Chapter 5
Approaching the Truth via Belief Change

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Starting from the 1960s of the past century, scientific change has become a main concern of philosophy of science. In particular, a great deal of attention has been devoted to *theory change*.[1] Two of the best known formal accounts of theory change are the post-Popperian theories of verisimilitude (for short: PPV)[2] and the AGM theory of belief change (for short: AGM).[3] In this paper, we will investigate the conceptual relations between PPV and AGM and, in particular, we will ask whether the AGM rules for theory change are effective means for approaching the truth, i.e., for achieving the cognitive aim of science pointed out by PPV.

PPV and AGM are characterized by strongly different assumptions concerning the aims of science. In fact, while all versions of PPV share the view that *verisimilitude* is the main cognitive aim of science, the only aims explicitly suggested by

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* Although we are separately responsible for particular sections (Gustavo Cevolani: Sections 5.1.2, 5.2 and 5.3; Francesco Calandra: Section 5.1.1), we have each benefited from regular discussions and the rereading of each other's contributions, which produced a unified exposition of all the subjects dealt with in the paper.

[1] For a discussion of the problem of rational theory change and its relations with the aims of science, see Cevolani and Festa (2009).
[2] In the present paper, the terms "verisimilitude" and "truth approximation" are used as synonymous. The first full-fledged account of verisimilitude was provided by Karl Popper (1963, 1972). Later, David Miller (1974) and Pavel Tichý (1974) showed that Popper's account was untenable, thus opening the way to the post-Popperian theories of verisimilitude, emerged since 1975. An excellent survey of the modern history of verisimilitude is provided by Niiniluoto (1998).
[3] In the literature, the terms "belief dynamics", "belief change", and "belief revision" are used as synonymous. AGM, which is named after Alchourrón, Gärdenfors, and Makinson (1985), was developed, starting from the 1970s, by researchers in philosophy of science, logic and Artificial Intelligence. The first monograph devoted to AGM was written by Gärdenfors (1988), and the first textbook presentation by Hansson (1999).

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AGM are consistency and informative content. In fact, truth and truth approximation play no role at all in AGM, as the following telling quote by Gärdenfors (1988, p. 20) clearly reveals:

> [. . .] the concepts of truth and falsity are irrelevant for the analysis of belief systems. These concepts deal with the relation between belief systems and the external world, which I claim is not essential for an analysis of epistemic dynamics. [. . .] My negligence of truth may strike traditional epistemologists as heretical. However, one of my aims is to show that many epistemological problems can be attacked without using the notions of truth and falsity.[4]

In spite of this, one may ask whether the AGM rules for belief change are effective means for approaching the truth (Niiniluoto 1999).

In Sections 5.1 and 5.2, the key ideas of PPV and AGM and their application to so called "propositional theories" will be illustrated. In Section 5.3 we will prove that, as far as propositional theories are concerned, AGM belief change is an effective tool for approaching the truth.

5.1 Post-Popperian Verisimilitude for Propositional Theories

5.1.1 Post-Popperian Theories of Verisimilitude

The intuitive idea underlying the concept of verisimilitude is that: a theory is highly verisimilar if it says many things about the domain under investigation and many of those things are true, or almost exactly true. One of the best known accounts of verisimilitude has been provided by Ilkka Niiniluoto (1987). Niiniluoto's approach can be applied to theories stated in many kinds of language, including propositional and first-order languages. In this paper, however, we will only be concerned with theories stated within a propositional language **L** with n atomic propositions $p_1, p_2, \ldots, p_n$. Given an atomic proposition p_m we will say that p_m and $\neg p_m$ are the *basic propositions* – or *b-propositions* – associated to p_m. The b-propositions of **L** form a set $\mathbf{B} = \{p_1, \neg p_1, p_2, \neg p_2, \ldots, p_n, \neg p_n\}$ including $2n$ members. The most informative propositions of **L** are called *constituents*. A constituent C_i is the most complete description of a possible world made by means of the expressive resources of **L**. In fact, for any atomic proposition p_m, C_i tells whether p_m is true or not. Hence, C_i can be written in the following form:

$$\pm p_1 \wedge \pm p_2 \wedge \ldots \wedge \pm p_n, \tag{5.1}$$

[4] Quite recently, however, some AGM theorists have criticized the lack of any concern for truth in AGM. For instance, Hans Rott argues that AGM "should worry more about truth" considered as one of the basic aims of scientific inquiry (see Rott 2000, pp 513, 518 and ff., and in particular note 38).

where "$\pm$" is either empty or the negation symbol "$\neg$". Any b-proposition occurring in Eq. 5.1 will be called a *basic claim* – or *b-claim* – of C_i. Moreover, we will say that each b-claim $\pm p_m$ of C_i *is true in* (the possible world described by) C_i. The constituents of **L** form a set $\mathbf{C} = \{C_1, C_2, \ldots, C_q\}$ including $q = 2^n$ members. Moreover, one can check that: (i) $C_1, C_2, \ldots, C_q$ are mutually exclusive and jointly exhaustive; (ii) there is an unique true constituent, which will be denoted by "C^*"; (iii) any sentence T of **L** can be expressed in its normal disjunctive form as follows:

$$T \equiv \bigvee_{j \in \mathbf{T}} C_j \tag{5.2}$$

where **T** is the index set of the constituents entailing T.

The so called "similarity approach" to verisimilitude is based on the idea that an appropriate measure of verisimilitude $Vs(T)$ should express the similarity between T and "the truth" C^* or, equivalently, the closeness of T to C^*. The basic intuition underlying Niiniluoto's version of the similarity approach is that the verisimilitude $Vs(T)$ of a theory $T \equiv \bigvee_{j \in \mathbf{T}} C_j$ can be defined as a function of the distances between the disjuncts C_j of T and C^*. The versions of the similarity approach based on this intuition may be called *disjunctive versions* (or *d-versions*).

In Niiniluoto's d-version of the similarity approach, $Vs(T)$ is defined as follows. First, a *distance function* Δ is defined on the ordered couples (C_i, C_j) of constituents of **C** by identifying $\Delta_{ij} \equiv \Delta(C_i, C_j)$ with the number of the differences in the $\pm$-signs between C_i and C_j, divided by n; i.e., with the number of the b-claims on which C_i and C_j *disagree*, divided by the total number of atomic propositions. This implies that $0 \leq \Delta_{ij} \leq 1$ and $\Delta_{ij} = 0$ iff $i = j$. Second, an *extended distance function* $\Delta(T, C_i)$ is defined on all the couples (T, C_i), where the distance $\Delta(T, C_i)$ of T from C_i is a function of the distances Δ_{ij} between the disjuncts C_j of T and C_i. Niiniluoto's favourite extended distance function $\Delta(T, C_i)$ is the so called *min-sum distance* function:

$$\Delta_{ms}^{\gamma\gamma'}(T, C_i) \equiv \gamma\Delta_{\min}(T, C_i) + \gamma'\Delta_{sum}(T, C_i), \tag{5.3}$$

with $0 < \gamma, \gamma' \leq 1$.[5] Since $\Delta_{ms}^{\gamma\gamma'}$ is normalized, the *similarity* $s(T, C_i)$ of T to C_i can be simply defined as:

$$s_{ms}^{\gamma\gamma'}(T, C_i) \equiv 1 - \Delta_{ms}^{\gamma\gamma'}(T, C_i). \tag{5.4}$$

[5] Distance $\Delta_{ms}^{\gamma\gamma'}$ is a weighted sum of two simpler (extended) distances, the *minimum distance* $\Delta_{\min}(T, C_i)$ and the *normalized sum distance* $\Delta_{sum}(T, C_i)$. The minimum distance of T from C_i is the distance from C_i of the *closest* constituent entailing T, defined as: $\Delta_{\min}(T, C_i) = \min_{j \in \mathbf{T}} \Delta_{ij}$. The normalized sum distance of T from C_i is the sum of the distances from C_i of all the constituents entailing T *normalized* with respect to the sum of the distances of *all* the elements of **C** from C_i : $\Delta_{sum}(T, C_i) = \sum_{j \in \mathbf{T}} \Delta_{ij} \Big/ \sum_{C_j \in \mathbf{C}} \Delta_{ij}$.

Finally, the *degree of verisimilitude* $Vs_{ms}^{\gamma\gamma'}(T)$ of T can be defined as the similarity between T and "the truth" C^*:

$$Vs_{ms}^{\gamma\gamma'}(T) \equiv s_{ms}^{\gamma\gamma'}(T, C^*) \equiv 1 - \Delta_{ms}^{\gamma\gamma'}(T, C^*). \tag{5.5}$$

One can prove that $Vs_{ms}^{\gamma\gamma'}$ satisfies a number of plausible principles. Among them, the following are especially important[6]:

(Vs.1) Among true statements, verisimilitude covaries with information.
(Vs.2) Among false statements, verisimilitude does not covary with information.
(Vs.3) Some false statements may be more verisimilar than some true statements.

5.1.2 Applying PPV to Propositional Theories

According to the d-version of the similarity approach, the verisimilitude $Vs(T)$ of a sentence T depends only on the distances between the states of affairs allowed by T – represented by the constituents C_i which entail T – and the true state of affairs C^*. On the other hand, according to a recently proposed version of this approach – which may be called the *conjunctive version* (or *c-version*) – $Vs(T)$ depends only on what T says about the "basic features" of the actual world C^*, where such features are expressed by the b-claims $\pm p_m$ which are true in C^*.[7]

The key concept of the c-version of the similarity approach is the notion of *conjunctive proposition* – or *c-proposition*. C-propositions are possibly the simplest kind of "propositional theories", i.e., of theories stated within a propositional language **L**.[8] While a constituent C_i specifies a complete list of the allegedly true b-propositions of **L**, a c-proposition T specifies a (possibly) incomplete list of such b-propositions. A c-proposition can be expressed in the following form:

$$\pm p_{1_T} \wedge \pm p_{2_T} \wedge \ldots \wedge \pm p_{k_T}, \tag{5.6}$$

where $k_T \leq n$. Constituents are nothing but a special kind of c-proposition with $k_T = n$; moreover, a tautology T can be seen as the c-proposition with $k_T = 0$.

Any b-proposition $\pm p_m$ occurring in Eq. 5.6 will be called a b-claim of T. The set T^+ of all the b-claims of a c-proposition T will be referred to as the *basic content* – or *b-content* – of T. Given a constituent C_i, T^+ can be partitioned into two subsets: (1) the subset $t(T, C_i)$ of the b-claims of T which are true in C_i, and

[6] There are good reasons to think that any plausible measure of verisimilitude should respect (Vs.1–Vs.3) (see Niiniluoto 1987, pp 232–233).
[7] The c-version of the similarity approach presented here has been developed by Festa (2007a,b,c), Cevolani and Festa (2009), and Cevolani et al. (2009) with respect to first-order and propositional languages.
[8] C-propositions are essentially identical to "descriptive statements" or "D-statements" (Kuipers 1982, pp 348–349) and to "quasi-constituents" (Oddie 1986, p 86).

(2) the subset $f(T, C_i)$ of the b-claims of T which are false in Ci. We may say that $t(T, C_i)$ is the *true b-content of T w.r.t.* C_i, while $f(T, C_i)$ is the *false b-content of T w.r.t.* C_i. Given a non-tautological c-proposition T, we will say that T is *true* in the case where $t(T, C^*) = T^+$ (and $f(T, C^*) = \varnothing$) and that T is *completely false* in the case where $t(T, C^*) = \varnothing$ (and $f(T, C^*) = T^+$). T is *false* if some of its b-claims are false. The c-proposition $\tilde{T}$, given by the conjunction of the *negations* of all T's b-claims, will be called the *specular of T*. It is easy to see that if T is true then $\tilde{T}$ is completely false and viceversa, whereas if T is false then $\tilde{T}$ too is false.

Starting from the qualitative notions of true and false b-content of T w.r.t. C_i, the corresponding quantitative notions of *degree of true b-content* $cont_t(T, C_i)$ and *degree of false b-content* $cont_f(T, Ci)$ *of T w.r.t.* C_i can be introduced as follows:

$$cont_t(T, C_i) = \frac{t(T, C_i)}{n} \text{ and } cont_f(T, C_i) = \frac{f(T, C_i)}{n} \tag{5.7}$$

The similarity $s_\tau(T, C_i)$ of T to C_i can then be defined as a weighted average of $cont_t(T, C_i)$ and $-cont_f(T, C_i)$, where $cont_t(T, Ci)$ is construed as the *prize* attributed to the true b-content of T w.r.t. C_i and $-cont_f(T, C_i)$ as the *penalty* attributed to the false b-content of T w.r.t. C_i:

$$\begin{aligned} s_\tau(T, C_i) &= \tau cont_t(T, C_i) + (1 - \tau)(-cont_f(T, C_i)) \\ &= \tau cont_t(T, C_i) - (1 - \tau) cont_f(T, C_i) \end{aligned} \tag{5.8}$$

where $0 < \tau \leq 1/2$. The verisimilitude of T, $Vs_\tau(T)$, is then identified with the similarity between T and the true constituent C^*:

$$Vs_\tau(T) = s_\tau(T, C^*). \tag{5.9}$$

In order to state some interesting features of Vs_τ, it is useful to introduce the notions of *verisimilar sentences* and of *sentences which are distant from the truth*—or *t-distant* sentences.[9] To this purpose, we shall say that a c-proposition T is *verisimilar* in the case where $Vs_\tau(T) > 0$ and that T is *t-distant* in the case where $Vs_\tau(T) < 0$. Some relevant consequences of Eqs. 5.8 and 5.9 can now be proved:

Theorem 1. *Given a c-proposition T of the form* $\pm p_{1_T} \wedge \pm p_{2_T} \wedge \ldots \wedge \pm p_{k_T}$*:*

1. $(\tau - 1) \leq Vs_\tau(T) \leq \tau$.
2. *If T is tautological, then* $Vs_\tau(T) = 0$.
3. *If T is true, then T is verisimilar.*

[9] A similar definition can be given with respect to any verisimilitude measure Vs, by selecting a suitable threshold value σ and calling "verisimilar" and "t-distant" those sentences whose verisimilitude is greater or lower than σ, respectively.

4. *If T is completely false, then T is t-distant.*
5. $Vs_\tau(T) = \sum_{p_m \in T^+} Vs_\tau(p_m)$.
6. *Vs_τ satisfies principles (Vs.1–Vs.3).*
7. $Vs_\tau(T) </=/> Vs_\tau(\tilde{T})$ *iff* $cont_t(T, C^*) </=/> cont_f(T, C^*)$.

5.2 AGM Belief Change for Propositional Theories

5.2.1 The AGM Theory of Belief Change

Within the AGM theory of belief change, the epistemic state of an ideal agent X is represented by a *belief set* or *theory*, i.e., by a deductively closed set of sentences. More precisely, given a language **L**, an operation of logical consequence *Cn* defined on **L**, and a set K of sentences within **L**, the notion of belief set is defined as follows:

$$K \text{ is a belief set if and only if } Cn(K) = K. \tag{5.10}$$

Although the notion of belief set in Eq. 5.10 includes also inconsistent belief sets, AGM theorists adopt the following *principle of consistency*:

(C) The belief set K of an ideal agent X should be consistent.

Suppose that the epistemic state of X is represented by a consistent belief set K. Then X can have one of the following epistemic attitudes towards a sentence A of **L**:

i X *accepts* A in the case where $A \in K$;
ii X *rejects* A in the case where $\neg A \in K$;
iii X *suspends the judgment on* A – or, equivalently, A is *undetermined for* X – in the case where both $A \notin K$ and $\neg A \notin K$.

The basic purpose of AGM is to provide a plausible account of how an ideal agent X should update his belief set K in response to certain epistemic inputs coming from some information source. Given a sentence A, two kinds of epistemic input concerning A are considered within AGM:

(a) *Additive inputs*, which can be expressed as orders of the form “Add A to your belief set!”.
(b) *Eliminative inputs*, which can be expressed as orders of the form “Remove A from your belief set!”.

Below, the additive input “Add A to your belief set!” and the eliminative input “Remove A from your belief set!” will also be denoted by the shorter expressions “additive input A” and “eliminative input A”, respectively.

Suppose that X receives the additive input "Add A to your belief set!". Of course, if A already belongs to K – i.e., if X already accepts A – then X's appropriate response is keeping K unchanged. However, there are two more interesting cases where $A \notin K$:

Expansion. A is *compatible* with K, i.e., $\neg A \notin K$. In this case, the epistemic operation by which X should update K by the addition of A is called *expansion*, and the expanded belief set is denoted by "K^+_A".

Revision. A is *incompatible* with K, i.e., $\neg A \in K$. In this case, the epistemic operation by which X should update K by the addition of A is called *revision*, and the revised belief set is denoted by "K^*_A".

Below, we will call "addition" the generic operation of updating K by an additive input A. Hence, the addition of A to K will be either the expansion of K by A, in the case where A is compatible with K, or the revision of K by A, in the case where A is incompatible with K.

Now suppose that X receives the eliminative input "Remove A from your belief set!". If A does not belong to K – i.e., X rejects, or suspends the judgment on, A–X's appropriate response consists is keeping K unchanged. However, the more interesting case where $A \in K$ may occur:

Contraction. If $A \in K$, the epistemic operation by which X should update K by the removal of A is called *contraction*, and the contracted belief set is denoted by "K^-_A".

AGM theorists have made systematic efforts aiming to show how, given a belief set K and a sentence A, an ideal agent X could specify the updated belief sets K^+_A, K^*_A and K^-_A. A basic intuition underlying the AGM approach is expressed by the following general principle of rationality, known as the *principle of minimal change*:

(MC) When the belief set K of an ideal agent X is updated in response to a given epistemic input, a *minimal change* of K should be accomplished. This means that X should continue to believe as many of the old beliefs as possible and start to believe as few new beliefs as possible.

There are many alternative ways of defining K^+_A, K^*_A and K^-_A in accordance with the general principles of consistency and minimal change. For this reason, Gärdenfors (1988) has proposed a number of adequacy conditions – the so called *Gärdenfors postulates* – that any appropriate definition of K^+_A, K^*_A and K^-_A should satisfy. For instance, the "Success" postulate for revision says that $A \in K^*_A$. However, it should be noted that the Gärdenfors postulates alone cannot fully determine the result of any belief change. Suppose, for example, that an agent X receives the additive input A. If X's theory K includes $\neg A$, then X has to revise K by A. This means that $\neg A$ must be removed from K, in order to guarantee both that $A \in K^*_A$ – as required by the Success postulate – and that K^*_A is consistent – in agreement with (C). Moreover, X has to remove from K not only $\neg A$ but – due to the definition (Eq. 5.10) of belief set – also any set of sentences entailing $\neg A$. Since there are normally many alternative ways to fulfill this task, the choice of one of them will depend on the relative "importance" that X attaches to the sentences in K. In this connection, one may assume that the elements of K are ordered with respect to their so called *epistemic entrenchment* (Gärdenfors and Makinson 1988). When X has to remove some sentences from K, he will choose the less entrenched in agreement with appropriate selection rules.

A well known method for defining the operations of expansion, revision and contraction in accordance with the Gärdenfors postulates and with entrenchment-based selection rules has been provided by Grove (1988). For the sake of brevity, below we will outline Grove's method only with reference to expansion and revision.[10] Grove shows that, given a propositional language **L**, any belief set K in **L** is identical to the set of all the logical consequences of some sentence T of **L** – i.e., is identical to the so called consequence class $Cn(T)$. Hence, a generic belief set or "theory" may be identified with the corresponding sentence T of **L**, expressed in its normal disjunctive form as $T \equiv \bigvee_{j \in \mathbf{T}} C_j$. An epistemic entrenchment relation can be defined on the sentences of **L** by ordering the constituents of **C** with respect to their relative closeness or similarity to the elements of T. Niiniluoto (1999) shows that such an ordering is easily obtained in the case where a suitable distance function Δ is defined on the constituents of **L** (see Section 5.1.1). In fact, the distance $\Delta_i(T)$ of a constituent C_i from a theory T may be defined as $\Delta_i(T) = \min_{j \in \mathbf{T}} \Delta_{ij} = \Delta_{\min}(C_i, T)$. Moreover, given an epistemic input A, the set $C_T(A)$ of the *closest* constituents to T entailing A is defined as: $C_T(A) = \{i \in \mathbf{A} : \Delta_i(T) \leq \Delta_j(T)\}$ for all $j \in \mathbf{A}$. By using these notions, Niiniluoto proves the following identities concerning expansion and revision[11]:

Theorem 2. *If the additive input A is compatible with T, in the sense that $\neg A \notin Cn(T)$, then T_A^+ is simply given by the conjunction of T and A:*

$$T_A^+ = T \wedge A = \bigvee_{i \in \boldsymbol{T} \cap \boldsymbol{A}} C_i .$$

Theorem 3. *If the additive input A is incompatible with T, in the sense that $\neg A \in Cn(T)$, then T_A^* is given by*

$$T_A^* = \bigvee_{i \in C_T(A)} C_i .$$

5.2.2 Applying AGM to Propositional Theories

Now we will show how the basic principles of AGM can be applied to the definition of T_A^+ and T_A^* in the case where both T and A are c-propositions. To this purpose, we have to introduce some preliminary notions concerning T and its logical relations with A.

[10] See Cevolani et al. (forthcoming) for a discussion of contraction.

[11] See Niiniluoto (1999), pp 7–9.

First of all, recall that T^+ is the set of all b-claims of T, i.e., the set of all b-propositions occurring in T. The set of the *negations* of the elements of T^+ will be denoted by "T^-", whereas the set of the b-propositions which occur neither in T^+ nor in T^- will be denoted by "$T^?$".[12] Note that the sets T^+, T^- and $T^?$ form a partition of the set $\mathbf{B}$ of the $2n$ b-propositions of $\mathbf{L}$. Suppose that the agent X receives the additive input A. In order to understand how X should update his belief set T in response to A, one should note that the logical relations between T and A depend on how A^+ overlaps the partition $\{T^+, T^-, T^?\}$. For this reason, it is useful to introduce the notions of the "redundant", "conflicting" and "extra" part of A with respect to T, as follows. Given two c-propositions T and A, the following related c-propositions are defined:

1. A_{rT}, the conjunction of the elements of $A^+ \cap T^+$, will be called the *redundant part of A w.r.t. T*.
2. A_{cT}, the conjunction of the elements of $A^+ \cap T^-$, will be called the *conflicting part of A w.r.t. T*.
3. A_{xT}, the conjunction of the elements of $A^+ \cap T^?$, will be called the *extra part of A w.r.t. T*.

Below, the conflicting and the extra parts of A w.r.t. T will be also referred to as the "non-redundant parts" of A w.r.t. T. Note that the three sets $A^+ \cap T^+$, $A^+ \cap T^-$ and $A^+ \cap T^?$ form a partition of A^+. Hence, A can be written as $A_{rT} \wedge A_{cT} \wedge A_{xT}$ and, in the same way, T can be written as $T_{rA} \wedge T_{cA} \wedge T_{xA}$. The following properties of the c-propositions A_{rT}, A_{cT} and A_{xT} defined above are worth noting. First, A_{rT} is identical to T_{rA}, by definition. Moreover, it is easy to see that $A_{cT} = \tilde{T}_{cA}$ and $T_{cA} = \tilde{A}_{cT}$ – i.e., that the conflicting part of A w.r.t. T is the specular of the conflicting part of T w.r.t. A, and vice versa. Finally, A_{xT} and T_{xA} share by definition no common conjuncts.

The above notions can be used to prove the following theorems concerning expansion and revision[13]:

Theorem 4. *If the additive input A is compatible with T, in the sense that $A^+ \cap T^- = \varnothing$, then $T^+_A = T \wedge A$.*

Theorem 5. *If the additive input A is incompatible with T, in the sense that $A^+ \cap T^- \neq \varnothing$, then $T^*_A = A \wedge T_{xA}$.*

A consequence of Theorem 4 is worth noting here. First, recalling that $A_{rT} = T_{rA}$, one can see that the information A_{rT} is already conveyed by T. Second, since A is *compatible* with T by hypothesis, the conflicting part of A w.r.t. T is empty – i.e., $A^+ \cap T^- = \varnothing$ and $A_{cT} = T_{cA} = \mathsf{T}$. From these two facts, it follows that the conjunction of T with A is identical to the conjunction of T with the extra part of A w.r.t. T. Hence, Theorem 4 implies that $T^+A = T \wedge A_{xT}$.

[12] If T is the theory of an agent X, then T^+, T^-, and $T^?$ can be seen as the set of the b-propositions which X *accepts*, *rejects*, and on which *suspends the judgment*, respectively.

[13] These theorems are proved in Cevolani et al. (forthcoming) together with a number of results about contraction.

5.3 Is AGM Belief Change a Road to Verisimilitude?

We can now come back to the question considered at the beginning of the paper, i.e., the question whether the AGM rules for belief change are effective means for approaching the truth. This question may be now rephrased as follows: are AGM expansion and revision effective means for approaching the truth?[14]

Niiniluoto (1999) investigates this problem with respect to his favored verisimilitude measure $Vs_{ms}^{\gamma\gamma'}$, introduced in Section 5.1.1. In particular, Niiniluoto asks in which cases expansion and revision lead our theories closer to the truth or, in other words, in which cases, given a theory T and an additive input A, T_A^+ and T_A^* are more verisimilar than T. In this connection, Niiniluoto can immediately prove the following result[15]:

Theorem 6. *Suppose that both T and A are true. Then $Vs_{ms}^{\gamma\gamma'}\left(T_A^+\right) \geq Vs_{ms}^{\gamma\gamma'}(T)$.*

It is not difficult to show that this result doesn't hold only for $Vs_{ms}^{\gamma\gamma'}$ but also for most of the existing verisimilitude measures. Indeed Theorem 3 holds for any verisimilitude measure satisfying the principle (Vs.1) according to which, among true statements, verisimilitude covaries with information.[16]

Unfortunately, Niiniluoto shows that Theorem 3 cannot be extended to more general cases. In particular, Niiniluoto proves that, even in the case where A is true, T_A^+ and T_A^* may be less verisimilar than T[17]:

Theorem 7. *Suppose that A is true. Then:*

1. *If T is false, T_A^+ may be less verisimilar than T.*
2. *T_A^* may be less verisimilar than T.*

Niiniluoto's results above concern the expansion and the revision of theories expressed in propositional and first-order languages. Theorem 3 shows that the simple addition of true epistemic inputs to such theories doesn't necessarily lead them closer to the truth. In this regard, one can say that expansion and revision are not effective means for approaching the truth, at least as far $Vs_{ms}^{\gamma\gamma'}$ is concerned.

However, a different conclusion can be reached if we restrict our attention to a special kind of propositional theories, i.e., c-propositions. In this case, we can specify various cases where expansion and revision are effective means for approaching the truth. Accordingly, from now on we will assume that both the theory T and the epistemic input A are c-propositions. The following theorems state the conditions under which expansion and revision increase the verisimilitude of a theory T with respect to the verisimilitude measure Vs_τ introduced in Section 5.1.2.

[14] The problem of the effectiveness of contraction for approaching the truth is considered in Cevolani et al. (forthcoming).
[15] See Niiniluoto (1999), Eq. 5.10.
[16] One of the few verisimilitude measures violating (Vs.1) has been proposed by Graham Oddie (1986).
[17] See Niiniluoto (1999), pp. 10–13, in particular equations 10, 17 and 20.

Theorem 8. *Given a theory T, suppose that A is compatible with T and $A^+ \not\subset T^+$.*[18] *Then:*

$$Vs_\tau\left(T_A^+\right) > Vs_\tau(T) \text{ iff } A_{xT} \text{ is verisimilar.}$$

Theorem 9. *Given a theory T, suppose that A is incompatible with T. Then:*

$$Vs_\tau\left(T_A^*\right) > Vs_\tau(T) \text{ iff } Vs_\tau(A_{xT}) > Vs_\tau(\tilde{A}_{cT}) - Vs_\tau(A_{cT}).$$

In order to grasp the intuitive meaning of Theorem 9, recall that, by hypothesis, A is incompatible with T, i.e., that the conflicting part of T w.r.t. A is not empty. According to Theorem 5, the revision of T by A replaces such conflicting part $T_{cA} = \tilde{A}_{cT}$ with A_{cT} and adds A_{xT} to T. Now suppose that $Vs_\tau(A_{cT}) < Vs_\tau(\tilde{A}_{cT})$. Then the difference $Vs_\tau(\tilde{A}_{cT}) - Vs_\tau(A_{cT})$ can be construed as the *loss* of verisimilitude due to the addition of the conflicting part of A to T. However, if the extra part of A *outweighs* this loss – i.e., if $Vs_\tau(A_{xT}) > Vs_\tau(\tilde{A}_{cT}) - Vs_\tau(A_{cT})$ – then the revised theory T_A^* will still be more verisimilar than T.

Recalling that, according to Theorem 1, if A is true then A is verisimilar, whereas if A is completely false then A is t-distant, one can now prove some interesting consequences of Theorems 8 and 9. First, the addition of true inputs to (false) theories always increases their verisimilitude:

Theorem 10. *Suppose that A is true. Then:*

1. $Vs_\tau\left(T_A^+\right) > Vs_\tau(T)$.
2. $Vs_\tau\left(T_A^*\right) > Vs_\tau(T)$.

Second, if the non-redundant parts of A w.r.t. T are verisimilar, then the addition of A to T leads T closer to the truth:

Theorem 11. *Suppose that A_{cT} and A_{xT} are verisimilar. Then:*

1. $Vs_\tau\left(T_A^+\right) > Vs_\tau(T)$.
2. $Vs_\tau\left(T_A^*\right) > Vs_\tau(T)$.

To sum up, expansion and revision are effective means for approaching the truth, as far as c-propositions and the verisimilitude measure Vs_τ are concerned, in the following sense. First, the addition of true inputs to (false) theories leads to more verisimilar theories. Second, the addition of inputs whose non-redundant parts are verisimilar also increases the verisimilitude of the original theory.

Finally, one may consider another aspect of AGM's effectiveness for approaching the truth which is not discussed by Niiniluoto (1999). In fact, Theorems 10 and 11 concern the expansion and the revision of T by true inputs or by inputs whose non-redundant parts are verisimilar. However, one might ask what happens in the case where T is expanded or revised by inputs which are completely false or whose non-redundant parts are t-distant. In such cases, it seems plausible to expect that the

[18] The proviso is needed in order to exclude the trivial case where A is already contained in T, i.e., the case where $A_{xT} = \top$ and $T_A^+ = T$.

expansion and the revision of T by A leads to theories which are less verisimilar than T. An answer to this question is provided by the following theorems. First, one can prove that the addition of completely false inputs to T leads to a less verisimilar theory, as the following result (which is the counterpart of Theorem 3) states:

Theorem 12. *Suppose that A is completely false. Then:*

1. $Vs_\tau\left(T_A^+\right) < Vs_\tau(T)$.
2. $Vs_\tau\left(T_A^*\right) < Vs_\tau(T)$.

Moreover, if the non-redundant parts of A are t-distant, the expansion of T by A is less verisimilar than T:

Theorem 13. *Suppose that A_{cT} and A_{xT} are t-distant. Then, $Vs_\tau\left(T_A^+\right) < Vs_\tau(T)$.*

Interestingly, however, this doesn't hold for revision; in fact:

Theorem 14. *T_A^* may be more verisimilar than T, even if both A_{cT} and A_{xT} are t-distant.*

The results illustrated in this paper suggest two further questions. The first is whether similar results may be obtained for the contraction of c-propositions by different kind of eliminative inputs. This problem is analyzed in Cevolani et al. (forthcoming). The second question is whether Theorems 10–14 can be extended to verisimilitude measures different from Vs_τ. In this connection, we advance the admittedly bold guess that the results proved in Theorems 10–14 hold for any plausible verisimilitude measure defined on propositional languages.

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Proofs.

Proof of Theorem 1. 1. The most verisimilar c-proposition T of $\mathbf{L}$ is the true constituent itself, i.e., C^*. If $T = C^*$, then $cont_t(T, C^*) = 1$, whereas $cont_f(T, C^*) = 0$. Then $Vs_\tau(T) = \tau cont_t(T, C^*) = \tau$ is the verisimilitude of the most verisimilar c-proposition of $\mathbf{L}$. On the other hand, the less verisimilar c-proposition T is the completely false constituent, i.e., the specular $\tilde{C}^*$ of C^*. If $T = \tilde{C}^*$, then $cont_t(T, C^*) = 0$, whereas and $cont_f(T, C^*) = 1$. Then $Vs_\tau(T) = -(1 - \tau) cont_f(T, C^*) = \tau - 1$ is the verisimilitude of the less verisimilar c-proposition of $\mathbf{L}$.

2. Recall that T is tautological iff $k_T = 0$, i.e., $T^+ = \varnothing$. Then, $cont_t(T, C^*) = cont_f(T, C^*) = 0$ and $Vs_\tau(T) = 0$.
3. If T is true (and non-tautological), then $t(T, C^*) = T^+ \neq \varnothing$ and $cont_t(T, C^*) > 0$, whereas $f(T, C^*) = \varnothing$ and $cont_f(T, C^*) = 0$. Consequently, since $\tau > 0$, $Vs_\tau(T) = \tau cont_t(T, C^*) > 0$.

4. If T is completely false, then $t(T, C^*) = \varnothing$ and $cont_t(T, C^*) = 0$, whereas $f(T, C^*) = T^+ \neq \varnothing$ and $cont_f(T, C^*) > 0$. Since $(1 - \tau) > 0$, $Vs_\tau(T) = -(1 - \tau)cont_f(T, C^*) < 0$.

   Note that a b-proposition p_m is also a c-proposition, whose unique b-claim is p_m. If p_m is true, then $t(p_m, C^*) = \{p_m\}$ and $cont_t(p_m, C^*) = 1/n$, whereas $f(p_m, C^*) = \varnothing$ and $cont_f(p_m, C^*) = 0$; moreover $Vs_\tau(p_m) = \tau/n$. Conversely, if p_m is false, then $f(p_m, C^*) = \{p_m\}$ and $cont_f(p_m, C^*) = 1/n$, whereas $t(p_m, C^*) = \varnothing$ and $cont_t(p_m, C^*) = 0$; moreover $Vs_\tau(p_m) = -(1 - \tau)/n$. It is now easy to see that, for any T, $cont_t(T, C^*) = \sum_{p_i \in t(T,C^*)} cont_t(p_i, C^*)$ and $cont_f(T, C^*) = \sum_{p_j \in f(T,C^*)} cont_f(p_j, C^*)$. Hence, $Vs_\tau(T) = \tau \times \sum_{p_i \in t(T,C^*)} cont_t(p_i, C^*) - (1 - \tau) \times \sum_{p_j \in f(T,C^*)} cont_f(p_j, C^*)$, i.e., since $t(T, C^*) \cup f(T, C^*) = T^+$, $Vs_\tau(T) = \sum_{p_i \in t(T,C^*)} Vs_\tau(p_i) + \sum_{p_j \in f(T,C^*)} Vs_\tau(p_j) = \sum_{p_m \in T^+} Vs_\tau(p_m)$.
5. Consider two c-propositions A and B such that A is logically stronger than B, i.e., such that $A \vdash B$ but $B \nvdash A$. This means that $B^+ \subset A^+$, i.e., that A contains all B's claims and at least one additional b-proposition $\pm p_m$. First, suppose that A and B are both true; it follows that $\pm p_m$ is true and $Vs_\tau(\pm p_m) = \tau/n$ by the lemma above. By the same lemma, $Vs_\tau(A) = Vs_\tau(B) + Vs_\tau(\pm p_m)$; moreover, since p_m is true, $Vs_\tau(\pm p_m) > 0$. Thus, $Vs_\tau(A) > Vs_\tau(B)$. Consequently, (Vs1) is satisfied: if A is logically stronger than B and both are true, A is more verisimilar than B. Suppose now that A and B are both false. If $\pm p_m$ is true, then A will be more verisimilar than B; however, if $\pm p_m$ is false, then A will be less verisimilar (but logically stronger) than B. Thus, (Vs2) is satisfied, since verisimilitude doesn't covary, among false c-propositions, with logical strength. Finally, to see that (Vs3) is satisfied, consider the measure Vs_τ with $\tau = 1/2$, defined on the language **L** with three atomic propositions p, q and r. Suppose that p, q and r are true and consider the two c-propositions $A \equiv p$ and $B \equiv p \wedge q \wedge \neg r$. Although A is true and B is false, $Vs_\tau(A) = \tau/n = 1/6$, whereas $Vs_\tau(B) = \tau 2/n - (1 - \tau)1/n = 1/3$. Thus, the false c-proposition B is more verisimilar than the true c-proposition A.
6. By definition, $cont_t(A, C^*) = cont_f(\tilde{A}, C^*)$ and $cont_f(A, C^*) = cont_t(\tilde{A}, C^*)$. Consequently, $Vs_\tau(A) </=/> Vs_\tau(\tilde{A})$ iff, by definition (9), $\tau cont_t(A, C^*) - (1-\tau)cont_f(A, C^*) </=/> \tau cont_f(A, C^*) - (1-\tau)cont_t(A, C^*)$, i.e., since $\tau > 0$, iff $cont_f(A, C^*) >/=/< cont_t(A, C^*)$.

Proof of Theorem 8. Let us prove the following result: $Vs_\tau\left(T_A^+\right) >/=/< Vs_\tau(T)$ iff $Vs_\tau(A_{xT}) >/=/< 0$. The expansion of T by A is $T_A^+ = T \wedge A$ by Theorem 4. As observed at the end of Section 5.2.2, since $A_{xT} = T_{xA}$ and $T_{cA} = A_{cT} = \top$ by hypothesis (since A is compatible with T), T_A^+ can be written as $T \wedge A_{xT}$. Consequently, $Vs_\tau\left(T_A^+\right) >/=/< Vs_\tau(T)$ iff $Vs_\tau(T \wedge A) >/=/< Vs_\tau(T)$.

By Theorem 1, $Vs_\tau(T \wedge A) = Vs_\tau(T) + Vs_\tau(A_{xT})$. Hence, $Vs_\tau\left(T_A^+\right) > / = / < Vs_\tau(T)$ iff $Vs_\tau(A_{xT}) > / = / < 0$.

Proof of Theorem 9. Let us prove the following result: $Vs_\tau\left(T_A^*\right) > / = / < Vs_\tau(T)$ iff $Vs_\tau(A_{xT}) > / = / < Vs_\tau(\tilde{A}_{cT}) - Vs_\tau(A_{cT})$. The revision of T by A is $T_A^* = A \wedge T_{xA}$ by Theorem 5. Recalling from Section 5.2.2 that $A = A_{rT} \wedge A_{cT} \wedge A_{xT}$, we have that $T_A^* = A_{rT} \wedge A_{cT} \wedge A_{xT} \wedge T_{xA}$. Since A is incompatible with T by hypothesis, i.e., A_{cT} is not empty, T may be expressed as $A_{rT} \wedge \tilde{A}_{cT} \wedge T_{xA}$. Thus, $Vs_\tau\left(T_A^*\right) > / = / < Vs_\tau(T)$ iff $Vs_\tau(A_{rT} \wedge A_{cT} \wedge A_{xT} \wedge T_{xA}) > / = / < Vs_\tau(A_{rT} \wedge \tilde{A}_{cT} \wedge T_{xA})$ iff (by Theorem 1) $Vs_\tau(A_{cT}) + Vs_\tau(A_{xT}) > / = / < Vs_\tau(\tilde{A}_{cT})$, i.e., iff $Vs_\tau(A_{xT}) > / = / < Vs_\tau(\tilde{A}_{cT}) - Vs_\tau(A_{cT})$.

Proof of Theorem 10. 1. If A is true, A_{xT} is also true (a fortiori) and $Vs_\tau(A_{xT})$ is verisimilar by Theorem 1. Thus, $Vs_\tau\left(T_A^+\right) > Vs_\tau(T)$ by Theorem 8.

2. If A is true, both A_{cT} and A_{xT} are true (a fortiori). Thus, $Vs_\tau(A_{xT}) > 0$ and $Vs_\tau(A_{cT}) > 0$ by Theorem 1. We want to prove, according to Theorem 8, that $Vs_\tau(A_{xT}) > Vs_\tau(\tilde{A}_{cT}) - Vs_\tau(A_{cT})$; to this purpose, it is then sufficient to prove that $Vs_\tau(\tilde{A}_{cT}) - Vs_\tau(A_{cT}) \leq 0$, i.e., $Vs_\tau(\tilde{A}_{cT}) \leq Vs_\tau(A_{cT})$. By Theorem 1.7, this holds iff $cont_t\,(A_{cT}, C^*) \geq cont_f\,(A_{cT}, C^*)$. To see that this is in fact the case, note that, since A_{cT} is true, $cont_t(A, C^*) > 0$ and $cont_f(A, C^*) = 0$. Consequently, by Theorem 8, $Vs_\tau\left(T_A^*\right) > Vs_\tau(T)$.

Proof of Theorem 11. 1. Note that since A is compatible with T by hypothesis, the only non-redundant part of A w.r.t. T is A_{xT}. If A_{xT} is verisimilar, then $Vs_\tau(A_{xT}) > 0$ by definition and $Vs_\tau\left(T_A^+\right) > Vs_\tau(T)$ by Theorem 8.

2. If A_{cT} and A_{xT} are verisimilar, then $Vs_\tau(A_{cT}) > 0$ by definition. In order to prove that $Vs_\tau(A_{xT}) > Vs_\tau(\tilde{A}_{cT}) - Vs_\tau(A_{cT})$, it is then sufficient to prove (see the proof of Theorem 10) that $cont_t\,(A_{cT}, C^*) \geq cont_f\,(A_{cT}, C^*)$. To see that this is in fact the case, note that, since $Vs_\tau(A_{cT}) > 0$, $\tau cont_t\,(A_{cT}, C^*) - (1-\tau) cont_f\,(A_{cT}, C^*) > 0$. Consequently, $cont_t\,(A_{cT}, C^*) > (1-\tau)/\tau cont_f\,(A_{cT}, C^*)$. Since $(1-\tau)/\tau \geq 1$, it follows that $cont_t\,(A_{cT}, C^*) \geq cont_f\,(A_{cT}, C^*)$. It follows from this that $Vs_\tau(A_{cT}) \geq Vs_\tau(\tilde{A}_{cT})$, i.e., $Vs_\tau(\tilde{A}_{cT}) - Vs_\tau(A_{cT}) \leq 0$, and thus that $Vs_\tau(A_{xT}) > Vs_\tau(\tilde{A}_{cT}) - Vs_\tau(A_{cT})$. Consequently, by Theorem 8, $Vs_\tau\left(T_A^*\right) > Vs_\tau(T)$.

Proof of Theorem 12. 1. If A is completely false, A_{xT} is also completely false (a fortiori) and $Vs_\tau(A_{xT}) < 0$ by Theorem 1. Thus, $Vs_\tau\left(T_A^+\right) < Vs_\tau(T)$ by the proof of Theorem 8.

2. If A is completely false, both A_{cT} and A_{xT} are completely false (a fortiori). Thus, $Vs_\tau(A_{xT}) < 0$ and $Vs_\tau(A_{cT}) < 0$ by Theorem 1. As observed above (see the proof of Theorem 8), to prove that $Vs_\tau(A_{xT}) < Vs_\tau(\tilde{A}_{cT}) - Vs_\tau(A_{cT})$ it is sufficient to prove that $cont_t\,(A_{cT}, C^*) \leq cont_f\,(A_{cT}, C^*)$. To see that this is in fact the case, note that, since A_{cT} is completely false, $cont_t\,(A_{cT}, C^*) = 0$ and $cont_f\,(A_{cT}, C^*) > 0$. Consequently, by the proof of Theorem 8, $Vs_\tau\left(T_A^*\right) < Vs_\tau(T)$.

Proof of Theorem 13. Note that since A is compatible with T by hypothesis, the only non-redundant part of A w.r.t. T is A_{xT}. If A_{xT} is t-distant, then $Vs_{\tau}(A_{xT}) < 0$ by definition and $Vs_{\tau}\left(T_A^+\right) < Vs_{\tau}(T)$ by Theorem 8.

Proof of Theorem 14. Consider the following counterexample to the claim that if A_{cT} and A_{xT} are *t-distant*, then $Vs_{\tau}\left(T_A^*\right) < Vs_{\tau}(T)$. Let $p_1, \ldots, p_6$ be *true* atomic propositions of **L** and let be $T \equiv \neg p_1 \wedge \neg p_2 \wedge \neg p_3 \wedge p_4 \wedge p_5$ a false theory. Let consider the (false) additive input $A \equiv p_1 \wedge p_2 \wedge p_3 \wedge \neg p_4 \wedge \neg p_5 \wedge \neg p_6$; the conflicting part of A w.r.t. T is $A_{cT} = p_1 \wedge p_2 \wedge p_3 \wedge \neg p_4 \wedge \neg p_5$ and the extra part of A w.r.t. T is $A_{xT} = \neg p6$. The revision of T by A will be, by Theorem 5, $T_A^* = p_1 \wedge p_2 \wedge p_3 \wedge \neg p_4 \wedge \neg p_5 \wedge \neg p_6$. Now consider a verisimilitude measure Vs_{τ} defined on **L** with $\tau = 1/3$. It is easy to calculate that $Vs_{\tau}(T) = -4/3n$. Moreover, $Vs_{\tau}(A_{cT}) = -1/3n$ and $Vs_{\tau}(A_{xT}) = -2/3n$, i.e., both A_{cT} and A_{xT} are t-distant. This notwithstanding, since $Vs_{\tau}\left(T_A^*\right) = -1/n,\ T_A^*$ is more verisimilar than T.

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