Intuition in Mathematics

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Abstract: The literature on mathematics suggests that intuition plays a role in it as a ground of belief. This article explores the nature of intuition as it occurs in mathematical thinking. Section 1 suggests that intuitions should be understood by analogy with perceptions. Section 2 explains what fleshing out such an analogy requires. Section 3 discusses Kantian ways of fleshing it out. Section 4 discusses Platonist ways of fleshing it out. Section 5 sketches a proposal for resolving the main problem facing Platonists—the problem of explaining how our experiences make contact with mathematical reality.

If you look at the literature on mathematics—the prefaces to math textbooks, discussion pieces by mathematicians, mathematical popularizations and biographies, philosophical works about the nature of mathematics, psychological studies of mathematical cognition, educational material on the teaching of mathematics—you will regularly find talk about intuition. This suggests that there is some role intuition plays in mathematics, specifically as a ground of belief about mathematical matters. The aim of the present chapter is to stake out some ideas about how best to understand intuition as it occurs in mathematics, i.e. about the nature of mathematical intuition.

A closer look at the textbooks, discussion pieces, popularizations and biographies, philosophical works, psychological studies, and educational material reveals, however, that there are a number of distinct notions that correspond to talk about mathematical intuition. The first order of business will be to draw some distinctions between these notions and pick an appropriate focus for our present inquiry. That is the aim of section 1. The notion I will focus on is one according to which mathematical intuition is a kind of experience that is like sensory perception in giving its subjects non-inferential access to a world of facts, but different from sensory perception in that the facts are about abstract mathematical objects rather than concrete material objects. Let us call this the perceptualist view of intuition. It has been the dominant conception of mathematical intuition in the western philosophical tradition since Plato, and the alternatives one finds all more or less derive from it, in ways to be indicated below.

After distinguishing the perceptualist view of intuition from some others to be set aside, the plan is as follows. In section 2, I will sketch some ideas about perception, by reference to which we can flesh out the analogy between mathematical intuition and perception. In sections 3 and 4, I explore the two main approaches to doing this in the philosophical literature—what I will call the Kantian and the Platonist views. Kantians face the problem that mathematical subject matter outstrips our sensory capacities. Platonists face the problem of accounting for how our experiences can be in contact with mathematical reality. In section 5, I sketch some ideas about how a Platonist might resolve this issue.

1. Preliminary Distinctions

Consider the following:

[a] …it is my opinion that, in our naïve intuition, when thinking of a point we do not picture to our mind an abstract mathematical point, but substitute something concrete for it. In imagining a line, we do not picture to ourselves 'length without breadth', but a strip of a certain width. Now such a strip has of course always a tangent, i.e. we can always imagine a straight strip having a small position (element) in common with the curved strip… (Felix Klein in Ewald 1996 pg 959).

[b] But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception…. (Gödel 2001 pg 268).

[c] I would like to argue, however, that recent research in numerical cognition fleshes out a concept of intuition, at least within the small domain of elementary arithmetic. The results indicate that a sense of number is part of *Homo sapiens*' core knowledge, present early on in infancy, and with a reproducible cerebral substrate…Its operation obeys three criteria that may

> be seen as definition of the term “intuition”: it is fast, automatic, and inaccessible to introspection. (Dehaene 2009 pg 233).

One point common to all of the passages is that mathematical intuitions are experiences in which a proposition seems true. Say you intuit that circles are symmetrical about their diameters. Then you have an experience in which it seems that circles are symmetrical about their diameters. Beyond this point of agreement, however, the kinds of experience described in passages [a], [b], and [c] are quite different.

According to Felix Klein in passage [a] when you intuit that p what you do is imagine—specifically visualize—concrete illustrations of the abstract subject matter of p. Let’s call this the view of intuition as concrete illustration. According to Kurt Gödel in passage [b] when you intuit that p you have an experience that is analogous to a sensory perception: in it the abstract subject matter of p itself—not merely a concrete illustration of the abstract subject matter of p—is present to mind. This is the perceptualist view of intuition mentioned in the introduction.

A proponent of the perceptualist view is taking on a stronger commitment than a proponent of the view of intuition as concrete illustration. Why bother? The main motivation, it seems to me, is that it allows us understand the ground of intuitive knowledge as analogous to the ground of perceptual knowledge. Suppose you know by sight that there is mail in the mailbox. A plausible account of the nature of your perceptual knowledge is that it derives in part from your visual awareness of the mail. Why the mail? Because that is, in part, what your knowledge is about.

Now suppose you know by mathematical intuition that circles are symmetrical about their diameters. An analogous account of the nature of your intuitive knowledge is that it derives in part from your intuitive awareness of something like circularity. Why circularity? Because that is, in part, what your knowledge is about. Why *not* some concrete illustration of circularity? Because that is not, even in part, what your knowledge is about.[1] If all you were aware of were a concrete illustration of circularity, then, plausibly, you would have to make some *inference* from what you are able to discern about it to the proposition about circularity itself. In this case your knowledge would not be wholly based on mathematical intuition. Compare the case in which instead of seeing mail in your mailbox you see the mailman driving away. You might still come to know that there is mail in your mailbox. In this case, however, your knowledge does not wholly derive from perception, but in part from inference: you infer that there is mail in your mailbox from your perceptual knowledge that the mailman is driving away. Surely we do make inferences both about our environment and about mathematical reality. But sometimes we also seem to know without having to make an inference. Traditionally, perception and mathematical intuition have been seen as sources of this non-inferential knowledge. So there is some motivation for taking on board the stronger commitment of the perceptualist view of intuition.

Proponents of the perceptualist view of intuition, however, might also privilege visual imagination, for they might think that it is always necessary to use

[1] The proposition that circles are symmetrical about their diameters *implies* propositions about concrete illustrations of circles—e.g. that concrete illustrations of circles will, to the extent that they are drawn accurately, be symmetrical about their diameters.

visual imagination as a means to gaining intuitive awareness of abstract objects. This is related to Kant's view of mathematical intuition, which I discuss below.

Finally, according to Stanislas Dehaene in passage [c] when you intuit that p what happens is that you have the spontaneous impression that p—an impression that is fast, automatic, and introspectively opaque. Let's call this the view of intuition as spontaneous impression. This view of intuition—as it occurs both inside and outside of mathematical contexts—is common among cognitive psychologists; for a helpful overview see Daniel Kahneman's Nobel Prize speech (Kahneman 2002) from which I have borrowed the term "impression." However, even if some of the experiences we rely on in forming mathematical beliefs come to us as spontaneous impressions, most do not. Consider, for example, the proposition that between any circle and any point outside of it there are exactly two tangents. Brief reflection should make this obvious—but note that it likely does take some reflection, unless, say, you are recalling it from memory.

The balance of this chapter focuses on mathematical intuitions conceived of as the perceptualist view suggests. The view of intuition as concrete illustration and the view of intuition as spontaneous impression both pick out real phenomena worth exploring. But here is a working hypothesis that seems plausible to me: the phenomena they pick out answer to partial rather than complete conceptions of mathematical intuition. The view of intuition as concrete illustration focuses on a *partial aspect* of *some* mathematical intuitions, namely the use of visual imagination as a means to awareness of the abstract subject matter of mathematical propositions. If we focused on this, we would be focusing on the means not the end, namely an intuition that involves awareness of mathematical subject matter. The view of intuition as spontaneous impression focuses on a *special subclass* of mathematical intuitions, namely mathematical intuitions that do not depend on those general purpose cognitive abilities we exercise in reflection, and so that are more amenable to the sort of investigation that has proved most fruitful in cognitive psychology. If we focused on this, we would be focusing on a special sort of mathematical intuition, not mathematical intuition in general.

2. Perception and Intuition

According to the perceptualist view of intuition, mathematical intuitions are similar to sensory perceptions in some respects, and different in other respects. The quote from Gödel gives some indications about these points of similarity and difference. The aim of this section is to bring them into better focus, and the natural place to start is with some observations about sensory perception.

Sensory perception is a way of gaining information about your immediate environment. For example, you might see that there is mail in your mailbox. Consider this perception. There are two features of it that I want to highlight.

The first is an aspect of its phenomenology—what the perception feels like from the inside. John Foster suggests a nice way to focus on the feature I have in mind. Imagine a blind person with the power of clairvoyance, limited, say, to what would be in his visual field if he weren't blind. He can't see his immediate environment, but he can immediately tell what is going on in it by appropriately directing his clairvoyant powers. His power of clairvoyance is, like sensory perception, a way of gaining information about his immediate environment. But there is a difference:

> When I seem to be clairvoyantly aware of some perception of the colour-arrangement in my environment, how do my experiences differ in character from the visual experiences which occur when I use my eyes? The answer is that, in the clairvoyant cases as envisaged, there is no provision for the *presentational feel* of phenomenal [i.e. perceptual] experience—for the subjective impression that an instance of the relevant type of environmental situation is directly presented. (Foster 2000 pg 112).

As Foster points out, there is a phenomenological difference between learning about your immediate environment by sight and learning about your immediate environment by clairvoyance. If you learn by clairvoyance that there is mail in the mailbox you just gain the conviction that this is so. It is like suddenly becoming convinced that the mail is there without even opening the mailbox. But if you learn by sight that there is mail in the mailbox you do not just gain the conviction, you also see what makes the conviction true—namely the mail, sitting there in the mailbox. That is the "relevant type of environmental situation."

In general, perceptual experiences have presentational phenomenology:

> Whenever you have a perceptual experience representing that p—e.g. that there is mail in the mailbox—your perceptual experience also makes it seem to you as if you are sensorily aware of items in your environment in virtue of which p is true—e.g. the mail, sitting there in the mailbox.[2]

This property of perception distinguishes it from guessing that p, having a premonition that p, supposing that p, receiving testimony that p, and knowing by clairvoyance that p. These other experiences do not have presentational phenomenology.

So far we have focused on what your perception feels like from the inside. Suppose you hallucinate that there is mail in your mailbox. From the inside this experience feels just like seeing that there is mail in your mailbox. So it also has presentational phenomenology, but its presentational phenomenology is not veridical: you *seem* to see the mail, sitting there in the mailbox, but you do not *really* see it there. Fortunately, this is not the norm. And the perception we started with was not a hallucination. What makes the difference?

At least part of the answer is that when you perceive the mail rather than merely hallucinate the mail your perceptual experience is caused by the mail. Here is how Peter Strawson puts it:

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[2] One might want to complicate the formulation of the idea to allow for the possibility that perceptual experiences lack presentational phenomenology with respect to some of their content. For present purposes, the formulation given will do. For further discussion of presentational phenomenology see (Chudnoff 2011 and 2012.

> The thought of my fleeting perception as a *perception* of a continuously and independently existing thing implicitly contains the thought that if the thing had not been there, I should not even have *seemed* to perceive it. It really should be obvious that with the distinction between independently existing objects and perceptual awareness of objects we already have the general notion of causal dependence of the latter on the former, even if this is not a matter to which we give much reflective attention in our pre-theoretical days. (Strawson 1979 reprinted in Dancy 1988 pgs 103 – 104)

It is worth emphasizing that this is only part of the answer. While the causal condition might be necessary for perception, it is not sufficient. For any given perceptual experience of yours is caused by events in your brain, but most of your perceptual experiences are not perceptions of events in your brain. We will not try to specify sufficient conditions for perception here.

In general, then, a perceptual experience is a genuine perception rather than a mere hallucination only if it meets a causal condition:

> If your perceptual experience representing that p is a genuine perception that p, then it is partly because the items in your environment in virtue of which p is true cause your perceptual experience.

The first feature of perception characterizes its phenomenology. This second feature of perception characterizes its metaphysical structure, specifically how it is hooked up to its subject matter.

Proponents of perceptualist views of intuition can appeal to these two features in specifying more exactly the similarities and differences between mathematical intuition and perception. The idea is that mathematical intuitions are phenomenologically like perceptions in possessing presentational phenomenology, but metaphysically different from perceptions in not hooking up to their subject matter causally. Anyone who wants to defend such a view must explain two things. The first is how your mathematical intuitions make their subject matter seem present to you given that it is not by representing it as standing before you in your immediate environment—e.g. as mail is represented, when it appears sitting there in your mailbox. The second is how your mathematical intuitions hook up to their subject matter given that their subject matter—e.g. circularity—is abstract and so causally inert.

3. Kantian Views

In broad outline, Kant's view of mathematical intuition has been more influential on both the philosophical and the mathematical tradition than that of any other writer. The aim of this section is to sketch his view, relate it to the perceptualist way of thinking about intuition, and briefly discuss its influence on early twentieth century developments in the foundations of mathematics. The first order of business will be to calibrate some terminology.

Suppose you come to know by intuition that circles are symmetrical about their diameters. In this case, the perceptualist would say:

- You have an intuition.
- It makes it seem to you that circles are symmetrical about their diameters.
- And in it you are aware of the items in virtue of which it is true that circles are symmetrical about their diameters.

Kant also makes a threefold distinction corresponding to the seeming, the awareness, and the whole experience that combines them, but he uses different terminology. He writes:

> Our cognition arises from two fundamental sources in the mind, the first of which is the reception of representations (the receptivity of impressions), the second the faculty for cognizing an object by means of these representations (spontaneity of concepts); through the former an object is **given** to us, through the latter it is **thought** in relation to that representation…Intuition and concepts therefore constitute the elements of all our cognition, so that neither concepts without intuition corresponding to

> them in some way nor intuition without concepts can yield a cognition. (Kant 1999 pg 193; A50/B74).

This suggests that if we were to use Kantian terminology, we should say that the seeming is a thought, the awareness is an intuition, and the whole that combines them is a cognition. Kant uses "intuition" for a part; I have been using "intuition" for the whole. In talking about Kant, I will use "mathematical intuition" for the whole/cognition in Kant's sense, "intuitive awareness" for the awareness part/intuition in Kant's sense, and "intuitive seeming" for the seeming part/that which corresponds to thought in a cognition for Kant.

Kant defends the following four theses about intuitive awareness:

(1) Intuitive awareness—in us—depends on our capacity for sensation.

Kant repeats (1) throughout the *Critique*, for example: "Objects are therefore **given** to us by means of sensibility, and it alone affords us **intuitions**…" (Kant 1999 pg 172; A19/B33). Kant believed this holds for us, but not for God. The difference is that God creates the objects of his intuitive awareness, whereas we are affected by the objects of our intuitive awareness. As we'll see, however, creation and affection are not the only options.

(2) Our capacity for sensation imposes forms on the objects of our intuitive awareness; space is the form of intuitable objects outside of us; time is the form of all intuitable objects.

This claim draws together a number of points developed in the Transcendental Aesthetic section of the *Critique*; (Kant 199 pgs 155 - 192; A20 – A49/B34 – B73).

(3 Mathematical subject matter—space and time themselves—must conform to the forms that our capacity for sensation imposes on the objects of our intuitive awareness.

This claim is associated with Kant's "Copernican Revolution." Here is a quote from the introduction where he sketches the main idea: "If intuition has to conform to the constitution of the objects, then I do not see how we can know anything of them *a priori*; but if the object…conforms to the constitution of our faculty of intuition, then I can very well represent this possibility to myself. (Kant 1999 pg. 110; Bxvi – Bxvii); see also (Kant 1999 pg 176; B41). Note that an object's conforming to the forms imposed on objects of our intuitive awareness is a different relation between it and intuitive awareness than either creation or affection. In creation the object causally depends on the mind; in affection the mind causally depends on the object. In the conforming relation Kant invokes the object *non-causally* depends on the mind. So Kant recognized a third possibility. In the next section we will consider a fourth.

(4) We are intuitively aware of mathematical subject matter via illustrations that draw on our capacity for sensation.

Kant develops this point in the section of the *Critique* on the Discipline of Pure Reason in its Dogmatic Use. He takes the case of reasoning about triangularity as an example: "Thus I construct a triangle by exhibiting an object corresponding to this concept, either through mere imagination, in pure intuition, or on paper, in empirical intuition…" (Kant 1999 pg. 630; A713/B741).

Recall that the first thing any perceptualist must explain is how mathematical intuitions make their subject matter seem present in intuitive awareness. From theses (1) and (4), we can see that Kant's view is that mathematical intuitions do this via sensory illustration. The Kantian view of presentational phenomenology as it occurs in mathematical intuitions might be put like this:

> Whenever you have a mathematical intuition representing that p your mathematical intuition also makes it seem to you as if you are intuitively aware of the items in virtue of which p is true and it does so via sensory illustration of them.

This is different from Felix Klein's view because Kant thinks that mathematical intuitions do make us intuitively aware of mathematical subject matter. It is just that they always do this via sensory illustration.

The second thing any perceptualist must explain is how mathematical intuitions hook up to their subject matter. From (2) and (3), we can see that Kant's view is that mathematical intuitions do this because mathematical subject matter must conform to the forms imposed on objects of our intuitive awareness. The Kantian view of the metaphysics of mathematical intuition might be put like this:

> If your mathematical intuition representing that p is a genuine (i.e. knowledge grounding) mathematical intuition that p, then it is partly because the items in virtue of which p is true must conform to the forms imposed on objects of our intuitive awareness.

The idea is that imagining a triangle, say, is a guide to the nature of triangularity, not because triangularity somehow influences our imagination or our imagination somehow influences triangularity, but because there are formal constraints on how we can imagine things and triangularity must also meet these formal constraints. Now one might wonder: how did we get so lucky, so that the formal constraints on how *we* imagine things are also constraints mathematical subject matter must meet? But there is no luck involved. Kant is a kind of idealist. Mathematical subject matter, at least insofar as it is knowable by us, is dependent on our minds—it lies in its nature to conform to the same constraints that govern our capacity for sensation.

Kant's view of mathematical intuition influenced early twentieth century developments in the foundations of mathematics. No major contributor to these developments accepted all that Kant thought about mathematical intuition. What most contributors accepted is the following general idea:

(K) Our capacity for sensory representation limits our capacity for intuitive awareness.

Different writers make the nature of these limits more exact in different ways. For discussion see the works by Brouwer and Hilbert in (Benacerraf and Putnam 1983), the articles on intuitionism and formalism in (Schapiro 2007), and (Parsons 1979, 2008).

(K)'s implications for the foundations of mathematics emerge when we consider the continuum of real numbers. Real analysis as developed in a standard textbook depends on reasoning about arbitrary sets of real numbers. Simple sets of real numbers do not obviously pose a problem for (K). Arguably, our capacity for sensory representation enables us to illustrate the real numbers in the unit interval [0, 1]: just imagine a line segment. But there are two worries. First, this illustration can be very misleading about the properties of the unit interval. It might suggest, for example, that the unit interval cannot be mapped onto the unit square, though really it can be. Second, once you admit the unit interval, more complicated sets of real numbers follow in its wake. Consider, for example, the set of real numbers that remains after the infinite process of first removing the middle third of the unit interval, the middle thirds of the two remaining intervals (i.e. [0, 1/3] and [2/3, 1], the middle thirds of the four remaining intervals (i.e. [0,1/9], [2/9, 1/3], [2/3], 7/9], and [8/9, 1]), etc. This is the Cantor Set. Even if our capacity for sensory representation enables us to illustrate the first few stages of the process that generates the Cantor Set, the Cantor Set itself defies illustration. The unit interval and the Cantor Set are significantly different from the point of view of illustration, but both are perfectly good sets of real numbers from the point of view of standard real analysis.

So intuitive awareness that conforms to (K) seems both *unreliable* about at least some of those mathematical objects it does represent and *limited* in what mathematical objects it can represent.

The three great early twentieth century schools of thought on the foundations of mathematics represent different reactions to the foregoing. Logicists tended to reject mathematical intuition as a source of mathematical knowledge. Intuitionists tended to reject the parts of standard mathematics—e.g. standard real analysis—that seemed to raise problems for mathematical intuition. Formalists tended to divide mathematics into a "real" part to which mathematical intuition has access and about which it is reliable, and an "unreal" part that must be developed in formal systems. See (Benacerraf and Putnam 1983) for primary readings, and (Schapiro 2007) for helpful secondary readings.

There is another possible reaction. That is to give up (K). Consider the Cantor Set again. Don't try to picture it, but just think about it and consider the question: does it contain any points? It should seem clear that it does. But this seeming does not derive from any illustration you might possess of the Cantor Set, since there is none. Rather, it derives from your thinking about the nature of the Cantor Set. It is a seeming based on thought, not sensory representation. One idea, then, is to develop an account of mathematical intuition according to which at least some mathematical intuitions are cognitive and not limited by our capacity for sensory representation. This would be a non-Kantian view of mathematical intuition. It is not really a reaction to Kant or Kantian views; rather, it is a return to a Pre-Kantian view of intuition that can be traced back to Plato.

4. Platonist Views

Though in outline the view of intuition we will consider in this section has ancient and medieval adherents, Descartes put it in its modern form. For our purposes two points are crucial.

First, in contrast to Kant, Descartes argues that the natures of mathematical objects are independent of our minds:

> When, for example, I imagine a triangle, even if perhaps no such figure exists, or has ever existed anywhere outside of my thought, there is still a determinate nature, or essence, or form of the triangle which is immutable and eternal, and not invented by me or dependent on my mind. (Descartes 1985b pg 45; AT 64).

So if we are intuitively aware of triangularity it is not because it affects us, we create it, or it must conform to forms determined by our capacity for sensation. There must be some fourth relation.

Second, in contrast to Kant, Descartes argues that intuitive awareness is independent of our capacity for sensation—even if it sometimes involves sensory experiences.

> [a] But if I want to think of a chiliagon, although I understand that it is a figure consisting of a thousand sides just as well as I understand the triangle to be a three-sided figure, I do not in the same way imagine the thousand sides or see them as if they were present before me. [b] It is true that since I am in the habit of imagining something whenever I think of a corporeal thing, I may construct in my mind a confused representation of some figure…(Descartes 1985b pg 50; AT 72) [c] In fact we have a clear understanding of the whole figure [i.e. the chiliagon], even though we cannot imagine it in its entirety all at once. And it is clear from this that the powers of understanding and imagining do not differ merely in degree but are two quite different kinds of mental operation. (Descartes 1985b pg 264; AT 385).

In [a] Descartes's chiliagon serves the same purpose as the Cantor Set above. In [b] Descartes notes that our thought might be associated with imagery. And in [c] Descartes emphasizes—in reply to Gassendi—that there is more than just imagery;

there is an awareness of the chiliagon based on thought and independent of imagery.

What all this shows is that the Platonist must offer non-Kantian explanations of how mathematical intuitions make their subject matter seem present, and in good cases succeed in hooking up to it.

One difficulty is that the concept of awareness based on thought can seem confused. There is a difference between thinking about something and being aware of it. You can think about all sorts of things that you are not aware of—e.g. the center of the sun. When we have in mind sensory awareness, this contrast is obvious. But what does the contrast consist in when we have in mind awareness based on thought? What could being aware of an object by thinking about it be other than just thinking about it?

To get a handle on this issue, we must distinguish between two kinds of thought. Suppose you are alone in your hotel room in France and you think:

(1) The tallest man in France is over 6ft tall.

Then you step outside and see a man who happens to be—though of course you do not know this—the tallest man in France, and you think:

(2) That man is over 6ft tall.

Even though (1) and (2) attribute the same property to the same guy, call him Jacques, they are quite different in nature. (1) attributes a property to Jacques because it attributes a property to whoever is the tallest man in France and Jacques is the tallest man in France. It picks Jacques out by description and is a descriptive thought. (2) attributes a property to Jacques because it is a thought you have that is grounded in the presence of Jacques himself. It picks Jacques out by demonstration and is a demonstrative thought.

Notice that you couldn't have entertained (2) had you not been visually aware of Jacques. Your awareness of Jacques is what enabled you to entertain a demonstrative thought about him. And this is a special property of awareness: being aware of something—visually or otherwise—enables demonstrative thoughts about that thing.[3] Now we can say what awareness based on thought is. If just by thinking about something enough—descriptively at first—you get yourself into a position to entertain demonstrative thoughts about that thing where before you were not in such a position, then you have succeeded in attaining an awareness of that thing that is based on thought.

So far we have been considering what awareness based on thought could be. But we still have to say something about its phenomenology and how it hooks you up to the object of awareness. About the first issue, let us note that there is such a thing as seeming to be in a state that enables demonstrative thought. Suppose when you step out of your room you do not really see Jacques but only hallucinate a very tall man. Your experience makes it seem to you as if you can pick someone out by

[3] This claim should be qualified in various ways. For discussion see (Snowdon and Robinson 1990), (Siegel 2006), (Johnston 2004), and (Tye 2010).

demonstration, but really you cannot. A similar thing can happen with thought. So if we want to say what it feels like from the inside to seem to be aware of the subject matter of a mathematical intuition, we should say that it feels like being in a state that enables demonstrative thoughts about that object:

> Whenever you have a mathematical intuition representing that p your mathematical intuition also makes it seem to you as if you are intuitively aware of the items in virtue of which p is true, and it does so in virtue of making it seem to you as if you are in a state that enables demonstrative thoughts about those items.

Notice that this characterization of the presentational phenomenology found in mathematical intuition leaves open the possibility that sometimes it substantively relies on imagery, sometimes it is merely accompanied by imagery, and sometimes it occurs without imagery at all and is a matter of pure thinking. This is just as the Platonist should expect.

Descartes does not discuss how intuitive awareness relates to its objects in detail, and what he says is misleading. When he discusses the “eternal truths” in his *Principles of Philosophy*, for example, he describes them as having “no existence outside our thought” and says of an example—that nothing comes from nothing—that it “resides within our mind.” (Descartes 1985a pg 208 – 209; AT 23 – 24). This makes it seem as if intuitive awareness should be assimilated to introspective awareness! Aside from its *prima facie* implausibility, it is in tension with the claim that mathematical objects are mind-independent with which we began this section.

There is, however, another way to interpret the idea. Plotinus, for example, calls (a part of) abstract reality Intellect and his view of what it is for us finite creatures to exercise our intellectual capacities is for us to be in accord with Intellect:

> The activities of Intellect are from above just as the activities arising from sense-perception are from below. We are this—the principal part of the soul, in the middle between two powers…Intellect is disputed, because we do not always use it, and because it is separate. And it is separate owing to its not inclining toward us, whereas we rather are looking upward to it. Sense-perception is our messenger, but Intellect "is our king."
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> But *we* are kings, too, whenever we are in accord with Intellect. We can be in accord with it in two ways: either by having, in a way, its writings written in us like laws or by being, in a way, filled up with it and then being able to see it or perceive it as being present. (From the *Enneads* excerpted in Dillon and Gerson 2004 pgs 89 – 90).

Intellect is not something that we create, nor something that affects us, nor something that must conform to forms determined by us. Rather Intellect is something that we conform to insofar as we succeed in exercising our intellectual capacities, such as the capacity for intuitive awareness of mathematical objects. So the fourth way for intellectual awareness to relate to its object is to be non-causally dependent on it. And this is the line that Platonists have historically taken.

We can frame it like this:

> If your mathematical intuition representing that p is a genuine (i.e. knowledge grounding) mathematical intuition that p, then it is partly because it is non-causally dependent on the items in virtue of which p is true.

One might wonder what the nature of this non-causal dependence relation is. Plotinus presents an inspiring picture, but does not provide us with any real understanding of what it is for our mathematical intuitions to be non-causally dependent on their subject matter. This is one of the main issues that any Platonist about intuition must address. The next section sketches a proposal.

5. The Constitution of Intuition

There are different ways for one thing to non-causally depend on another. Our first aim, then, will be to pick out the right non-causal dependence relation. After that, we will explore how intuition experiences might bear that relation to mathematical objects.

Consider the following claims:

(1) Xantippe became a widow because Socrates died.

(2) My car is parked illegally because it is parked next to a fire hydrant.

(3) This bicycle exists because these items are so arranged to enable locomotion on two wheels by pedaling.

Xantippe's widowhood depends on Socrates' death, but not because Socrates' death causes Xantippe's widowhood. (1) is a non-causal dependence claim. So are (2) and (3).

Let us focus on (3). It is what we might call a form and matter explanation. It explains why an object of a certain kind—a bicycle—exists by citing the fact that some matter—a group of items such as pedals, wheels, seat, etc—possesses a certain form—being so arranged to enable locomotion on two wheels by pedaling. According to (3), the bicycle's existence non-causally depends on its matter possessing the right form.

Suppose we want to give a form and matter explanation for the existence of a mathematical intuition. For this to work, we would have to identify two things: the intuition's matter and the intuition's form. Consider the matter. Clearly it will not consist of physical items, such as pedals, wheels, and seat. Instead it will consist of other experiences, such as thoughts and imaginings. The idea that intuitions consist of other experiences derives from the phenomenologist, Edmund Husserl. He writes:

> In the sense of the *narrower, 'sensuous' perception*, an object is directly apprehended or is itself present, if it is set up in an act of perception *in a straightforward manner*. What this means is this: that the object is also an

> *immediately given object* in the sense that…it is not *constituted* in relational, connective, or otherwise articulated acts, *acts founded on other acts which bring other objects to perception*…[In the case of awareness of "ideal objects" e.g. mathematical objects] *new objects are based on older ones, they are related to what appears in the basic acts*. Their manner of appearance is essentially determined by this relation. We are here dealing with a sphere of objects, *which can only show themselves 'in person' in such founded acts*. (Husserl 2001 pgs 282 - 283, italics in the original)

According to Husserl, sensory awareness is different from what we are calling intuitive awareness in that sensory awareness can be a basic experience and intuitive awareness must be a non-basic experience that is constituted out of other experiences, such as thoughts and imaginings. Seeing a hula-hoop, for example, is not constituted out of other experiences. Becoming intuitively aware of circularity itself, however, is constituted out of other experiences, such as the experience of imagining concrete illustrations.

Now consider the form—i.e. the form that some experiences must exhibit in order to constitute an intuition that makes its subject aware of some mathematical object. We can take bicycles as a model. Their parts must exhibit a form that enables a certain physical activity, specifically locomotion on two wheels by pedaling. In the previous section we considered the connection between awareness of something and the enabling of a certain mental activity, specifically entertaining demonstrative thoughts about that thing. So a natural idea is this: in order for some experiences to constitute an intuition that makes its subject aware of some mathematical object those experiences must exhibit a form that enables their subject to entertain demonstrative thoughts about that mathematical object.

Consider, then, the following possible form and matter explanation for the existence of a mathematical intuition:

(4) This mathematical intuition—e.g. that circles are symmetrical about their diameters—exists because these experiences—e.g. imagining folding circles over their diameters—are so arranged to enable demonstrative thoughts about circularity.

Suppose claim (4) is true of some particular intuition. In this case the intuition non-casually depends on some experiences enabling demonstrative thoughts about circularity. But experiences cannot enable demonstrative thoughts about circularity if circularity does not exist (for recall: we are considering real demonstrative thought, not just seeming demonstrative thought). So the intuition non-causally depends on circularity. And that is the result we were looking for.

This is just a sketch of a proposal. One might wonder: What does “arranged” mean in (4) mean? Can we say more about what some experiences must be like in order to enable demonstrative thoughts about an abstract object? Why believe that we ever really entertain demonstrative thoughts about abstract object, instead of just seeming to do so? These are good questions. A fuller account should address them and others. For further discussion of intuitive awareness along the lines pursued in this section see (Chudnoff forthcoming-a and forthcoming-b).

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