

The metrical space X is rendered a topological space by defining an open base B as follows: $B := \{y \mid d(x,y) < 1/n \text{ for some } x \in X \text{ and } n \in \mathbf{N}\}$. In the following, all metric spaces are assumed to be endowed with this topology. There may be different metrics defined on X defining the same topology. Hence, the topological structure of a metrical space does not uniquely determine its metrical structure.

Definition 2.3 Let $\langle X, \mathcal{O}(X)\rangle$ be a topological space.

[1] One should note that a set may be open and closed. This is only linguistically a paradox. For instance, for every topological space $\langle X, \mathcal{O}(X)\rangle$ the sets $\emptyset$ and X are open and closed. An open and closed set is often called *clopen*.

(i) X is *Hausdorff* if & only if distinct points α and β have disjoint open neighbourhoods U and V, respectively, that is, U and V are disjoint with $\alpha \in U$ and $\beta \in V$.

(ii) X is *normal* if & only if for disjoint closed sets A and B there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

(iii) X is *compact* if & only if every family of open sets A_i such that $X = \bigcup A_i$ has a finite subfamily $A_1, \ldots, A_n$ such that $X = A_1 \cup \cdots \cup A_n$.

The set of real numbers $\mathbf{R}$ endowed with the standard topology is a non-compact normal Hausdorff space. The closed unit interval $[0, 1]$ is a compact Hausdorff space.[2] More generally, metrical spaces are Hausdorff spaces, but they need not be compact. A topological space is *metrizable* if and only if its topology is induced by a metric. That is to say, it has a base defined by some metric. With respect to metrizability we need the following fundamental theorem (cp. Nagata 1985, pp. 253f.):

Theorem 2.4 (Urysohn) A second countable compact Hausdorff space is metrizable.

A refined version of this theorem lies at the heart of the theory of truth approximation to be formulated in this paper. More precisely, it will be shown that the logical space $T(L)$ of theories of a countable language L is a topological space to which a refined version of Theorem 2.4 applies. This refinement concerns logical order. Theories are naturally ordered with respect to logical strength, and somehow or other one has to deal with it. Generally speaking, the interplay between topological, metrical, and order structures turns out to be essential for a working theory of truth approximation. The following definition is useful to fix the relation between topological, metrical, and order structures (cp. Gierz & others 1980, p. 272):

Definition 2.5 A (partial) order relation on a set X is a binary relation that is reflexive, antisymmetric, and transitive.

(i) A set X endowed with a partial order $\leq$ is called a *poset*.

(ii) A topological space X is a *pospace* (partially ordered space) if and only if X is a poset *and* the order relation $\leq$ is a *closed* set with respect to the product topology on $X \times X$; that is, the set $\{\langle x, y\rangle \mid x, y \in X \text{ and } x \leq y\}$ is closed in $X \times X$.

Next let us assume that the pospace X is also endowed with a metrical structure d in the sense that the topological structure is induced by a metric d. Then, it is natural to ask how the order structure is related to the metrical structure. Somehow the order structure should respect the metric structure and vice versa. The following definition formulates a reasonable connection between metric and order structure:

[2] As is well-known, a compact Hausdorff space is normal.

Definition 2.6 Let X be a poset endowed with a metric d. The metric d is *radially convex* if $x \leq y \leq z$ implies $d(x,y) + d(y,z) = d(x,z)$.

For instance, the standard metric on the real line $\mathbf{R}$ is radially convex. For $n \geq 2$, one may define an order on $\mathbf{R}^n$ by $x \leq z$: if & only if $x_i \leq z_i$ where $x = (x_i)$ and $z = (z_i)$ for each i. With respect to this order the standard Euclidean metric of $\mathbf{R}^n$ is no longer radially convex. A radially convex metric on $\mathbf{R}^n$ is defined by $d(x,y) = \sum |x_i - y_i|/2^i$ for each $x, y \in \mathbf{R}^n$. This metric is equivalent to the standard metric in the sense that it induces the same topology on $\mathbf{R}^n$. Hence, it is natural to ask if each metrical pospace admits a radially convex metric equivalent to the original metric. For compact pospaces, this question is answered in the affirmative by the following theorem of Carruth (1968, p. 229):

Theorem 2.7 (Urysohn-Carruth metrization theorem) A compact pospace X with a metric d has an equivalent metric $d^\star$ that is *radially convex*.

The aim of this paper is to show that the logical space $T(L)$ of theories of a countable language L is indeed a compact metric pospace, and therefore may be assumed to have a radially convex metric. As far as I know, the notion of a radially convex metric does not appear in the literature dealing with matters of truth approximation, but related notions have been discussed by several authors. In (1984) Miller develops a pseudo-metrical[3] theory of Boolean algebras $\mathfrak{B}$ that is based on the following two axioms for a real-valued function $d : \mathfrak{B} \times \mathfrak{B} \longmapsto \mathbf{R}$ (ibidem, p. 99):

$$(0) \qquad d(x,y) + d(y,z) \geq d(x,z)$$

$$(1) \qquad d(x,y) \geq d(zx,zy) + d(x+z,x+y)$$

The inequality (0) is just the familiar 'triangle inequality' for distance functions. The meaning of (1) is more intricate. (1) may be interpreted as the essential link between the lattice structure and the metrical structure. As is shown by Miller, (0) and (1) imply that d is indeed a radially convex pseudo-metric which satisfies some further special properties. On the other hand, it seems that one cannot show that radial convexity entails (1). The relation between Miller's account and the one developed in this paper may roughly described as follows: Miller *assumes* that the Boolean algebra $\mathfrak{B}$ is endowed with a real-valued function d satisfying (0) and (1). Then he shows that this function is a radially convex pseudo-metric. In this paper it is shown that any countable Boolean algebra can be endowed with a radially convex metric. Hence, we show that for countable Boolean algebras Miller's assumption can be partially satisfied.

[3] In contrast to a metrical distance function, for a pseudo-metric it may happen that for two distinct elements x and y one has $d(x,y) = 0$. Actually, Miller's account is somewhat more general in that d need not be real-valued, but this is of no concern for the present paper.

3 Theories as deductively closed sets of the Lindenbaum algebra

In this section we begin to build up the logical space of theories formulated in a countable propositional language L. Then the true theory W may be thought to be located in such a way that the 'truthlikeness' of a theory can be conceived as 'distance from the truth'. In order to keep the technical apparatus as lean as possible let us concentrate on theories $F, G, \dots$ (among them the true theory W) that can be formulated in a classical propositional language L having a countable alphabet. That is to say, the logic of L is classical bivalent logic, L has the usual logical constants $\wedge, \vee, \neg$, and so on, and at most countably many propositional constants $p_1, p_2, \dots, p_n, \dots$.[4] The first thing to note is that under these assumptions the class of (finite) well-formed formulas of L is still countable, since finite strings can be coded by natural numbers.[5]

If we want to deal with the logical relations of theories and propositions we need not work with the propositions, that is, the well-formed formulas of L. Rather, we can restrict our attention to the equivalence classes of *logically equivalent* propositions. In other words, we can replace the algebra of well-formed formulas of L by the Lindenbaum algebra LINDA(L):

Definition and Lemma 3.1 Let L be a propositional language of classical logic. Two formulas a and b are said to be logically equivalent if and only if $a \leftrightarrow b$ is a tautology of propositional logic. The set of equivalence classes $[a]$ is denoted by LINDA(L) and is called the Lindenbaum algebra of L. LINDA(L) is a Boolean algebra whose operations are defined by the operations of L:

$$[a] \wedge [b] := [a \wedge b], [a] \vee [b] := [a \vee b], \neg[a] := [\neg a].$$

The order relation $\leq$ of LINDA as a Boolean algebra is just the order of logical consequence. That is to say $[a] \leq [b] := [a] \Rightarrow [b]$. The unit of LINDA(L) is denoted by 1 and the bottom element by 0. For typographical reasons we write in the following 'a' instead of '$[a]$'.

LINDA(L) may be characterized as the algebraic structure that determines the logical relations between propositions, and, as we shall see in a moment, also the logical relations between theories formulated in the language L. Following Tarski, a theory is defined as a set of propositions of L closed under logical consequence. In the language of algebraic logic this can be expressed as follows:

Definition 3.2 Let LINDA(L) be the Lindenbaum algebra of a countable language L. A *theory* F on L is a subset of L that is closed with respect to

[4] It may be shown (not in this paper) that the restriction to propositional logic is not necessary. Virtually the same argument goes through for a countable first-order language. Even the assumption of classical logic can be dropped in favour of more general logics that have a well-behaved Lindenbaum algebra. Finally, even the first-order assumption may be weakened, if one is prepared to give up the strict correspondence between the syntactic and the semantic level. The only non-negotiable assumption for the argument of this paper is that the language is countable. In this paper I am content to treat the simplest case, to wit, classical propositional logic with countably many propositional constants.

[5] For a detailed proof see, for example, Dunn & Hardegree (2001), p. 139.

logical consequence. Hence, algebraically, theories are characterized as *filters* of the Boolean lattice LINDA(L), that is, as subsets of LINDA(L) satisfying the following two conditions:

(i) F is upward closed: $a \leq b$ and $a \in F$ implies $b \in F$.

(ii) If $a, b \in F$, then $a \wedge b \in F$.

The set of theories of L is denoted by $T(L)$. For later use, the following special class of filters will turn out to be important.

(iii) F is a prime filter if & only if $a \vee b \in F$ implies $a \in F$ or $b \in F$.

One may distinguish at least the following three other kinds of theories (or filters):

$(\text{iv})_1$ The theory F is a finitely axiomatizable theory if & only if $F = \{a \mid b \leq a\}$ for some $b \in F$. Such a theory is called the principal filter generated by b.

$(\text{iv})_2$ The theory F is a complete theory if & only if for all $a \in$ LINDA(L) either $a \in F$ or $\neg a \in F$. A complete theory is also called an ultrafilter.[6]

$(\text{iv})_3$ The theory F is consistent if & only if $F \neq$ LINDA(L). The Boolean algebra LINDA(L) is called the trivial filter or the trivial theory on L.

At first glance, it is not at all obvious that complete theories exist. Indeed, the existence of sufficiently many ultrafilters, that is, complete theories, presupposes the validity of the axiom of choice or some similar principle (see Davey & Priestley 1990, Chapter 9). The axiom of choice is assumed throughout the rest of this paper without further mention. Then the true theory W in the sense of Popper may be conceptualized as a distinguished ultrafilter, that is, as a distinguished complete theory W for which either $a \in W$ or $\neg a \in W$ for all $a \in$ LINDA(L).

There is a natural map $r :$ LINDA(L) $\longmapsto T(L)$ defined by $r(a) := \{b \mid a \leq b\}$. In order to render this map order-preserving one defines an order relation $\leq$ on $T(L)$ by $F \leq G := G \subseteq F$. Here, $\subseteq$ is the standard set-theoretical order defined on $T(L)$ due to the fact that filters are subsets of LINDA(L). As is well known, $T(L)$ endowed with this order $\leq$ is not just an ordered structure but is a complete co-Heyting algebra $\langle T(L), \leq \rangle$ (Tarski 1935, Johnstone 1982).[7] Since $r :$ LINDA(L) $\longmapsto T(L)$ is order-preserving, one may consider the order relation $\leq$ on $T(L)$ as a natural generalization of the logical order of propositions of LINDA(L).

For $\langle T(L), \leq \rangle$ one can define a theory of truthlikeness, or, more generally, a qualitative (order-theoretical) theory of theory comparison that is essen-

[6]Obviously, the notion of a filter can be defined for general lattices. For Boolean algebras the notions of ultrafilter and prime filter coincide. This does not hold for more general lattices: if LINDA(L) is not Boolean, ultrafilters are prime filters, but the reverse does not hold.

[7]These authors consider the set-theoretical order $\subseteq$ on $T(L)$ that is opposite to the 'logical' order $\leq$. Hence for them, the structure of $T(L)$ is that of a Heyting algebra.

tially equivalent to the 'naive approach' of Kuipers (2002). This can be seen as follows: recall that the intersection $\bigcap F_i$ of an arbitrary family of filters is again a filter. More precisely, it is the supremum of the F_i with respect to the order relation $\leq$. Hence, for any subset $S \subseteq$ LINDA(L) there is a maximal filter F with $F \leq S$, to wit, the set-theoretical intersection of all filters including S. Hence, for any filter F its quasi-complement $F^{\#}$ is defined by $F^{\#} := \bigcap\{F_i \mid \mathbf{C}F \subseteq F_i\}$, $\mathbf{C}F$ being the set-theoretical complement of F. Moreover, for two filters F and G there is a unique largest filter $F \vee G$ included in both, to wit, the intersection of all filters that include F and G. Hence one may define:

Definition 3.3 Let U, V, and R be theories on L. Then U is at least as close to the target theory R as V is ($U \leq_R V$) if and only if

$$(U \cap R^{\#}) \cup (U^{\#} \cap R) \quad \subseteq \quad (V \cap R^{\#}) \cup (V^{\#} \cap R)$$

This definition corresponds to Kuipers's 'naive approach'. The only difference is that $T(L)$ is not a Boolean algebra of sets but only a complete co-Heyting algebra. This means essentially that the quasi-complement $V^{\#}$ of V satisfies only some weaker versions of the familiar laws of a Boolean complement:

$$(3.4) \qquad (U \wedge U^{\#})^{\#} = 1 \qquad U^{\#\#} \leq U \qquad U^{\#} = U^{\#\#\#}$$

Since Definition 3.3 depends only on the logical structure, one may call it an objective qualitative measure of similarity. Or, in the case the target theory R is the true theory W, the relation $\leq_W$ defines an objective logical measure of truthlikeness.[8] The shortcomings of this 'naive account' are well-known and need not be rehearsed here (cp. Miller 1984, Kuipers 2000, Chapter 10). As an important flaw of Definition 3.3 I take the fact that according to it most theories are not comparable. As will be shown, this shortcoming can be repaired. Any theory of truthlikeness that intends to improve on the naive approach by rendering more theories comparable should preserve the naive account as far as possible; that is, assuming that one has a metric d measuring the distances between theories these distances should be compatible with the logical order relations between theories established by the naive account. In the following it will be shown that this can be achieved by radially convex metrics.

4 The Stone Representation Theorem

In order to endow the set of theories $T(L)$ of L with an appropriate metric, that is, with a metric that respects the existing logical order structure, in this section we are going to construct a faithful topological representation of $T(L)$ as the co-Heyting algebra of closed sets of a topological space. The

[8] It should be noted that for Kuipers, though not for Popper or Miller, the true theory W is not an ultrafilter.

underlying set of the topological space to be constructed is the set $ST(L) := \{F_p \mid Fp$ is a proper prime filter of LINDA$(L)\}$. The topology on $ST(L)$ is defined by the lattice $\mathcal{C}(ST(L)))$ of closed sets as follows: For every filter $V \in T(L)$ define the range $r(V) \subseteq ST(L)$ by $r(V) := \{F_p \mid V \subseteq F_p\}$. The proof that $\langle r(V), V \in T(L)\rangle$ defines a topology on $ST(L)$ is a routine exercise. More precisely we get:

Theorem 4.1 Let L be a countable language with classical logic. The topological space $\langle ST(L), \mathcal{O}(ST(L))\rangle$ is a second countable, compact Hausdorff space, and the finitely axiomatizable theories a are represented by the clopen subsets $r(a)$ of $ST(L)$.

Proof: This theorem is nothing but a specialization of the famous Stone Representation Theorem (cp. Halmos 1963, Johnstone 1982) to the case of countable Boolean algebras.

Invoking Theorem 2.4 we get:

Corollary 4.2 The topological space $\langle ST(L), \mathcal{O}(ST(L))\rangle$ is metrizable.

Let us pause for a moment and take stock of what we have achieved so far. We have shown that by intrinsic logical means, that is, without introducing further primitives, the space of prime theories may be endowed with a metric that is compatible with the underlying topological structure of that space. This metric is in no way unique. Moreover, it is logically rather useless, since evidently all prime theories are logically incompatible. What we should like to have, however, is a metric not only for the rather inaccessible prime theories, but for all theories. This metric should be compatible with the logical relations existing between theories. In the next section, these desiderata are shown to be satisfiable. In order to understand why and how this can be done, it is useful to consider in some more detail the proof of Urysohn's metrization theorem.

The basic idea of the proof is to metrize a given topological space X by showing that it can be embedded in a certain metric space in such a way that this embedding induces a metrical structure on X. Denote by I the closed unit interval $[0, 1]$ endowed with the standard topology. The embedding space just mentioned is the so-called Hilbert cube $I^{\mathbf{N}}$ defined as follows:

Definition 4.3 Let $I := [0, 1]$ be the closed unit interval endowed with the standard metric, that is, $d(x, y) := |x - y|$. Denote the countable product of I with itself by $I^{\mathbf{N}}$. The space $I^{\mathbf{N}}$ is called the *Hilbert cube*. The elements of $I^{\mathbf{N}}$ are countable sequences x of real numbers with $0 \leq x_i \leq 1$. A metric d on $I^{\mathbf{N}}$ is defined by $d(x, y) := \sum d(x_i, y_i)/2^i$. The topology defined by d renders $I^{\mathbf{N}}$ a second countable, compact Hausdorff space.

The embedding of X into $I^{\mathbf{N}}$ is done is the following way: one constructs countably many continuous functions ('Urysohn functions') $f_n : X \longmapsto I$, $n \in \mathbf{N}$, which separate the points of X, in the sense that for distinct points α

and β there is always a function f_n with $f_n(\alpha) \neq f_n(\beta)$. The product function $f = (f_n)_{n \in \mathbf{N}}$ defines an embedding $e : X \longmapsto I^{\mathbf{N}}$ by $e(x) := (f_n(x))_{n \in \mathbf{N}}$. Then the metric structure of the Hilbert cube $I^{\mathbf{N}}$ may then be pulled back by e to X rendering it a metric space:

The metric structure of $ST(L)$ allows us to measure the distance between the points of $ST(L)$, that is, prime theories. This is not very useful in itself, since prime theories are rather elusive entities. What we are interested in, however, is measuring the distances between general theories in such a way that their order relations are respected.

To cope with this task, first note that $I^{\mathbf{N}}$ is not just a metric space but even a pospace with respect to the order relation $\leq$ defined by $x \leq y : x_i \leq y_i$ for all $i \in \mathbf{N}$. Secondly, the metric of Definition 4.3 is radially convex with respect to this order relation. Moreover, it satisfies the condition $d(x,z) \geq d(x \wedge y, y \wedge z) + d(x \vee y, y \vee z)$. In the next section it will be shown that the space of theories $T(L)$ is still a second countable compact Hausdorff space. Hence it can be embedded into the Hilbert cube. Moreover, due to the Urysohn-Carruth metrization theorem this can be done in such a way that the space of theories inherits the radially convex metric of $I^{\mathbf{N}}$.

5 The logical space of theories as a Vietoris space

The final step for constructing a metric suitable for truth approximation is to endow the set of theories, that is, the set $\mathcal{C}(ST(L))$ of closed sets of $ST(L)$ with an appropriate metric, that is, with a metric that respects the existing topology and the order structure on $\mathcal{C}(ST(L))$. Recall that $\mathcal{C}(ST(L))$ can be conceived as an order structure $\langle \mathcal{C}(ST(L)), \leq \rangle$ where $\leq$ is the opposite order to set-theoretical inclusion $\subseteq$.[9] In order to apply the Urysohn-Carruth metrization Theorem 2.4 one has to show that $\langle \mathcal{C}(ST(L)), \leq \rangle$ is a pospace. That is to say, it has to be endowed with a topology, and this topology has to be compatible with the order relation $\leq$ in the sense that the graph of $\leq$ is closed with respect to the product topology of $\mathcal{C}(ST(L)) \times \mathcal{C}(ST(L))$ (see Definition 2.5 (ii)). The recipe for endowing $\mathcal{C}(ST(L))$ with a topological structure may be traced back to Vietoris and Hausdorff in the twenties of the last century. It goes like this.

Theorem 5.1 Assume $\langle X, \mathcal{O}(X) \rangle$ to be a second countable compact Hausdorff space. Denote the set of closed subsets of X by $\mathcal{C}(X)$, and define a set of subsets of $\mathcal{C}(X)$ by

$$t(U) := \{F \mid F \in \mathcal{C}(X) \text{ and } F \subseteq U\}$$
$$m(U) := \{F \mid F \in \mathcal{C}(X) \text{ and } F \cap U \neq \emptyset\}$$

where U ranges over all open subsets of X. Then $\{t(U) \mid U \in \mathcal{O}(X)\} \cup \{m(U) \mid U \in \mathcal{O}(X)\}$ defines a subbase of a second countable compact Hausdorff topol-

[9] We rely on the opposite order so as to have the map LINDA $\mapsto ST$(LINDA) an order preserving map instead of a order reversing one.

ogy on $\mathcal{C}(X)$. The set $\mathcal{C}(X)$ endowed with this topology is called the *Vietoris* space of $\langle \mathcal{C}(X), \mathcal{O}(\mathcal{C}(X))\rangle$ of X.

With respect to the natural order of set-theoretical inclusion $\subseteq$ of closed subsets of X the Vietoris space $\mathcal{C}(X)$ can be conceived as an order structure. Using the following elementary reformulation of the concept of a pospace (cp. Gierz & others 1980, p. 273) one can show that it is indeed a pospace:

Theorem 5.2 Let $\langle X, \leq\rangle$ be a poset with topology $\mathcal{O}(X)$. X is a pospace if & only if whenever $a, b \in X$ and NOT $(a \leq b)$, there exist open sets $U, V \in O(X)$ with $a \in U$ and $b \in V$ such that if $x \in U$ and $y \in V$, then NOT $(x \leq y)$.

Theorem 5.3 Let X be a compact Hausdorff space. Then the Vietoris space $\mathcal{C}(X)$ of closed subsets of X is a pospace.

Proof: Assume $a, b \in \mathcal{C}(X)$ and NOT $(a \leq b)$. Hence there are $\alpha \in a$ and $\beta \in b$. Since X is normal, the closed sets $\{\alpha\}$ and b have disjoint open neighbourhoods $V(\alpha)$ and $R(b)$. Let $R(a)$ be an open set containing a. Note that $a \in m(V(\alpha))$. Hence $R(a) \cap m(V(a))$ is an open neighbourhood of a. More precisely, $t(R(a)) \cap m(V(\alpha))$ and $R(b)$ are open neighbourhoods of a and b such that for all $x \in t(R(a)) \cap m(V(\alpha))$ and $y \in t(R(b))$ we have NOT $(x \leq y)$. The reason is simply that the elements y of $R(b)$ do not contain the point α while the elements x of $t(R(a)) \cap m(V(\alpha))$ are open neighbourhoods of α by definition. Hence, $\mathcal{C}(X)$ is a pospace.

Thanks to Theorem 5.3, we may apply the Urysohn-Carruth metrization theorem (2.4) to the space of theories $\langle \mathcal{C}(ST(L)), \leq\rangle$ obtaining a radially convex metric d on $\mathcal{C}(ST(L))$. Putting things together we finally have reached the main theorem of this paper:

Theorem 5.4 Let L be a countable language. Then the logical space $T(L)$ of theories on L can be conceived of as the Vietoris space $\mathcal{C}(ST(L))$ of closed sets of $ST(L)$.[10] Then $T(L)$ is a *pospace* that can be endowed with a radially convex metric.

This metric can be used for truth approximation, or, more generally, for a metrical theory of theory comparison. It is, as it should be, in line with the underlying topological structure, and at least partially with the relevant logical structure, to wit the order structure $\leq$. For the time being, I do not know if it satisfies the further condition Miller (1984) set up for a good metric of truthlikeness In order to understand how this deficit may be overcome it is useful to dwell in some more detail upon Carruth's proof of his metrization theorem. It mimics exactly the proof of the classical theorem of Urysohn according to which a second countable compact Hausdorff space X is metrizable

[10]In a somewhat different manner, Brink, Pretorius, & Vermeulen (1992) use the order structure of the Vietoris space to define an order-theoretical notion of truth approximation. The main difference to the present paper is that they do not consider the possibility of endowing T(L) with a (radially convex) metrical structure.

since there is an embedding $e : X \longmapsto I^{\mathbf{N}}$. Explicitly, X is rendered a metric space by restricting the standard metric of $I^{\mathbf{N}}$to $e(X)$ and then pulling it back to X itself. The essential point is that thanks to technical results of Nachbin (1965) Carruth proves that the embedding function e is *order-preserving*, that is, for $a, b \in X$ and $a \leq b$ one has $e(a) \leq e(b)$. What he does *not* prove is that in the case of the Vietoris space $\mathcal{C}(X)$ is that the embedding e is also a lattice homomorphism. He does not prove that the embedding e satisfies also $e(a \wedge b) = e(a) \wedge e(b)$, and $e(a \vee b) = e(a) \vee e(b)$. If this could be done one could immediately derive Miller's requirement $d(a, b) \geq d(c \wedge a, c \wedge b) + d(c \vee a, c \vee b)$. As is easily checked, this is true for the standard metric of the Hilbert cube, and could be pulled back to $\mathcal{C}(X)$. Hence, if the embedding $e : \mathcal{C}(X) \longmapsto I^{\mathbf{N}}$ can be proved to be a lattice homomorphism one may pull back the 'good' metric of $I^{\mathbf{N}}$ to a 'good' metric on $\mathcal{C}(X)$.

6 Geometric conventionalism and logical spaces

The radially convex metric d constructed for the logical space of theories $T(L)$ in no way is unique. The construction of the order-preserving embedding $e : T(L) \longmapsto I^{\mathbf{N}}$ by the Urysohn-Carruth proof is highly non-constructive and involves several choices. This does not mean that d is arbitrary: recall that it has to respect the topological *and* the order structure. This ensures that the trivial metric (for example) is excluded. More precisely one may say that for the construction of d the ordered topological structure of $T(L)$ has to be taken as a priori, since it flows directly from the language L and its logic. On the other hand, moving from the ordered topological structure to the metric structure of $T(L)$ involves a conventional choice. This should not be considered as a flaw. Quite the contrary. It is very plausible that questions of truthlikeness and theory approximation do not totally depend on 'objective' facts but are also a matter of pragmatically motivated decisions. (Miller 1994, Chapter 10, § 4).

It may be useful to compare the problem of a priori and conventional structural strata of the logical space $T(L)$ with the corresponding problem of physical space. As is well known, a hundred years ago Poincaré subscribed to a sort of partial conventionalism with respect to the structure of physical space. According to him, the underlying topological structure of space was a priori while the metrical structure was conventional. $T(L)$, the results of this paper suggest an analogous result: there is an underlying objective topological structure defined by logic alone. On the other hand, there is some choice in constructing a metric for $T(L)$ via its embedding in the Hilbert cube $I^{\mathbf{N}}$. This may be considered as a conventional component in matters of truthlikeness.

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