Multi-Keyword Multi-Click Option Contracts for Sponsored Search Advertising

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In sponsored search, advertising slots are usually sold by a search engine to an advertiser through an auction mechanism in which advertisers bid on keywords. In theory, an auction mechanism encourages the advertisers to truthfully bid for keywords. However, keyword auctions have a number of problems including: (i) volatility in revenue, (ii) uncertainty in the bidding and charged prices for advertisers' keywords, and (iii) weak brand loyalty between the advertiser and the search engine. To address these issues, we study the possibility of creating a special option contract that alleviates these problems. In our proposal, an advertiser purchases an option in advance from a search engine by paying an upfront fee, known as the option price. The advertiser then has the right, but no obligation, to then purchase specific keywords for a fixed cost-per-click (CPC) for a specified number of clicks in a specified period of time. Hence, the advertiser has increased certainty in sponsored search while the search engine could raise the customers' loyalty. The proposed option contract can be used in conjunction with traditional keyword auctions. As such, the option price and corresponding fixed CPC price must be set such that there is no arbitrage opportunity. In this paper, we discuss an option pricing model tailored to sponsored search that deals with spot CPCs of targeted keywords in a generalized second price (GSP) auction. We show that the pricing model for keywords is closely related to a special exotic option in finance that contains multiple underlying assets (multi-keywords) and is also multi-exercisable (multi-clicks). Experimental results on real advertising data verify our pricing model and demonstrate that advertising options can benefit both advertisers and search engines.

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1. INTRODUCTION

Sponsored search is an important online advertising format [PWC 2013]. A search engine sells ad slots in the search engine results pages (SERPs) generated in response to a user query. The terms in the query are referred to as keywords and the price of an ad slot is usually determined by a keyword auction [Börgers et al. 2008; Fain and Pedersen 2006; Maillé et al. 2012; Varian 2009] such as the widely used generalized second price (GSP) auction [Edelman et al. 2007; Varian 2007]. In the GSP auction, advertisers bid on keywords present in the query, and the highest bidder pays the price associated with the second highest bid.

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Despite the success of the keyword auctions, there are two major drawbacks. First, the uncertain and volatile bidding prices make it difficult for advertisers to predict their campaign costs and thus complicate their business planning [Wang and Chen 2012]. Second, the pay-as-you-go nature of the auctions does not encourage stable relationships between search engines and advertisers [Jank and Yahav 2010]: advertisers can switch from one search engine to another in the next bidding with near zero cost.

To alleviate these problems we propose a *multi-keyword multi-click option contract*. This is a contract between an advertiser and a search engine. It consists of a non-refundable upfront fee, known as the option price, paid by the advertiser, in return for the right, but not the obligation, to subsequently purchase a fixed number of clicks for particular keywords for a specified fixed CPC during a specified period of time. The fact that the contract covers a specific period of time should encourage a more stable relationship between the advertiser and the search engine. From the advertiser's perspective, fixing the CPC significantly reduces the uncertainty in cost of an advertising campaign. Moreover, if the CPC price set by the auction falls below the fixed CPC price, the advertiser is not obligated to exercise the option, but can, instead, participate in the auction. Thus, the option can be considered as establishing an upper limit on the cost of the campaign. From the search engine's perspective, the option is an additional service that acts as an insurance policy for advertisers. We show that the search engine can, in fact, increase their ad revenue by the option price.

Our proposed option differs from a standard option in two respects. First, the option can be exercised not once, but multiple times during the contract period. Second, the option is not for a single asset (i.e. keyword), but a basket of assets. Both types of exotic options have been studied within the economics community and the novelty of this work and the comparisons with the related work are discussed in Section 2. The primary question for us is how to determine the option price and the associated fixed CPC associated with each keyword in the advertiser's list. Clearly if the option is priced too low, then significant loss in revenue may ensure. Moreover, this may create an arbitrage opportunity where the buyer of the option sells their advertising slots at a risk-less profit. Conversely, if the fixed CPC prices are too high, then the advertiser will either not purchase the option or not exercise their right to purchase ad slots at the agreed fixed price. This paper has the following contributions. We propose a new mechanism for selling future keywords in sponsored search, which it can be implemented in conjunction with keyword auctions. We show that in the broader environment, advertisers can easily adjust their advertising strategies by interacting an option contract with keyword auctions. We then illustrate how the proposed ad option can be effectively priced so that arbitrage between the spot (auction) market and the option market are eliminated. Finally, we show that ad option can serve as an insurance service, and provide an additional revenue stream for a search engine.

2. RELATED WORK

We first review the prior work on options, and then discuss the related work in online advertising.

2.1. Options and Their Pricing

A standard option is a contract in which the seller grants the buyer the right, but not the obligation, to enter into a transaction with the seller to either buy or sell an underlying asset at a fixed price on or before a fixed date. The fixed price is called *strike price* and the fixed date is called *expiration date*. The seller grants this right in exchange for a certain amount of money, called *option price*. An option is called a *call option* or a *put option* depending on whether the buyer is purchasing a right to buy or sell the asset. The simplest option is the *European option* [Hull 2009], which can be exercised only

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on the expiration date. This differs from an *American option* [Hull 2009], which can be exercised at any time during the contract period. Both European and American options are *standard options*. In the beginning of the 1980's, the standard options became better understood and their trading volume exploded; financial institutions began to search for alternative forms of options, known as *exotic options* [Zhang 1998], to meet their particular business needs. Among them, two types of options, multi-asset options and multi-exercise options, are particularly relevant to this paper.

Multi-asset options are options with payoffs affected by at least two underlying instruments [Zhang 1998]. These instruments can be assets such as stocks, bonds, currencies, and indices such as the S&P-100 or FTSE-100. There are two types of multi-asset options that can be employed and extended for sponsored search. First, the *basket call option*, whose the payoff is affected by the weighted sum of the prices of a basket of instruments. The structure of a basket call option [Wilmott 2006] can be used to describe the setting of keyword broad match, where the weights are interpreted as the probabilities that sub-phrases occur as queries. Second, the *dual-strike call option* has two strike prices written on two underlying instruments [Zhang 1998]. The dual-strike option provides a framework for a multi-keyword option; the advertiser is able to switch target keywords during the contract lifetime, but in online advertising, the number of candidate keywords for advertisers to choose from is usually more than 2. We extend the dual cases [Zhang 1998] to higher-dimensional cases and proposes to use Monte Carlo simulation to provide a tractable solution.

Multi-exercise options are a generalization of American options that provide the buyer more than one exercise right and sometimes control over one or more other variables, e.g. the amount of the underlying instrument exercised at a certain time. Multi-exercise options have become more prevalent over the past decade especially in the energy industry [Villinski 2004; Marshall et al. 2011; Marshall 2012]. Our proposed option contract is also a simple case of multi-exercise options because it permits the option buyer to repeatedly exercise the right to obtain clicks on targeted keywords. However, for sponsored search, the multi-exercise opportunity is more flexible, compared to the energy industry. We allow advertisers to (i) exercise options at any time in the option lifetime, i.e. the exercise time is not pre-specified, and (ii) no minimum number of clicks is required for each exercise, i.e. there is no penalty fee if the advertiser does not exercise the minimum clicks. Also, there is no transaction fee in sponsored search. Thus in this paper, the value of an m-click option is the sum of m independent and identical (i.i.d.) single-click options.

A key issue for an option contract is how to price it. In fact, option pricing models constitute one of the most import building blocks of asset pricing theory. In 1900, Bachelier [Bachelier 1900] first proposed to use a continuous-time random walk process to price option contracts. Sixty five years later, Samuelson [Samuelson 1965b] replaced Bachelier's assumption on the asset price with a geometric form, called *geometric Brownian motion (GBM)*, thereby solving the problem of a negative asset price in option pricing. The research of Samuelson highly affected Black and Scholes [Black and Scholes 1973] and Merton [Merton 1973], who then studied risk-neutral pricing for European options. The *Black-Scholes-Merton (BSM) option pricing formula* spurred research in this area. Our pricing model and its derivations are based on the above development in finance and its suitability is discussed in the experiment using real data sets.

2.2. Related Work in Online Advertising

Meinl and Blau introduced financial derivatives (including options) into web services, but did not formally discuss their application to online advertising [Meinl and Blau 2009]. To the best of our knowledge, Moon and Kwon in [Moon and Kwon 2010] first

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proposed a hybrid option contract for online advertising. Although it is called an option, it in fact deals with a different problem than ours. It is the option between choosing two payment methods: the cost-per-mille (CPM, CPM is calculated based on the number of advertisements displayed, i.e. cost per thousand of displayed ads) and CPC (cost-per-click). The option buyer is guaranteed the right to choose the minimum payment between CPM and the CPC after the click-through rate (CTR is the ratio of the number of advertisements clicked on to the number of advertisements displayed) is realized. This option contract is similar to an *option paying the worst and cash* [Zhang 1998], where the option payoff depends on the minimum of the two underlying assets. The option price proposed by Moon and Kwon is determined by a negotiation between the advertiser and the publisher under the framework of a *Nash bargaining game*. There are two utility functions: one is for the advertiser and another is for the publisher. The objective function is to maximize the product of the two utility functions with corresponding negotiation powers. Thus, the final option price is the optimal solution maximizing the negotiated join utility but does not rule out arbitrage opportunity.

Wang and Chen [Wang and Chen 2012] studied an option structure for a different problem, capping the CPM price in advance. They discussed a general pricing framework based on the one-step binomial tree [Cox et al. 1979]. They showed that the calculated option price can help the ad service provider manage revenue volatility over a certain period of time. The proposed option contract only corresponds to a single ad exposure, i.e. single click or impression, and the more realistic situation of fixing multiple keywords with multiple clicks is not covered.

This paper develops an option contract tailored to the unique environment of sponsored search, where the keyword broad match, the switch between multiple keywords, and exercising multiple clicks are considered. Our work is therefore significantly different to the earlier studies. It is also worth mentioning that our proposed option contract is more flexible than the guarantee contracts studied in online advertising [Salomatin et al. 2012] as in our proposed option, the advertisers need not exercise their rights should the prices go down in the keyword auction. The proposed scheme in [Salomatin et al. 2012] enables an advertiser to send the click demand and ad budget in his/her request to a search engine. It is the search engine, not the advertiser, who will decide the guaranteed ad delivery service according to the query and position. As such the advertiser has less control of the ad exposure time and position.

3. MULTI-KEYWORD MULTI-CLICK AD OPTION CONTRACT

In this section, we first explain how our multi-keyword, multi-click ad option works. We then introduce a pricing model to price the ad option such that there is no arbitrage.

To illustrate our idea, consider the following example. Suppose that a Computer Science department creates a new M.Sc. on “Web Science and Big Data Analytics” and is interested in an advertising campaign based around relevant search terms such as ‘MSc Web Science’, ‘MSc Big Data Analytics’, ‘Data Mining’, etc. The campaign is to start immediately and last for three months and the goal is to generate at least 1000 clicks on the ad which directs users to the M.Sc.’s homepage. The department (i.e., advertiser) does not know how the clicks will be distributed among the keywords, nor how much the campaign will cost if based on keyword auctions. However, with ad option, the advertiser can submit a request to the search engine to lock-in the ad cost. The request consists of the candidate keywords, the overall number of clicks needed, and the duration of the contract. The search engine responds with a price table for the option contract, as shown in Figure 1. It contains the option price and the fixed CPC for each keyword (with its sub-phrases for the board match case). The CPCs are fixed yet different across the candidate keywords. The contract is entered into when the advertiser pays the option price.

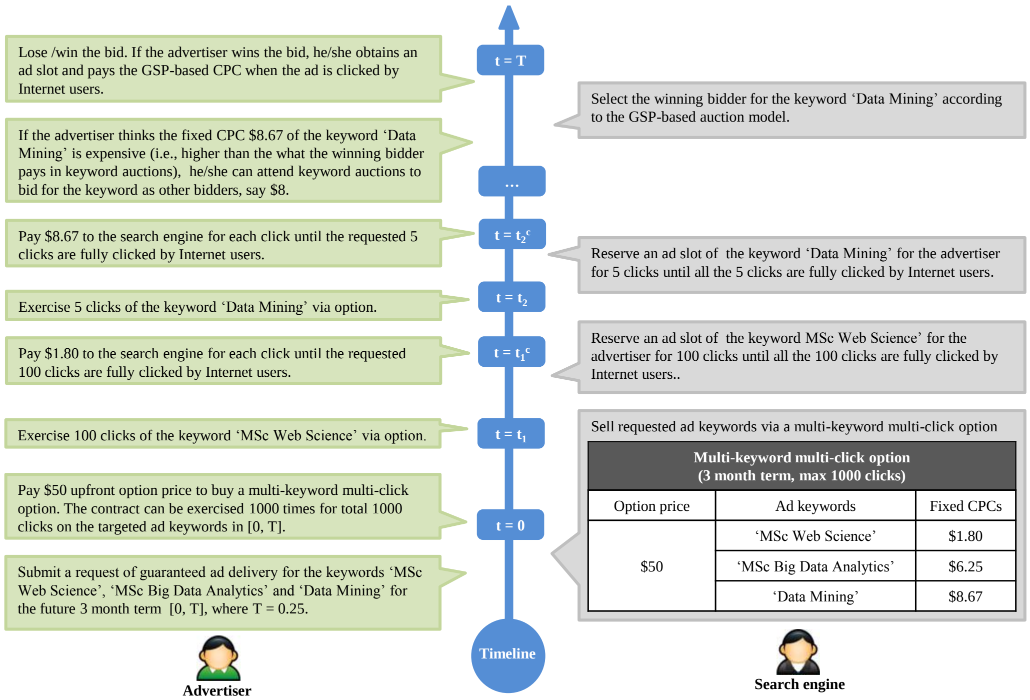

Fig. 1. A schematic view of buying, selling and using a multi-keyword, multi-click ad option contract.

During the contract period $[0, T]$, where T represents the contract expiration date (in terms of year format and is three months in our example), the advertiser has the right, at any time, to exercise portions of the contract, for example, to buy a requested number of clicks for a specific keyword. This right expires after time T or when the total number of clicks have been purchased, whichever is sooner. For example, at time $t_1 \leq T$ the advertiser may exercise the right for 100 clicks on the keyword ‘MSc Web Science’. After receiving the exercise request, the search engine immediately reserves ad slots for the keyword for the advertiser until the ad is clicked by 100 times. In our current design, the search engine decides which rank position the ad should be displayed as long as the required number of clicks is fulfilled - we assume there are adequate search impressions within the period. It is also possible to define a rank specific option where all the parameters (CPCs, option prices etc.) become rank specific.

The advertiser can switch among the candidate keywords and also monitor the keyword auction market for the candidate keywords. If, for example, the CPC for the keyword “MSc Web Science” drops below the fixed CPC, then the advertiser may choose to participate in the auction rather than exercise the option for the keyword. If later in the campaign, the spot price for the keyword ‘MSc Web Science“ exceeds the fixed CPC, the advertiser can then exercise the option.

The above example illustrates the flexibility of the proposed ad option. Specifically, (i) the advertiser does not have to use the option and can participate in keyword auctions as well, (ii) the advertiser can exercise the option at any time during the contract period, (iii) the advertiser can exercise the option up to the maximum number of clicks, (iv) the advertiser can request any number of clicks in each exercise provided the accumulated number of exercised clicks does not exceed the maximum number, and (v)

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the advertiser can switch between keywords at each exercise with no additional cost. Of course, this flexibility complicates the pricing of the option, which is discussed next.

3.1. Ad Option Pricing

The proposed ad options are more flexible than the conventional impression-guaranteed[1] contracts. The ad options enable advertisers to fix their ad campaign costs beforehand, yet leaves the decision of selecting suitable keywords and the exact timing to place the ad to later. Because of the extra flexibility, there is an *intrinsic* value associated with an ad option, and advertisers are required to pay an upfront fee to buy an ad option.

We denote the value of an n-keyword m-click ad option contract at time t as a function $V(t, \boldsymbol{C}(t); T, \boldsymbol{F}, m)$, where T is the contract exercisable period (or lifetime), m is the number of allowed clicks specified in the contract, and $\boldsymbol{F} = \{F_1, \cdots, F_n\}$ are the agreed fixed CPCs of the n candidate keywords. It is worth noting that the advertiser does not need to decide on what specific keyword (asset) will be used in advance. Instead, the ad option contract specifies a set of candidate keywords (potentially unlimited in a predefined domain) and their fixed prices. During the campaign, the advertiser can decide which keywords they intend to purchase and freely switch among them on the basis of their business needs. The value of an ad option also changes over time and depends on the current time t and the candidate keyword prices in the existing keyword auction market $\boldsymbol{C}(t)$.

We adopt a widely used stochastic process [Samuelson 1965a] to model $\boldsymbol{C}(t)$ by assuming that the GSP-based CPC of a keyword satisfies a multivariate log-normal distribution. Specifically, keyword K_i's CPC movement can be modeled as a multivariate Geometric Brownian Motion (GBM) as follows

$$dC_i(t) = \mu_i C_i(t)dt + \sigma_i C_i(t)dW_i(t), \quad i = 1, \ldots, n, \tag{1}$$

where μ_i and σ_i are the drift and volatility of the CPC respectively, and $W_i(t)$ is a standard Brownian motion satisfying the conditions:

$$\mathbb{E}(dW_i(t)) = 0, \tag{2}$$

$$\mathrm{var}(dW_i(t)) = \mathbb{E}(dW_i(t)dW_i(t)) = dt, \tag{3}$$

$$\mathrm{cov}(dW_i(t), dW_j(t)) = \mathbb{E}(dW_i(t)dW_j(t)) = \rho_{ij}dt, \tag{4}$$

where ρ_{ij} is the correlation coefficient between the ith and jth keywords, such that $\rho_{ii} = 1$ and $\rho_{ij} = \rho_{ji}$. The correlation matrix is denoted by $\boldsymbol{\Sigma}$, so that the covariance matrix is simply $\boldsymbol{M\Sigma M}$, where $\boldsymbol{M}$ is the matrix with the σ_i along the diagonal and zeros everywhere else. For a detailed explanation of such stochastic settings, see [Björk 2009; Wilmott 2006]. Later in Section 4, we discuss GBM parameter estimation, its fitness and robustness within our settings. The limitations of GBM model are discussed in Section 5.

We need to determine how much an advertiser should pay the search engine in order to obtain an ad option for a set of requested candidate keywords. In other words, we need to know the initial value of V when time $t = 0$. This is discussed next.

3.1.1. Proposed Pricing Model. To determine the option price, we first note that the value of a m-click ad option is equal to m number of 1-click ad option; that is

$$V(t, \boldsymbol{C}(t); T, \boldsymbol{F}, m) = mV(t, \boldsymbol{C}(t); T, \boldsymbol{F}, 1). \tag{5}$$

This is because, by definition, advertisers who own a proposed m-click option have the freedom of exercising the m clicks *separately* and *independently* within the period.

[1]Here, the term "guaranteed" means something happening with probability 1.

Thus, there is no difference between an m-click ad option and a set of m 1-click ad options in terms of exercise right and restriction. Their values are the same. In our proposed scheme, we allow the m-click ad option to be exercised up to m number of times over the lifetime of the contract.

Second, when $t = T$ we have

$$V(T, \boldsymbol{C}(T); T, \boldsymbol{F}, m) = m \max\{C_1(T) - F_1, \ldots, C_n(T) - F_n, 0\}. \tag{6}$$

That is, at the end of the option period, T, the value of the option is determined by the maximum difference between the spot CPC and the fixed CPC of the candidate keywords. This is due to the fact that the optimal decision of the advertiser is to exercise the option to buy the keyword which has the maximun difference between its market value and the value specified by the option. Thus, Eq. (6) can also be considered as the *payoff* of the ad option at time T.

The proposed multi-keyword multi-click ad option complements the existing keyword auctions and provides a risk management tool for advertisers. Thus, it is vitally important that the ad option value $V(t, \boldsymbol{C}(t); T, \boldsymbol{F}, m)$ should not generate any *arbitrage* opportunity [Varian 1987] between the two markets. We must therefore price the ad option such that no one is be able to make profits by taking the price differences between the two markets without taking any risk.

The No-Arbitrage Principle has been widely used to price financial option models [Björk 2009; Wilmott 2006]. We now show how the principle can be applied in our case. Suppose that an arbitrager (a person who hopes to make guaranteed profits from the difference between the keyword auction market and the option market) borrows some money from a bank with a fixed interest rate r and uses the borrowed money to buy the proposed multi-keyword multi-click ad option from a search engine. The arbitrager then immediately decomposes the m clicks and sells some clicks of the candidate keywords to other advertisers on the keyword auction market. The revenue is

$$R(t) = V(t, \boldsymbol{C}(t); \boldsymbol{F}, T, m) - \sum_{i=1}^{n} \psi_i(t) C_i(t), \tag{7}$$

where $\psi_i(t)$ represents the number of the clicks for the keyword C_i sold on the spot market, and $\sum_i \psi_i(t) = m$. Before rigorously modelling the scenario, we slightly simplify the above description. As shown in Eq. (5) we consider the value of a multi-keyword m-click option as the sum of m multi-keyword single-click options. For mathematical convenience, we assume the clicks can be infinitely divisible. Using Eq. (5) gives

$$R(t) = m\Big(V(t, \boldsymbol{C}(t); \boldsymbol{F}, T, 1) - \sum_{i=1}^{n} \Delta_i \times C_i(t)\Big), \tag{8}$$

where Δ_i is the portion of the clicks going for the keyword C_i at time t and $\sum_i \Delta_i = 1$. The changes of $R(t)$ over a short time dt is then given

$$dR(t) = m\Big(\frac{\partial V}{\partial t}dt + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}C_iC_j\frac{\partial^2 V}{\partial C_i \partial C_j}dt + \sum_{i=1}^{n}\frac{\partial V}{\partial C_i}dC_i - \sum_{i=1}^{n}\Delta_i dC_i\Big). \tag{9}$$

The above equation shows that the change of the revenue (when investing in the price differences in the two markets) depends on time and is related to the uncertainties of the candidate keywords CPCs in the auction market. If we, however, choose $\Delta_i = \partial V / \partial C_i$, we in fact eliminate the uncertainties in $dR(t)$ (i.e., the parts which have

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Brownian motions disappear), i.e.,

$$dR(t) = m\bigg(\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}C_iC_j\frac{\partial^2 V}{\partial C_i \partial C_j}\bigg)dt. \tag{10}$$

This is called *delta hedging* in financial option pricing [Black and Scholes 1973; Merton 1973]. Under these conditions, revenue growth is deterministic and there is no risk.

Of course, if we put the same amount money in a bank, it grows with interest r. That is

$$dR(t) = rR(t)dt = rm\bigg(V - \sum_{i=1}^{n}C_i\frac{\partial V}{\partial C_i}\bigg)dt, \tag{11}$$

where we replaced $R(t)$ using Eq. (8) and replaced Δ_i with $\partial V/\partial C_i$. If at each time, the arbitrager adjusts the portion of clicks to each keyword in order to make revenue $R(t)$ so that the uncertainties of the candidate keywords' CPCs in the auction market is eliminated, i.e. chooses $\Delta_i = \partial V/\partial C_i$, he/she can obtain the guaranteed revenue with zero variance. In such a case, the revenue growth should equal the bank's interest rate - otherwise there is an arbitrage. This gives a parabolic partial differential equation (PDE) for the no-arbitrage equilibrium as

$$\frac{\partial V}{\partial t} + r\sum_{i=1}^{n}C_i\frac{\partial V}{\partial C_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_i\sigma_j\rho_{ij}C_iC_j\frac{\partial^2 V}{\partial C_i \partial C_j} - rV = 0. \tag{12}$$

where Eq. (12) satisfies the boundary condition specified in Eq. (6). For simplicity, we denote the boundary condition as follows

$$V(T, \boldsymbol{C}(T); T, \boldsymbol{F}, 1) = \Phi(\boldsymbol{C}(T)). \tag{13}$$

Applying the Multidimensional Feynman-Kač Stochastic Representation [Björk 2009] to Eq. (12) gives the solution

$$V(t, \boldsymbol{C}(t); \boldsymbol{F}, T, 1) = e^{-r(T-t)}\mathbb{E}_t^{\mathbb{Q}}[\Phi(\boldsymbol{C}(T))], \tag{14}$$

where $\mathbb{E}_t^{\mathbb{Q}}[\cdot]$ is the conditional expectation with respect to time t under the risk-neutral probability $\mathbb{Q}$ [Björk 2009]. Under this, the process $C_i(t)$ is rewritten as

$$dC_i(t) = rC_i(t)dt + \sigma_iC_i(t)dW_i^{\mathbb{Q}}(t), \tag{15}$$

where $W_i^{\mathbb{Q}}(t)$ is the $\mathbb{Q}$-Brownian motion under the risk-neutral probability $\mathbb{Q}$. Finally, the option price π is given as

$$\begin{aligned}
\pi &= mV(0, \boldsymbol{C}(0); \boldsymbol{F}, T, 1)\\
&= me^{-rT}\mathbb{E}_0^{\mathbb{Q}}[\Phi(\boldsymbol{C}(T))]\\
&= me^{-rT}\big(2\pi T\big)^{-n/2}|\boldsymbol{\Sigma}|^{-1/2}\bigg(\prod_{i=1}^{n}\sigma_i\bigg)^{-1}\\
&\quad\times\int_0^{\infty}\cdots\int_0^{\infty}\frac{\Phi(\widetilde{\boldsymbol{C}})}{\prod_{i=1}^{n}\widetilde{C}_i}\exp\bigg\{-\frac{1}{2}\boldsymbol{\zeta}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{\zeta}\bigg\}d\widetilde{\boldsymbol{C}},
\end{aligned} \tag{16}$$

where $\zeta_i = \frac{1}{\sigma_i\sqrt{T}}[\ln\{\widetilde{C}_i/C_i(0)\} - (r - \frac{\sigma_i^2}{2})T]$. Next, we discuss the solution of Eq. (16) under various situations.

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3.1.2. Solutions and Discussion. If we only have one candidate keyword, $n = 1$, Eq. (16) is actually equivalent to the Black-Scholes-Merton pricing formula for an European call option [Black and Scholes 1973; Merton 1973]; that is,

$$\pi = mC(0)\mathscr{N}[\zeta_1] - mFe^{-rT}\mathscr{N}[\zeta_2], \tag{17}$$

where the notation $\mathscr{N}[\cdot]$ represents the cumulative probability distribution function of a standard normal distribution, where

$$\zeta_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\{C(0)/F\} + (r + \frac{\sigma^2}{2})T\right], \tag{18}$$

$$\zeta_2 = \zeta_1 - \sigma\sqrt{T}. \tag{19}$$

If we have two candidate keywords, $n = 2$, Eq. (16) contains a bivariate normal distribution; π can be obtained by employing the pricing formula for an European dual-strike call option [Zhang 1998] as follows

$$\pi = mC_1(0)\phi_1 + mC_2(0)\phi_2 - me^{-rT}(F_1\phi_3 + F_2\phi_4), \tag{20}$$

where

$$\phi_1 = \int_{-\infty}^{\zeta_1+\sigma_1\sqrt{T}} f(u)\mathscr{N}\left[\frac{q_1(u+\sigma_1\sqrt{T}) - \rho\sigma_1\sqrt{T} + \rho u}{\sqrt{1-\rho^2}}\right]du, \tag{21}$$

$$\phi_2 = \int_{-\infty}^{\zeta_2+\sigma_2\sqrt{T}} f(v)\mathscr{N}\left[\frac{q_2(u+\sigma_2\sqrt{T}) - \rho\sigma_1\sqrt{T} + \rho v}{\sqrt{1-\rho^2}}\right]dv, \tag{22}$$

$$\phi_3 = \int_{-\infty}^{\zeta_1} f(u)\mathscr{N}\left[\frac{q_1(u) + \rho u}{\sqrt{1-\rho^2}}\right]du, \tag{23}$$

$$\phi_4 = \int_{-\infty}^{\zeta_2} f(v)\mathscr{N}\left[\frac{q_2(v) + \rho v}{\sqrt{1-\rho^2}}\right]dv, \tag{24}$$

$$\begin{aligned} q_1(u) = {} & \frac{1}{\sigma_2\sqrt{T}}\left[\ln\left\{\frac{F_2 - F_1 + C_1(0)e^{(r-\frac{1}{2}\sigma_1^2)T - u\sigma_1\sqrt{T}}}{C_2(0)}\right\}\right] \\ & - \frac{1}{\sigma_2\sqrt{T}}\left[(r - \frac{1}{2}\sigma_2^2)T\right], \end{aligned} \tag{25}$$

$$\begin{aligned} q_2(u) = {} & \frac{1}{\sigma_1\sqrt{T}}\left[\ln\left\{\frac{F_1 - F_2 + C_2(0)e^{(r-\frac{1}{2}\sigma_2^2)T - v\sigma_2\sqrt{T}}}{C_1(0)}\right\}\right] \\ & - \frac{1}{\sigma_1\sqrt{T}}\left[(r - \frac{1}{2}\sigma_1^2)T\right], \end{aligned} \tag{26}$$

$$\zeta_1 = \frac{1}{\sigma_1\sqrt{T}}\left[\ln\{C_1(0)/F_1\} + (r - \frac{1}{2}\sigma_1^2)T\right], \tag{27}$$

$$\zeta_2 = \frac{1}{\sigma_2\sqrt{T}}\left[\ln\{C_2(0)/F_2\} + (r - \frac{1}{2}\sigma_2^2)T\right]. \tag{28}$$

Unfortunately, in most our cases, we have more than two candidate keywords ($n \geq 3$), and taking integrals in Eq. (16) is computationally difficult. Thus, we resort to a Monte Carlo method rather than a closed-form solution. The detailed computations are shown in Algorithm 1. The solution generates $\widetilde{n}$ possible price paths for each targeted keyword over the time T horizon (i.e., each path contains $365 \times T$ discrete values). Notice that the generated price paths are correlated under a given risk-neutral

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ALGORITHM 1: Calculating an n-keyword m-click ad option price π by using Monte Carlo method.

Input:
$\boldsymbol{K} = [K_1, \ldots, K_n]$; # the targeted keywords in an option
$\boldsymbol{C}(0) = [C_1(0), \ldots, C_n(0)]$; # the current CPCs of targeted keywords
$\boldsymbol{\Sigma}$; # the correlation matrix of change rates of historical CPCs
$\sigma_i, i = 1, \ldots, n$; # the volatility of historical change rates of CPCs
m; # the maximum number of clicks
T; # the option expiration date (in terms of year, i.e., 365 days)
r; # the constant risk-less bank interest rate
$\widetilde{n}$; # the number of simulations
`// STEP I: Generate CPCs of keywords under a multivariate GBM`
for $k = 1$ *to* $\widetilde{n}$ **do**

$$[z_{1,k}, \ldots, z_{n,k}]' \sim \mathbf{MVN}(0, \boldsymbol{\Sigma});$$

for $i = 1$ *to* n **do**
for $j = 1$ *to* $T \times 365$ **do**

$$C_{i,k,j} \leftarrow C_i(0) \exp\left\{(r - \frac{1}{2}\sigma_i^2)\frac{j}{365} + \sigma_i z_{i,k}\sqrt{\frac{j}{365}}\right\}. \tag{29}$$

`// STEP II: Calculate the option price by approximation`
for $k = 1$ *to* $\widetilde{n}$ **do**

$$G_k \leftarrow \Phi([C_{1,k,T\times 365}, \ldots, C_{n,k,T\times 365}]);$$

$$\pi \leftarrow me^{-rT}\mathbb{E}_0^{\mathbb{Q}}\big[\Phi(\boldsymbol{C}(T))\big] \approx me^{-rT}\left(\frac{1}{\widetilde{n}}\sum_{k=1}^{\widetilde{n}} G_k\right). \tag{30}$$

Output:
π; # the option price

multivariate GBM. As there are n candidate keywords in an option, we have a cube $n\times\widetilde{n}\times(365\times T)$ of generated price data, denoted by $C_{i,k,j}$, where $i = 1, \ldots, n$, $k = 1, \ldots, \widetilde{n}$ and $j = 1, \ldots, 365 \times T$. Then, the option values at time T are calculated and their time-discounted mean value to time $t = 0$ is the option price. The estimation of the parameters and their impact will be discussed in the experiment section later.

3.2. Effects on Search Engine's Revenue

One way of thinking of the ad option is to consider it as an *insurance* for advertisers. They pay an insurance premium (i.e., upfront option price) and obtain the right to purchase keywords with fixed CPCs in the future. Thus, the ad option caps ad costs even if the spot CPCs go up in auction markets. Since this insurance does not come without a cost (i.e., the option price), the ad option is also beneficial to the search engine's revenue.

Let us separately analyze the revenues of a search engine from selling keywords in auctions and via an ad option. Suppose there are n candidate keywords (and each is with a single click) to sell. If the search engine sells the keywords in auctions, his/her total revenue $R^a(T)$ at time T is $\sum_{i=1}^{n} C_i(T)$. If the search engine sells these n candidate keywords via an ad option, his/her total revenue would be

$$R^o(T) = \begin{cases} \pi e^{rT}, & p_0, \\ \pi e^{rT} + F_i + \sum_{j\neq i}^{n} C_j(T), & p_i, i = 1, \ldots, n. \end{cases} \tag{31}$$

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where F_i denotes the agreed fixed CPC for keyword i. The notation p_0 represents the probability that the ad option is not exercised while the p_i is the probability that the ad option is exercised for the keyword K_i for the given spot CPCs $\boldsymbol{C}(T) = \{C_1(T), \ldots, C_n(T)\}$. Hence, the expected difference between these two revenues is

$$\mathbb{E}_t^{\mathbb{Q}}[R^o(T) - R^a(T)] = \pi e^{rT} - \sum_{i=1}^{n} (\mathbb{E}_t^{\mathbb{Q}}[C_i(T)] - F_i)\widetilde{p}_i, \tag{32}$$

where $\widetilde{p}_i$ is the probability that the advertiser exercises the option through the keyword K_i for the given expected spot CPCs $\mathbb{E}_t^{\mathbb{Q}}[\boldsymbol{C}(T)] = \{\mathbb{E}_t^{\mathbb{Q}}[C_1(T)], \ldots, \mathbb{E}_t^{\mathbb{Q}}[C_n(T)]\}$. Eq. (32) can be evaluated numerically as $\mathbb{E}_t^{\mathbb{Q}}[C_i(T)]$ and $\widetilde{p}_i$ are able to be calculated based on simulated paths.

To simplify our discussion, we consider a single keyword, i.e. $n = 1$. If we let $D(F) = \mathbb{E}_t^{\mathbb{Q}}[R^o(T) - R^a(T)]$, denote the difference in revenue, then Eq. (32) can be rewritten as

$$\begin{aligned} D(F) &= C(0)e^{rT}\mathcal{N}[\zeta_1] - F\mathcal{N}[\zeta_2] \\ &\quad - (\mathbb{E}_t^{\mathbb{Q}}[C(T)] - F) \times \mathbb{P}(\mathbb{E}_t^{\mathbb{Q}}[C(T)] \geq F). \end{aligned} \tag{33}$$

We now first discuss the boundary values. First, if $F = 0$, the option price achieves its maximum $e^{-rT}\mathbb{E}_t^{\mathbb{Q}}[C(T)]$, so $D(F) \to 0$. Second, if $\pi = 0$, the fixed CPC F is as large as possible, and the probability $\mathbb{P}(\mathbb{E}_t^{\mathbb{Q}}[C(T)] \geq F) \to 0$ and $D(F) \to 0$. Let us take a closer look at $\mathbb{P}(\mathbb{E}_t^{\mathbb{Q}}[C(T)] >= F)$. We have

$$\begin{aligned} \mathbb{P}(\mathbb{E}_t^{\mathbb{Q}}[C(T)] \geq F) &= \mathbb{P}(C(0)\exp\{(r - \frac{1}{2}\sigma^2)T\} \geq F) \\ &= \mathbb{P}(\ln\{F/C(0)\} - (r - \frac{1}{2}\sigma^2)T \leq 0) \\ &= \mathbb{P}(\ln\{C(T)/C(0)\} - (r - \frac{1}{2}\sigma^2)T \\ &\qquad \leq \ln\{C(0)/F\} + (r - \frac{1}{2}\sigma^2)T + \sigma W(T)) \\ &\approx \mathcal{N}\left[\frac{1}{\sigma\sqrt{T}}[\ln\{C(0)/F\} + (r - \frac{1}{2}\sigma^2)T]\right] \\ &= \mathcal{N}[\zeta_2]. \end{aligned} \tag{34}$$

Substituting Eq. (34) into Eq. (33) gives

$$\begin{aligned} D(F) &= C(0)e^{rT}\mathcal{N}[\zeta_1] - \mathbb{E}_t^{\mathbb{Q}}(C(T))\mathcal{N}[\zeta_2] \\ &\geq C(0)e^{rT}\mathcal{N}[\zeta_1] - C(0)e^{(r-\frac{1}{2}\sigma^2)T}\mathcal{N}[\zeta_1] > 0, \end{aligned} \tag{35}$$

which shows that $D(F)$ is always larger than zero, suggesting that the search engine will increase their revenue if selling a click as an option rather than through auction. We next take the derivative of $D(F)$ with respect to the agreed fixed CPC F and assign its value to zero:

$$\begin{aligned} \frac{\partial D(F)}{\partial F} &= C(0)e^{rT}\frac{\partial \mathcal{N}[\zeta_1]}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial F} - \mathcal{N}[\zeta_2] - F\frac{\partial \mathcal{N}[\zeta_2]}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial F} \\ &\quad - (\mathbb{E}_t^{\mathbb{Q}}[C(T)] - F)\frac{\partial \mathbb{P}(\mathbb{E}_t^{\mathbb{Q}}[C(T)] \geq F)}{\partial F} \\ &\quad + \mathbb{P}(\mathbb{E}_t^{\mathbb{Q}}[C(T)] \geq F) = 0. \end{aligned} \tag{36}$$

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Table I. Overview of experimental settings of data.

Market	Group	Training set (31 days)	Deve&test set (31 days)
US	1	25/01/2012-24/02/2012	24/02/2012-25/03/2012
	2	30/03/2012-29/04/2012	29/04/2012-31/05/2012
	3	10/06/2012-12/07/2012	12/07/2012-17/08/2012
	4	10/11/2012-11/12/2012	11/12/2012-10/01/2013
UK	1	25/01/2012-24/02/2012	24/02/2012-25/03/2012
	2	30/03/2012-29/04/2012	29/04/2012-31/05/2012
	3	12/06/2012-13/07/2012	13/07/2012-19/08/2012
	4	18/10/2012-22/11/2012	22/11/2012-24/12/2012

Since $\partial \mathscr{N}(x)/\partial x = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, the following equation holds

$$\frac{\partial \mathscr{N}[\zeta_2]}{\partial \zeta_2} \Big/ \frac{\partial \mathscr{N}[\zeta_1]}{\partial \zeta_1} = \exp\left\{\frac{1}{2}(\zeta_1^2 - \zeta_2^2)\right\} = \frac{C(0)e^{rT}}{F}. \tag{37}$$

Taking the derivative of ζ_1 and ζ_2 with respect to F gives

$$\frac{\partial \zeta_1}{\partial F} = \frac{\partial \frac{1}{\sigma\sqrt{T}}\left[\ln\{C(0)/F\} + (r + \frac{1}{2}\sigma^2)T\right]}{\partial F} = -\frac{1}{F\sigma\sqrt{T}}, \tag{38}$$

$$\frac{\partial \zeta_2}{\partial F} = \frac{\partial \zeta_1}{\partial F} - \frac{\partial \sigma\sqrt{T}}{\partial F} = -\frac{1}{F\sigma\sqrt{T}}. \tag{39}$$

Substituting Eqs (34), (37), (38) and (39) into Eq. (36) shows that $D(F)$ achieves its maximum or minimum value if $F = \mathbb{E}_t^{\mathbb{Q}}[C(T)]$. Further, taking the second derivative of $D(F)$ at the point $F = \mathbb{E}_t^{\mathbb{Q}}[C(T)]$ gives

$$\begin{aligned}\frac{\partial^2 D(F)}{\partial F^2} &= \frac{\partial \mathbb{P}(\mathbb{E}_t^{\mathbb{Q}}[C(T)] \geq F)}{\partial F} = \frac{\partial \mathscr{N}[\zeta_2]}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial F} \\ &= -\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\zeta_2{}^2} \frac{1}{F\sigma\sqrt{T}} < 0,\end{aligned} \tag{40}$$

which shows that $D(F)$ has a maximum value when the fixed price $F = \mathbb{E}_t^{\mathbb{Q}}[C(T)]$. Thus, if the search engine sets the agreed fixed CPC, $F = \mathbb{E}_t^{\mathbb{Q}}[C(T)]$, then the difference in revenue between the option and auction is maximized. In Section 4.4, we experimentally examine the revenue differences between the keyword option and auction under various settings.

4. EXPERIMENTS

We first examine if the proposed ad option can be fairly priced with real sponsored search data. We then investigate the effects of the option on a search engine's revenue.

4.1. Data and Experimental Design

Our data[2] are collected from Google AdWords [Google 2013] by using its Traffic Estimation service. When an advertiser submits his/her ad keywords, budget, and other settings (e.g., keyword match type and targeted ad location), the service will return a list of sponsored search daily data, including estimated CPC, clicks, global and local impressions and position. This information is recorded for the period from 26/11/2011

[2]The raw data is available at:
http://www.computational-advertising.org [Yuan and Wang 2012].

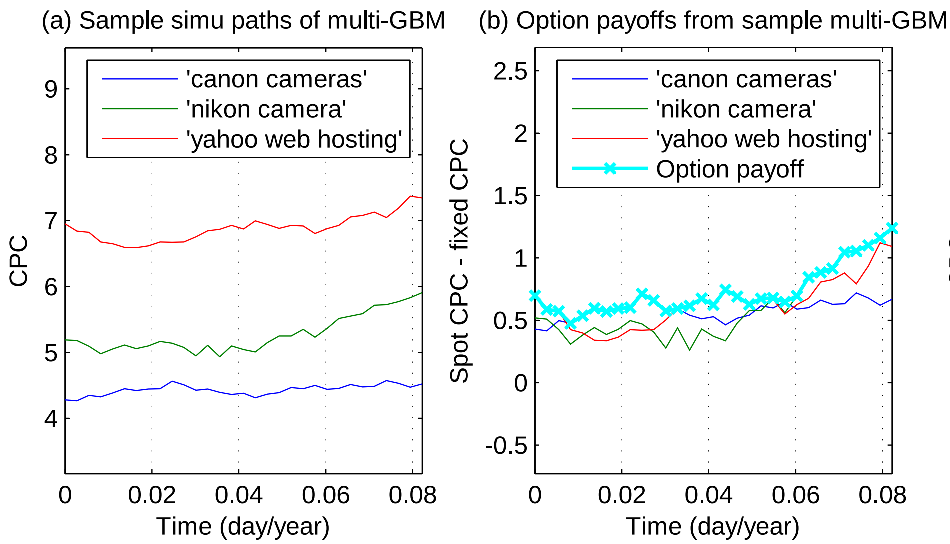

Fig. 2. Empirical example of generating sample paths under GBM for a 3-keyword single-click option and calculating corresponding payoffs.

to 14/01/2013, for a total of 557 keywords across US and UK markets (in which 21 keywords have missing data and 115 keywords all have zero CPCs).

The data of each market is split into 4 experimental groups and each group has one training, one development, and one test set, as illustrated in Table I. The training set is used to: (i) select the keywords with non-zero CPCs; (ii) test the statistical properties of the underlying dynamic and estimate the model parameters. We then price ad options and simulate the buying and selling transactions in the development set. Finally, the test set is used as the baseline to examine the developed option pricing models.

4.2. Parameter Estimation and Option Pricing

To estimate the GBM parameters, we use the method suggested by Wilmott (Sec. 11.3) [Wilmott 2006]. For the ad keyword K_i, the volatility σ_i is the sample standard deviation of change rates of log CPCs and the correlation is

$$\rho_{ij} = \frac{\sum_{k=1}^{\widetilde{m}} \big(y_i(k) - \bar{y}_i\big)\big(y_j(k) - \bar{y}_j\big)}{\sqrt{\sum_{k=1}^{\widetilde{m}} \big(y_i(k) - \bar{y}_i\big)^2 \sum_{k=1}^{\widetilde{m}} \big(y_j(k) - \bar{y}_j\big)^2}}, \tag{41}$$

where $\widetilde{m}$ is the size of the training data and $y_i(t_k)$ is the kth change rate of log CPCs. Figure 2 illustrates an empirical example of the keywords 'canon cameras', 'nikon camera' and 'yahoo web hosting', where the parameters are estimated as follows

$$\boldsymbol{\sigma} = \begin{pmatrix} 0.2263 \\ 0.4521 \\ 0.2136 \end{pmatrix}, \ \boldsymbol{\Sigma} = \begin{pmatrix} 1.0000 & 0.2341 & 0.0242 \\ 0.2341 & 1.0000 & -0.0540 \\ 0.0242 & -0.0540 & 1.0000 \end{pmatrix}.$$

A high contextual relevance of ad keywords normally means that they have a high substitutional degree to each other, such as 'canon cameras' and 'nikon camera', whose CPCs move in the same direction with correlation 0.2341. The other keyword 'yahoo web hosting' is contextually less relevant to the formers and also has very low price correlations to them. This example shows that the contextual relevance of ad keywords

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Table II. Testing arbitrage of options under the GBM: n is the number of keywords, N is the number of options priced in a group, $\mathbb{P}(\alpha)$ is % of options in a group identified arbitrage, and the $\mathbb{E}[\alpha]$ is the average arbitrage value of the options identified arbitrage, where the arbitrage α is defined by Eq. (44).

n	Group	US market			UK market		
		N	$\mathbb{P}(\alpha)$	$\mathbb{E}[\alpha]$	N	$\mathbb{P}(\alpha)$	$\mathbb{E}[\alpha]$
1	1	94	0.00%	0.00%	76	0.00%	0.00%
	2	64	0.00%	0.00%	45	0.00%	0.00%
	3	94	1.06%	0.75%	87	0.00%	0.00%
	4	112	0.89%	-0.37%	53	0.00%	0.00%
2	1	47	4.26%	1.63%	38	0.00%	0.00%
	2	32	9.38%	0.42%	22	4.55%	13.41%
	3	47	4.26%	0.84%	43	4.65%	0.82%
	4	56	5.36%	3.44%	26	23.08%	-6.22%
3	1	31	0.00%	0.00%	25	4.00%	0.00%
	2	21	4.76%	-1.38%	15	0.00%	0.00%
	3	31	0.00%	0.00%	29	3.45%	-1.12%
	4	37	10.81%	3.87%	17	35.29%	-2.54%

has an impact on the keywords CPCs movement. Based on the estimated parameters, we draw a sample of simulated paths of 3-dimensional GBM in Figure 2(a) for 31 days (where the x-axis is expressed in terms of year value). Recall that the option payoff at any time t in the contract lifetime is $\max\{C_1(t) - F_1, \ldots, C_n(t) - F_n, 0\}$. In Figure 2(b), we plot the price difference between the spot CPC and the fixed CPC of each targeted keyword (i.e., $C_i(t) - F_i$, $i = 1, \ldots, n$) and also indicate the corresponding option daily payoffs (shown by the cyan curve) where $F_1 = 3.8505$, $F_2 = 4.6704$ and $F_3 = 6.2520$. Figure 2(b) suggests that switching between keywords would help advertisers maximise the benefits of an ad option. Repeating the above simulations 100 times generates 100 simulated vales of each keyword for each day, and their daily mean values are calculated. We also calculate 100 option payoffs and calculate their daily mean values to obtain the final option price according to Algorithm 1.

To examine the fairness, i.e. no-arbitrage, of the calculated option price, we construct a risk-less advertising strategy by delta hedging $\partial V/\partial C_j$ and check if any arbitrage exists. Since a single-keyword single-click option can be priced by the Black-Scholes-Merton European option model [Black and Scholes 1973; Merton 1973] (see Eq. (17)), the hedging delta is

$$\Delta = \frac{\partial V}{\partial C} = \mathscr{N}\left[\frac{1}{\sigma\sqrt{T}}\left(\ln\left\{\frac{C(0)}{F}\right\} + (r + \frac{\sigma^2}{2})T\right)\right]. \tag{42}$$

For the multi-keyword single-click option, the hedging delta of each keyword can be computed by the Monte Carlo method, i.e., $\partial V/\partial C_i = \mathbb{E}^{\mathbb{Q}}[\partial V(T, C(T))/\partial C_i(T)]$. To examine the arbitrage, we compare the risk-less bank investment return to the actual advertising strategy investment return.

Table II presents the overall results of our arbitrage test under the GBM, in which the continuous compounded risk-less bank interest rate $r = 5\%$ (equivalent to $\widetilde{r} = 4.12\%$ per 31 days return[3]). We generate 100 simulated paths for each keyword and examine the options using delta hedging. Let $\widehat{r}$ be the return of advertising strategy over 31 days. The arbitrage detection criteria is

$$\widehat{r} \leq |\widetilde{r} \pm \varepsilon| \ ? \ \textbf{arb doesn't exist : arb exists}, \tag{43}$$

[3]The relationship between the continuous compounding r and the return per 31 days $\widetilde{r}$ is: $1 + \widetilde{r} = e^{r \times 30/365}$. For detail information, please refer to Hull (sec 4.2 [Hull 2009]).

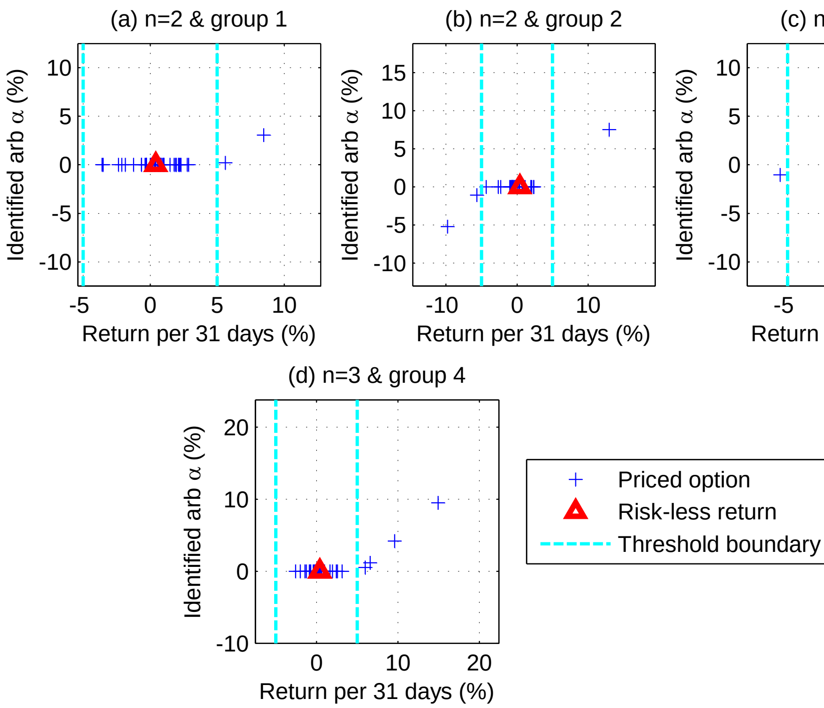

Fig. 3. Empirical example of the arbitrage analysis under GBM dynamic for the US market.

where the notation ε is the model variation threshold (where we set $\varepsilon = 5\%$ in our experiments). We define the identified arbitrage α as the excess return between $\widehat{r}$ and $|\widetilde{r} \pm \varepsilon|$. Therefore, if arbitrage exists, we have

$$\alpha = \begin{cases} \widehat{r} - (\widetilde{r} + \varepsilon), & \text{if } \widehat{r} \geq \widetilde{r} + \varepsilon, \\ \widehat{r} - (\widetilde{r} - \varepsilon), & \text{if } \widehat{r} \leq \widetilde{r} - \varepsilon, \end{cases} \tag{44}$$

where a positive α means that the advertiser who buys an option can obtain guaranteed profits while the negative α indicates the case of making profits by selling an option. As shown in Table II, there are 99.76% (1-keyword), 93.06% (2-keyword) and 92.71% (3-keyword) options fairly priced if GBM is valid. Only a small number of options exhibits arbitrage and most of the mean arbitrage values lie within 5%, such as shown in Figure 3.

The existence of small arbitrage in our tests under the GBM dynamics may be due to the following two reasons: (i) the stability of process simulations in both option pricing and arbitrage test; (ii) the ad keywords are randomly selected for the 2 or 3-keyword options, so a significant difference of keywords CPCs generates a large variation of calculated option payoffs that triggers a certain arbitrage.

4.3. Model Validation and Robustness Test

We now investigate if the GBM assumptions are satisfied by all keywords and examine if there are and how many arbitrage exists in the priced options for non-GBM keywords.

4.3.1. Checking the Underlying GBM Assumptions. There are two GBM assumptions that need to be verified [Marathe and Ryan 2005]: (i) normality of change rates of log

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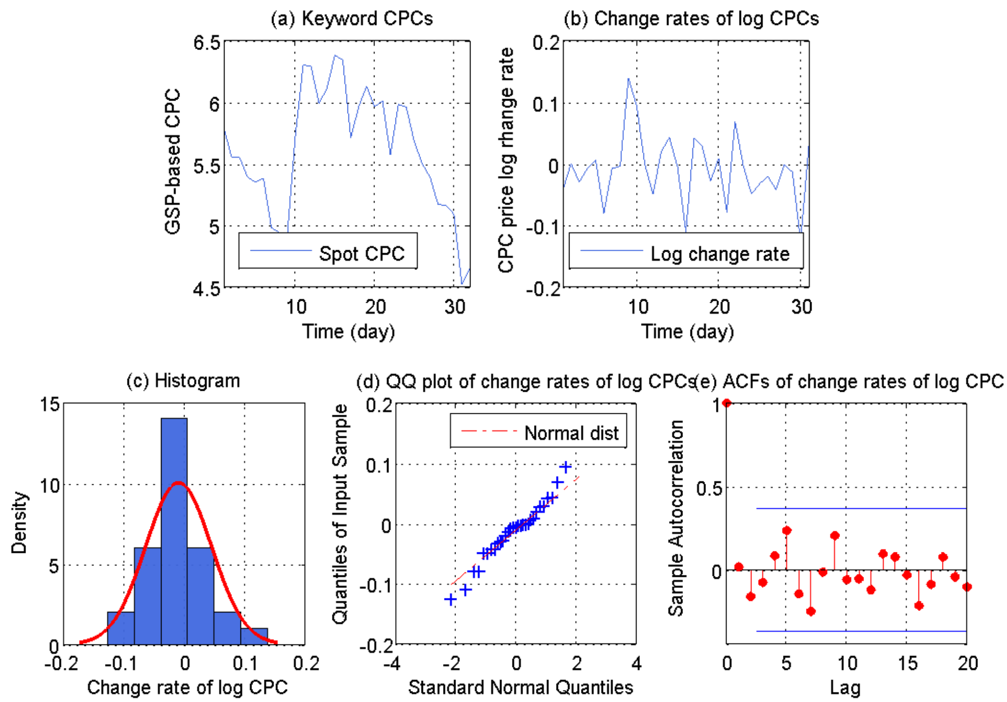

Fig. 4. Empirical example of the GBM assumptions checking for the keyword 'insurance', where the Shapiro-Wilk test is with p-value 0.2144 and the Ljung-Box test is with p-value 0.6971.

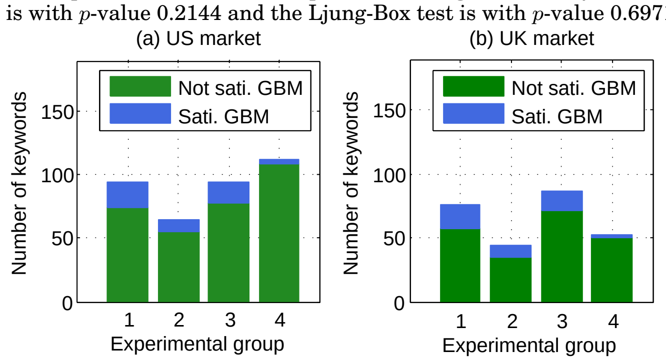

Fig. 5. Overview of the GBM assumptions checking for all ad keywords of experimental groups of both US and UK markets.

CPCs; and (ii) independence from previous data. Normality can be either graphically checked by histogram/Q-Q plot or statistically verified by the Shapiro-Wilk test [Shapiro and Wilk 1965]. To examine independence, we employ the autocorrelation function (ACF) [Tsay 2005] and the Ljung-Box statistic [Ljung and Box 1978]. To illustrate the procedure, Figure 4 gives an example of the keyword 'insurance'. Figure 4 (a)-(b) exhibit the movement of CPCs and log change rates while Figure 4 (c)-(d) show that the stated two assumptions are satisfied in this case.

The GBM assumptions were checked for the training data of all ad keywords. As shown in Figure 5, there are 14.25% and 17.20% of keywords in US and UK markets that satisfy the GBM, respectively. Thus about 15.73% of keywords that can be effectively priced into an option contract under the assumption of GBM dynamics. It is

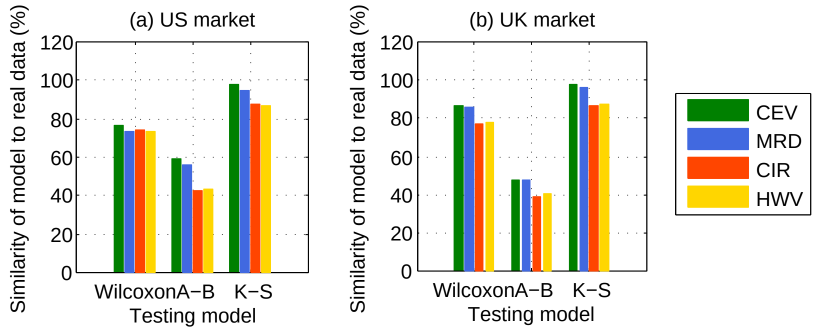

Fig. 6. Overview of model similarity testing: Wilcoxon test, Ansari-Bradley (A-B) test and Two-sample Kolmogorov-Smirnov (K-S) test.

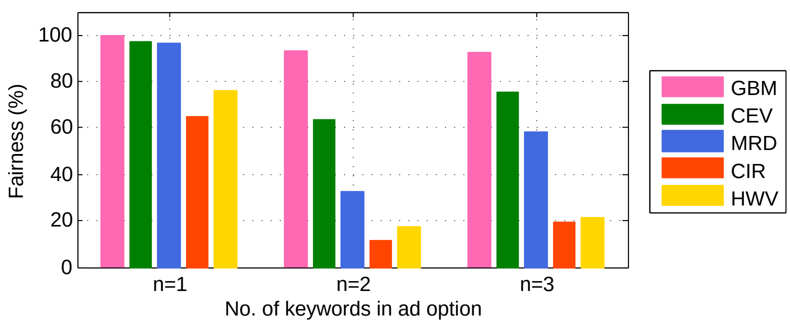

Fig. 7. Overview of pricing model robust tests.

worth mentioning that not all keywords follow the GBM. We use the GBM dynamics in our model mainly for the mathematical convenience; but nonetheless, we give our investigation on the arbitrage opportunities under non-GMB dynamics in order to test how robust of our pricing model is for non-GMB keywords.

4.3.2. Examining Arbitrage for Non-GBM Dynamics. We now test several popular stochastic processes (together with the real data) to check the arbitrage from the options of non-GBM keywords. Table III shows the candidate dynamics (the CEV, the MRD, the CIR and HWV models). Each dynamic model represents a certain feature of time series data, such as mean-reverting, constant volatility and square root volatility [Hull 2009]. The arbitrage tests here are slightly different from that of GBM. We estimate the dynamic parameters from the data in the test sets instead of the learning sets and treat the real data as one single path of the dynamics. Therefore, the simulated data has the same drift, volatility and correlations as the real test data. Since the simulation parameters are significantly different to the pricing parameters, we are able to examine the arbitrage multiple times when the real-world environment does not follow the GBM. Also for the candidate dynamics, several hypothesis tests have been employed to check if the simulated path and real data come from a same distribution. These tests include the Wilcoxon test [Wilcoxon 1945], Ansari-Bradley test [Mood et al. 1974] and Two-sample Kolmogorov-Smirnov test [Justel et al. 1997]. Figure 6 summarizes results of the dynamics's fitness testing, where the y-axis represents the mean percentage of simulated paths not rejected by the hypothesis tests. Even though the three tests give different absolute percentages, the dynamics' performance is similar,

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i.e. the CEV model has the best simulations of the real data, followed by the MRD model. The CIR and HWV models are very close.

Table IV presents the arbitrage testing results for non-GBM dynamics, where most of experimental groups exhibit arbitrage. The CEV model gives the best no-arbitrage performance, showing that 78.65% of CEV-based keywords can be fairly priced by using the GBM-based option pricing model. About 53.05% of CIR and about 43% of MRD or HWV based options have no arbitrage. For single-keyword options, the fairness percentage is more than 85% across all groups. However, this figure drops to around 38% for multi-keyword options (36.27% for 2-keyword options and 42% for 3-keyword options). For the identified arbitrage, many groups (especially single-keyword options) show small arbitrage values around 10% while arbitrage explodes in some groups.

In summary, Tables II and IV have illustrated that our option pricing model is effective and robust for the real sponsored search data. As in Figure 7, when the keywords satisfy the GBM (15.73%), the pricing model ensures that 95.17% options are fairly priced under 5% arbitrage precision. For the non-GBM keywords, the best CEV model gives 78.65% fairness while the worst CIR model is with 31.97%. Overall, the best expected fairness of option pricing for all keywords is 81.25% while the worst is 41.91%. Also, the increase of the number of ad keywords in an ad option increases the likelihood of arbitrage. This is confirmed by the fact that expected fairness drops from 86.83% (99.76% GBM and 83.60% non-GBM for single-keyword options) to 43.69% (2-keyword options) and 53.39% (3-keyword options), respectively.

4.4. Effects of Ad Options to Search Engine

We continue the discussion from Section 3.2 and investigate the effects of ad options on the search engine's revenue.

Let's start with the case of single-keyword options. The example of keyword 'equity loans' in Figure 8(a) illustrates (other keywords exhibit the similar pattern) the conclusions from our theoretical analysis in Section 3.2 that (i) the revenue difference between option and auction is always positive and (ii) that when the fixed CPC $F = \mathbb{E}_t^{\mathbb{Q}}[C(T)] = 4.5022$, the revenue difference $D(F)$ achieves its maximum (the corresponding option price $\pi = 0.1088$) and the two boundary values are approximately zero.

We further examine non-GBM cases. Figure 8(b)-(e) shows that when the fixed CPC is close to zero, the revenue difference $D(F) \to 0$. This is because when the fixed CPC approximates zero, it is almost certain that the option will be used in the contract period. As such, the only income for the keyword is from the option price, which in this case is close to the CPC in the auction market (discounted back to t=0). On the other hand, if the fixed CPC is very high, it is almost certain that the option won't be used. In this case, the option price $\pi \to 0$ and the probability of exercising the option $\mathbb{P}(\mathbb{E}_t^{\mathbb{Q}}[C(T)] \geq F) \to 0$. Hence, $D(F)$ would be zero. However, under the non-GBM dynamics, the point $F = \mathbb{E}_t^{\mathbb{Q}}[C(T)]$ is not the optimal value that gives the maximum $D(F)$, which indicates that arbitrage may occur.

Table III. Tested non-GBM dynamics: $k_i = 0.5$ while other parameters are learned from the training data.

Dynamic	Stochastic differential equation (SDE)
Constant elasticity of variance (CEV) model [Cox and Ross 1976]	$dC_i(t) = \mu_i C_i(t)dt + \sigma_i(C_i(t))^{1/2}dW_i(t)$
Mean-reverting drift (MRD) model [Hull 2009; Wilmott 2006]	$dC_i(t) = k_i(\mu_i - C_i(t))dt + \sigma_i(C_i(t))^{1/2}dW_i(t)$
Cox-Ingersoll-Ross (CIR) model [Cox et al. 1985]	$dC_i(t) = k_i(\mu_i - C_i(t))dt + (\sigma_i)^{1/2}C_i(t)dW_i(t)$
Hull-White/Vasicek (HWV) model [Hull 2009]	$dC_i(t) = k_i(\mu_i - C_i(t))dt + \sigma_i dW_i(t)$

Table IV. Overview of delta hedging arbitrage testing for non-GBM dynamics: same notations as in Table II.

Market	n	Group	N	real data + CEV simu		real data + MRD simu		real data + CIR simu		real data + HWV simu	
				$\mathbb{P}$(arb)	Mean arb	$\mathbb{P}$(arb)	Mean arb	$\mathbb{P}$(arb)	Mean arb	$\mathbb{P}$(arb)	Mean arb
US	1	1	74	2.70%	-1.97%	8.11%	-0.38%	75.68%	1.94%	56.76%	-0.01%
		2	55	0.00%	0.00%	1.82%	1.80%	16.36%	1.86%	9.09%	-0.93%
		3	77	9.09%	-10.26%	5.19%	-7.93%	42.86%	0.85%	23.38%	-1.51%
		4	108	3.70%	1.38%	2.78%	4.27%	7.41%	3.99%	8.33%	2.84%
	2	1	37	24.32%	3.63%	81.08%	-4.09%	97.30%	-16.24%	97.30%	-13.24%
		2	27	37.04%	6.01%	70.37%	5.36%	85.19%	11.01%	85.19%	10.51%
		3	38	31.58%	5.97%	31.58%	-0.41%	73.68%	-6.91%	57.89%	-5.96%
		4	54	29.63%	5.95%	81.48%	6.61%	94.44%	16.78%	94.44%	16.25%
	3	1	24	45.83%	-1.36%	79.17%	-6.04%	100.00%	-19.99%	100.00%	-17.44%
		2	18	11.11%	-2.00%	22.22%	-5.11%	55.56%	-4.39%	72.22%	-2.22%
		3	25	24.00%	7.01%	32.00%	-3.71%	84.00%	-11.14%	76.00%	-10.26%
		4	36	16.67%	3.91%	30.56%	2.66%	83.33%	2.73%	88.89%	3.38%
UK	1	1	58	0.00%	0.00%	0.00%	0.00%	74.14%	1.95%	55.17%	-1.60%
		2	35	0.00%	0.00%	2.86%	1.51%	22.86%	1.65%	14.29%	2.29%
		3	72	5.56%	-5.78%	1.39%	-10.38%	29.17%	1.03%	18.06%	-0.32%
		4	50	4.00%	5.55%	6.00%	4.47%	10.00%	4.56%	8.00%	3.45%
	2	1	29	37.93%	-0.76%	62.07%	-5.52%	89.66%	-14.55%	72.41%	-11.39%
		2	17	47.06%	2.18%	82.35%	5.00%	100.00%	9.87%	100.00%	8.62%
		3	36	19.44%	2.71%	33.33%	-1.78%	75.00%	-5.24%	61.11%	-3.58%
		4	25	64.00%	6.71%	96.00%	9.01%	92.00%	21.17%	92.00%	20.04%
	3	1	19	26.32%	-1.56%	84.21%	-5.21%	100.00%	-16.33%	78.95%	-16.34%
		2	11	18.18%	0.40%	18.18%	-1.09%	63.64%	-1.28%	63.64%	-1.05%
		3	24	16.67%	3.45%	25.00%	-1.61%	79.17%	-9.14%	66.67%	-9.23%
		4	16	37.50%	7.83%	43.75%	7.37%	81.25%	0.68%	81.25%	8.64%

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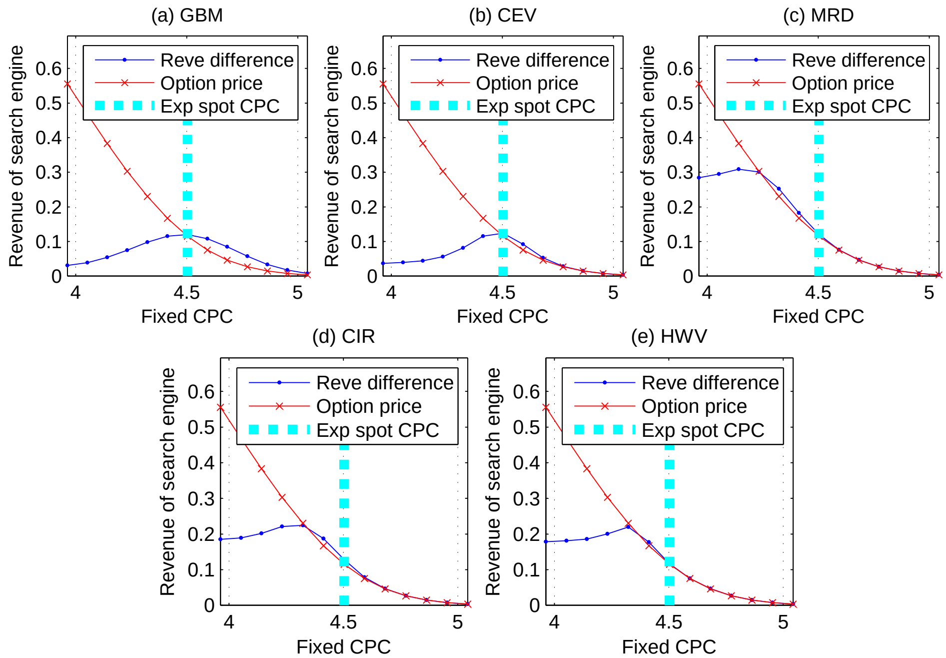

Fig. 8. Empirical example of search engine's revenue for the keyword 'equity loans'.

Next, Figure 9 illustrates the case where we have 2 candidate keywords: 'iphone4' and 'dar martens', where $r = 5\%$ and $\rho = 0.0259$. In Figure 9(a), we see that the higher the fixed CPCs the lower is the option price. This property is the same as for the single-keyword options. Also, the calculated option price achieves maximum when all the fixed CPCs are zeros. Figure 9(b) then shows the revenue difference curve of the search engine, where the red star represents the value when $F_1 = \mathbb{E}_t^{\mathbb{Q}}[C_1(T)]$ and $F_2 = \mathbb{E}_t^{\mathbb{Q}}[C_2(T)]$. The expected revenue differences are all above zero, showing that this 2-keyword ad option is beneficial to the search engine's revenue. However, an interesting point to discuss is that the red star point is not the maximum difference revenue, which is different from single-keyword options. This may be due to the fact that the underlying CPCs move in a correlated manner and the advertiser switches his/her exercising from one to another. The revenues' difference curve in Figure 9(b) is very smooth while Figure 10(b) shows a bit volatile pattern because the underlying correlation increases. Above all, the properties of the revenue difference are similar to those of single-keyword options and they are all positive.

It would be impossible to graphically examine the revenue difference for higher dimensional ad options (i.e., $n \geq 3$). However, based on the earlier discussions, we can summarize two properties. First, there are boundary values of the revenue differences. If every $F_i \to 0$, $D(\mathbf{F}) \to 0$; if every $F_i \to \infty$, $D(\mathbf{F}) \to 0$. Second, there exists a maximum revenue difference value even though this may not at the point $F_i = \mathbb{E}_t^{\mathbb{Q}}[C_i(T)]$. Overall, we are able to say that a proper setting of fixed CPCs by a search engine can increase the ad revenue compared to keyword auctions.

5. CONCLUDING REMARKS

In this paper, we proposed a new ad selling mechanism for sponsored search that benefits both advertisers and search engine. On the one hand, advertisers are able to secure

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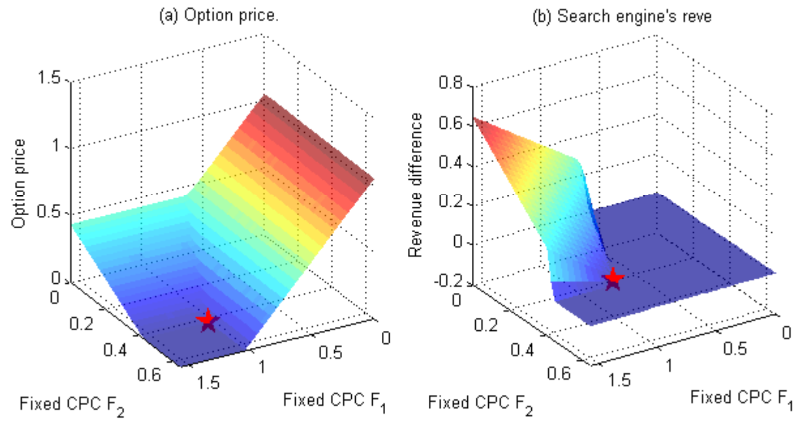

Fig. 9. Empirical example of option pricing and search engine's revenue for the keywords 'iphone4' and 'dr martens', where $\rho = 0.0259$.

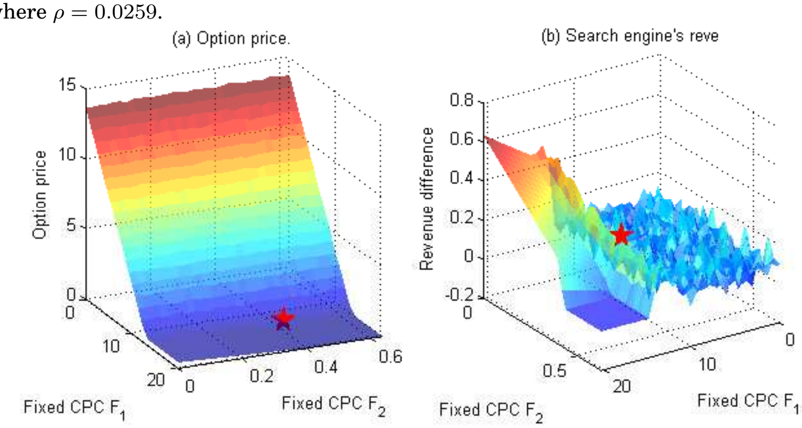

Fig. 10. Empirical example of option pricing and search engine's revenue for the keywords 'non profit debt consolidation' and 'canon 5d', where $\rho = 0.2247$.

ad service delivery in the future and can be released from auction campaigns, thereby reducing uncertainty in the cost of a campaign. On the other hand, a search engine benefits by (i) being able to sell the inventory of ad slots in advance and (ii) generating a more stable and predictable revenue over a long-term period. The search engine may also increase customer (advertisers) loyalty through contractual relationships, which has the potential to improve revenue further. Thus, we believe the proposed ad option contract is a good complement to keyword auctions for sponsored search.

There are three major limitations of our ad option framework. First, like all methods based on the GBM [Samuelson 1973; Marathe and Ryan 2005], the price movement of the underlying ad keywords may not follow GBM dynamics. Some time series features, such as price jumps and volatility clustering, cannot be captured effectively. However, as the first ad option framework, our experimental results have shown that the GBM assumption is reasonably robust to deviations from this assumption. Second, other model assumptions, such as infinitely divisible clicks, continuous-time perfect delta hedging, constant bank interest rate etc, are borrowed from the *perfect market* settings in economics [Black and Scholes 1973; Merton 1973], but some of these assumptions may not be valid in a real-world sponsored search environment. Third, we did not explicitly cover the modelling of the supply and demand into the framework;

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our economic objective in this paper has been focused on reducing the risk of price movements and defining the "fair" price in that regard. One can certainly look at the problem from other perspectives; for instance, regarding the pricing as an optimisation problem based on the supply and demand and their impact on the prices [Gallego and Van Ryzin 1994].

Our work leaves several directions for future research. A stochastic process tailored to some specific ad keywords is worth studying (e.g., jump-diffusion models and volatility models). Also, dynamically pricing and allocating ad options (with considering the market changes) may further consolidate our work as we didn't discuss how to manipulate the limited inventory in front of the uncertain demand [Gallego and Van Ryzin 1994]. In addition, to address the competitions among advertisers who have the similar campaign needs, pricing ad options based on a game-theoretic perspective may be of interest.

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