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一类半线性双曲系统的状态观测问题****

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提 要

本文考虑一类半线性双曲系统的状态观测问题，给出了一个 (求解) 算法并证明了其在某种相当强的意义上的有效性.

关键词 状态观测，半线性双曲系统，算法
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§1. 引 言

对半线性双曲型偏微分方程

$$\begin{cases} y''-a(x)y_{xx}=f(y,y',y_x)+b(t,x), & (t,x)\in(0,+\infty)\times I, \\ y(t,0)=y(t,l)=0, & t\in(0,+\infty) \end{cases} \tag{1.1}$$

(其中 $l\in(0,+\infty)$, $I\triangleq(0,l)$, $a(\cdot)\in C^\infty(I)$ 且 $\min\limits_{x\in I}a(x)>0$, $b(\cdot,\cdot)\in C([0,+\infty),L^2(I))$, $f(\cdot,\cdot,\cdot)\in C^1(\mathbb{R}^3,\mathbb{R}^1)$ 且满足 (整体) Lipschitz 条件; $\forall(z_0(\cdot),z_1(\cdot))\in H_0^1(I)\times L^2(I)$, 记 (1.1) 对应于初值 $(z_0(\cdot),z_1(\cdot))$ 的 (弱) 解为 $y(\cdot,\cdot;z_0,z_1)$.) 与初值条件

$$y(0,x)=y_0(x),\quad y'(0,x)=y_1(x),\qquad x\in I \tag{1.2}$$

($(y_0(\cdot),y_1(\cdot))\in H_0^1(I)\times L^2(I)$) 描述的动态系统，考虑如下状态观测问题:

(SOP) 取定 $T\in(0,+\infty)$, 试用系统 (1.1)–(1.2) 的可测量边界数据

$$\left\{\frac{\partial y}{\partial x}(t,0)\mid t\in[0,T]\right\} \tag{1.3}$$

在 $H_0^1(I)\times L^2(I)$ 中确定出其初态 $(y_0(\cdot),y_1(\cdot))$.

本文给出了可以求得 (SOP) 的一个 (足够好) 近似解的算法，然后在 §3 中证得的结果的基础上，讨论了此算法在空间 $H_0^1(I)\times L^2(I)$ 中强意义上的有效性.

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已有不少学者研究过动态微分系统的状态观测问题：Kalman 对线性 (常) 微分系统提出了 (状态) 能观与能控概念，此后有许多工作致力于将这两个在现代控制理论中有着基本的重要性的概念推广到更复杂或更一般系统情形，并对系统在何种条件下能观或能控作了理论上的深入探讨 (见 [1-6, 9, 11-13, 15-19] 及它们所引的有关参考文献). 众所周知：对线性动态微分系统，能观 (性) 与能控 (性) 常常是两个互为对偶的概念 (因而往往研究其中之一就足够了，参见 [9, 11, 12, 16]); 但对非线性动态微分系统，能观与能控这两个概念之间一般说来并不存在对偶关系 (因此人们不得不分别研究非线性系统的能观性和能控性). (对于半线性动态微分系统能控性方面的讨论，读者可见 [13, 18, 19] 和它们所引的丰富的有关参考文献). 至于半线性动态微分系统的能观性问题，[1, 4, 6] 和它们所引的某些参考文献对有限维系统情形作了讨论，而对无限维系统情形，文 [5, 15] 讨论了 $f(u,v,w)=g(u)$, $\forall z_1,z_2,z_3\in\mathbb{R}^1$, $g(\cdot)\in \mathrm{C}^1(\mathbb{R}^1,\mathbb{R}^1)$ 整体 Lipschitz 连续且 Lipschitz 常数充分小的特殊情形，但没有给出任何求解 (其所考虑的状态观测) 问题的算法.

§2. 主要结果的表述及应用

我们重申下列基本假定：

(H1) $a(\cdot)\in \mathrm{C}^\infty(I)$ 且 $\min\limits_{x\in I} a(x)>0$;

(H2) $b(\cdot,\cdot)\in \mathrm{C}([0,+\infty),\mathrm{L}^2(I))$;

(H3) $f(\cdot,\cdot,\cdot):\mathbb{R}^3\to\mathbb{R}^1$ 连续可微且存在常数 $L>0$, 使得

$$|f(u_1,v_1,w_1)-f(u_2,v_2,w_2)|\le L(|u_1-u_2|+|v_1-v_2|+|w_1-w_2|),\qquad \forall (u_i,v_i,w_i)\in\mathbb{R}^3,\ i=1,2.$$

用 [10, 14] 中的结果与方法和所谓的能量估计方法容易证得如下定理.

定理 2.1 设 (H1)–(H3) 成立，$T>0$. 则

(1) $\forall$ 初值 $(z_0(\cdot),z_1(\cdot))\in \mathrm{H}_0^1(I)\times \mathrm{L}^2(I)$, (1.1) 在 $\mathrm{C}([0,T],\mathrm{H}_0^1(I))\cap \mathrm{C}^1([0,T],\mathrm{L}^2(I))$ 中有唯一弱解 $y(\cdot,\cdot;z_0,z_1)$, 且存在常数 $C=C(T,L)>0$, 使得

$$\begin{aligned}&\max_{0\le t\le T}[\|y(t,\cdot;\tilde z_0,\tilde z_1)-y(t,\cdot;\bar z_0,\bar z_1)\|^2_{\mathrm{H}_0^1(I)}+\|y'(t,\cdot;\tilde z_0,\tilde z_1)-y'(t,\cdot;\bar z_0,\bar z_1)\|^2_{\mathrm{L}^2(I)}]\\ &\le C^2[\|\tilde z_0(\cdot)-\bar z_0(\cdot)\|^2_{\mathrm{H}_0^1(I)}+\|\tilde z_1(\cdot)-\bar z_1(\cdot)\|^2_{\mathrm{L}^2(I)}],\\ &\qquad\forall (\tilde z_0(\cdot),\tilde z_1(\cdot)),(\bar z_0(\cdot),\bar z_1(\cdot))\in \mathrm{H}_0^1(I)\times\mathrm{L}^2(I);\end{aligned}\tag{2.1}$$

(2) $\forall (z_0(\cdot),z_1(\cdot))\in \mathrm{H}_0^1(I)\times \mathrm{L}^2(I)$,

$$\frac{\partial y}{\partial x}(\cdot,0;z_0,z_1),\ \frac{\partial y}{\partial x}(\cdot,l;z_0,z_1)\in \mathrm{L}^2(0,T)\tag{2.2}$$

且存在常数 $C=C(T,L)>0$, 使得

$$\begin{aligned}&\int_0^T\Big[\Big|\frac{\partial y}{\partial x}(t,0;\tilde z_0,\tilde z_1)-\frac{\partial y}{\partial x}(t,0;\bar z_0,\bar z_1)\Big|^2+\Big|\frac{\partial y}{\partial x}(t,l;\tilde z_0,\tilde z_1)-\frac{\partial y}{\partial x}(t,l;\bar z_0,\bar z_1)\Big|^2\Big]\mathrm{d}t\\ &\le C^2[\|\tilde z_0(\cdot)-\bar z_0(\cdot)\|^2_{\mathrm{H}_0^1(I)}+\|\tilde z_1(\cdot)-\bar z_1(\cdot)\|^2_{\mathrm{L}^2(I)}],\\ &\qquad\forall (\tilde z_0(\cdot),\tilde z_1(\cdot)),(\bar z_0(\cdot),\bar z_1(\cdot))\in \mathrm{H}_0^1(I)\times\mathrm{L}^2(I);\end{aligned}\tag{2.3}$$

(3) 若 $b(\cdot,\cdot)\in C^1([0,+\infty)\times I)$, $z_0(\cdot)\in H^2(I)\cap H_0^1(I)$, $z_1(\cdot)\in H_0^1(I)$ 并且 $\forall u,v,w\in\mathbb{R}^1$, $f(u,v,w)=h(u)$, $h(\cdot):\mathbb{R}^1\to\mathbb{R}^1$ 连续可微且存在 $L_h>0$, 使 $\forall u_1,u_2\in\mathbb{R}^1$, $|h(u_1)-h(u_2)|\le L_h|u_1-u_2|$, 则

$$\frac{\partial y}{\partial x}(\cdot,0;z_0,z_1),\frac{\partial y}{\partial x}(\cdot,l;z_0,z_1)\in H^1(0,T)(\subseteq C[0,T]). \tag{2.4}$$

注意到 (2.2), 我们可以定义一个非线性映照 Γ $(:H_0^1(I)\times L^2(I)\to L^2(0,T))$ 如下

$$\Gamma(z_0,z_1)\ (\cdot)\triangleq\frac{\partial y}{\partial x}(\cdot,0;z_0,z_1),\quad \forall(z_0(\cdot),z_1(\cdot))\in H_0^1(I)\times L^2(I). \tag{2.5}$$

我们有如下结果 (本文两个主要结果中的一个，其证明见本文 §3):

定理 2.2 设 (H1)–(H3) 成立，$T>2\hat{l}$, $\hat{l}\triangleq\int_0^l\frac{\mathrm{d}x}{\sqrt{a(x)}}$. 则存在常数 $C=C(T,l,L)>0$, 使得

$$\begin{aligned}&\frac{1}{C}\|\Gamma(\tilde{z}_0,\tilde{z}_1)(\cdot)-\Gamma(\bar{z}_0,\bar{z}_1)(\cdot)\|_{L^2(0,T)}\\ \le&\|\tilde{z}_0(\cdot)-\bar{z}_0(\cdot)\|_{H_0^1(I)}+\|\tilde{z}_1(\cdot)-\bar{z}_1(\cdot)\|_{L^2(I)}\le C\|\Gamma(\tilde{z}_0,\tilde{z}_1)(\cdot)-\Gamma(\bar{z}_0,\bar{z}_1)(\cdot)\|_{L^2(0,T)},\\ &\forall(\tilde{z}_0(\cdot),\tilde{z}_1(\cdot)),(\bar{z}_0(\cdot),\bar{z}_1(\cdot))\in H_0^1(I)\times L^2(I).\end{aligned} \tag{2.6}$$

注 2.1 当 $T\in(0,2\hat{l})$ 时，(2.6) 中的第 2 个不等式未必成立 (反例见 [9]).

由 (2.6) 立知 (2.5) 定义的映照 Γ 是单射，因而 (SOP) 有唯一解；并且由 (2.6) 即知 (SOP) 的解 ((1.1)–(1.2) 中的 $(y_0(\cdot),y_1(\cdot))$) 连续地依赖于系统 (1.1)–(1.2) 的可测量边界数据 $\{\frac{\partial y}{\partial x}(t,0;y_0,y_1)\mid t\in[0,T]\}$. 我们将基于定理 2.1, 定理 2.2 和后面 (将会) 陈述的定理 2.4 给出一个求 (SOP) 的近似解的算法. 为此，定义泛函

$$J(z_0(\cdot),z_1(\cdot))\triangleq\frac{1}{2}\int_0^T\left|\frac{\partial y}{\partial x}(t,0;z_0,z_1)-\frac{\partial y}{\partial x}(t,0;y_0,y_1)\right|^2\mathrm{d}t,\qquad \forall(z_0(\cdot),z_1(\cdot))\in H_0^1(I)\times L^2(I). \tag{2.7}$$

由 (2.6) 即知 (SOP) 中待求的初态 $(y_0(\cdot),y_1(\cdot))$ 就是下述问题 (MP) 的唯一解.

(MP) 求 $(\hat{z}_0(\cdot),\hat{z}_1(\cdot))\in H_0^1(I)\times L^2(I)$, 使得

$$J(\hat{z}_0(\cdot),\hat{z}_1(\cdot))=\hat{J}\triangleq\inf\{J(z_0(\cdot),z_1(\cdot))\mid(z_0(\cdot),z_1(\cdot))\in H_0^1(I)\times L^2(I)\}. \tag{2.8}$$

取 I 的一列分划

$$0=x_0^j<x_1^j<\cdots<x_{N_j}^j=l,\quad j=1,2,\cdots$$

使得 $\{x_0^j,x_1^j,\cdots,x_{N_j}^j\}_{j=1}^\infty$ 满足

$$\Delta_j\triangleq\max_{1\le i\le N_j}(x_i^j-x_{i-1}^j)\to 0\ (j\to\infty), \tag{2.9}$$

并取一列正数 $\{\varepsilon_j\}_{j=1}^\infty$, 使得其满足

$$\varepsilon_j\to 0\quad(j\to\infty). \tag{2.10}$$

(对任意自然数 j) 考虑如下有限维优化问题：

$(\mathbf{MP})_j$ 求 $\{\hat{u}_1^j,\cdots,\hat{u}_{N_j}^j;\hat{v}_1^j,\cdots,\hat{v}_{N_j}^j\}\subseteq\mathbb{R}^1$, 使得

$$J_j(\hat{u}_1^j,\cdots,\hat{u}_{N_j}^j;\hat{v}_1^j,\cdots,\hat{v}_{N_j}^j)\le\hat{J}_j+\varepsilon_j,\tag{2.11}$$

$$\begin{aligned}&J_j(u_1^j,\cdots,u_{N_j}^j;v_1^j,\cdots,v_{N_j}^j)\\&\triangleq\frac{1}{2}\|\Gamma(z_0(\cdot,\{u_i^j\}_{i=1}^{N_j}),z_1(\cdot,\{v_i^j\}_{i=1}^{N_j}))-\Gamma(y_0(\cdot),y_1(\cdot))\|_{\mathrm{L}^2(0,T)}^2,\end{aligned}\tag{2.12}$$

$z_0(\cdot,\{u_i\}_{i=1}^{N_j})$ 与 $z_1(\cdot,\{v_i\}_{i=1}^{N_j})$ 的定义式为

$$\begin{cases}z_0(x,\{u_i^j\}_{i=1}^{N_j})\triangleq\displaystyle\int_0^x\sum_{i=1}^{N_j}u_i^j\chi_{(x_{i-1}^j,x_i^j]}(\xi)\mathrm{d}\xi-\frac{x}{l}\int_0^l\sum_{i=1}^{N_j}u_i^j\chi_{(x_{i-1}^j,x_i^j]}(\xi)\mathrm{d}\xi,\\ z_1(x,\{v_i^j\}_{i=1}^{N_j})\triangleq\displaystyle\sum_{i=1}^{N_j}v_i^j\chi_{(x_{i-1}^j,x_i^j]}(x),\quad 0\le x\le l.\end{cases}\tag{2.13}$$

显见 $z_0(\cdot,\{u_i^j\}_{i=1}^{N_j})\in\mathrm{H}_0^1(I)$ 且

$$\begin{aligned}&\dot{z}_0(\cdot,\{u_i^j\}_{i=1}^{N_j})\\&=\sum_{i=1}^{N_j}u_i^j\chi_{(x_{i-1}^j,x_i^j]}(\cdot)-\frac{1}{l}\int_0^l\sum_{i=1}^{N_j}u_i^j\chi_{(x_{i-1}^j,x_i^j]}(\xi)\mathrm{d}\xi,\quad\forall\{u_i^j\}_{i=1}^{N_j},\quad\{v_i^j\}_{i=1}^{N_j}\subseteq\mathbb{R}^1,\end{aligned}$$

$$\hat{J}_j\triangleq\inf\{J_j(u_1^j,\cdots,u_{N_j}^j;v_1^j,\cdots,v_{N_j}^j)\mid u_i^j,\ \ v_i^j\in\mathbb{R}^1,i=1,\cdots,N_j\}.\tag{2.14}$$

借助于定理 2.2 可证得下述结果 (本文两个主要结果中的另一个, 其证明见本文 §4):

定理 2.3 若定理 2.2 的假设条件都成立, 则

$$\lim_{j\to\infty}\hat{J}_j=0(=J(y_0(\cdot),y_1(\cdot))=\hat{J}),\tag{2.15}$$

从而有

$$\begin{cases}\|z_0(\cdot,\{\hat{u}_i^j\}_{i=1}^{N_j})-y_0(\cdot)\|_{\mathrm{H}_0^1(I)}\to 0,\\ \|z_1(\cdot,\{\hat{v}_i^j\}_{i=1}^{N_j})-y_1(\cdot)\|_{\mathrm{L}^2(I)}\to 0,\end{cases}\quad j\to\infty.\tag{2.16}$$

至此, 以定理 2.2 和定理 2.3 为理论基础, 给出解 (状态观测问题)(SOP) 的 4 步算法 (其有效性和有限终止性由定理 2.2 和定理 2.3 保证):

第 1 步 适当选取 (一个正数) T, 通过对系统 (1.1)–(1.2) 所作的测量获知 (1.3).

第 2 步 $j:=1$.

第 3 步 适当选取 (一个) 正整数 N_0 和 (一个) 正数 ε_0,

$$N_j:=jN_0,\quad\varepsilon_j:=\frac{\varepsilon_0}{j},\quad x_i^j:=\frac{il}{N_j},\quad i=0,1,\cdots N_j,$$

求得 (有限维优化问题) $(\mathrm{MP})_j$ 的 (一个) 解 $\{\hat{u}_1^j,\cdots,\hat{u}_{N_j}^j;\hat{v}_1^j,\cdots,\hat{v}_{N_j}^j\}$.

第 4 步 设 δ_0 为一个反映了预定精度要求的正数. 检验 $\{\hat{u}_1^j,\cdots,\hat{u}_{N_j}^j;\hat{v}_1^j,\cdots,\hat{v}_{N_j}^j\}$ 是否满足

$$\int_0^T\Big|\frac{\partial y}{\partial x}(t,0;z_0(\cdot,\{\hat{u}_i^j\}_{i=1}^{N_j}),z_1(\cdot,\{\hat{v}_i^j\}_{i=1}^{N_j}))-\frac{\partial y}{\partial x}(t,0;y_0(\cdot),y_1(\cdot))\Big|^2\mathrm{d}t\le\delta_0.$$

若是, 则置

$$\begin{cases}\tilde{y}_0(x)=z_0(x),\\ \tilde{y}_1(x)=z_1(x),\end{cases}\quad x\in I,$$

分别以 $\tilde{y}_0(\cdot)$ 与 $\tilde{y}_1(\cdot)$ 作为 $y_0(\cdot)$ 与 $y_1(\cdot)$ 的 (足够好的) 近似，终止；若否，则 $j := j+1$, 然后转到第 3 步.

§3. 定理 2.2 的证明

为证定理 2.2, 需要如下引理 (它是 [8] 中定理 3.1 的简单改进和推广，在此我们略去了其证明).

引理 3.1 设 Ω 是 $\mathbb{R}^n$ 中的有界区域, $x_0 \in \mathbb{R}^n \setminus \bar{\Omega}$, $T \in \left(2\sup_{x\in\Omega}|x_0 - x|, +\infty\right)$; 记 $Q \triangleq (0,T)\times\Omega$, $\Sigma \triangleq (0,T)\times\partial\Omega$; 记 $\partial\Omega$ 在 $x(\in\partial\Omega)$ 处的指向 Ω 外部的单位法向量为 $\nu(x)$,

$$\Gamma_0 \triangleq \{x\in\partial\Omega \mid (x-x_0)\cdot\nu(x) > 0\}, \quad \Sigma_0 \triangleq (0,T)\times\Gamma_0; \tag{3.1}$$

假定 $v \in \mathrm{H}^2(Q)$ 且对之存在一个非负常数 A, 使得

$$\begin{cases} |v'' - \triangle v| \le A(|v| + |v'| + |\nabla v|), & (t,x)\in Q, \\ v(t,x) = 0, & (t,x)\in\Sigma. \end{cases} \tag{3.2}$$

则存在常数 $B = B\left(A, \sup_{x\in\Omega}|x_0 - x|, T, x_0\right) > 0$, 使得

$$\|v\|^2_{\mathrm{H}^1(Q)} \le B\left\|\frac{\partial v}{\partial\nu}\right\|^2_{\mathrm{L}^2(\Sigma_0)}. \tag{3.3}$$

上述引理有如下直接推论:

推论 3.1 设 $l_1 \in (0,+\infty)$, $x_0 \in (l_1,+\infty)$, $T\in(2x_0,+\infty)$; 记 $I_1 \triangleq (0,l_1)$, $Q_1 \triangleq (0,T)\times I_1$, 设 $r_i \in \mathrm{L}^\infty(Q_1)$, $i=1,2,3$. 若 $v\in\mathrm{H}^2(Q_1)$ 适合

$$\begin{cases} v'' - v_{xx} = r_1 v + r_2 v' + r_3 v_x, & (t,x)\in Q_1, \\ v(t,0) = v(t,l_0) = 0, & t\in(0,T), \end{cases} \tag{3.4}$$

则存在常数 $B = B(A,T,x_0) > 0$, 其中 $A \triangleq \sum_{i=1}^{3}\|r_i\|_{\mathrm{L}^\infty(Q_1)}$, 使得

$$\|v\|_{\mathrm{H}^1(Q_1)} \le B\left\|\frac{\partial v}{\partial x}(\cdot,0)\right\|^2_{\mathrm{L}^2(0,T)}. \tag{3.5}$$

由推论 3.1, 通过适当的极限过程和通常的能量估计方法，可得

推论 3.2 设 $l_1\in(0,+\infty)$, $x_0\in(l_1,+\infty)$, $T\in(2x_0,+\infty)$; 记 $I_1\triangleq(0,l_1)$, $Q_1\triangleq(0,T)\times I_1$, 设 $r_i\in\mathrm{L}^\infty(Q_1)$, $i=1,2,3$; 设 $v_0(\cdot)\in\mathrm{H}^1_0(I_1)$, $v_1(\cdot)\in\mathrm{L}^2(I_1)$. 若 $v\in\mathrm{C}([0,T],\mathrm{H}^1_0(I_1))\cap\mathrm{C}^1([0,T],\mathrm{L}^2(I_1))$ 是 (偏微分) 方程

$$\begin{cases} v'' - v_{xx} = r_1 v + r_2 v' + r_3 v_x, & (t,x)\in Q_1, \\ v(t,0) = v(t,l_0) = 0, & t\in(0,T), \\ v(0,x) = v_0(x), & v'(0,x) \in v_1(x), \quad x\in I_1 \end{cases} \tag{3.6}$$

的弱解，则有常数 $B = B(\|r_1\|_{\mathrm{L}^\infty(Q_1)}, \|r_2\|_{\mathrm{L}^\infty(Q_1)}, \|r_3\|_{\mathrm{L}^\infty(Q_1)}, T, x_0) > 0$, 使得

$$\|v_0\|^2_{\mathrm{H}^1_0(I_1)} + \|v_1\|^2_{\mathrm{L}^2(I_1)} \le B\left\|\frac{\partial v}{\partial x}(\cdot,0)\right\|^2_{\mathrm{L}^2(0,T)}. \tag{3.7}$$

定理 2.2 的证明 在下面的讨论中均设 $T > 2\hat{l}$. 在 $H_0^1(I) \times L^2(I)$ 中任意取定两个元素 $(\tilde{z}_0(\cdot), \tilde{z}_1(\cdot))$ 与 $(\bar{z}_0(\cdot), \bar{z}_1(\cdot))$, 记

$$w(\cdot,\cdot) \triangleq y(\cdot,\cdot;\tilde{z}_0,\tilde{z}_1) - y(\cdot,\cdot;\bar{z}_0,\bar{z}_1), \tag{3.8}$$

易知 w 是如下 (偏微分) 方程

$$\begin{cases} w'' - a(x)w_{xx} = q_1 w + q_2 w' + q_3 w_x, & (t,x) \in (0,T)\times I, \\ w(t,0) = w(t,l) = 0, & t \in (0,T), \\ w(0,x) = w_0(x), \quad w'(0,x) = w_1(x), & x \in I \end{cases} \tag{3.9}$$

的 (唯一) 弱解, (3.9) 中,

$$\begin{aligned} q_i(t,x) \triangleq \int_0^1 & f_i'(y(t,x;\bar{z}_0,\bar{z}_1) + \theta w(t,x), y'(t,x;\bar{z}_0,\bar{z}_1) + \theta w'(t,x), y_x(t,x;\bar{z}_0,\bar{z}_1) \\ & + \theta w_x(t,x))\mathrm{d}\theta, \quad (t,x) \in (0,T)\times I, \quad i = 1,2,3 \end{aligned} \tag{3.10}$$

(由 (H3) 与上式立知

$$\|q_i\|_{L^\infty((0,T)\times I)} \le L, \quad i = 1,2,3), \tag{3.11}$$

$$\begin{cases} w_0(x) \triangleq \tilde{z}_0(x) - \bar{z}_0(x), \\ w_1(x) \triangleq \tilde{z}_1(x) - \bar{z}_1(x), \end{cases} \quad x \in I. \tag{3.12}$$

记 $\hat{I} \triangleq (0,\hat{l})$, 引进 (空间变量的一个) 变换

$$x \leftrightarrow \hat{x} = \int_0^x \frac{\mathrm{d}\xi}{\sqrt{a(\xi)}} : I \leftrightarrow \hat{I}, \tag{3.13}$$

定义

$$u(t,\hat{x}) \triangleq w(t,x), \quad \forall (t,\hat{x}) \in [0,T]\times \hat{I}, \tag{3.14}$$

则由 w 是 (3.9) 在 $C([0,T],H_0^1(I)) \cap C^1([0,T],L^2(I))$ 中的弱解, 易知 $u \in C([0,T],H_0^1(\hat{I})) \cap C^1([0,T],L^2(\hat{I}))$ 且适合

$$\begin{cases} u'' - u_{\hat{x}\hat{x}} = p_1 u + p_2 u' + p_3 u_{\hat{x}}, & (t,\hat{x}) \in \hat{Q} \triangleq (0,T)\times \hat{I}, \\ u(t,0) = u(t,\hat{l}) = 0, & t \in (0,T), \\ u(0,\hat{x}) = w_0(x), \quad u'(t,\hat{x}) = w_1(x), & \hat{x} \in \hat{I}, \end{cases} \tag{3.15}$$

其中

$$\begin{cases} p_1(t,\hat{x}) \triangleq q_1(t,x), \\ p_2(t,\hat{x}) \triangleq q_2(t,x), \\ p_3(t,\hat{x}) \triangleq \frac{1}{\sqrt{a(x)}}\left[q_3(t,x) - \frac{a_x(x)}{2}\right], \end{cases} \quad (t,\hat{x}) \in \hat{Q}. \tag{3.16}$$

根据上式与 (3.11) 即知

$$\|p_1\|_{L^\infty(\hat{Q})}, \quad \|p_2\|_{L^\infty(\hat{Q})} \le L, \quad \|p_3\|_{L^\infty(\hat{Q})} \le \frac{L + \frac{1}{2}\max\limits_{x\in I}|a_x(x)|}{\sqrt{\min\limits_{x\in L} a(x)}}, \tag{3.17}$$

再注意到 (3.15) 和 (3.13) 等，对 u 应用推论 3.2 便知存在常数 $\hat{B}=\hat{B}(T,l,L)>0$, 使得

$$\|w_0(\cdot)\|^2_{\mathrm{H}^1_0(I)}+\|w_1(\cdot)\|^2_{\mathrm{L}^2(I)}\le \hat{B}\Big\|\frac{\partial u}{\partial \hat{x}}(\cdot,0)\Big\|^2_{\mathrm{L}^2(0,T)}. \tag{3.18}$$

由上式，(3.12)–(3.14) 和 (3.8) 易知有常数 $C_1=C_1(T,l,L)>0$, 使得

$$\|\tilde{z}_0(\cdot)-\bar{z}_0(\cdot)\|^2_{\mathrm{H}^1_0(I)}+\|\tilde{z}_1(\cdot)-\bar{z}_1(\cdot)\|^2_{\mathrm{L}^2(I)}\le C_1^2\Big\|\frac{\partial y}{\partial x}(\cdot,0;\tilde{z}_0,\tilde{z}_1)-\frac{\partial y}{\partial x}(\cdot,0;\bar{z}_0,\bar{z}_1)\Big\|^2_{\mathrm{L}^2(0,T)}; \tag{3.19}$$

另外，由定理 2.1 之 (2) 即知存在常数 $C_2=C_2(T,L)>0$, 使得

$$\begin{aligned}&\Big\|\frac{\partial y}{\partial x}(\cdot,0;\tilde{z}_0,\tilde{z}_1)-\frac{\partial y}{\partial x}(\cdot,0;\bar{z}_0,\bar{z}_1)\Big\|^2_{\mathrm{L}^2(0,T)}\\ &\le C_2^2[\|\tilde{z}_0(\cdot)-\bar{z}_0(\cdot)\|^2_{\mathrm{H}^1_0(I)}+\|\tilde{z}_1(\cdot)-\bar{z}_1(\cdot)\|^2_{\mathrm{L}^2(I)}].\end{aligned} \tag{3.20}$$

取 $C\triangleq\max(\sqrt{2}C_1,C_2)$, 则由 (3.19)–(3.20) 就知道 (2.6) 成立.

§4.　定理 2.3 的证明

利用 $\mathrm{C}_0^\infty(I)$ 在 $\mathrm{H}^1_0(I)$ 中与 $\mathrm{C}(I)$ 在 $\mathrm{L}^2(I)$ 中的稠密性可知对任意正数 δ 存在 $y_{0,\delta}(\cdot)\in \mathrm{C}_0^\infty(I)$ 和正整数 K_δ, 使得

$$\|y_0(\cdot)-y_{0,\delta}(\cdot)\|_{\mathrm{H}^1_0(I)}\le\delta, \tag{4.1}$$

存在 $\{\bar{u}_i^j(\delta)\}_{i=1}^{N_j},\{\bar{v}_i^j(\delta)\}_{i=1}^{N_j}\subseteq\mathbb{R}^1$,

$$\begin{aligned}&\Big\|\dot{y}_{0,\delta}(\cdot)-\sum_{i=1}^{N_j}\bar{u}_i^j(\delta)\chi_{(x^j_{i-1},x^j_i]}(\cdot)\Big\|_{\mathrm{L}^2(I)}+\Big\|y_1(\cdot)-\sum_{i=1}^{N_j}\bar{v}_i^j(\delta)\chi_{(x^j_{i-1},x^j_i]}(\cdot)\Big\|_{\mathrm{L}^2(I)}\le\delta,\\ &\qquad j=K_\delta,K_\delta+1,\cdots.\end{aligned} \tag{4.2}$$

记

$$\bar{z}^j_\delta(\cdot)\triangleq\int_0^{\cdot}\sum_{i=1}^{N_j}\bar{u}_i^j(\delta)\chi_{(x^j_{i-1},x^j_i]}(\xi)\mathrm{d}\xi,\quad j=K_\delta,K_\delta+1,\cdots,\quad\forall\delta>0, \tag{4.3}$$

则 (由上式和 (4.2))

$$\begin{aligned}|\bar{z}^j_\delta(x)-y_{0,\delta}(x)|&=\Big|\int_0^x\Big[\sum_{i=1}^{N_j}\bar{u}_i^j(\delta)\chi_{(x^j_{i-1},x^j_i]}(\xi)-\dot{y}_{0,\delta}(\xi)\Big]\mathrm{d}\xi\Big|\\ &\le\sqrt{x}\Big\|\sum_{i=1}^{N_j}\bar{u}_i^j(\delta)\chi_{(x^j_{i-1},x^j_i]}(\cdot)-\dot{y}_{0,\delta}(\cdot)\Big\|_{\mathrm{L}^2(I)}\le\sqrt{l}\delta,\\ &\qquad\forall x\in I,\quad j=K_\delta,K_\delta+1,\cdots,\quad\forall\delta>0.\end{aligned} \tag{4.4}$$

故有 (注意到 $y_{0,\delta}(\cdot)\in\mathrm{C}_0^\infty(I)$ 因而 $y_{0,\delta}(l)=0$)

$$|\bar{z}^j_\delta(l)|=|\bar{z}^j_\delta(l)-y_{0,\delta}(l)|\le\sqrt{l}\delta,\quad j=K_\delta,K_\delta+1,\cdots,\quad\forall\delta>0. \tag{4.5}$$

接下来记

$$\bar{z}^j_{\delta,0}(\cdot)\triangleq z_0(\cdot,\{\bar{u}_i^j(\delta)\}_{i=1}^{N_j})=\bar{z}^j_\delta(\cdot)-\frac{x}{l}\bar{z}^j_\delta(l),\quad j=K_\delta,K_\delta+1,\cdots,\quad\forall\delta>0 \tag{4.6}$$

(则显然 $\bar{z}^j_{\delta,0}(\cdot)\in \mathrm{H}^1_0(I)$, $j=K_\delta,K_\delta+1,\cdots,\ \forall\delta>0$); 由上式与 (4.3), (4.5), (4.4) 和 (4.2), 有

$$\begin{aligned}
&\|\bar{z}^j_{\delta,0}(\cdot)-y_{0,\delta}(\cdot)\|^2_{\mathrm{H}^1_0(I)}\\
&=\int_0^l|\bar{z}^j_{\delta,0}(x)-y_{0,\delta}(x)|^2\mathrm{d}x+\int_0^l|\dot{\bar{z}}^j_{\delta,0}(x)-\dot{y}_{0,\delta}(x)|^2\mathrm{d}x\\
&=\int_0^l\Big|\bar{z}^j_\delta(x)-\frac{x}{l}\bar{z}^j_\delta(l)-y_{0,\delta}(x)\Big|^2\mathrm{d}x\\
&\quad+\int_0^l\Big|\sum_{i=1}^{N_j}\bar{u}^j_i(\delta)\chi_{(x^j_{i-1},x^j_i]}(x)-\frac{1}{l}\bar{z}^j_\delta(l)-\dot{y}_{0,\delta}(x)\Big|^2\mathrm{d}x\\
&\le 2\Big[\int_0^l|\bar{z}^j_\delta(x)-y_{0,\delta}(x)|^2\mathrm{d}x+|\bar{z}^j_\delta(l)|^2l\Big]\\
&\quad+2\Big\{\int_0^l\Big|\sum_{i=1}^{N_j}\bar{u}^j_i(\delta)\chi_{(x^j_{i-1},x^j_i]}(x)-\dot{y}_{0,\delta}(x)\Big|^2\mathrm{d}x+\Big[\frac{1}{l}|\bar{z}^j_\delta(l)|\Big]^2l\Big\}\\
&\le 2[(\sqrt{l}\delta)^2l+(\sqrt{l}\delta)^2l]+2\Big[\delta^2+\frac{1}{l}(\sqrt{l}\delta)^2\Big]=4(l^2+1)\delta^2,\\
&\qquad j=K_\delta,K_\delta+1,\cdots,\quad\forall\delta>0.
\end{aligned}\tag{4.7}$$

所以 (由 (2.12)–(2.13),(4.3) 与 (4.6),(2.6), 上式和 (4.1)–(4.2))

$$\begin{aligned}
&J_j(\bar{u}^j_1(\delta),\cdots,\bar{u}^j_{N_j}(\delta);\bar{v}^j_1(\delta),\cdots,\bar{v}^j_{N_j}(\delta))\\
&=\frac{1}{2}\Big\|\Gamma\Big(\bar{z}^j_{\delta,0}(\cdot),\sum_{i=1}^{N_j}\bar{v}^j_i(\delta)\chi_{(x^j_{i-1},x^j_i]}(\cdot)\Big)-\Gamma(y_0(\cdot),y_1(\cdot))\Big\|^2_{\mathrm{L}^2(0,T)}\\
&\le C^2\Big[\|\bar{z}^j_{\delta,0}(\cdot)-y_0(\cdot)\|^2_{\mathrm{H}^1_0(I)}+\Big\|\sum_{i=1}^{N_j}\bar{v}^j_i(\delta)\chi_{(x^j_{i-1},x^j_i]}(\cdot)-y_1(\cdot)\Big\|^2_{\mathrm{L}^2(I)}\Big]\\
&\le C^2\Big\{2[\|\bar{z}^j_{\delta,0}(\cdot)-y_{0,\delta}(\cdot)\|^2_{\mathrm{H}^1_0(I)}+\|y_{0,\delta}(\cdot)-y_0(\cdot)\|^2_{\mathrm{H}^1_0(I)}]\\
&\quad+\Big\|\sum_{i=1}^{N_j}\bar{v}^j_i(\delta)\chi_{(x^j_{i-1},x^j_i]}(\cdot)-y_1(\cdot)\Big\|^2_{\mathrm{L}^2(I)}\Big\}\\
&\le C^2\{2[4(l^2+1)\delta^2+\delta^2]+\delta^2\}=C^2(8l^2+11)\delta^2,\\
&\qquad j=K_\delta,K_\delta+1,\cdots,\quad\forall\delta>0.
\end{aligned}\tag{4.8}$$

上式 (与 (2.12)–(2.14)) 蕴涵

$$0\le\hat{J}_j\le C^2(8l^2+11)\delta^2,\quad j=K_\delta,K_\delta+1,\cdots,\quad\forall\delta>0.\tag{4.9}$$

故 (2.15) 成立. 最后, 由 (2.6), (2.11), 上式和 (2.10), 便知

$$\begin{aligned}
&[\|z_0(\cdot,\{\hat{u}^j_i\}_{i=1}^{N_j})-y_0(\cdot)\|_{\mathrm{H}^1_0(I)}+\|z_1(\cdot,\{\hat{v}^j_i\}_{i=1}^{N_j})-y_1(\cdot)\|_{\mathrm{L}^2(I)}]^2\\
&\le C^2\|\Gamma(z_0(\cdot,\{\hat{u}^j_i\}_{i=1}^{N_j}),z_1(\cdot,\{\hat{v}^j_i\}_{i=1}^{N_j}))-\Gamma(y_0(\cdot),y_1(\cdot))\|^2_{\mathrm{L}^2(0,T)}\\
&=2C^2J_j(\hat{u}^j_1,\cdots,\hat{u}^j_{N_j};\hat{v}^j_1,\cdots,\hat{v}^j_{N_j})\\
&\le 2C^2(\hat{J}_j+\varepsilon_j)\to 0\quad(j\to\infty).
\end{aligned}\tag{4.10}$$

因此 (2.16) (亦) 成立.

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STATE OBSERVATION PROBLEM FOR A CLASS OF SEMI-LINEAR HYPERBOLIC SYSTEMS

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Abstract

This paper considers a state observation problem for a class of semi-linear hyperbolic systems. The authors give a (conceptual) algorithm to solve the (state observation) problem and show that it is effective in a certain strong sense.

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