

TOWARDS SUPREMUM-SUM SUBDIFFERENTIAL CALCULUS
FREE OF QUALIFICATION CONDITIONS*R. CORREA[†], A. HANTOUTE[‡], AND M. A. LÓPEZ[§]

Abstract. We give a formula for the subdifferential of the sum of two convex functions where one of them is the supremum of an arbitrary family of convex functions. This is carried out under a weak assumption expressing a natural relationship between the lower semicontinuous envelopes of the data functions in the domain of the sum function. We also provide a new rule for the subdifferential of the sum of two convex functions, which uses a strategy of augmenting the involved functions. The main feature of our analysis is that no continuity-type condition is required. Our approach allows us to unify, recover, and extend different results in the recent literature.

Key words. sum and pointwise supremum of convex functions, Fenchel and approximate sub-differentials

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1. Introduction. Let X be a locally convex topological vector space, $f_t : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$, $t \in T$, be a nonempty family of extended real-valued proper convex functions, indexed by a (nonnecessarily finite) set T , and $g : X \rightarrow \overline{\mathbb{R}}$ be a convex function. If we denote by $f : X \rightarrow \overline{\mathbb{R}}$ the pointwise supremum function,

$$(1) \quad f := \sup_{t \in T} f_t,$$

our purpose is to construct the Fenchel subdifferential set of the sum $f + g$ by means exclusively of the subdifferentials of the data functions f_t and g . Our first main result, provided in Theorem 4 under a natural closure condition satisfied by the lower semicontinuous (lsc) envelopes of the functions f_t and g , states that

$$(2) \quad \partial(f + g)(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{L \cap \text{dom } f})(x) \right\},$$

where $\mathcal{F}(x)$ is the family of finite-dimensional subspaces containing x , $T_\varepsilon(x)$ is the set of ε -active indices at x , and $\delta_{L \cap \text{dom } f}$ is the indicator function of $L \cap \text{dom } f$. The intersection over L in (2) is omitted in many special cases, particularly in the finite-dimensional setting, where $X \in \mathcal{F}(x)$, and in the infinite-dimensional setting when

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the domain of the function $f + g$ fulfills some interiority condition (see Corollary 8 below). Obviously, the formula above covers both the sum of two functions and the supremum of functions. For instance, for $f_t \equiv f$, for all $t \in T$, we establish a new formula for the subdifferential of the sum, while for $g \equiv 0$ we obtain a formula for the subdifferential of the supremum function, which improves the one given in [9, Theorem 4]. In the particular case when g is the indicator function of a convex set, this problem has been approached by using different techniques in [16, Theorem 3.2].

Let us remark that formula (2) cannot be obtained just by applying consecutively first the subdifferential rule for the sum and next the rule for the pointwise supremum. It is true that in some specific cases, for instance, when all the data functions are lsc, one may express $\partial(f + g)(x)$ in terms of $\partial_\varepsilon(\sup_{t \in T} f_t)(x) + \partial_\varepsilon g(x)$, $\varepsilon > 0$, but this approach fails since there is no explicit characterization of $\partial_\varepsilon(\sup_{t \in T} f_t)(x)$.

In a second stage, by exploiting an idea of augmenting the functions, we establish the second main result of this paper, Theorem 5, which gives rise to a representation of the subdifferential of the sum of two proper convex functions using the (exact) subdifferential of them.

Concerning the subdifferential of the supremum function, there are different achievements in the literature, dealing with various situations depending on the structure of the space X , the algebraic/topological properties of the index set T , the continuity of the supremum function f defined in (1), and the two-variables function $h(x, t) := f_t(x)$, etc. See, for instance, [1, 6, 7, 9, 19, 21, 23], among many others. More specifically, provided that X is a normed space and the supremum function f is finite and continuous at x , a remarkable result due to Valadier [22] asserts that the subdifferential of f at x is completely characterized by means of the subdifferential of the data functions f_t at nearby points; that is,

$$\partial f(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x), y \in B_\varepsilon(x)} \partial f_t(y) \right\},$$

where $B_\varepsilon(x)$ is the ball centered at x of radius ε . In the particular case when T is a compact topological space, X is locally convex, and the function h is continuous in $U \times T$ for some open set $U \subset X$, Valadier showed (see, e.g., [15, Theorem 6.4.9]) that, for every $x \in U$,

$$\partial f(x) = \overline{\text{co}} \left\{ \bigcup_{t \in T_0(x)} \partial f_t(x) \right\}.$$

More recently, using the ε -subdifferential in the setting of locally convex spaces, it has been established in [9, Theorem 4] that

$$(3) \quad \partial f(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + N_{L \cap \text{dom } f}(x) \right\},$$

where the functions f_t are assumed to satisfy the closure condition

$$\overline{f} = \sup_{t \in T} \overline{f}_t$$

(here, \overline{f} and \overline{f}_t denote the lsc envelopes of the respective functions). The intersection over L in the previous formula is omitted when $\text{ri}(\text{dom } f)$ is nonempty [9, Lemma 3]

and we obtain

$$(4) \quad \partial f(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + N_{\text{dom } f}(x) \right\}.$$

On the other hand, if T is finite and $T = T_0(x)$ (that is, all the functions are active at x), then the foregoing formula (3) reduces to [9, Corollary 12]

$$(5) \quad \partial f(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T} \partial_\varepsilon f_t(x) \right\}.$$

This characterization is the well-known Brøndsted formula given in [1] for lsc proper convex functions. This was the first rule for subdifferential calculus without qualification conditions. Other exact rules have been given under qualification conditions; for instance, assuming that all but one of the functions are continuous at a point of the domain of the others, we have [18]

$$\partial f(x) = \text{co} \left\{ \bigcup_{t \in T_0(x)} \partial f_t(x) \right\} + N_{\text{dom } f}(x).$$

Concerning the subdifferential of the sum of two proper convex functions $\varphi, \psi : X \rightarrow \overline{\mathbb{R}}$, the simplest rule stating that

$$(6) \quad \partial(\varphi + \psi)(x) = \partial\varphi(x) + \partial\psi(x),$$

requires qualification conditions, in particular, those introduced by Moreau–Rockafellar, Robinson, Attouch–Brézis (in Banach spaces), among others (see, e.g., [23, Theorem 2.8.7]). In the absence of a qualification condition, the following rule [11] (see, also, [10]) always holds provided that the functions are lsc:

$$(7) \quad \partial_\varepsilon(\varphi + \psi)(x) = \bigcap_{\delta > 0} \text{cl} \left(\bigcup_{\substack{\varepsilon_1 + \varepsilon_2 \leq \varepsilon + \delta \\ \varepsilon_1, \varepsilon_2 \geq 0}} \partial_{\varepsilon_1} \varphi(x) + \partial_{\varepsilon_2} \psi(x) \right).$$

Indeed, this formula also holds if the lsc requirement is relaxed to the following condition (see [8, Theorem 13] for the case $\varepsilon = 0$)

$$(8) \quad \overline{\varphi + \psi} = \overline{\varphi} + \overline{\psi}.$$

For instance, according to [5, Lemma 11], the last equality holds provided that

$$\text{ri}(\text{dom } \varphi) \cap \text{ri}(\text{dom } \psi) \neq \emptyset,$$

and the restrictions of the functions φ and ψ to the affine hulls of their corresponding domains are continuous relative to $\text{ri}(\text{dom } \varphi)$ and $\text{ri}(\text{dom } \psi)$, respectively. Other calculus rules can be found in [2, 3, 13, 20], among others.

Related closedness conditions will be crucial in this paper, and they are addressed to relax the lsc property. For instance, it has been shown in [5] (see Propositions 1 and 2 below) that for every $x \in X$

$$\partial(\varphi + \psi)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon \varphi(x) + \partial\psi(x)),$$

provided that the following asymmetric conditions hold:

$$\overline{\varphi + \psi} = \overline{\varphi} + \psi, \quad \text{dom } \varphi \cap \text{ri}(\text{dom } \psi) \neq \emptyset,$$

and the restriction of ψ to the affine hull of its domain is continuous on $\text{ri}(\text{dom } \psi)$.

One way to avoid the requirement of qualification conditions consists of augmenting the involved functions, an idea which is intensively exploited throughout this paper (see, for instance, Theorems 4 and 5). To anticipate this approach let us analyze here the finite-dimensional case. The classical Rockafellar result [17, Theorem 23.8] asserts that the condition

$$\text{ri}(\text{dom } \varphi) \cap \text{ri}(\text{dom } \psi) \neq \emptyset$$

ensures the fulfillment of (6). Then, if we introduce the augmented functions

$$\tilde{\varphi} := \varphi + \delta_{\text{dom } \psi}, \quad \tilde{\psi} := \psi + \delta_{\text{dom } \varphi},$$

it follows that

$$\text{ri}(\text{dom } \tilde{\varphi}) \cap \text{ri}(\text{dom } \tilde{\psi}) = \text{ri}(\text{dom } \varphi \cap \text{dom } \psi),$$

which is nonempty, provided that we are in the nontrivial case $\text{dom } \varphi \cap \text{dom } \psi \neq \emptyset$. Then, it suffices to apply the rule above to get

$$\partial(\varphi + \psi)(x) = \partial(\tilde{\varphi} + \tilde{\psi})(x) = \partial(\varphi + \delta_{\text{dom } \psi})(x) + \partial(\psi + \delta_{\text{dom } \varphi})(x),$$

which is a rule free of qualification conditions.

The summary of the paper is as follows. After introducing the notation and antecedents in section 2, we give the main results in section 3; they are Theorems 4 and 5. Some consequences of these two theorems, as well as related results, are stated in the final section in order to unify, recover, and extend different results in the recent literature [8, 12, 16]. The results of this work will be applied in a forthcoming paper to the integration of the approximate subdifferential of nonconvex functions in the line of recent papers like [4].

2. Preliminaries. In this paper, X stands for a (real) separated locally convex space whose topological dual is denoted by X^* and, unless otherwise specified, it is endowed with the w^* -topology. Hence, the pair (X, X^*) forms a dual topological pair by means of the canonical bilinear form $\langle x, x^* \rangle = \langle x^*, x \rangle := x^*(x)$, $(x, x^*) \in X \times X^*$. The zero vectors in the involved spaces are all denoted by θ , and the convex closed balanced neighborhoods of θ are called θ -neighborhoods. The family of such θ -neighborhoods is denoted by \mathcal{N} .

Given a nonempty set A in X (or in X^*), by $\text{co } A$ and $\text{aff } A$ we denote the *convex hull* and the *affine hull* of A , respectively. Moreover, $\text{cl } A$ and \overline{A} are indistinctly used for denoting the *closure* of A (*w*-closure* if $A \subset X^*$). Thus, $\overline{\text{co } A} := \text{cl}(\text{co } A)$, $\overline{\text{aff } A} := \text{cl}(\text{aff } A)$, etc. We use $\text{ri } A$ to represent the (topological) *relative interior* of A (i.e., the interior of A in the topology relative to $\text{aff } A$ if $\text{aff } A$ is closed, and the empty set otherwise). Associated with $A \neq \emptyset$, we consider the (one-sided) *polar cone* of A defined by

$$A^\circ := \{x^* \in X^* \mid \langle x^*, x \rangle \geq -1 \text{ for all } x \in A\}.$$

We say that a convex function $\varphi : X \rightarrow \overline{\mathbb{R}}$ is proper if its (*effective*) *domain*,

$$\text{dom } \varphi := \{x \in X \mid \varphi(x) < +\infty\},$$

is nonempty. We denote by $\varphi|_A$ the restriction of φ to A . We say that φ is lsc if its epigraph,

$$\text{epi } \varphi := \{(x, \lambda) \in X \times \mathbb{R} \mid \varphi(x) \leq \lambda\},$$

is closed. The lsc envelope of φ is the function $\overline{\varphi}$ such that $\text{epi } \overline{\varphi} = \text{cl}(\text{epi } \varphi)$.

For $\varepsilon \geq 0$, the ε -subdifferential of φ at a point $x \in \text{dom } \varphi$ is the (w^* -closed convex) set

$$\partial_\varepsilon \varphi(x) := \{x^* \in X^* \mid \varphi(y) - \varphi(x) \geq \langle x^*, y - x \rangle - \varepsilon \text{ for all } y \in X\}.$$

If $\varphi(x) = +\infty$, then we set $\partial_\varepsilon \varphi(x) = \emptyset$. In particular, for $\varepsilon = 0$ we get the Fenchel subdifferential of φ at x , $\partial \varphi(x) := \partial_0 \varphi(x)$. We denote $\text{dom } \partial \varphi := \{x \in X \mid \partial \varphi(x) \neq \emptyset\}$. When $x \in \text{dom } \partial \varphi$, we know that

$$(9) \quad \varphi(x) = \overline{\varphi}(x) \text{ and } \partial \varphi(x) = \partial \overline{\varphi}(x).$$

We also use in this paper the following well-known relations, which are satisfied at every $x \in X$:

$$(10) \quad \partial \varphi(x) = \bigcap_{\varepsilon > 0} \partial_\varepsilon \varphi(x),$$

and, for any other function $\psi : X \rightarrow \overline{\mathbb{R}}$,

$$(11) \quad \partial_\varepsilon \varphi(x) + \partial_\delta \psi(x) \subset \partial_{\varepsilon+\delta}(\varphi + \psi)(x) \text{ for all } \varepsilon, \delta \geq 0.$$

The support and the indicator functions of $A \subset X$ are defined, respectively, as

$$\sigma_A(x^*) := \sup\{\langle x^*, a \rangle \mid a \in A\} \text{ for } x^* \in X^*,$$

with $\sigma_\emptyset \equiv -\infty$ and

$$\delta_A(x) := \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \in X \setminus A. \end{cases}$$

If A is convex and $\varepsilon \geq 0$, we define the ε -normal set to A at x by

$$N_A^\varepsilon(x) := \begin{cases} \{x^* \in X^* \mid \langle x^*, y - x \rangle \leq \varepsilon \text{ for all } y \in A\} & \text{if } x \in A, \\ \emptyset & \text{if } x \in X \setminus A \end{cases}$$

with $N_\emptyset^\varepsilon \equiv \emptyset$. If $\varepsilon = 0$, we omit the reference to ε and write $N_A(x)$, which corresponds to the usual normal cone of A at x .

The following two propositions given in [5] are essential in our analysis.

PROPOSITION 1 (see [5, Theorem 12]). *Let f and g be two proper convex functions defined on X and satisfying $f + g = f + g$, together with*

$$\text{dom } f \cap \text{ri}(\text{dom } g) \neq \emptyset \text{ and } g|_{\text{aff}(\text{dom } g)} \text{ is continuous on } \text{ri}(\text{dom } g).$$

Then for every $x \in X$

$$\partial(f + g)(x) = \bigcap_{\varepsilon > 0} \text{cl}(\partial_\varepsilon f(x) + \partial g(x)).$$

PROPOSITION 2 (see [5, Theorem 15]). *Let f and g be two proper convex functions defined on X . We assume that*

$$\text{ri}(\text{dom } f) \cap \text{ri}(\text{dom } g) \neq \emptyset,$$

and that both functions $f|_{\text{aff}(\text{dom } f)}$ and $g|_{\text{aff}(\text{dom } g)}$ are continuous on $\text{ri}(\text{dom } f)$ and $\text{ri}(\text{dom } g)$, respectively. Then, for every $x \in X$,

$$\partial(f + g)(x) = \text{cl}(\partial f(x) + \partial g(x)).$$

3. Main results. In this section, we give our main results. The first one, Theorem 4, establishes a general rule for the subdifferential of the sum of two convex functions where one of them is the supremum of an arbitrary family of convex functions. This will be done under a weak assumption expressing a natural relationship between the lsc envelopes of the data functions in the domain of the sum function. The second result given in Theorem 5 provides a rule for the subdifferential of the sum of two convex functions and uses the idea of augmenting the involved functions. The main feature of our approach is that no continuity-type condition will be required.

The following result, which is a consequence of Proposition 1, is needed for the proof of Theorem 4 below.

LEMMA 3. *Let $(A_\varepsilon)_{\varepsilon>0}$ be a nondecreasing family of nonempty closed convex sets of X^* ; that is,*

$$\varepsilon_1 \leq \varepsilon_2 \implies A_{\varepsilon_1} \subset A_{\varepsilon_2}.$$

Consider a proper convex function $g : X \rightarrow \overline{\mathbb{R}}$ and a fixed $x \in X$. We assume that $g|_{\text{aff}(\text{dom } g)}$ is continuous on $\text{ri}(\text{dom } g)$ (assumed nonempty) and, for all small $\varepsilon > 0$,

$$(\text{ri}(\text{dom } g) - x) \cap \text{dom } \sigma_{A_\varepsilon} \neq \emptyset.$$

Then for all $x \in X$

$$\bigcap_{\varepsilon>0} \text{cl}(A_\varepsilon + \partial_\varepsilon g(x)) = \bigcap_{\varepsilon>0} \text{cl}(A_\varepsilon + \partial g(x)).$$

Proof. First, observe that the inclusion “ \supset ” is valid (and obvious). For the opposite inclusion we may suppose that $x = \theta$ and that $\bigcap_{\varepsilon>0} \text{cl}(A_\varepsilon + \partial_\varepsilon g(\theta))$ is nonempty. Then $\partial_\varepsilon g(\theta) \neq \emptyset$ for all $\varepsilon > 0$ and this implies that g is lsc at θ . It follows that $\partial_\varepsilon g(\theta) = \partial_\varepsilon \bar{g}(\theta)$ for all $\varepsilon \geq 0$. Moreover, since $\text{ri}(\text{dom } g) = \text{ri}(\text{dom } \bar{g})$, \bar{g} satisfies the same continuity assumption as g and, so, we may (and do) assume that g is lsc. Let $\delta > 0$ such that $\text{ri}(\text{dom } g) \cap \text{dom } \sigma_{A_\delta} \neq \emptyset$. Then for $0 < \varepsilon \leq \delta$ we have

$$A_\varepsilon + \partial_\varepsilon g(\theta) \subset A_\delta + \partial_\varepsilon g(\theta) = \partial \sigma_{A_\delta}(\theta) + \partial_\varepsilon g(\theta) \subset \partial_\varepsilon (\sigma_{A_\delta} + g)(\theta),$$

whence $\text{cl}(A_\varepsilon + \partial_\varepsilon g(\theta)) \subset \partial_\varepsilon (\sigma_{A_\delta} + g)(\theta)$, and so

$$\begin{aligned} \bigcap_{\varepsilon>0} \text{cl}(A_\varepsilon + \partial_\varepsilon g(\theta)) &\subset \bigcap_{\varepsilon>0} \partial_\varepsilon (\sigma_{A_\delta} + g)(\theta) \\ &= \partial(\sigma_{A_\delta} + g)(\theta) \\ &= \bigcap_{\varepsilon>0} \text{cl}(\partial_\varepsilon \sigma_{A_\delta}(\theta) + \partial g(\theta)) \quad (\text{by Proposition 1}) \\ &= \text{cl}(A_\delta + \partial g(\theta)). \end{aligned}$$

The desired inclusion follows by intersecting over $\delta > 0$. □

Next, we establish the main theorem of this section, which constitutes the desired extension of (3). We shall also derive other variants in the next section. As before, $\mathcal{F}(x)$ denotes the family of finite-dimensional subspaces containing $x \in X$.

THEOREM 4. *Let f, g , and $f_t : X \rightarrow \overline{\mathbb{R}}$, $t \in T$, be proper convex functions with $f = \sup_{t \in T} f_t$, and let $L \subset X$ be a finite-dimensional subspace. Assume that*

$$\overline{(f + g)}(x) = \sup_{t \in T} \overline{f}_t(x) + g(x) \quad \text{for all } x \in \text{dom } \partial(f + g + \delta_L).$$

Then, for every $x \in L \cap \text{dom } \partial(f + g)$,

$$\partial(f + g)(x) \subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{L \cap \text{dom } f})(x) \right\}.$$

Moreover, under the assumption

$$\overline{(f + g)}(x) = \sup_{t \in T} \overline{f}_t(x) + g(x) \quad \text{for all } x \in \text{dom } f \cap \text{dom } g,$$

we have, for every $x \in X$,

$$\partial(f + g)(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{L \cap \text{dom } f})(x) \right\}.$$

Proof. The idea of the proof is to look for an appropriate family of lsc convex functions, which yields a tight approximation of the subdifferential of the sum $f + g$. To this aim we denote

$$\tilde{f} := \sup_{t \in T} \overline{f}_t, \quad \phi := g + \delta_{L \cap \text{dom } f}, \quad \psi := \overline{\phi},$$

and observe that

$$(12) \quad L \cap \text{dom } f \cap \text{dom } g = \text{dom } \phi \subset \text{dom } \psi \subset \overline{L \cap \text{dom } f \cap \text{dom } g}.$$

Step 1. Given an x in $\overline{L \cap \text{dom } f \cap \text{dom } g}$ we pick $z_0 \in \text{ri}(\text{dom}(f + g + \delta_L))$ ($\subset \text{dom } \partial(f + g + \delta_L)$). Then for $\lambda \in (0, 1)$ we have that $x_\lambda := \lambda z_0 + (1 - \lambda)x \in \text{ri}(\text{dom}(f + g + \delta_L)) \subset \text{dom } \partial(f + g + \delta_L)$ and, so, using the first assumption of the theorem,

$$\overline{(f + g)}(x_\lambda) = \tilde{f}(x_\lambda) + \phi(x_\lambda) = \tilde{f}(x_\lambda) + \psi(x_\lambda) \leq (1 - \lambda)\tilde{f}(x) + \lambda\tilde{f}(z_0) + \psi(x_\lambda).$$

Since the function $\lambda \rightarrow \psi(x_\lambda)$ is continuous, by taking the limit over $\lambda \rightarrow 0^+$ we get

$$\overline{(f + g)}(x) \leq \tilde{f}(x) + \psi(x);$$

hence,

$$(13) \quad \overline{f + g} \leq \tilde{f} + \psi =: \varphi \leq f + g + \delta_{L \cap \text{dom } f}.$$

Step 2. From (12) and (13) one gets $\text{dom } \phi \subset \text{dom } \varphi \subset \text{dom } \psi \subset \overline{\text{dom } \phi}$, and so

$$(14) \quad \text{ri}(\text{dom } \phi) = \text{ri}(\text{dom } \varphi) = \text{ri}(\text{dom } \psi) \neq \emptyset$$

and

$$(15) \quad \overline{\text{dom } \phi} = \overline{\text{dom } \varphi} = \overline{\text{dom } \psi}.$$

Take $x \in L \cap \text{dom } \partial(f + g)$ ($\subset \text{dom } \phi \subset \text{dom } \varphi \subset \text{dom } \psi$). Then $\overline{(f + g)}(x) \in \mathbb{R}$ and

$$\overline{(f + g)}(x) = (f + g)(x) = f(x) + \phi(x) \geq \tilde{f}(x) + \psi(x) = \varphi(x).$$

Using (12) we get

$$(16) \quad \overline{(f+g)}(x) = \varphi(x) = f(x) + g(x), \quad f(x) = \tilde{f}(x), \quad g(x) = \phi(x) = \psi(x).$$

Using (14) and (15), it follows that, for all $\varepsilon > 0$,

$$(17) \quad N_{\text{dom } \phi}(x) = N_{\text{dom } \varphi}(x) = N_{\text{dom } \psi}(x) = (\partial_\varepsilon \phi(x))_\infty = (\partial_\varepsilon \varphi(x))_\infty = (\partial_\varepsilon \psi(x))_\infty,$$

where for $A \subset X^*$ the set A_∞ represents the recession cone of A . Since $\overline{f+g} \leq \varphi$ we deduce that $\partial(\overline{f+g})(x) \subset \partial\varphi(x)$.

Step 3. Now, take x^* in $\partial(f+g)(x) \subset \partial\varphi(x)$. Since φ is the pointwise supremum of the proper lsc convex functions $\varphi_t := f_t + \psi$, and $\text{ri}(\text{dom } \varphi) \neq \emptyset$ by (14) and (15), according to (4) we obtain

$$\begin{aligned} x^* \in \partial\varphi(x) &= \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(x)} \partial_\varepsilon \varphi_t(x) + N_{\text{dom } \varphi}(x) \right\} \\ &= \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(x)} \partial_\varepsilon \varphi_t(x) + N_{L \cap \text{dom } f \cap \text{dom } g}(x) \right\}, \end{aligned}$$

where $\tilde{T}_\varepsilon(x) := \{t \in T \mid \varphi_t(x) \geq \varphi(x) - \varepsilon\}$. Moreover, since (recall (7) and (16))

$$\partial_\varepsilon \varphi_t(x) \subset \text{cl}(\partial_{2\varepsilon} \bar{f}_t(x) + \partial_{2\varepsilon} \psi(x)) = \text{cl}(\partial_{2\varepsilon} \bar{f}_t(x) + \partial_{2\varepsilon} g(x)),$$

we get

$$\begin{aligned} x^* &\in \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(x)} \text{cl}(\partial_{2\varepsilon} \bar{f}_t(x) + \partial_{2\varepsilon} g(x)) + N_{L \cap \text{dom } f \cap \text{dom } g}(x) \right\} \\ &= \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(x)} \partial_{2\varepsilon} \bar{f}_t(x) + \partial_{2\varepsilon} g(x) + N_{L \cap \text{dom } f \cap \text{dom } g}(x) \right\} \\ &= \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(x)} \partial_\varepsilon \bar{f}_t(x) + \partial_\varepsilon g(x) + N_{L \cap \text{dom } f \cap \text{dom } g}(x) \right\} \\ (18) \quad &= \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(x)} \partial_\varepsilon \bar{f}_t(x) + \partial_\varepsilon g(x) \right\}, \end{aligned}$$

where in the last equality we used the fact that $(\partial_\varepsilon g(x))_\infty = N_{L \cap \text{dom } f \cap \text{dom } g}(x)$ (see (17)).

Set $A_\varepsilon := \overline{\text{co}}\{\bigcup_{t \in \tilde{T}_\varepsilon(x)} \partial_\varepsilon \bar{f}_t(x)\}$, $\varepsilon > 0$. Then for every

$$z \in \text{ri}(\text{dom } \psi) = \text{ri}(L \cap \text{dom } f \cap \text{dom } g)$$

and $\varepsilon > 0$ we have

$$\begin{aligned} \sigma_{A_\varepsilon}(z-x) &= \sigma_{\bigcup_{t \in \tilde{T}_\varepsilon(x)} \partial_\varepsilon(\bar{f}_t)(x)}(z-x) \leq \sup_{t \in \tilde{T}_\varepsilon(x)} (\bar{f}_t(z) - \bar{f}_t(x) + \varepsilon) \\ &\leq \sup_{t \in \tilde{T}_\varepsilon(x)} (f(z) - \bar{f}_t(x) + \varepsilon) \leq f(z) - f(x) + 2\varepsilon < \infty, \end{aligned}$$

which shows that $(\text{ri}(\text{dom } \psi) - x) \subset \text{dom } \sigma_{A_\varepsilon}$ and, thus,

$$(\text{ri}(\text{dom } \psi) - x) \cap \text{dom } \sigma_{A_\varepsilon} = \text{ri}(\text{dom } \psi) - x \neq \emptyset.$$

Consequently, according to Lemma 3, by (18) and (16) we get

$$\begin{aligned} x^* &\in \bigcap_{\varepsilon>0} \text{cl}(A_\varepsilon + \partial_\varepsilon g(x)) = \bigcap_{\varepsilon>0} \text{cl}(A_\varepsilon + \partial_\varepsilon \psi(x)) = \bigcap_{\varepsilon>0} \text{cl}(A_\varepsilon + \partial \psi(x)) \\ &= \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(x)} \partial_\varepsilon(\bar{f}_t)(x) + \partial g(x) \right\}. \end{aligned}$$

Moreover, since (recall (16))

$$\begin{aligned} \tilde{T}_\varepsilon(x) &= \{t \in T \mid \bar{f}_t(x) + \psi(x) \geq f(x) + \psi(x) - \varepsilon\} \\ &\subset \{t \in T \mid f_t(x) + \psi(x) \geq f(x) + \psi(x) - \varepsilon\} = T_\varepsilon(x), \end{aligned}$$

for each $t \in \tilde{T}_\varepsilon(x)$ we have

$$\bar{f}_t(x) \geq f(x) - \varepsilon \geq f_t(x) - \varepsilon,$$

so that $\partial_\varepsilon \bar{f}_t(x) \subset \partial_{2\varepsilon} f_t(x)$. On the other hand, because

$$\overline{(g + \delta_{L \cap \text{dom } f})(x)} = (g + \delta_{L \cap \text{dom } f})(x)$$

we have that $\partial \overline{(g + \delta_{L \cap \text{dom } f})(x)} = \partial(g + \delta_{L \cap \text{dom } f})(x)$ and, so (since $\tilde{T}_\varepsilon(x) \subset T_\varepsilon(x) \subset T_{2\varepsilon}(x)$),

$$\begin{aligned} x^* &\in \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{ \bigcup_{t \in \tilde{T}_\varepsilon(x)} \partial_{2\varepsilon} f_t(x) + \partial(g + \delta_{L \cap \text{dom } f})(x) \right\} \\ &\subset \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{ \bigcup_{t \in T_{2\varepsilon}(x)} \partial_{2\varepsilon} f_t(x) + \partial(g + \delta_{L \cap \text{dom } f})(x) \right\} \\ &= \bigcap_{\varepsilon>0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{L \cap \text{dom } f})(x) \right\}. \end{aligned}$$

This finishes the proof of the first statement.

Step 4. The second hypothesis of the theorem implies the first one; that is, for every $L \in \mathcal{F}(x)$

$$(19) \quad \overline{f + g} = \sup_{t \in T} \bar{f}_t + g \quad \text{on } \text{dom } \partial(f + g + \delta_L).$$

Then we infer that

$$\partial(f + g)(x) \subset \bigcap_{\substack{\varepsilon>0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{L \cap \text{dom } f})(x) \right\},$$

and the first inclusion follows. To prove the converse inclusion we take

$$z^* \in \bigcap_{\substack{\varepsilon>0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{L \cap \text{dom } f})(x) \right\}.$$

Observe that $\partial_\varepsilon f_t(x) \subset \partial_{2\varepsilon} f(x)$ for all $\varepsilon > 0$ and $t \in T_\varepsilon(x)$ so that

$$\begin{aligned} z^* &\in \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \{ \partial_{2\varepsilon} f(x) + \partial(g + \delta_{L \cap \text{dom } f})(x) \} \\ &\subset \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \partial_{2\varepsilon}(f + g + \delta_{L \cap \text{dom } f})(x) = \bigcap_{L \in \mathcal{F}(x)} \partial(f + g + \delta_L)(x). \end{aligned}$$

Now, if $z \in X$, for L being the (finite-dimensional) linear subspace generated by $\{x, z\}$ we have $z^* \in \partial(f + g + \delta_L)(x)$ and, so,

$$\langle z - x, z^* \rangle \leq f(z) + g(z) - f(x) - g(x) + \delta_L(z) - \delta_L(x) = f(z) + g(z) - f(x) - g(x).$$

Since z is arbitrary we conclude that $z^* \in \partial(f + g)(x)$. \square

The following theorem provides a characterization of the subdifferential of the sum using only exact subdifferentials. Here we use the idea of augmenting the functions in order to avoid the requirement of continuity-type conditions.

THEOREM 5. *Given two proper convex functions $f, g : X \rightarrow \overline{\mathbb{R}}$, for every $x \in X$*

$$\partial(f + g)(x) = \bigcap_{L \in \mathcal{F}(x)} \partial(f + \delta_{L \cap \text{dom } g})(x) + \partial(g + \delta_{L \cap \text{dom } f})(x).$$

Moreover, when $\text{ri}(\text{dom } f \cap \text{dom } g)$ is nonempty and the restrictions of f and g on $\text{aff}(\text{dom } f \cap \text{dom } g)$ are continuous on $\text{ri}(\text{dom } f \cap \text{dom } g)$, we obtain

$$\partial(f + g)(x) = \partial(f + \delta_{\text{dom } g})(x) + \partial(g + \delta_{\text{dom } f})(x).$$

Proof. We may assume that $x \in \text{dom } f \cap \text{dom } g$. First, we verify that

$$(20) \quad \partial(f + g)(x) = \bigcap_{L \in \mathcal{F}(x)} \partial(f + g + \delta_L)(x).$$

The inclusion “ \subset ” is a straightforward consequence of $\partial(f + g)(x) \subset \partial(f + g + \delta_L)(x)$ for every $L \in \mathcal{F}(x)$, while the converse inclusion follows similarly as in the proof of Step 4 in Theorem 4.

Now, by (20),

$$\partial(f + g)(x) = \bigcap_{L \in \mathcal{F}(x)} \partial(f + g + \delta_L)(x) = \bigcap_{L \in \mathcal{F}(x)} \partial(f + \delta_{L \cap \text{dom } g} + g + \delta_{L \cap \text{dom } f})(x).$$

Since for every $L \in \mathcal{F}(x)$

$$\text{ri}(\text{dom}(f + \delta_{L \cap \text{dom } g})) = \text{ri}(L \cap \text{dom } f \cap \text{dom } g) = \text{ri}(\text{dom}(g + \delta_{L \cap \text{dom } f})),$$

by applying Proposition 2 we get

$$\partial(f + g)(x) = \bigcap_{L \in \mathcal{F}(x)} \text{cl}(\partial(f + \delta_{L \cap \text{dom } g})(x) + \partial(g + \delta_{L \cap \text{dom } f})(x)).$$

To conclude the first part of the proof we only need to show the closedness of

$$\partial(f + \delta_{L \cap \text{dom } g})(x) + \partial(g + \delta_{L \cap \text{dom } f})(x).$$

Let $x_0 \in \text{ri}(L \cap \text{dom } f \cap \text{dom } g)$, $V \subset X$ be a θ -neighborhood, and $m > 0$ such that

$$V_0 := V \cap (\text{aff}(L \cap \text{dom } f \cap \text{dom } g) - x_0) \subset (L \cap \text{dom } f \cap \text{dom } g) - x_0$$

and

$$g(x_0 + y) + \delta_{L \cap \text{dom } f}(x_0 + y) \leq m \text{ for all } y \in V_0.$$

Now we pick z^* a (weak*-) limit of a net $(x_i^* + y_i^*)_{i \in I}$, where $(x_i^*)_{i \in I} \subset \partial(f + \delta_{L \cap \text{dom } g})(x)$ and $(y_i^*)_{i \in I} \subset \partial(g + \delta_{L \cap \text{dom } f})(x)$. Then, for each i and $y \in V_0$, from the inequality above we get

$$(21) \quad \langle y_i^*, y + x_0 - x \rangle \leq g(y + x_0) + \delta_{L \cap \text{dom } f}(y + x_0) - g(x) - \delta_{L \cap \text{dom } f}(x) \leq m - g(x).$$

On the other hand, we may assume that for all $i \in I$

$$\langle y_i^*, x - x_0 \rangle = \langle x_i^*, x_0 - x \rangle + \langle x_i^* + y_i^*, x - x_0 \rangle \leq f(x_0) - f(x) + \langle z^*, x - x_0 \rangle + 1,$$

so that (21) implies, for all $i \in I$ and all $y \in V_0$,

$$\langle y_i^*, y \rangle \leq m - g(x) - f(x) + f(x_0) + \langle z^*, x - x_0 \rangle + 1 < \infty.$$

Consider

$$\rho := \max\{1, m - g(x) - f(x) + f(x_0) + \langle z^*, x - x_0 \rangle + 1\},$$

so that $\rho > 0$ and $(y_i^*)_{i \in I} \subset (\rho^{-1}V_0)^\circ$. Since $(\rho^{-1}V_0)^\circ$ is weak*-compact by the Alaoglu–Bourbaki theorem, we find a subnet of $(y_i^*)_{i \in I}$ which (weak*-) converges to a point \tilde{y}^* in the topological dual of $\text{aff}(L \cap \text{dom } f \cap \text{dom } g) - x_0$ (the use of finite-dimensional arguments would be sufficient to show the convergence of a subnet of $(y_i^*)_{i \in I}$, but they do not apply in the proof of the second part of the theorem). Moreover, using the Hahn–Banach theorem, we extend \tilde{y}^* to $y^* \in X^*$, which coincides with \tilde{y}^* on $\text{aff}(L \cap \text{dom } f \cap \text{dom } g) - x_0$. Then we write, without loss of generality,

$$\langle y_i^*, y - x \rangle \rightarrow \langle y^*, y - x \rangle \text{ for all } y \in \text{aff}(L \cap \text{dom } f \cap \text{dom } g).$$

Next, by taking the limit on i in the following inequality (recall that $y_i^* \in \partial(g + \delta_{L \cap \text{dom } f})(x)$)

$$\langle y_i^*, y - x \rangle \leq g(y) - g(x) \text{ for all } y \in L \cap \text{dom } f,$$

we get

$$\langle y^*, y - x \rangle \leq g(y) - g(x) \text{ for all } y \in L \cap \text{dom } f,$$

which means that $y^* \in \partial(g + \delta_{L \cap \text{dom } f})(x)$. Now, by setting $x^* := z^* - y^*$, we have for all $y \in L \cap \text{dom } f \cap \text{dom } g$

$$\langle x^*, y - x \rangle = \langle z^* - y^*, y - x \rangle = \lim_{i \in I} \langle x_i^*, y - x \rangle \leq f(y) - f(x),$$

which shows that $x^* \in \partial(f + \delta_{L \cap \text{dom } g})(x)$. In other words,

$$z^* = x^* + y^* \in \partial(f + \delta_{L \cap \text{dom } g})(x) + \partial(g + \delta_{L \cap \text{dom } f})(x),$$

as we wanted to prove.

We proceed with the proof of the second statement of the theorem. Observe that, by the current hypothesis,

$$\text{ri}(\text{dom}(f + \delta_{\text{dom } g})) = \text{ri}(\text{dom}(g + \delta_{\text{dom } f})) = \text{ri}(\text{dom } f \cap \text{dom } g) \neq \emptyset,$$

and the restrictions of the functions f and g to $\text{aff}(\text{dom } f \cap \text{dom } g)$ are continuous on $\text{ri}(\text{dom } f \cap \text{dom } g)$. Then, by applying Proposition 2, we get

$$\begin{aligned}\partial(f+g)(x) &= \partial((f + \delta_{\text{dom } g}) + (g + \delta_{\text{dom } f}))(x) \\ &= \text{cl}(\partial(f + \delta_{\text{dom } g})(x) + \partial(g + \delta_{\text{dom } f})(x)).\end{aligned}$$

To conclude the proof we follow the same arguments as above to show the closedness of the set $\partial(f + \delta_{\text{dom } g})(x) + \partial(g + \delta_{\text{dom } f})(x)$. \square

For comparative purposes, Theorem 4 asserts that for any pair of proper convex functions f and g , satisfying

$$\overline{f+g} = \overline{f} + g \quad \text{on } \text{dom } f \cap \text{dom } g,$$

it holds, for every $x \in X$,

$$\partial(f+g)(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \text{cl}(\partial_\varepsilon f(x) + \partial(g + \delta_{L \cap \text{dom } f})(x)).$$

Remark 1. As was observed by one of the referees, the second statement of Theorem 5 can also be obtained as a consequence of the Moreau–Rockafellar result for the subdifferential of the sum. However, the current proof is based on Proposition 2, which yields an almost exact calculus rule for the sum, namely, $\partial(f+g)(x) = \text{cl}(\partial f(x) + \partial g(x))$, without appealing to Moreau–Rockafellar’s theorem. Thus, the objective of our proof of Theorem 5 was to show that the previous closure is superfluous under the current conditions.

4. Consequences and related results. In this section we give some consequences of Theorem 4, which cover and improve different results in the literature.

Recall that the hypothesis of Theorem 4 states that

$$(22) \quad \overline{(f+g)}(x) = \sup_{t \in T} \overline{f}_t(x) + g(x) \quad \text{for all } x \in \text{dom } f \cap \text{dom } g.$$

The first result in this section simplifies [16, Theorem 3.1], both in the statement and in the proof.

COROLLARY 6. *Let f and $f_t : X \rightarrow \overline{\mathbb{R}}$, $t \in T$, be the same as in Theorem 4, and let $D \subset \text{dom } f$ be a nonempty convex set. Assume that*

$$\overline{(f + \delta_D)}(x) = \sup_{t \in T} \overline{f}_t(x) \quad \text{for all } x \in D.$$

Then we have, for every $x \in X$,

$$\partial(f + \delta_D)(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + N_{L \cap D}(x) \right\}.$$

In particular, if $D = \text{dom } f$, this corollary provides a remarkable extension of (3); that is, under the condition

$$(23) \quad \overline{f}(x) = \sup_{t \in T} \overline{f}_t(x) \quad \text{for all } x \in \text{dom } f,$$

it holds that

$$\partial f(x) = \bigcap_{\substack{\varepsilon > 0 \\ L \in \mathcal{F}(x)}} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + N_{L \cap \text{dom } f}(x) \right\}.$$

This also extends [16, Corollary 3.2], where the equality in (23) was required to be held on $\text{cl}(\text{dom } f)$ instead of $\text{dom } f$.

It is also worth observing that Corollary 6 can be obtained from [16, Theorem 3.1] under the following assumption:

$$(\overline{f + \delta_D})(x) = \sup_{t \in T} \overline{f}_t(x) \text{ for all } x \in \bigcup_{L \in \mathcal{F}} \text{cl}(L \cap D),$$

where \mathcal{F} is the family of finite-dimensional subspaces of X . This fact motivates the following proposition addressed to compare both assumptions.

PROPOSITION 7. *Let f and D be the same as in Corollary 6. Then the following statements are equivalent:*

- (i) $(\overline{f + \delta_D})(x) = \sup_{t \in T} \overline{f}_t(x)$ for all $x \in D$.
- (ii) $(\overline{f + \delta_D})(x) = \sup_{t \in T} \overline{f}_t(x)$ for all $x \in \bigcup_{L \in \mathcal{F}} \text{cl}(L \cap D)$.

Proof. The implication (ii) \implies (i) is obvious. To prove the opposite one we take x in $\text{cl}(L \cap D)$ for certain $L \in \mathcal{F}$. Since $L \cap D \subset L$ and is nonempty, we pick $x_0 \in \text{ri}(L \cap D)$. Then $x_\lambda := \lambda x_0 + (1 - \lambda)x \in L \cap D \subset D$ for $\lambda \in (0, 1)$ and, so,

$$(\overline{f + \delta_D})(x) = \lim_{\lambda \rightarrow 0^+} (\overline{f + \delta_D})(x_\lambda) = \lim_{\lambda \rightarrow 0^+} \sup_{t \in T} \overline{f}_t(x_\lambda) = \sup_{t \in T} \overline{f}_t(x). \quad \square$$

The intersection over L in the formulas of Theorem 4 is obviously omitted in the finite-dimensional setting. The following corollary provides another situation in the infinite-dimensional setting where this fact also occurs.

COROLLARY 8. *With the same notation as in Theorem 4, we assume that (22) holds. If $\text{ri}(\text{dom } f \cap \text{dom } g) \neq \emptyset$ and $g|_{\text{aff}(\text{dom } f \cap \text{dom } g)}$ is continuous on $\text{ri}(\text{dom } f \cap \text{dom } g)$, then for every $x \in X$ we have that*

$$\partial(f + g)(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{\text{dom } f})(x) \right\}.$$

Proof. Fix $x \in \text{dom } f \cap \text{dom } g$. Given a θ -neighborhood $U \subset X^*$ we choose $L \in \mathcal{F}(x)$ such that $L^\perp \subset U$, and take $L_1 \in \mathcal{F}(x)$ verifying $L \subset L_1$ and $L_1 \cap \text{ri}(\text{dom } f \cap \text{dom } g) \neq \emptyset$. Then by Theorem 4 we have that

$$(24) \quad \partial(f + g)(x) \subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left(\bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{L_1 \cap \text{dom } f})(x) \right).$$

Then we only need to get a simplified expression for the set $\partial(g + \delta_{L_1 \cap \text{dom } f})(x)$. To this aim we check that the functions

$$\varphi := \delta_{L_1} \text{ and } \psi := g + \delta_{\text{dom } f}$$

satisfy the conditions of Proposition 2. On one hand, by the choice of L_1 we have that

$$\text{ri}(\text{dom } \varphi) \cap \text{ri}(\text{dom } \psi) = L_1 \cap \text{ri}(\text{dom } f \cap \text{dom } g) \neq \emptyset.$$

On the other hand, $\varphi|_{L_1}$ is continuous on L_1 , while $\psi|_{\text{aff}(\text{dom } \psi)} = g|_{\text{aff}(\text{dom } f \cap \text{dom } g)}$ is continuous on $\text{ri}(\text{dom } \psi)$ by assumption. Then, by Proposition 2,

$$\partial(g + \delta_{L_1 \cap \text{dom } f})(x) = \partial(g + \delta_{\text{dom } f} + \delta_{L_1})(x) = \text{cl}(\partial(g + \delta_{\text{dom } f})(x) + L_1^\perp).$$

Plugging this equality into (24) yields

$$\begin{aligned} \partial(f + g)(x) &\subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left(\bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \text{cl}(\partial(g + \delta_{\text{dom } f})(x) + L_1^\perp) \right) \\ &= \bigcap_{\varepsilon > 0} \overline{\text{co}} \left(\bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{\text{dom } f})(x) + L_1^\perp \right) \\ &\subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left(\bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{\text{dom } f})(x) + U \right). \end{aligned}$$

Finally, since that U was arbitrarily chosen we get

$$\partial(f + g)(x) \subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left(\bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{\text{dom } f})(x) \right).$$

Thus the inclusion “ \subset ” of the statement follows. This finishes the proof since the other inclusion is straightforward, taking into account that

$$\partial(g + \delta_{\text{dom } f})(x) \subset \partial(g + \delta_{L \cap \text{dom } f})(x) \quad \text{for all } L \in \mathcal{F}(x). \quad \square$$

Along the lines of [12], we see that the intersection over L in the formula of Theorem 4 is also omitted in the infinite-dimensional setting provided that some closedness condition is satisfied.

COROLLARY 9. *With the same notation as in Theorem 4, we assume that (22) holds. If the function $g + \delta_{\text{dom } f}$ is lsc, then for every $x \in X$*

$$\partial(f + g)(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial_\varepsilon(g + \delta_{\text{dom } f})(x) \right\}.$$

Moreover, in the particular case where $g \equiv 0$, if $\mathbb{R}_+(\text{dom } f - x)$ is closed, we get

$$\partial f(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + N_{\text{dom } f}(x) \right\}.$$

Proof. Let $U, W \subset X^*$ be θ -neighborhoods and choose $L \in \mathcal{F}(x)$ such that $L^\perp + W \subset W + W \subset U$. Then, by (7) we get, for every $\varepsilon > 0$,

$$\begin{aligned} \partial(g + \delta_{L \cap \text{dom } f})(x) &= \partial(g + \delta_{\text{dom } f} + \delta_L)(x) \subset \text{cl}(\partial_\varepsilon(g + \delta_{\text{dom } f})(x) + L^\perp) \\ &\subset \partial_\varepsilon(g + \delta_{\text{dom } f})(x) + L^\perp + W \subset \partial_\varepsilon(g + \delta_{\text{dom } f})(x) + U. \end{aligned}$$

Then, according to Theorem 4,

$$\begin{aligned} \partial(f + g)(x) &\subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial(g + \delta_{L \cap \text{dom } f})(x) \right\} \\ &\subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial_\varepsilon(g + \delta_{\text{dom } f})(x) + U \right\}. \end{aligned}$$

Since U was arbitrarily chosen we get

$$\partial(f + g)(x) \subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + \partial_\varepsilon(g + \delta_{\text{dom } f})(x) \right\}.$$

The opposite inclusion is immediate.

Assume now that $g \equiv 0$ and $\mathbb{R}_+(\text{dom } f - x)$ is closed. Then, again using (7), as $\delta_{\mathbb{R}_+(\text{dom } f - x)}$ is lsc by assumption, for all $\varepsilon > 0$

$$\begin{aligned} N_{L \cap \text{dom } f}(x) &= N_{L \cap (\mathbb{R}_+(\text{dom } f - x))}(\theta) \subset \text{cl} \left(N_{\mathbb{R}_+(\text{dom } f - x)}^\varepsilon(\theta) + L^\perp \right) \\ &= \text{cl} \left(N_{\mathbb{R}_+(\text{dom } f - x)}(\theta) + L^\perp \right) \subset N_{\mathbb{R}_+(\text{dom } f - x)}(\theta) + L^\perp + W \\ &\subset N_{\mathbb{R}_+(\text{dom } f - x)}(\theta) + U = N_{\text{dom } f}(x) + U. \end{aligned}$$

Thus, by Theorem 4,

$$\begin{aligned} \partial f(x) &\subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + N_{L \cap \text{dom } f}(x) \right\} \\ &\subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + N_{\text{dom } f}(x) + U \right\}, \end{aligned}$$

which gives us

$$\partial f(x) \subset \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{t \in T_\varepsilon(x)} \partial_\varepsilon f_t(x) + N_{\text{dom } f}(x) \right\}. \quad \square$$

Observe that the second part of the previous corollary applies when f is a polyhedral function, since in this case the set $\mathbb{R}_+(\text{dom } f - x)$ is closed [5]. A characterization of this closedness property can be found in [14].

We close this paper with a refinement of the classical result due to Brøndsted (see (5)). The proof of the following corollary combines Theorem 4 and the arguments used in [8, Corollary 12].

COROLLARY 10. *Let f_1, \dots, f_k be proper convex functions, and let $x \in X$ such that $f_1(x) = \dots = f_k(x)$. Assume that*

$$\overline{\max\{f_1, \dots, f_k\}} = \max\{\overline{f_1}, \dots, \overline{f_k}\} \quad \text{on } \bigcap_{i=1}^k \text{dom } f_i.$$

Then we have that

$$\partial(\max\{f_1, \dots, f_k\})(x) = \bigcap_{\varepsilon > 0} \overline{\text{co}} \left\{ \bigcup_{i=1}^k \partial_\varepsilon f_i(x) \right\}.$$

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