Presuppositions, Logic, and Dynamics of Belief

SLAVKO BRKIĆ

University of Zadar, Department of Philosophy, K. P. Krešimira IV, 2
slbrkic@unizd.hr / slavko.brkic@zd.htnet.hr

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ABSTRACT: In researching presuppositions dealing with logic and dynamic of belief we distinguish two related parts. The first part refers to presuppositions and logic, which is not necessarily involved with intentional operators. We are primarily concerned with classical, free and presuppositonal logic. Here, we practice a well known Strawson's approach to the problem of presupposition in relation to classical logic. Further on in this work, free logic is used, especially Van Fraassen's research of the role of presupposition in supervaluations logical systems. At the end of the first part, presuppositional logic, advocated by S.K. Thomason, is taken into consideration. The second part refers to the presuppositions in relation to the logic of the dynamics of belief. Here the logic of belief change is taken into consideration and other epistemic notions with immanent mechanism for the presentation of the dynamics. Three representative and dominant approaches are evaluated. First, we deal with new, less classical, situation semantics. Besides Strawson's theory, the second theory is the theory of the belief change, developed by Alchourron, Gärdenfors, and Makinson (AGM theory). At the end, the oldest, universal, and dominant approach is used, recognized as Hintikka's approach to the analysis of epistemic notions.

KEY WORDS: Epistemic logic, presuppositions, common/mutual/joint knowledge, presupposition-based relation of inference, belief change, belief revision.

> In addition to this application, I believe that a large part of the discussion within linguistics and philosophy concerning presuppositions of sentences can be given a more unified treatment in terms of the expectations of the speaker.
>
> P. Gärdenfors

Speaking about Strawson's approach we can start with the claim that something is presuppositional, and it is the same as saying that something is different from the assumption.

It was Max who broke the glass.[1]

[1] Cf. Soames, 1989

It presupposes that the glass is broken and Max is the one who did it. The speaker is related to the presupposition and the assumption. Presuppositions are inherited, while the assumptions are not.[2]

For the purpose of the research dealing with logic and presuppositions, the form of a sentence is not relevant, e.g.

It wasn't Max who broke the glass.
Maybe it was Max who broke the glass.
It is unlikely that it was Max who broke the glass.
If it was Max who broke the glass, he will buy the new one.

We can use (P) to define presupposition, and (A) for assumption.

(P) Somebody broke the glass.
(A) Max broke the glass.

Some questions are imposed here: What is a presupposition? What do we mean by saying, A presupposes B? What is the function of the presupposition in the presentation of information? In which way do semantic rules, which are defined by information, influence presupposition? In which way pragmatic rules, which specify the way in which expressions magnify the set of presuppositions commonly used by the speakers, relate to presuppositions?[3]

In formal approach, many authors point out communicational intention, and plead for the autonomy of the usage of the sentence in relation to its reference, meaning and truth.

> To give the meaning of an expression (in the sense in which I am using the word) is to give *general directions* for its use to refer to or mention particular objects or persons.[4]

> The meaning of expression cannot be identified with the object it is used, on a particular occasion, to refer to. The meaning of a sentence cannot be identified with the assertion it is used, on a particular occasion, to make. For to talk about the meaning of an expression or sentence is not to talk about its use on a particular occasion, but about the rules, habits, conventions governing its correct use, on all occasions, to refer or to assert.[5]

The relationship of the communicational theory and the theory of assumptions with negative reference is neither false nor truth, and it can be summarized: listener's knowledge is a relevant element of communicational situation, and it structures expectations. First, speaker's expectations refer

[2] Discrepancy between the scope of logic and common sense reasoning can span a nonmonotonic logic. But, according to Thomason, systems of nonmonotonic inheritance tend to be expressively weak, and their relations to the more powerful nonmonotonic logic has never been fully clarified. Cf. Thomason, 2003.

[3] Cf. Soames, 1989: 555 ("Descriptive questions" and "Fundational questions").

[4] Cf. Strawson, 1950: 327.

[5] Ibid.

to what he knows or supposes about the listener's previous knowledge. Second, listener's expectations are directly dependent on what he knows and what he wants to know (knows [*presupposes*] that speaker knows).

So, the basic presupposition of the research is that speech situation presupposes minimum of common (mutual or joint) knowledge.[6] It influences and directs the research of the key logic theories of knowledge, especially of the dynamic of knowledge.

Van Fraassen is the leading figure of the theory of presupposition.[7] Bas C. van Fraassen shows semantic relation presupposing between the sentences. He considers it important to see the difference between implications and presuppositions, searching for the relation of presupposition and truth, which, according to him, opens the problem of the paradox of the liar and the self-reference.

Taking into consideration Strawson's assumption that the main features of the non-existing object cannot be recognized as false or true, we finally come to the question of the false or true sentence in relation to its interpretation or intention or something else. Having in mind artificial languages we can expect correct interpretation of it.

A presupposes B iff

(a) If A is true, B is true.
(b) If A is false, B is true.

According to Van Fraassen, it seems that presuppositions are trivial semantic relations. If all sentences presuppose universally valid sentences, the principle of two-values is not acceptable: each sentence is either true or false. This imposes the question of the relationship of presuppositions and implications. It seems that we can say

'The Present King of France is bald' (1968),

implicating that

'The King of France exists'.

We can, now, define relation of presupposing in this structure (including negation) as:

A presupposes B iff $A \Vdash B \vee \sim A \Vdash B$;

One of the most applicable ways of the presuppositional logic, advocated by S.K. Thomson, is not deprived of the baroque–like logic apparatus.

[6] For common sense reasoning we need reasoning about the attitudes of other agents. Cf. McCarthy, 1959.

[7] Cf. Van Fraassen, 1966; 1968; 1969; 1972; 1980; 1982; 1988; Van Fraassen and Lambert, 1967.

Thomason, besides his valid reasons for the different treatment of identity, in comparison to van Fraassen, has to enrich the language so that material presupposition might look like the following:

ϕ presupposes ψ if whenever ϕ is either true or false then ψ is true.

So, for a meta-language model of presupposition and structure A, Thomason has the language augmented by an existence predicate, definite description operator, and restricted generalization operator. In this way:[8]

$\phi \Rightarrow_{\mathrm{p}} \psi$ iff $(\forall A)(A(\phi) \in \{\perp, T\}$ then $A(\phi) = T$.

In the construction object-language models for presuppositions, Thomason follows Woodruff's term of "material presupposition":[9]

$\phi \rightarrow \psi =_{\mathrm{df}} (\phi \vee \sim\phi) \rightarrow \psi$

If there is a strict presupposition with the permitted truth-value gaps, then the structure is a non-empty set of possible worlds and evaluation functions. We assign all propositional variables the pair of disjunctive sets of possible worlds in the way that there is a set of possible worlds with truth variables, and the other with the false variables.

Presuppositional determination of the valuation function, quite contrary to Thomason, might be presented in the framework of the suggested presuppositional logic in the following Hintikka-style way:

$\alpha \rightarrow_{a} \beta =_{df.} K_{a} ((\alpha \vee \sim\alpha \rightarrow \beta),$

where "$\rightarrow$" is for presupposes, and "$_{a}$" is a term for a knower;

Accepting the above-mentioned determination of presuppositions, we can come to intentional logic and clear attitudes towards the problem of presuppositions.

For the sake of this research we have to take into consideration two things: common knowledge and logic representation of the dynamics of belief.

In this context the prominent part of analysis is the Barwise-style situation semantics.The notion of common knowledge in an analysis is probably introduced by David Lewis (1969). It is defined as an unlimited hierarchy of reciprocal knowledge.[10] If *p* is common knowledge of the two cognitively capable individuals (man and machine), *a* and *b*, then the sequence of truthful judgements follows.

8 Cf. Thomason, 1979: 360.
9 Cf. Woodruff, 1970.
10 Cf. Lewis, 1969.

$$K_a p \qquad K_b p$$
$$K_a K_b p \qquad K_b K_a p$$
$$K_a K_b K_a p \qquad K_b K_a K_b p$$

In the previous literature it has been confirmed that this notion is formally introduced by R. Aumann (1976), who suggested iterative together with circular or fixed-point definiton.[11] This suggestion helps us to compress unlimited number of syllogisms from the above mentioned part, into a single syllogism.. Individual knowledge or belief refers to 'propositions' percieved as a subset of the world views. This universe is divided into individual parts: it can't make a difference among the states of the world which are in the same element of its partition.

Aumann's circular definiton, referring to two cognitions, is the following: One event is part of the common knowledge if it consists of events pertaining to the subset of both partitions.

Aumann does not show clearly the equivalence between this definition and the other, non-limited iterative definition.

Similar situation is in situation semantics, although some authors insist on specificity and originality of their own approach. In this way, Barwise reaches trans-finite iteration but in different framework.[12]

In the analysis of epistemic notions, from the point of view of situation semantics, the problem of presuppositions is avoided by classical approach. Quantification of intentional contexts is accepted with all specifics in their approach The background knowledge as a basis of presuppositional problem is solved *via* the foundation of the fixed-point.

Definition of common knowledge is as infinite hierarchy of crossed knowledges[13]

$$E\varphi$$
$$E(\varphi \ \& \ E\varphi)$$
$$E(\varphi \ \& \ E\varphi \ \& E(\varphi \ \& \ E\varphi)$$
$$.$$
$$.$$
$$\ldots,$$

where, E stands for "everybody knows (that)".

Definition E: $E\varphi \leftrightarrow {}_{a\in A} K_a\varphi$

Fixed point axiom: $C\varphi \rightarrow E(C\varphi \wedge \varphi)$,

[11] For constructing paradox-free concepts of truth we have Russell's Ramified Type Theory, Tarski's Hierarchy of Truth Predicates, and the Fixed Points of Robert L. Martin, Peter Woodruff, Saul Kripke, and others. Cf. Martin und Woodruff, 1975; Kripke, 1975; Aumann, 1976. Theorem of the uniquenes might be found in Bernardi, 1975; 1976; Sambin, 1976.

[12] Cf. Barwise, 1989.

[13] Cf. Lismont and Mongin, 1993; Lismont, 1995.

where C stands for "common knowledge".

Induction Rule: $\dfrac{\varphi \to E\varphi}{E\varphi \to C\varphi}$

Monotonicity Rules: $\dfrac{\varphi \to \psi}{\mu\varphi \to \mu\psi}$ $(\mu \neq E)$.

What to say about presuppositions in this frame? In situation semantics we have similar example as *The King of France*: The President of U.S. is sneezing.

> Russell would say that the statement in question is true if there is one and only one president of the United States, and that person is sneezing.There are various ways this idea might be built into situation semantics, but one can see result would always interpret the statement with courses of events in which various individuals satisfied the condition of being president. Some of these would not have Reagan in them at all, and so he would not be a constituent of the interpretation. On the other hand, being president would be a constituent. Each of the courses of events would be defined at the present time, and each would consider the property of being president at that time. Russell's theory puts the describing condition into the interpretation, but not the described individual. Strawson made just the opposite decisions. Described individual, but not the describing condition, is a constituent of the interpretation.[14]

The key for the understanding of the relationship between the presuppositions and the logic of belief change is the theory of the change of belief (AGM Theory).[15] This theory gives appropriate records of rational postulates referring to belief change, but now in the form of functional language. In this case the most prominent thesis of the belief change model refers to the acceptance of the epistemic input in the old state of belief. The definition of functions, expansions, contractions and revisions depends on the time of acceptance of input. Let's see Gärdenfors's definition:

> In passing, I present a simple application of the definition of a proposition. Let A be a proposition, that is, a function defined on epistemic states. A is said to be *accepted as known* in epistemic state K if and only if $A(K) = K$. In other words, this identity says that, if the function corresponding to the epistemic input A is applied to K, then the resulting state of belief is K itself: Adding A has no effect on K. The relation 'A is accepted in K' will play a central role in the theory to be presented, parallel to the role of the relation 'A is true in the world w' for the possible worlds analysis of propositions.[16]

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[14] Cf. Barwise and Perry, 1983: 145–146.

[15] Harper can be considered as the founder of the theory. Cf. Harper, 1975; 1976; 1977; 1978; Harper and Hooker, 1976. Before this theory, we have investigators from inductive logic and theory of probabability: Y. Bar-Hillel, R. Chisholm, R. Carnep, N. Goodman, C. G. Hempel, K. J. J. Hintikka, R. Jeffery, H. E. Jr. Kyburg, K. Lehrer, I. Levi, K. R. Popper, H. Reichenbach, N. Resher, W. Salmon, L. Savage, P. Suppes, G. H. von Wright.

For AGM Theory cf. Alchourron and Makinson 1982; 1985a, 1985b, Alchourron et al. 1986; Gärdenfors, 1988; 1992.

[16] Cf. Gärdenfors, 1988: 133–134.

Since the linear acceptance of the propositions (one by one) in reference to epistemic state (as a set of propositions) brings iteration and iteration together with cognizance brings common knowledge[17], this all together mirrors Gärdenfors's strategy. His assertion that proposition A (firmly believed or accepted) is known in K if $A(K) = K$. He said that proposition A is accepted in K only if $K \in A^f$, where A^f denotes the set of fixed points of the function A, i.e. $\{x: A(x) = x\}$.[18]

Gärdenfors presents set of postulates for belief changes which are supposed to be rationality criteria for revisions (and contractions) of belief sets as:

Gärdenfors assumes that for every belief set K and every sentence A in L, there is unique belief set K^*A representing the revision of K with respect to A. He presents set of postulates for belief changes which are supposed to be rationality criteria for revisions (and contractions) of belief sets as:

(K*1) For any sentence A and any belief set K, K*A is a belief set.

	("Closure")
(K*2) $A \in K^*A$	("Success")
(K*3) $K^*A \subseteq K^+A$	("Expansion 1")
(K*4) If $\neg A \notin K$, then $K^+A \subseteq K^*A$	("Expansion 2")
(K*5) $K^*A = K_\perp$ iff $\vdash \neg A$	("Consistency preservation")
(K*6) If $\vdash$ A $A \leftrightarrow B$, then $K^*A = K^*B$.	("Extensionality")
(K*7) $K^*A \ \& \ B \subseteq (K^*A)^+B$	("Conjunction 1")
(K*8) If $\neg B \notin K^*A$, then $(K^*A)^+B \subseteq K^*A \ \& \ B$	("Conjunction 2 rational monotony")
(K*M) If $H \subseteq K$, then $H^*A \subseteq K^*A$;	

Gärdenfors presents some consequences of those postulates:

1* If $A \in K$, then $K = K^*A$

2* $K^*A = (K \cap K^*A)^+A$

3* $K^*A = K^*B$ iff $B \in K^*A$ and $A \in K^*B$ ("Reciprocity")

4* $K^*A \cap K^*B \subseteq K^*A \vee B$

5* $\neg B \notin K^*A \vee B$, then $K^*A \vee B \subseteq K^*B$

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[17] Common knowledge is, as it is confirmed, the point on which the presuppositions are clearly represented. Because of it the previous Gärdenfors's determination of the acceptance of the epistemic input (propositions) is the most important in the whole theory of belief change. Evidently, this definition avoids non-limited iteration *via* the fixed point.

[18] Cf. Smorynski, 1985: ch. 5; Gupta and Belnap, 1993: chs. 2–3; Pearce and Rautenberg, 1987.

6* $K^*A \cap K^*B \subseteq K^*A\ V\ B$

7* $K^*A\ V\ B = K^*A$ or $K^*A\ V\ B = K^*B$ or $K^*A\ V\ B = K^*A \cap K^*B$

8* $K^*A\ V\ B \subseteq Cn(K^*A \cup K^*B)$ ("Disjunction")

9* If $B \in K$, then $B \notin K^*A$ or $\neg B \in K^*A$;

10* If K maximal, then for any A, K^*A is maximal

11* K^*A is maximal for any sentence A such that $\neg A \in K$

12* If $\vdash A \rightarrow B$, then $K^*A \subseteq K^*B$

13* If $B \in K^*A$, then $K^*A\&B\ K^*A$ ("Cut")

14* If $B \in K^*A$, then $K^*A\&B$ ("Cautious monotony")

At the end stands the question how to revise the base of facts or the set of beliefs? It is necessary to have the *selectional mechanism* or additional information to make a decision which sentence is to be deleted and which one is to be kept in the set of beliefs. According to Gärdenfors there are five possibilities for the delivering of the missing information for the concrete process of the changes, referring to the set of beliefs. One of them is contractive function of the epistemic *entrenchment*.

> Even if all sentences in a belief set are accepted or considered as facts (so that they are assigned maximal probability), this does not mean that all sentences are of equal value for reasoning in planning and problem-solving. Certain pieces of our knowledge and beliefs about the world are more important than others when planning future actions, conducting scientific investigations, or reasoning in general. We will say that some sentences in a belief system have a higher degree of epistemic entrenchment than others.[19]

Epistemic entrenchment owes its existence to the development of the inductive logic, and in its framework it has its doublets: "the measure of plausibility", "potential surprise", "modal categories", "possibility distribution", "inductive probability", "priority of the base information"[20], "the level of belief"… Besides that, entrenchment is a notion defined in the field mentioned above. Nelson Goodman is the first who pointed out the external parts of logic in the conclusion, criticising qualitative theory of confirmation advocated by Carnap.

Besides the fact that he introduced entrenchment, Goodman was involved with one other important thesis dealing with the usage of language. If the usage of language includes significant epistemic asymmetry between different predicative notions in science, than all systems of the inductive logic, which include primitive predicate, treated epistemologically, are equally ex-

[19] Cf. Gärdenfors, 1992: 30.
[20] Cf. Fagin, et al., 1983.

posed to failure. It is quite clear why Goodman situated his organization of the phenomenon in the human habits. Our inductions refer to the future events and the structure of the world, and human habits and beliefs are of the utmost importance in the formation of the *expectations* dealing with it.

We, following Gärdenfors's works, have:

If A and B are recognized as the sentences of the involved language, we are supposed to write A<B for "B is epistemic entrenchment of A"

Gärdenfors proposes the basic set of postulates for relation of epistemic entrenchment:

(EE1)	Non A<A:	(irreflexivity)
(EE2↑)	If A<B and B $\vdash$ C, then A<C;	(continuing up)
(EE2↓)	If A<B and C $\vdash$ A, then C<B	(continuing down)
(EE3↑)	If A<B and A<C, then A<B&C	(conjunction up)
(EE3↓)	If A&B<B then A<B	(conjunction down)

Therefore, if epistemic entrenchment is a relation over the set of formulas which satisfying (EE1)-(EE3), then we have following:

(EEi)	If A<B, Cn(A) = Cn(A') and Cn(B) = Cn(B), then A'<B'	(extensionality)
(EEii)	If A&C<B&C, then A<B	
(EEiii)	If A<B and C<D, then A&C<B&D	
(EEiv)	If A<B then non B<A	(asymmetry)
(EEv)	If A<B and B<C, then A<C	(transitivity)
(EEvi)	A<A$\vee$B Iff A$\vee\neg$B<A$\vee$B	

Relation of epistemic entrenchment can satisfy following added postulates:

(EE4)	If A<B then or A<C or C<B	(virtual connectivity)
(EE5)	If K $\neq$ $K_\perp$, then A $\in$ K iff B<A for any B	(minimality)
(EE6)	If non $\vdash$ A, then A<B for any B	(maximality)
(EE7)	If A $\in$ K and B $\notin$ K, then B<A	(K-representation)
(EE8)	If A<T and non B<T, then A<B	(top equivalence)
(EE8)	If B $\vdash$ A then A plain $\nless$ B	(singleton non-covering)
(EE9)	If A&B<A then A&B $\nless$ B	(conjunctiveness)

The axiomatic basis for the present logical system:

Primitive symbols

– propositional variables: p, q, r, s,…

– monadic operators: ¬, K_a;

– dyadic operators: V

(,). parentheses here serve to group the statements to which '¬' is applied.[21]

Definitions:

def.&	(α&β)=df.	¬(¬ αV¬β)
def.→	(α→β)=df.	(¬ αV β)
def. ↔	(α↔β)=df.	(α→ β)&(β→α)
		(¬αV β) & (¬βV α)
		¬(¬(¬αV β) V ¬(¬βV α)
		(α&β) V (¬α & ¬β)
def.[+]	$[^{+}\alpha]_a\beta$ =*df*.	$K_a(\alpha\rightarrow\beta)$[22]

Formation rules:

FR1:	Any sentence letter is *wff*.
FR2:	If α is a *wff* , so is ¬α i K_aα *wff*.
FR3:	If α and β are *wff*, so is (αV β) *wff*.

Axioms (selected set of *wff*s – from *Principia Mathematica*):

PMA1 (pVp)→p

PMA2 q→(pVq)

PMA3 (pVq)→(qVp)

PMA4 (q→r) →((pVq)→(pVr)

Transformation rules:

US: (*Uniform Substitution*): If a is a theorem, so is every substitution-instance of α.

Eq: (*Substitution of equivalent*): If ├ (α ↔ β), and if and if γ and δ differ only in that g may have a in one or more places where δ has β, then ├ (γ ↔δ) (and hence if ├ α then ├ β).

[21] Cfr. Quine, W. V. O. 1980 (1998), *Elementary Logic*, Harvard University Press, pp. 11, 27, 53 and Quine, W. V. O. 1982, *Methods of Logic*, Harvard University Press, pp. 29–30, 143.

[22] Where "$_a$" stands for "knower".

MP:	*(Modus ponens):*	$\alpha, \alpha \to \beta / \beta$
$R^{-}N$:	(*Necessitation*):	$\alpha / [^{-}\alpha]_a \alpha$
$R^{*}N$:	(*Necessitation*):	$\alpha / [^{*}\alpha]_a \beta$
R_kN:	(*Necessitation*):	$\alpha / K_a \alpha$

"Derived" rules:

$C^{-}E$:	Extensionality	$\alpha \leftrightarrow \beta / [^{-}\alpha]_a \gamma \leftrightarrow [^{-}\beta]_a \gamma$
$C^{*}E$:	Extensionality	$\alpha \leftrightarrow \beta / [^{*}\alpha]_a \gamma \leftrightarrow [^{*}\beta]_a \gamma$
$R^{-}M$:		$\beta \to \gamma / [^{-}\alpha]_a \beta \to [^{-}\alpha]_a \gamma$

Levy has suggested that revisions should be defined in terms of contractions:

LI:	$[^{*}\alpha]_a \beta =_{df.} [^{-}\neg\alpha]_a K_a (\alpha \_ \beta)$

Some proving formulas from propositional calculus:

PMT 1	$(pVq) \leftrightarrow \neg(\neg p \& \neg q)$	
PMT 2	$p \leftrightarrow \neg\neg p$	DN
PMT 3	$p \to (pVq)$	
PMT 4	$(p \& q) \to p$	
PMT 5	$(p \& q) \to q$	
PMT 6	$p \to (q \to p)$	
PMT 7	$\neg p \to (p \to q)$	
PMT 8	$p \to (q \to (p \& q))$	Adj
PMT 9	$(p \to q) \to ((q \to r) \to (p \to r))$	Syll
PMT10	$(p \to (q \to r)) \to ((p \& q) \to r)$	Imp
PMT11	$(p \to q) \to ((p \to r) \to (p \to (q \& r)))$	Comp
PMT12	$(p \to q) \to ((r \to s) \to ((p \& r) \to (q \& s)))$	
PMT13	$(p \to r) \to ((q \to r) \to ((pVq) \to r))$	
PMT14	$(\neg p \leftrightarrow q) \leftrightarrow (p \leftrightarrow \neg q)$	
PMT15	$(\neg p \to p) \leftrightarrow p$	
PMT16	$((q \to p) \& (\neg q \to p)) \leftrightarrow p$	
PMT17	$((p \to q) \& (p \to \neg q)) \leftrightarrow \neg p$	
PMT18	$(p \to q) \to ((p \& r) \to (q \& r))$	
PMT19	$(p \to (q \& r)) \to ((p \to q) \& (p \to r)$	

We obtain system of logic of belief change by adding some axioms to the above defined basis:

PMA1-PMA4 from PM plus:

K_k	$K_a(p\rightarrow q)\rightarrow(K_a p\rightarrow K_a q)$	
K^-	$[^-p]_a(q\rightarrow r)\rightarrow([^-p]_a q\rightarrow[^-p]_a r)$	
K^*	$[^*p]_a(p\rightarrow q)\rightarrow([^*p]_a p\rightarrow[^*p]_a q)$	
T-	$K_a p\rightarrow[^-T]_a p$,	(where T is theorem)
C/M	$[^-p]_a q\&[^-p]_a r\leftrightarrow[^-p]_a(q\rightarrow r)$	
CL_r	$K_a[^-p]_a q\rightarrow[^-p]_a q$	
CR	$[^-p]K_a q\leftrightarrow[^-p]_a q$	
C2	$[^-p]_a q\rightarrow K_a q$	
C3	$\neg K_a p\& K_a q\rightarrow[^-p]_a q$	
C4	$[^-p]_a p\rightarrow[^-q]_a p$	
C5	$K_a q\rightarrow[^-p]_a(p\rightarrow q)$	
C7	$[^-p]_a r\&[^-q]_a r\rightarrow[^-p\& q]_a r$	
C8	$\neg[^-p\& q]_a p\&[^-p\& q]_a r\rightarrow[^-p]_a r$	
RL_r	$K_a[^*p]_a q\rightarrow[^*p]_a q$	
RR	$[^*p]_a K_a q\leftrightarrow[^*p]_a q$	
R2	$[^*p]_a p$	
R3	$[^*p]_a q\rightarrow[^+p]_a q$	
R4	$\neg K_a\ \neg p\&[^+p]_a q\rightarrow[^*p]_a q$	
R5	$[^*p]_a{}^\wedge\rightarrow[^*q]_a\neg p$	
R7	$[^*p\& q]_a r\rightarrow[^*p]_a(q\rightarrow r)$	

Now we can prove some formulas for logic of belief change:

T_1 $[^-\neg p]_a(p\rightarrow q)\rightarrow K_a(p\rightarrow q)$

p r o o f:

(1)	$[^*p]_a q\rightarrow[^+p]_a q$	R3
(2)	$[^*a]_a b =_{df.} [^-\neg\ a]_a[^+\ a]_a\ b$	LI
(3)	$[^-\neg p]_a[^+p]_a q\rightarrow[^+p]_a q$ (1) ×	(2)
(4)	$[^-\neg p]_a K_a(p\rightarrow q)\rightarrow K_a(p\rightarrow q)$	(3) × def. [+]

(5) $[^{-}\neg p]_a(p\to q)\to K_a(p\to q)$ (4) × CR × Eq

Q.E.D.

T$_2$ $\neg K_a\neg p\&K_a(p\to q)\to[^{-}\neg p]_a(p\to q)$

p r o o f:

(1) $\neg K_a\neg p\&[^{+}p]_a q\to[^{*}p]_a q$ R4

(2) $[^{*}a]_a b=_{df.}[^{-}\neg a]_a[^{+}a]_a b$ LI

(3) $\neg K_a\neg p\&K_a(p\to q)\to[^{-}\neg p]_a[^{+}p]_a q$ (1) × (2)

(4) $\neg K_a\neg p\&K_a(p\to q)\to[^{-}\neg p]_a K_a(p\to q)$ (3) × def. [+]

(5) $\neg K_a\neg p\&K_a(p\to q)\to[^{-}\neg p]_a(p\to q)$ (4) × CR × Eq

Q.E.D.

DR1 If $\vdash(\alpha\to\beta)$, then $\vdash(K_a\alpha\to K_a\beta)$

d e r i v a t i o n:

(1) $(\alpha\to\beta)$ hypothesis

(2) $K_a(\alpha\to\beta)$ (1) × R_kN

(3) $K_a(p\to q)\to(K_a p\to K_a q)$ K_k

(4) $(K_a(\alpha\to\beta)\to(K_a\alpha\to K_a\beta))$ K_k [α/p, β/q]

(5) $(K_a\alpha\to K_a\beta)$ (2),(4) × MP

Q.E.D.

T$_3$ $K_a(p\leftrightarrow q)\to(K_a p\leftrightarrow K_a q)$

p r o o f:

(1) $K_a(p\to q)\to(K_a p\to K_a q)$ K_k

(2) $K_a(q\to p)\to(K_a q\to K_a p)$ (1)[q/p,p/q]

(3) $(p\to q)\to((r\to s)\to((p\&r)\to(q\&s)))$ PMT12

(4) $(K_a(p\to q)\&K_a(q\to p))\to((K_a p\to K_a q)\&$
$(K_a q\to K_a p))$ (1),(2) × (3)

(5) $K_a(p\leftrightarrow q)\to(K_a p\leftrightarrow K_a q)$ (4) × def. ↔

Q.E.D.

T$_4$ $K_a(p\&q)\to(K_a p\&K_a q)$

p r o o f:

(1)	$K_a(p\&q)\rightarrow K_ap$	PMT4 × DR1
(2)	$K_a(p\&q)\rightarrow K_aq$	PMT5 × DR1
(3)	$(p\rightarrow q)\rightarrow((p\rightarrow r)\rightarrow(p\rightarrow(q\&r))$	PMT11
(4)	$(K_a(p\&q)\rightarrow K_ap)\rightarrow((K_a(p\&q)\rightarrow K_aq)$ $((K_a(p\&q)\rightarrow(K_ap\&K_aq)))$	(3) × $[K_a(p\&q)/p, K_ap/q, K_aq/r]$
(5)	$(K_a(p\&q)\rightarrow K_aq)\rightarrow((K_a(p\&q)\rightarrow(K_ap\&K_aq))$	(1),(4) × MP
(6)	$K_a(p\&q)\rightarrow(K_ap\&K_aq)$	(2),(5) × MP
		Q.E.D.

T_5 $((K_ap\&K_aq)\rightarrow K_a(p\&q))$

p r o o f:

(1)	$p\rightarrow(q\rightarrow(p\&q))$	PMT8
(2)	$K_ap\rightarrow K_a(q\rightarrow(p\&q))$	(2) × **DR1**
(3)	$K_a(p\rightarrow q)\rightarrow(K_ap\rightarrow K_aq)$	K_k
(4)	$K_a(q\rightarrow(p\&q))\rightarrow(K_aq\rightarrow K_a(p\&q))$	$K_k[q/p, p\ \&\ q/q]$
(5)	$(p\rightarrow q)\rightarrow((q\rightarrow r)\rightarrow(p\rightarrow r))$	PMT9
(6)	$(K_ap\rightarrow K_a(q\rightarrow(p\&q)))\rightarrow((K_a(q\rightarrow(p\&q)))\rightarrow$ $\rightarrow(K_aq\rightarrow K_a(p\&q)))\rightarrow(K_ap\rightarrow(K_aq\rightarrow K_a(p\&q)))$	(5) × $[K_ap/p, K_a(q\rightarrow$ $\rightarrow(p\&q)/q, K_aq\rightarrow$ $\rightarrow K_a(p\ \&q)/r]$
(7)	$(K_a(q\rightarrow(p\&q))\rightarrow(K_aq\rightarrow K_a(p\&q)))\rightarrow$ $\rightarrow(K_ap\rightarrow(K_aq\rightarrow K_a(p\&q)))$	(2),(6) × MP
(8)	$(K_ap\rightarrow(K_aq\rightarrow K_a(p\&q)))$	(4),(7) × MP
(9)	$(p\rightarrow(q\rightarrow r)\rightarrow((p\&q)\rightarrow r)$	PMT10
(10)	$(K_ap\rightarrow(K_aq\rightarrow K_a(p\&q))\rightarrow((K_ap\&K_aq)\rightarrow$ $\rightarrow K_a(p\&q))$	(9) × $[K_ap/p, K_aq/q, K_a(p\ \&\ q)/r]$
(11)	$((K_ap\&K_aq)\rightarrow K_a(p\&q))$	(8),(10) × MP
		Q.E.D.

T_6 $K_a(p\&q)\leftrightarrow(K_ap\&K_aq)$

p r o o f:

(1)	$((K_ap\&K_aq)\rightarrow K_a(p\&q))$	**T_5**
(2)	$K_a(p\&q)\rightarrow(K_ap\&K_aq)$	**T_4**
(3)	$p\rightarrow(q\rightarrow(p\&q)$	PMT8
(4)	$(K_a(p\&q)\rightarrow(K_ap\&_aq))\rightarrow(((K_ap\&K_aq)\rightarrow$ $\rightarrow K_a(p\&q))\rightarrow((K_a(p\&q)\rightarrow(K_ap\&K_aq))$ $((K_ap\&K_aq)\rightarrow K_a(p\&q)))$	
		$(3) \times [K_a(p\&q)\rightarrow$ $\rightarrow(K_ap\&K_aq)/p, ((K_ap\&$ $K_aq)\rightarrow K_a(p\&q))/q]$
(5)	$(((K_ap\&K_aq)\rightarrow K_a(p\&q))\rightarrow((K_a(p\&q)\rightarrow$ $\rightarrow(K_ap\&K_aq))\&((K_ap\&K_aq)\rightarrow K_a(p\&q)))$	(1),(4) × MP
(6)	$((K_a(p\&q)\rightarrow(K_ap\&K_aq))\&((K_ap\&K_aq)\rightarrow$ $K_a(p\&q)))$	(2),(5) × MP
(7)	$K_a(p\&q)\leftrightarrow(K_ap\&K_aq)$	(6) × def.$\leftrightarrow$
		Q.E D.

DR2 If $\vdash$ ($a\leftrightarrow b$), then $\vdash$ ($K_a\ \alpha\leftrightarrow K_a\ \beta$)

d e r i v a t i o n:

(1)	$\alpha\leftrightarrow\beta$	hypothesis
(2)	$K_a(\alpha\leftrightarrow\beta)$	(1) × (R_kN)
(3)	$K_a(p\leftrightarrow q)\rightarrow(K_ap\leftrightarrow K_aq)$	**T_3**
(4)	$K_a(\alpha\leftrightarrow\beta)\rightarrow(K_a\alpha\leftrightarrow K_a\beta)$	(4)[α/p, β/q]
(5)	$K_a\alpha\leftrightarrow K_a\beta$	(2),(4) × MP
		Q.E.D.

T_7 $(K_apVK_aq)\rightarrow K_a(pVq)$

p r o o f:

(1)	$p\rightarrow(pVq)$	PMT3
(2)	$K_ap\rightarrow K_a(pVq)$	(1) × **DR1**
(3)	$q\rightarrow(pVq)$	PMA2

(4)	$K_a q \rightarrow K_a(pVq)$	(3) × **DR1**
(5)	$(p\rightarrow r)\rightarrow((q\rightarrow r)\rightarrow((pVq)\rightarrow r))$	PMT13
(6)	$(K_a p\rightarrow K_a(pVq))\rightarrow((K_a q\rightarrow K_a(pVq))\rightarrow$	
	$\rightarrow(K_a pVK_a q)\rightarrow K_a(pVq))$	(5)[$K_a p/p$, $K_a q/q$, $K_a(pVq)/r$]
(7)	$((K_a q\rightarrow K_a(pVq)\rightarrow(K_a pVK_a q)\rightarrow K_a(pVq))$	(2),(6) × MP
(6)	$(K_a pVK_a q)\rightarrow K_a(pVq)$	(4), (7) × MP
		Q.E.D.

T_8 $K_a(\neg p\rightarrow p) \leftrightarrow K_a p$

p r o o f:

(1)	$(\neg p\rightarrow p) \leftrightarrow p$	PMT15
(2)	$K_a(\neg p\rightarrow p)\leftrightarrow K_a p$	(1) × **DR2**
		Q.E.D.

[T_9] $[^+\neg p]_a p\leftrightarrow K_a p$

T_{10} $K_a(p\rightarrow\neg p)\leftrightarrow K_a\neg p$

p r o o f:

(1)	$(\neg p\rightarrow p)\leftrightarrow p$	PMT15
(2)	$(p\rightarrow\neg p)\leftrightarrow\neg p$	(1)[$p/\neg p$, $\neg p/p$]
(3) $K_a(p\rightarrow\neg p)\leftrightarrow K_a\neg p$		(2) × **DR2**
		Q.E.D.

[T_{11}] $[^+p]_a\neg p\leftrightarrow K_a\neg p$

T_{12} $(K_a(q\rightarrow p)\&K_a(\neg q\rightarrow p)) \leftrightarrow K_a p$

p r o o f:

(1)	$((q\rightarrow p)\&(\neg q\rightarrow p))\leftrightarrow p$	PMT16
(2)	$K_a((q\rightarrow p)\&(\neg q\rightarrow p))\leftrightarrow K_a p$	(1) × DR2
(3)	$K_a(p\&q) \leftrightarrow(K_a p\&K_a q)$	**T_6**
(4)	$K_a((q\rightarrow p)\&(\neg q\rightarrow p))\leftrightarrow(K_a(q\rightarrow p)\&$ $K_a(\neg q\rightarrow p))$	(3) × [$(q\rightarrow p)/p$,$(\neg q\rightarrow p)/q$]

(5)	$(K_a(q\rightarrow p)\&K_a(\neg q\rightarrow p))\leftrightarrow K_a p$	(4) × Eq
		Q.E.D.

[T$_{13}$] $[^+q]_a p\&[^+\neg q]_a p\leftrightarrow K_a p$

T$_{14}$ $(K_a(p\rightarrow q)\&(p\rightarrow\neg q))\leftrightarrow K_a\neg p$

p r o o f:

(1)	$((p\rightarrow q)\&(p\rightarrow\neg q)) \leftrightarrow \neg p$	PMT17
(2)	$K_a((p\rightarrow q)\&(p\rightarrow\neg q)) \leftrightarrow K_a\neg p$	(1) × **DR2**
(3)	$K_a(p\&q)\leftrightarrow(K_a p\&K_a q)$	**T$_6$**
(4)	$K_a((p\rightarrow q)\&(p\rightarrow\neg q))\leftrightarrow(K_a(p\rightarrow q)\&$ $K_a(p\rightarrow\neg q))$	$(3)[(p\rightarrow q)/p,(p\rightarrow\neg q)/q]$
(5)	$(K_a(p\rightarrow q)\&K_a(p\rightarrow\neg q))\leftrightarrow K_a\neg p$	(4) × Eq
		Q.E.D.

[T$_{15}$] $[^+p]_a\, q\&[^+p]_a\neg q\leftrightarrow K_a\neg p$

T$_{16}$ $K_a p\rightarrow K_a(q\rightarrow p)$

p r o o f:

(1)	$p\rightarrow(q\rightarrow p)$	PMT6
(2)	$K_a p\rightarrow K_a(q\rightarrow p)$	(1) × **DR1**
		Q.E.D.

[T$_{17}$] $K_a p\rightarrow[^+q]_a p$

T$_{18}$ $K_a\neg p\rightarrow K_a(p\rightarrow q)$

p r o o f:

(1)	$\neg p\rightarrow(p\rightarrow q)$	PMT7
(2)	$K_a\neg p\rightarrow K_a(p\rightarrow q)$	(1) × **DR1**
		Q.E.D.

[T$_{19}$] $K_a\neg p\rightarrow[^+p]_a q$

T$_{20}$ $(K_a(p\rightarrow q)\&_a(q\rightarrow r))\rightarrow K_a(p\rightarrow r)$

p r o o f:

(1)	$(p\to q)\to((q\to r)\to(p\to r))$	PMT9
(2)	$K_a(p\to q)\to K_a((q\to r)\to(p\to r))$	(1) × **DR1**
(3)	$K_a(p\to q)\to(K_ap\to K_aq)$	K_k
(4)	$K_a(p\to q)\to(K_a(q\to r)\to K_a(p\to r))$	(2),(3) × Eq
(5)	$(p\to(q\to r)\to((p\&q)\to r)$	PMT10
(6)	$(K_a(p\to q)\to(K_a(q\to r)\to K_a(p\to r))\to$ $\to((K_a(p\to q)\&K_a(q\to r))\to K_a(p\to r)$	(5) × $[K_a(p\to q)/p,$ $K_a(q\to r)/q,\ K_a(p\to r)/r]$
(7)	$(K_a(p\to q)\&K_a(q\to r))\to K_a(p\to r)$	(4),(6) × MP
		Q.E.D.

[T$_{21}$] $[^+p]_aq\&[^+q]_ar\to[^+p]_ar$

T$_{22}$ $(K_ap\&K_aq)\to K_a(p\leftrightarrow q)$

p r o o f:

(1)	$K_ap\to K_a(q\to p)$	**T$_{16}$**
(2)	$K_aq\to K_a(p\to q)$	(1) × [q/p, p/q]
(3)	$(p\to q)\to((r\to s)\to((p\&r)\to(q\&s)))$	PMT12
(4)	$(K_ap\to K_a(q\to p))\to((K_aq\to K_a(p\to q))\to$ $\to((K_ap\&K_aq)\to K_a(q\to p)\&K_a(p\to q)))$	(3) × $[K_ap/p,\ K_a(q\to p)/q,$ $K_aq/r,\ K_a(p\to q)/s]$
(5)	$((K_aq\to K_a(p\to q))\to((K_ap\&K_aq)\to$ $\to K_a(q\to p)\&K_a(p\to q)))$	(1),(4) × MP
(6)	$(K_ap\&K_aq)\to(K_a(q\to p)\&K_a(p\to q))$	(2),(5) × MP
(7)	$K_a(p\&q)\leftrightarrow(K_ap\&K_aq)$	**T$_6$**
(8)	$(K_ap\&K_aq)\to K_a((q\to p)\&(p\to q))$	(6),(7) × Eq
(9)	$(K_ap\&K_aq)\to K_a(p\leftrightarrow q)$	(8) × def. $\leftrightarrow$
		Q.E.D.

[T$_{23}$] $(K_ap\&K_aq)\to K_a([^+p]_a\,q\&[^+q]_a\,p)$

T$_{24}$ $K_a\neg(p\to\neg q)\leftrightarrow(K_ap\&K_aq)$

p r o o f:

(1)	$K_a(p\&q)\leftrightarrow(K_ap\&K_aq)$	**T$_6$**

(2)	$(p\&q)\leftrightarrow\neg(\neg p\vee\neg q)$	PMT1 × def. &
(3)	$K_a\neg(p\rightarrow\neg q)\leftrightarrow(K_ap\&K_aq)$	(2) × [p/×, q/×] × × Eq × (1) × def →
		Q.E.D.

T_{25} $K_a\neg p\rightarrow(\neg K_a\neg q\vee K_a\neg(p\vee q))$

p r o o f:

(1)	$p\rightarrow(q\rightarrow(p\&q))$	PMT8
(2)	$K_ap\rightarrow K_a(q\rightarrow(p\&q))$	(1) × **DR1**
(3)	$K_a(p\rightarrow q)\rightarrow(K_ap\rightarrow K_aq)$	K_k
(4)	$K_ap\rightarrow(K_aq\rightarrow K_a(p\&q))$	(2),(3) × Eq
(5)	$(p\&q)\leftrightarrow\neg(\neg p\vee\neg q)$	PMT1 × def. &
(6)	$K_ap\rightarrow(K_aq\rightarrow K_a\neg(\neg p\vee\neg q))$	(4) × PMT2 × Eq
(7)	$K_ap\rightarrow(\neg K_aq\vee K_a\neg(\neg p\vee\neg q))$	(6) × def. →
(8)	$K_a\neg p\rightarrow(\neg K_a\neg q\vee K_a\neg(p\vee q))$	(7)[¬p/q, ¬q/q, p/¬p, q/¬q]
		Q.E.D.

T_{26} $\neg K_a((p\&q)\leftrightarrow(K_ap\rightarrow\neg K_aq)$

p r o o f:

(1)	$(\neg p\leftrightarrow q)\leftrightarrow(p\leftrightarrow\neg q)$	PMT14
(2)	$K_a(p\&q)\leftrightarrow(K_ap\&K_aq)$	**T_6**
(3)	$(p\&q)\leftrightarrow\neg(\neg p\vee\neg q)$	PMT1 × def. &
(4)	$K_a(p\&q)\leftrightarrow\neg(\neg K_ap\vee\neg K_aq)$	(3) × Eq
(5)	$K_a(p\&q)\leftrightarrow\neg(\neg K_ap\rightarrow\neg K_aq)$	(4) × def. → [p/α, q/β]
(6)	$p\leftrightarrow\neg\neg p$	PMT2
(7)	$K_a(p\&q)\leftrightarrow\neg(K_ap\rightarrow\neg K_aq)$	(5) × Eq × (6)
(8)	$\neg K_a((p\&q)\leftrightarrow(K_ap\rightarrow\neg K_aq)$	(7) × (1)
		Q.E.D.

T_{27} $\neg K_ap\vee\neg K_aq\vee K_a(p\&q)$

p r o o f:

(1)	$((K_ap\&K_aq)\rightarrow K_a(p\&q))$	**T_5**

(2)	$(\neg(K_a p \& K_a q) V K_a(p\&q))$	(1) × def.$\rightarrow$[$(K_a p\&$ $K_a q)/\alpha$, $K_a(p \,\&\, q)/\beta$]
(3)	$(p\&q)\leftrightarrow\neg(\ \neg p V \neg q)$	PMT1 × def. &
(4)	$\neg\neg(\neg K_a p V \neg K_a q) V K_a(p\&q)$	(2) × (3) × Eq
(5)	$p\leftrightarrow\neg\neg p$	PMT3
(4)	$\neg K_a p V \neg K_a q V K_a(p\&q)$ (4) × (5) × Eq	
		Q.E.D.

T_{28} $K_a\neg(p\&\neg(q\&r))\rightarrow((K_a p\rightarrow K_a q)\&(K_a p\rightarrow K_a r))$

p r o o f:

(1)	$((p\rightarrow(q\&r))\rightarrow((p\rightarrow q)\&(p\rightarrow r))$	PMT19
(2)	$K_a(p\rightarrow(q\&r))\rightarrow K_a((p\rightarrow q)\&(p\rightarrow r)$	(1) × **DR1**
(3)	$K_a(p\&q)\leftrightarrow(K_a p\& K_a q)$	**T_6**
(4)	$K_a(p\rightarrow(q\&r))\rightarrow(K_a(p\rightarrow q)\& K_a(p\rightarrow r))$	(2) × (3) × Eq
(5)	$K_a(p\rightarrow(q\&r))\rightarrow((K_a p\rightarrow K_a q)\&(K_a p\rightarrow K_a r)$	(4) × **DR1** × Eq
(6)	$K_a(\neg p V(q\&r))\rightarrow((K_a p\rightarrow K_a q)$ $(K_a p\rightarrow K_a r)$	(5) × def.$\rightarrow$[p/α, $(q\&r)/\beta$]
(7)	$(p\&q)\leftrightarrow\neg(\neg p V \neg q)$	PMT1 × def. &
(8)	$K_a\neg(\neg\neg p\&\neg(q\&r))\rightarrow((K_a p\rightarrow K_a q)\&$ $\&(K_a p\rightarrow K_a r)$	(6) × (5) × Eq
(9)	$p\leftrightarrow\neg\neg p$	PMT2
(10)	$K_a\neg(p\&\neg(q\&r))\rightarrow((K_a p\rightarrow K_a q)$ $\&(K_a p\rightarrow K_a r))$	(8) × (9) × Eq
		Q.E.D.

T_{29} $(K_a(p\rightarrow q)\& K_a(p\&r))\rightarrow K_a(q\&r)$

p r o o f:

(1)	$(p\rightarrow q)\rightarrow((p\&r)\rightarrow(q\&r))$	PMT18
(2)	$K_a(p\rightarrow q)\rightarrow K_a((p\&r)\rightarrow(q\&r))$	(1) × **DR1**
(3)	$K_a(p\rightarrow q)\rightarrow(K_a(p\&r)\rightarrow K_a(q\&r))$	(2) × **DR1** × Eq
(4)	$(p\rightarrow(q\rightarrow r))\rightarrow((p\&q)\rightarrow r)$	PMT10

(5)	$K_a(p\to q)\to(K_a(p\&r)\to K_a(q\&r))\to$ $\to(K_a(p\to q)\&(K_a(p\&r)\to K_a(q\&r))$	(4)[$K_a(p\to q)$/ p,$K_a(p\&r)$/q, $K_a(q\&r)$/r]
(6)	$(K_a(p\to q)\&K_a(p\&r))\to K_a(q\&r)$	(3),(5) × MP
		Q.E.D.

$[T_{30}][^+p]_a\ q\&K_a(p\&r)\to K_a(q\&r)$

The above mentioned basis of the (update) epistemic logic, in accordance with presuppositional consideration, can be enlarged with nonmonotonic and presupposition-based relation of inference, $|\approx$:[23]

(PP) $\alpha\ |\approx_a \beta$ iff $K_a\neg\alpha < K_a(\alpha\Rightarrow\beta) \vee K_a\alpha < K_a(\neg\alpha\Rightarrow\beta)$

In other words, α presupposes (for $_a$) β if the knowledge ($_a$'s) of $\neg\alpha$ is not more entrenched by the knowledge of implication from α to β, or the knowledge of α is less entrenched than the fact that the knowledge of $\neg\alpha$ brings the knowledge of β.

Through the definition of expansion we have:

(PP') $|\approx_a \beta$ iff $K_a\neg\alpha < [^+\alpha]_a\beta \vee K_a\alpha < [^+\alpha]_a\beta$

where $K_a\neg\alpha$, i.e. $K_a\alpha$ means that $\alpha/\neg\alpha$ in the set of beliefs *K* is accepted as known. So we have a determination of presuppositions in Gärdenfors-style:

(PP") $|\approx_a \beta$ iff $\neg\alpha < [^+\alpha]_a\beta \vee \alpha < [^+\alpha]_a\beta$

This satisfies, we think, Strawson's approach to presupposition. It also confirms Fraassen's consideration of free logic. It is also compatible with Thomason's presuppositions. This approach corrects Gärdenfors's unnatural perception of negation, since it satisfies the system of Hintikka's development of epistemic logic with reference to informational independence, which derives from the set of axioms which cover postulates for the dynamics of non-belief. Iteration of operators is permitted through cognizance with respect to algebra and fixed points, which Gärdenfors takes in the form of his definition of the acceptance of the new input in the revision of the set of beliefs. Natural perception of the common knowledge in the form of explicit group (in the sense of a social community) knowledge is evident here.

[23] Richmond Thomason in his "Logic and Artificial Intelligence" says: "Although there are strong affinities to nonmonotonic logic, nonmonotonic logic relies more heavily on graph-based representations than on traditional logical ideas, and seems to provide a much finer-grained approach to nonmonotonic reasoning that raises entirely new issues, and which quickly becomes problematic" (Thomason, 2003).

In other words, Tuomela's perception of the common knowledge is accepted.[24]

Condition related presuppositions, i.e. logical determination of presuppositions makes Gardefors's definition of nonmonotonic, expectation-based relation of inference, in the opposite direction.[25] In this way we can easily get rid of the unnecessary baroque language style and at the same time we are introducing presuppositions into logical analysis. The price we pay is the decline of logical rigidity. We are breaking the principle of excluded middle, together with Tarski's semantic theory, taking into consideration the question: Whether we are ready to devastate standard logical analysis for the benefit of its application?

Finally we remain within the framework of epistemic logic, advocated by Hintikka, hoping that it will successfully support our above mentioned presuppositional relations. In Hintikka's approach to the epistemic logic, which serves as a framework for the analysis of common knowledge, i.e. presuppositions, we can recognize three directions.

First, besides the presentation of axioms and/or theorems, and their interrelationship, existential presuppositions and the presuppositions of uniqueness are presented. Referring to existential presuppositions, we are dealing here with the supposition of the construction of the set model, where free symbol refers to some actual and existing individual. Presuppositions of the uniquenes refer to the conditions of the unique reference of the free singular term in different worlds, whose members are treated as variables in the sentence. In this way without direct construction of presuppositional logic we have *presuppositional solution* in the framework of standard logical analysis, i.e. we have a situation where 'the ball is thrown' into the field of quantification.

> Mutually dependent quantifiers are especially instructive because among other reasons they show the futility of trying to formulate a half-way natural compositional semantics for IF [independence-friendly *op. cit. S. B.*] logic. This futility is illustrated by the fate of attempts to restore compositionality by allowing other than sentence-initial occurrences of the contradictory negation. An example of such an attempt is found in Janssen 1997. Such attempts falter on the so-called strong lair paradox.[26]

Second, in the framework of game-theory semantics we are concerned with the informational independence of quantification. Starting with Henkin's prefixes and Skolem's functions, this enables the slashing of quantifier and operator.[27] Clearly, in this approach, linear structure of quantifications

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[24] Tuomela, 1991.

[25] Gärdenfors, 1993: 7.

[26] Hintikka, 2000.

[27] Information based independence of the quantifiers is based on the works connected with the logical analysis of the natural language where branched quantifications are introduced.

is substituted with partial structure, as in all presuppositional logics, and the consequences are recognized as the absence of Frege's principle of composition (and in some cases Tarski's semantic theory). But, we have also established interrelations between the quantifier and intentional operator, i.e. there is a possibility of the problem solution referring to the iteration of the operator through different individuals. In other words, there is no need to establish the definition of the fixed point, since quantifications carry the main burden of the common knowledge, i.e. presuppositions.[28]

Third, we are trying to analyse one of the approaches, which is based on Hintikka's achievements in the field of epistemic notions. New dimension is added in the definition of the *common knowledge,* considered to be a predecessor of presuppositions. Regarding the fact that something is truth for all existing individuals, it is not necessarily the truth for the individual referred as, for example, *b*, since it does not exist. To conclude, it means that every free individual symbol refers to actually existing individuals. All that might be specified as singular term, must exist, except the empty one.

"Since our semantical treatment of quantification is almost equivalent to the traditional deductive systems of quantification theory, all these systems are based on the same presuppositions. Empty singular terms are in the same way ruled out in all of them. We shall call the presuppositions we have thus found *existential presuppositions*."[29]

This approach to logical analysis of epistemic notions asks for additional corrections and definitions of common knowledge in the sense of rational belief. According to definition *E*, we have:

$\boxed{K_a p}$	$\boxed{K_b p}$	$K_c p$	…
$K_a K_a p$	$\boxed{K_b K_a p}$	$K_c K_a p$	…
$\boxed{K_a K_b p}$	$K_b K_b p$	$K_c K_b p$	…
$K_a K_c p$	$K_b K_c p$	$K_c K_c p$	…
$K_a K_a K_a p$	$K_b K_a K_a p$	$K_c K_a K_a p$	…
$K_a K_a K_b p$	$K_b K_a K_b p$	$K_c K_a K_b p$	…
$K_a K_a K_c p$	$K_b K_a K_c p$	$K_c K_a K_c p$	…

Besides Hintikka, the theory of branched quantifiers is associated with the name of Jon Barwise, and in the same time Patton associates it with Quine. Cf. Patton 1991.

[28] Cf. Hintikka, 1990; Hintikka [to appear], "Independence-Friendly Logic as a Medium of Information Representation and Reasoning About Knowledge"; Hintikka, [to appear], "Defining Truth, the Whole Truth and Nothing but the Truth"; Hintikka and Sandu, 1996; Maunu, [to appear], "Questions and Answers in Independence-Friendly Logic"; Sandu, 1993; Barwise, 1979.

[29] Cf. Hintikka, 1969: 27.

$K_aK_bK_ap$	$K_bK_bK_ap$	$K_cK_bK_ap$	…
$K_aK_bK_bp$	$K_bK_bK_bp$	$K_cK_bK_bp$	…
$K_aK_bK_cp$	$K_bK_bK_cp$	$K_cK_bK_cp$	…
$K_aK_cK_ap$	$\boxed{K_bK_cK_ap}$	$K_cK_cK_ap$	…
$K_aK_cK_bp$	$K_bK_cK_bp$	$K_cK_cK_bp$	…
$\boxed{K_aK_cK_cp}$	$K_bK_cK_cp$	$K_cK_cK_cp$	…
…	…	…	…

If that is so, we can have:[30]

$$E\varphi =_{df.} \&_{a_k \in A} (K^i_{a_k}\varphi \ \& \ K^j_{a^k}\neg \ K^i_{a^k}\varphi)$$

where $K^j_{a_k}\neg$ - means the absence of disbelief, and j stands for iterative function, so that $j = i + 1$ (k stands for any knowers, $k = 1, 2, 3, \ldots$);[31]

The absence of non-belief can be recorded as $\neg K^j_{ak}\neg$, only if we do not want to introduce some limitations to the increased iteration or if we do not want to introduce operators of the following type: ″he is aware that *p*″ etc.. We can write

$$E\varphi =_{df.} \&_{a_k \in A} (K^i_{a_k}\varphi \ \& \ \neg K^j_{a_k}\neg K^i_{a_k}\varphi)^{32}$$

If that is so, we represent common knowledge for two knowers as:

$$\begin{aligned} E\varphi = {} & K_a\varphi \wedge K_b\varphi \wedge K_a\, K_a\, \varphi \wedge K_a\, K_b\varphi \wedge K_b\, K_a\varphi \wedge K_b\, K_b\, \varphi \wedge \\ & \wedge \neg K_a\neg\, K_a\, \varphi \wedge \neg K_a\neg\, K_b\, \varphi \wedge \neg K_b\neg\, K_a\, \varphi \wedge \neg K_b\neg\, K_b\, \varphi \wedge \\ & \wedge \neg K_a\neg K_a\, K_a\, \varphi \wedge \neg K_a\neg\, K_a\, K_b\, \varphi \wedge \neg K_a\neg\, K_b\, K_a\, \varphi \wedge \neg K_a\neg\, K_b\, K_b\, \varphi \wedge \\ & \wedge \neg K_b\neg\, K_a\, K_a\, \varphi \wedge \neg K_b\neg\, K_a\, K_b\, \varphi \wedge \neg K_b\neg\, K_b\, K_a\, \varphi \wedge \neg K_b\neg\, K_b\, K_b\, \varphi; \end{aligned}$$

We consider this approach to epistemic logic useful since it covers the above mentioned definitions of presuppositions; it also includes presuppositional relations, it rejects the role of expectation in the theory of the belief change. This logic is, we hope, a revised and enlarged version of part of Hintikka's epistemic logic (*viz. IF* logic).

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[30] Cf. Tuomela, 1991.

[31] Hopefully, this way covers the systems of disbeliefs.

[32] Cf. Eφ above.

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Presupozicije, logika i dinamika vjerovanja

SLAVKO BRKIĆ

SAŽETAK: U istraživanju presupozicija u odnosu na logiku i dinamiku vjerovanja razlikujemo dva povezana dijela. Prvi dio se odnosi na presupozicije i logiku koja ne mora biti povezana s intenzionalnim operatorima. Tu se primarno koncentriramo na klasičnu, slobodnu i presupozicijsku logiku. U odnosu na klasičnu logiku razmatramo dobro poznati Strawsonov pristup problemu presupozicija. Nadalje, razmatramo slobodne logike, posebice van Fraassenovo istraživanje uloge presupozicija u supervaluacijskim logičkim sistemima. Na kraju prvog dijela razmatramo izvornu Thomasonovu izgradnju presupozicijske logike. Drugi dio se odnosi na povezanost presupozicija i logike dinamike vjerovanja. Ovdje razmatramo logiku promjene vjerovanja u okviru epistemičkih pojmova imanentnih mehanizmu dinamičke logike. Tri razmatrana pristupa su situacijska semantika (Barwise, Perry), teorija promjene vjerovanja, odnosno, Alchourron/Gärdenfors/Makinsonova (AGM) teorija, te na kraju Hintikkin pristup u izgradnji epistemičke logike.

KLJUČNE RIJEČI: Epistemička logika, presupozicije, zajedničko/skupno/opće znanje, presupozicijski bazirane relacije zaključivanja, promjena vjerovanja, revizija vjerovanja.
