The Voronoi inverse mapping

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Abstract

Given an arbitrary set T in the Euclidean space whose elements are called sites, and a particular site s, the Voronoi cell of s, denoted by $V_T(s)$, consists of all points closer to s than to any other site. The Voronoi mapping of s, denoted by ψ_s, associates to each set $T \ni s$ the Voronoi cell $V_T(s)$ of s w.r.t. T. These Voronoi cells are solution sets of linear inequality systems, so they are closed convex sets. In this paper we study the Voronoi inverse problem consisting in computing, for a given closed convex set $F \ni s$, the family of sets $T \ni s$ such that $\psi_s(T) = F$. More in detail, the paper analyzes relationships between the elements of this family, $\psi_s^{-1}(F)$, and the linear representations of F, provides explicit formulas for maximal and minimal elements of $\psi_s^{-1}(F)$, and studies the closure operator that assigns, to each closed set T containing s, the largest element of $\psi_s^{-1}(F)$, where $F = V_T(s)$.

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1 Introduction

Let $T \subset \mathbb{R}^n$, with $n \geq 1$, be a set whose elements are called *Voronoi sites*. The *Voronoi cell of* $s \in T$, denoted by $V_T(s)$, consists of all points closer to s than to any other site, i.e.,

$$V_T(s) := \{x \in \mathbb{R}^n : d(x,s) \leq d(x,t),\ t \in T\},$$

where d denotes the Euclidean distance on $\mathbb{R}^n$. In other terms, denoting by $P_T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ the metric projection on T, i.e., $P_T(x) := \{t \in T : d(x,t) = d(x,T)\}$, then $V_T(s) = P_T^{-1}(s)$. The *Voronoi diagram* of T is $\operatorname{Vor}(T) := \{V_T(t), t \in T\}$. This family of closed convex sets is a tesselation of $\mathbb{R}^n$ if and only if T is closed [6, Proposition 1], which is the general assumption of this paper. When T is finite, $\operatorname{Vor}(T)$ is formed by full dimensional convex polyhedral sets.

Voronoi cells of finite sets of sites in two, three, and $n > 3$ dimensions were introduced by Descartes, Dirichlet, and Voronoi, in 1644, 1850, and 1908, respectively. Voronoi cells of finite sets are widely applied in computational geometry, operations research, data compression, economics, marketing, etc. (see, e.g., [1] and [9]). In 1934 Delaunay [2, General Lemma] introduced Voronoi cells of discrete sets of sites (i.e., sets without accumulation points) with applications in crystallography. Recently, Voigt and Weis, [12,13], obtained results on Voronoi cells of discrete sets exploiting the fact that $V_T(s)$ is the solution set of the linear system

$$\upsilon_T(s) := \left\{(t-s)' x \leq \tfrac{\|t\|^2 - \|s\|^2}{2}, t \in T \setminus \{s\}\right\}, \tag{1}$$

where t' and $\|t\|$ denote the transpose of t and the Euclidean norm of t, respectively. A similar approach was used in [6] in order to obtain, in a systematic way, geometric information on $V_T(s)$ in terms of the data (T and s) for different types of infinite sets of sites such as curves, closed convex sets, etc. The latter work shows that Voronoi cells of discrete sets are useful in location decision making while Voronoi cells of non-discrete sets can be used to assign clients for services delivered along curves (such as rivers or highways) or for services delivered at a finite set of uncertain sites (robust approach). Let us mention here that Voronoi cells of infinite sets are of potential use for delimiting international waters among neighbor countries by assuming that each point should be assigned to the closest country. Then each country should get the union of the Voronoi cells corresponding to the points of the national shore together with a proportional part of the Voronoi cells of the border points. For instance, in the simple case of a compact convex island T shared by two countries occupying non-overlapping territories A, $B \subset T$, the first country would get

$$\bigcup \{V_T(s) : s \in \operatorname{bd} T \cap (A \backslash B)\}$$

together with one half of

$$\bigcup \{V_T(s) : s \in \operatorname{bd} T \cap A \cap B\},$$

where the boundary of T, $\operatorname{bd} T$, represents the coastline of the island, $A \cap B$ denotes the border between both countries, and $A \backslash B$ stands for the set of points of A not in the border. Observe that, as shown in [6, Proposition 18], given $s \in \operatorname{bd} T$, $V_T(s)$ is an angle equal to the sum of s with the normal cone to T at s, so that the international waters assigned to both countries would be limited by the bisector lines to the angles $V_T(s)$, $s \in \operatorname{bd} T \cap A \cap B$. In conclusion, the assignment of international waters by the proximity criterion only depends on a small number of Voronoi cells (just two in the typical case that A and B are connected sets).

Since the Voronoi cell of a set $T \ni s$ has the same Voronoi cell as its closure, this paper is focused on closed sets of sites (as in [6], [12] and [13]). In order to simplify the notation we introduce the families of sets

$$\mathcal{T}(s) := \{T \subset \mathbb{R}^n : \ T \text{ is closed, } s \in T\}$$

and

$$\mathcal{V}(s) := \{V \subset \mathbb{R}^n : \ V \text{ is closed and convex, } s \in V\}.$$

The *Voronoi mapping* of s, denoted by $\psi_s : \mathcal{T}(s) \rightarrow \mathcal{V}(s)$, associates to each set $T \in \mathcal{T}(s)$ its Voronoi cell $V_T(s)$. This mapping arises in a natural way in the study of Voronoi cells. Indeed, several results in the recent paper [6] can be interpreted in terms of the images by ψ_s of elements of $\mathcal{T}(s)$, e.g., $\psi_s(T)$ is a polyhedral convex set (a quasipolyhedral convex set, a closed convex cone, a manifold) whenever T is a finite set (a discrete set, a closed convex set, a manifold, respectively). The Voronoi mapping ψ_s has also been studied from a topological perspective. Reem [10] analyzes the stability of Voronoi diagrams with respect to perturbations of the sites, while in [7] the stability of a given Voronoi cell $\psi_s(T) = V_T(s)$ under small perturbations of s in $\mathbb{R}^n$ is studied, as well as either global perturbations of compact subsets $P \subset T \backslash \{s\}$, or individual perturbations of the elements of P, or simultaneous perturbations of s and P. The stability of the combinatorial structure of Voronoi diagrams was studied by Vyalyi, Gordeyev and Tarasov [14].

In this paper we characterize $\psi_s^{-1}(F)$ (the so-called Voronoi inverse problem) for an important subclass of sets $F \in \mathcal{V}(s)$, which includes the polyhedral convex sets, providing formulas for the computation of the largest and the smallest elements of $\psi_s^{-1}(F)$ (when the latter element does exist). Notice that the largest element of $\psi_s^{-1}(\psi_s(T))$ is the unique one for which any enlargement causes the Voronoi cell to shrink; similarly, the smallest element of $\psi_s^{-1}(\psi_s(T))$, when it exists, is the unique set belonging to $\psi_s^{-1}(\psi_s(T))$ for which the elimination of any of its elements provokes a growth in the

Voronoi cell. We also analyze some set-theoretic properties of ψ_s and the existence of fixed sets for this mapping and for the mapping η_s associating with each $T \in \mathcal{T}(s)$ the largest element of $\psi_s^{-1}(\psi_s(T))$. Moreover, we characterize those sets $T \in \mathcal{T}(s)$ that coincide with the largest or the smallest element of $\psi_s^{-1}(\psi_s(T))$. Finally, we apply all these results to obtain properties about Voronoi diagrams and extensions to arbitrary sets of sites.

This paper is organized as follows. Section 2 introduces the necessary notation and preliminaries while the next two sections deal with Voronoi cells of closed sets. Section 3 characterizes $\psi_s^{-1}(F)$ and discusses maximal and minimal elements of $\psi_s^{-1}(\psi_s(T))$, Section 4 analyzes relationships between $\psi_s^{-1}(F)$ and linear representations of F. Section 5 studies the closure operator that assigns, to each closed set T containing s, the largest element in $\psi_s^{-1}(\psi_s(T))$. Finally, extensions of the results provided in Section 3 to arbitrary sets of sites are discussed in Section 6.

2 Notation and preliminaries

Throughout the paper we use the following notation. The zero vector, the open unit ball and the unit sphere in $\mathbb{R}^n$ are denoted by $\mathbf{0}$, $\mathbf{B}$ and $\mathbf{S}^{n-1}$, respectively. Given $X \subset \mathbb{R}^n$, we denote by $\operatorname{aff} X$, $\operatorname{conv} X$, $\operatorname{cone} X = \mathbb{R}_+ \operatorname{conv} X$, $\operatorname{int} X$, $\operatorname{cl} X$, $\operatorname{bd} X$, and $\operatorname{rint} X$ the affine hull of X, the convex hull of X, the convex conical hull of X, the interior of X, the closure of X, the boundary of X, and the relative interior of X, respectively. The *negative polar* of a convex cone $X \subset \mathbb{R}^n$ is $X^\circ := \{y \in \mathbb{R}^n : y'x \leq 0 \ \forall x \in X\}$, and the *orthogonal subspace* to a linear subspace X is $X^\perp := \{y \in \mathbb{R}^n : y'x = 0 \ \forall x \in X\}$. The *support function* of X is $\delta_X^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, defined by $\delta_X^*(x) := \sup_{y \in X} y'x$. It is known that δ_X^* is a lower semicontinuous (lsc in brief) positively homogeneous convex function; moreover, δ_X^* is finite-valued (and so continuous on $\mathbb{R}^n$) whenever X is bounded. We will also use the metric projection onto a subset $X \subset \mathbb{R}^n$, $P_X(y)$.

We now recall some useful concepts and results about consistent linear systems (observe that s is the trivial feasible solution of the system $\upsilon_T(s)$ in (1)). Let $\sigma := \{a_k'x \leq b_k, \ k \in K\}$ be a linear consistent system with solution set $F \subset \mathbb{R}^n$, $F \neq \emptyset$ (we also say that σ is a linear representation of F). The *conic representation* of F,

$$C(F) := \left\{(a, b) \in \mathbb{R}^{n+1} : a'x \leq b, \ x \in F\right\},$$

coincides, by Farkas' Lemma ([5, Theorem 1.2]), with the closed convex cone

$$\operatorname{cl} \operatorname{cone} \{(a_k, b_k), k \in K; \ (\mathbf{0}, 1)\}.$$

In particular, the conic representation of the solution set, $\psi_s(T) = V_T(s)$, of the system $\upsilon_T(s)$ in (1) is the closed convex cone generated by the vertical vector upwards, $(\mathbf{0}, 1)$, and the *parabolic lift* of T at s, defined as the image of T by the mapping $t \mapsto \left(t - s, \frac{\|t\|^2 - \|s\|^2}{2}\right)$, i.e.,

$$C(\psi_s(T)) = \operatorname{cl}\operatorname{cone}\left\{\left(t - s, \frac{\|t\|^2 - \|s\|^2}{2}\right), t \in T;\ \ (\mathbf{0}, 1)\right\}.$$

Parabolic lifts were introduced in computational geometry as an alternative to the widely used stereographic map in [3]. In particular, parabolic lifts at $\mathbf{0}$ have already been used in relation to Voronoi Diagrams (e.g., in [3] and the section of [1] entitled "Back to geometry") and geometric separators [11].

An element $\widehat{x} \in \mathbb{R}^n$ satisfying $a_k'\widehat{x} < b_k$ for all $k \in K$ is called *Slater point* of σ. Notice that s is a Slater point of the system $\upsilon_T(s)$ because one always has

$$\|x - s\| < \|x - t\| \Longleftrightarrow (t - s)' x < \tfrac{\|t\|^2 - \|s\|^2}{2},$$

so, given $t \in T \backslash \{s\}$ and considering $x = s$, it follows that $(t - s)' s < \frac{\|t\|^2 - \|s\|^2}{2}$.

A given inequality is *redundant* in a consistent system $\sigma := \{a_k'x \leq b_k,\ k \in K\}$ whenever its elimination preserves the solution set; in particular, any inequality $a_k'x \leq b_k$ such that $a_k = \mathbf{0}$ is redundant in σ. A consistent linear system σ is said to be *minimal* when it does not contain redundant inequalities, in which case the index set K is countable [5, Theorem 6.4]. The solution sets of minimal systems are called *generalized polyhedral sets* (G-*polyhedral* in brief). If F is a full dimensional G-polyhedral set and we denote by $G(F)$ the set of vectors $(a, b) \in C(F)$ such that $a'x = b$ defines a facet (i.e., an $n - 1$ dimensional face) of F and $\|(a, b)\| = 1$, then $\sigma_G := \{a'x \leq b,\ (a, b) \in G(F)\}$ is a linear representation of F, and any minimal representation of F is a subsystem of σ_G up to re-scaling [5, Theorem 6.6]. Notice that, as a consequence, the closed unit ball in $\mathbb{R}^n$ is not G-polyhedral for $n \geq 2$. Infinite minimal systems may have Slater points which are not interior points of their solution sets.

Let F be a polyhedral convex set. Then F is G-polyhedral (as one can get an equivalent minimal system by sequential elimination of redundant inequalities from any finite representation of F), and $G(F)$ is finite. If, additionally, $\dim F = n$, then a finite linear representation σ of F is minimal if and only if each inequality of σ defines a different facet of F [5, Theorem 6.5]. Indeed, σ_G is the unique minimal representation of F up to re-scaling [5, Theorem 6.7]. Observe that σ_G can be formed by selecting one point at the relative interior of each of the facets of F and a corresponding hyperplane supporting F at each of these points, followed by the normalization of the vector of coefficients.

To finish this section, we present some new properties of ψ_s, which complement those already appeared in the literature. In particular we provide a sufficient condition for the Voronoi cell of a set T of sites to coincide with that of $\mathrm{bd}\, T$.

Proposition 1 *The mapping ψ_s transforms unions into intersections and has no fixed element.*

PROOF. Given an arbitrary family $T_i \in \mathcal{T}(s)$, $i \in I$, we have

$$\begin{aligned}\psi_s\left(\bigcup_{i\in I} T_i\right) &= \left\{x \in \mathbb{R}^n : (t-s)' x \leq \tfrac{\|t\|^2 - \|s\|^2}{2},\ t \in \bigcup_{i\in I} T_i\right\} \\ &= \bigcap_{i\in I}\left\{x \in \mathbb{R}^n : (t-s)' x \leq \tfrac{\|t\|^2 - \|s\|^2}{2},\ t \in T_i\right\} \\ &= \bigcap_{i\in I} \psi_s(T_i),\end{aligned}$$

so that ψ_s transforms unions into intersections.
Since $T \cap \psi_s(T) = \{s\}$, if $T \in \mathcal{T}(s)$ is a fixed point of ψ_s, i.e., $T = \psi_s(T)$, then the preceding equality means that $T = \{s\}$. Hence $\{s\} = \psi_s(\{s\})$, but $\psi_s(\{s\}) = \mathbb{R}^n \neq \{s\}$. This is a contradiction. □

Proposition 2 *Let $s \in \mathbb{R}^n$ and $T \in \mathcal{T}(s)$. Then s is an isolated point of T if and only if $s \in \mathrm{int}\, \psi_s(T)$, and in this case $\psi_s(T) = \psi_s(\mathrm{bd}\, T)$.*

PROOF. If s is an isolated point of T, then $d(s, T \setminus \{s\}) > 0$ and $s + \frac{1}{2} d(s, T \setminus \{s\}) \mathbf{B} \subset \psi_s(T)$, which shows that $s \in \mathrm{int}\, \psi_s(T)$. Conversely, if $s \in \mathrm{int}\, \psi_s(T)$, then, from $\mathrm{int}\, \psi_s(T) \cap (T \setminus \{s\}) \subset \psi_s(T) \cap (T \setminus \{s\}) = \emptyset$ it follows that s is an isolated point of T. (This equivalence appears in [6])

Assume now that s is an isolated point of T. We will show that $\psi_s(T) = \psi_s(\mathrm{bd}\, T)$; the set $\psi_s(\mathrm{bd}\, T)$ is well defined because $s \in \mathrm{bd}\, T$. Since $\mathrm{bd}\, T \subset T$, we have $\psi_s(T) \subset \psi_s(\mathrm{bd}\, T)$. To prove the opposite inclusion, let $x \in \psi_s(\mathrm{bd}\, T)$. We will first prove that $x \notin T \setminus \{s\}$. Assume, to the contrary, that $x \in T \setminus \{s\}$, and consider the metric projection $P_{[s,x] \cap (T \setminus \{s\})}(s)$ of s onto the closed set $[s,x] \cap (T \setminus \{s\})$ (a singleton set). Clearly, $P_{[s,x] \cap (T \setminus \{s\})}(s) \in \mathrm{bd}\,(T \setminus \{s\}) = \mathrm{bd}\, T \setminus \{s\}$; since $d\left(x, P_{[s,x] \cap (T \setminus \{s\})}(s)\right) < d(x,s)$, we get a contradiction with $x \in \psi_s(\mathrm{bd}\, T)$. Thus, either $x = s$ or $x \notin T$. In the first case, we have $x = s \in \psi_s(T)$. In the second case, for every $t \in T$ we have $P_{[x,t] \cap T}(x) \in \mathrm{bd}\, T$, and therefore from $x \in \psi_s(\mathrm{bd}\, T)$ it follows that $d(x,s) \leq d\left(x, P_{[x,t] \cap T}(x)\right) \leq d(x,t)$, which shows that $x \in \psi_s(T)$. □

When s is not an isolated point of T, the equality $\psi_s(T) = \psi_s(\mathrm{bd}\, T)$ may or may not hold, as the following examples show.

Example 3 *For* $T := \mathbb{R} \times \mathbb{R}_+$ *and* $s := (0,0)$, *one has* $\psi_s(T) = \{0\} \times \mathbb{R}_-$ *and* $\psi_s(\operatorname{bd} T) = \{0\} \times \mathbb{R}$.

Example 4 *For* $T := \mathbb{R}^n_+$ *and* $s := \mathbf{0}$, *one has* $\psi_s(T) = \mathbb{R}^n_- = \psi_s(\operatorname{bd} T)$.

The next corollary is a straightforward consequence of Proposition 2 and the fact that $\psi_s^{-1}(F) \neq \emptyset$ for all $F \in \mathcal{V}(s)$ [6, Theorem 2].

Corollary 5 *Let* $s \in \mathbb{R}^n$ *and* $F \in \mathcal{V}(s)$. *Then the following statements are equivalent:*

(i) $s \in \operatorname{int} F$.

(ii) s *is an isolated point of* T, *for every* $T \in \psi_s^{-1}(F)$.

(iii) There exists $T \in \psi_s^{-1}(F)$ *such that* s *is an isolated point of* T.

3 The Voronoi inverse problem

Given $F \in \mathcal{V}(s)$, the family of sets $\psi_s^{-1}(F)$ is always non-empty [6]; we will consider this family to be partially ordered by inclusion. Observe that $\psi_s^{-1}(F)$ contains the closure of the union of every collection of its elements; thus, there always exists a largest element in $\psi_s^{-1}(F)$, which is

$$T^+(F,s) := \operatorname{cl}\left(\bigcup_{T \in \psi_s^{-1}(F)} T\right).$$

A set $T \in \psi_s^{-1}(F)$ is *minimal* whenever $\psi_s^{-1}(F)$ contains no proper subset of T and it is the *smallest* element in $\psi_s^{-1}(F)$ when it is contained in any element of $\psi_s^{-1}(F)$; in particular, we denote by $T^-(F,s)$ such a smallest element when it exists . According to (1), if $T_1, T_2 \in \psi_s^{-1}(F)$ and $T_1 \subset T \subset T_2$, then $\operatorname{cl} T \in \psi_s^{-1}(F)$. If the smallest set $T^-(F,s)$ exists, then $\psi_s^{-1}(F)$ is a complete lattice:

$$\psi_s^{-1}(F) = \left\{T \in \mathcal{T}(s) : T^-(F,s) \subset T \subset T^+(F,s)\right\}. \tag{2}$$

The next example shows that $\psi_s^{-1}(F)$ may have no minimal element (in particular, no smallest element) even when F is polyhedral.

Example 6 *Let* $F := \mathbb{R} \times \mathbb{R}_+$ *and* $s := (0,0) = \mathbf{0}$. *Let* $T \in \psi_s^{-1}(F)$, *and consider any* $t := (t_1, t_2) \in T$. *Since* $F \subset \left\{x \in \mathbb{R}^2 : t'x \leq \frac{\|t\|^2}{2}\right\}$ *and* $(\pm\mu, 0), (0,\mu) \in F$ *for all* $\mu > 0$, *one has* $t_1 = 0$ *and* $t_2 \leq 0$, *so that* $T \subset \{0\} \times \mathbb{R}_-$. *If* $\mathbf{0}$ *is isolated in* T, *then* $T \setminus \{\mathbf{0}\}$ *is closed. Hence* $V_T(\mathbf{0}) = \mathbb{R} \times \left[\frac{\max_{t \in T \setminus \{\mathbf{0}\}} t_2}{2}, +\infty\right[\neq F$. *So,* T *is an infinite set having* $\mathbf{0}$ *as an accumula-*

tion point, and for any closed set $T' \subset T$ with this property it holds $\psi_s(T') = F$, i.e., T cannot be minimal.

From (1) it follows that $V_T(s) = s + V_{T-s}(\mathbf{0})$, i.e.,

$$\psi_s(T) = s + \psi_{\mathbf{0}}(T - s), \tag{3}$$

where $T - s := \{t - s : t \in T\}$; this allows us to reduce many proofs to the case $s = \mathbf{0}$.

3.1 The set $T^+(F, s)$

Theorem 7 *Let $s \in \mathbb{R}^n$ and $F \in \mathcal{V}(s)$. Then*

$$\begin{aligned} T^+(F,s) &= \left\{t \in \mathbb{R}^n : \delta^*_{F-s}(t-s) \leq \tfrac{\|t-s\|^2}{2}\right\} \\ &= \mathbb{R}^n \setminus \bigcup_{x \in F \setminus \{s\}} (x + \|x - s\| \mathbf{B}). \end{aligned} \tag{4}$$

If $F \neq \mathbb{R}^n$ then $T^+(F,s)$ is an unbounded set and, furthermore, the boundary of $T^+(F,s)$ is bounded whenever F is bounded.

PROOF. By (3), it is sufficient to prove this result for $s = \mathbf{0}$.

We have to prove that, if $\mathbf{0} \in F \subset \mathbb{R}^n$, the largest set $T \subset \mathbb{R}^n$ such that $V_T(\mathbf{0}) = F$ is

$$\widetilde{T} := \left\{t \in \mathbb{R}^n : \delta^*_F(t) \leq \frac{\|t\|^2}{2}\right\} = \mathbb{R}^n \setminus \bigcup_{x \in F \setminus \{\mathbf{0}\}} (x + \|x\| \mathbf{B}). \tag{5}$$

First, observe that $\widetilde{T}$ is closed (as the function $t \mapsto \delta^*_F(t) - \frac{\|t\|^2}{2}$ is lsc), and $\mathbf{0} \in \widetilde{T}$. Moreover, according to [4, Proposition 7], $\operatorname{epi} \delta^*_F = C(F)$, so

$$t \in \widetilde{T} \Leftrightarrow \left(t, \frac{\|t\|^2}{2}\right) \in C(F). \tag{6}$$

So, on the one hand, $\widetilde{T} \in \psi_{\mathbf{0}}^{-1}(F)$. On the other hand, if $T \in \psi_{\mathbf{0}}^{-1}(F)$, then

$$\operatorname{cl} \operatorname{cone} \left\{\left(t, \frac{\|t\|^2}{2}\right), t \in T; (\mathbf{0}, 1)\right\} = C(F), \tag{7}$$

so that

$$\left(t, \frac{\|t\|^2}{2}\right) \in C(F) \ \ \forall t \in T,$$

which implies that $T \subset \widetilde{T}$. Thus, $\widetilde{T}$ is the largest element of $\psi_{\mathbf{0}}^{-1}(F)$, i.e., $T^{+}(F,\mathbf{0}) = \widetilde{T}$. The second equation of (5) is established by the following chain of equivalences:

$$\begin{aligned} t \notin \widetilde{T} &\Leftrightarrow \exists x \in F : x't > \tfrac{\|t\|^2}{2} \\ &\Leftrightarrow \exists x \in F : \|t\|^2 - 2x't < 0 \\ &\Leftrightarrow \exists x \in F : \|t-x\|^2 < \|x\|^2 \\ &\Leftrightarrow t \in \bigcup_{x \in F\setminus\{\mathbf{0}\}} (x + \|x\| \mathbf{B}). \end{aligned}$$

We now prove that $T^{+}(F,\mathbf{0})$ is unbounded when $F \neq \mathbb{R}^n$. Since $C(F) \neq \mathbb{R}_{+}(\mathbf{0},1)$ (as $F \neq \mathbb{R}^n$), we can take $(a,b) \in C(F)$, with $a \neq \mathbf{0}$. Thus,

$$\left(\lambda a, \frac{\|\lambda a\|^2}{2}\right) = \lambda(a,b) + \left(\frac{\|\lambda a\|^2}{2} - \lambda b\right)(\mathbf{0},1) \in C(F) \;\; \forall \lambda \geq \frac{2|b|}{\|a\|^2},$$

so that $\left\{\lambda a : \lambda \geq \frac{2|b|}{\|a\|^2}\right\} \subset T^{+}(F,\mathbf{0})$, by (6), and $T^{+}(F,\mathbf{0})$ is unbounded. Finally, if $F \subset \rho \operatorname{cl} \mathbf{B}$, then

$$\operatorname{bd} T^{+}(F,\mathbf{0}) \subset \operatorname{cl}\left(\bigcup_{x \in F\setminus\{\mathbf{0}\}} (x + \|x\| \mathbf{B})\right) \subset F + \rho \operatorname{cl} \mathbf{B} \subset 2\rho \operatorname{cl} \mathbf{B}.$$

□

Consider again Example 6, where $F := \mathbb{R} \times \mathbb{R}_{+}$ and $s := (0,0) = \mathbf{0}$. By Theorem 7, we get

$$T^{+}(F,\mathbf{0}) = \mathbb{R}^2 \setminus \bigcup_{x \in F\setminus\{\mathbf{0}\}} (x + \|x\| \mathbf{B}) = \{0\} \times \mathbb{R}_{-}.$$

Example 8 *Theorem 7 applied to* $F := \operatorname{cl} \mathbf{B}$ *and* $s := \mathbf{0}$ *gives*

$$T^{+}(\operatorname{cl} \mathbf{B},\mathbf{0}) = \mathbb{R}^n \setminus \bigcup_{x \in \operatorname{cl} \mathbf{B}\setminus\{\mathbf{0}\}} (x + \|x\| \mathbf{B}) = (\mathbb{R}^n \setminus 2\mathbf{B}) \cup \{\mathbf{0}\}. \tag{8}$$

The next corollary provides simple formulas for computing $T^{+}(F,s)$ when F is a *Motzkin decomposable set* (that is, the sum of a compact convex set with a closed convex cone [4]); in particular, when it is a polyhedral convex set, or a closed convex cone with apex s, or a linear manifold.

Corollary 9 *If* $s \in C + D$*, where* $C \subset \mathbb{R}^n$ *is a compact convex set and* $D \subset \mathbb{R}^n$ *is a closed convex cone, then*

$$T^{+}(C+D,s) = \left\{t \in s + D^{\circ} : (u-s)'(t-s) \leq \frac{\|t-s\|^2}{2}, \; u \in C\right\}.$$

In particular:
(i) If $s \in F := \operatorname{conv}\{u_1, .., u_p\} + \operatorname{cone}\{v_1, .., v_q\}$, *with* $u_1, .., u_p, v_1, .., v_q \in \mathbb{R}^n$, *then* $T^+(F, s)$ *is the intersection of the reverse-convex set* $\bigcap_{i=1,\ldots,p} \left\{t \in \mathbb{R}^n : -\|t - s\|^2 + 2(u_i - s)'(t - s) \leq 0\right\}$ *with the translated finitely generated convex cone* $s + [\operatorname{cone}\{v_1, .., v_q\}]^\circ$.
(ii) If $D \subset \mathbb{R}^n$ *is a closed convex cone and* $s \in \mathbb{R}^n$, *then* $T^+(s + D, s) = s + D^\circ$.
(iii) If $F \subset \mathbb{R}^n$ *is a linear manifold and* $s \in F$, *then* $T^+(F, s) = s + (F - s)^\perp$.

PROOF. Let $F := C + D$, where C is a compact convex set and D is a closed convex cone. Since $F - s = (C - s) + D$, one has

$$\delta^*_{F-s}(t) = \begin{cases} \max_{u \in C-s} u't, \ t \in D^\circ, \\ +\infty, \qquad\qquad \text{otherwise}. \end{cases} \tag{9}$$

The conclusion follows from Theorem 7.
Statements (i) and (ii) are immediate consequences, with $C := \operatorname{conv}\{u_1, .., u_p\}$ and $D := \operatorname{cone}\{v_1, .., v_q\}$, and $C := \{s\}$, respectively.
Statement (iii) follows from (ii), with $D := F - s$, since $D^\circ = (F - s)^\perp$. $\square$

3.2 The set $T^-(F, s)$

With the aim of providing a sufficient condition for the existence of a smallest element of $\psi_s^{-1}(F)$, with each $s \in \mathbb{R}^n$ and $F \in \mathcal{V}(s)$ we associate the following set:

$$L(F, s) := \left\{t \in \operatorname{bd} T^+(F, s) : P_{T^+(F,s)\setminus\{s\}}(x) = \{t\} \text{ for some } x \in \operatorname{bd} F\right\}.$$

Proposition 10 *Let* $s \in \mathbb{R}^n$ *and* $F \in \mathcal{V}(s)$. *Then*

$$L(F, s) \subset \bigcap\nolimits_{T \in \psi_s^{-1}(F)} T.$$

PROOF. Let $t_0 \in L(F, s)$ and $T \in \psi_s^{-1}(F)$, and suppose that $t_0 \notin T$. Take $x_0 \in \operatorname{bd} F$ such that $P_{T^+(F,s)\setminus\{s\}}(x_0) = \{t_0\}$. Since $T\setminus\{s\} \subset T^+(F, s)\setminus\{s, t_0\}$, by $P_{T^+(F,s)\setminus\{s\}}(x_0) = \{t_0\}$ we have $d(x_0, T \setminus \{s\}) > d(x_0, t_0) \geq d(x_0, s)$. Consider the strictly positive number $\rho := (d(x_0, T \setminus \{s\}) - d(x_0, s))/2$ and take any $x \in x_0 + \rho\mathbf{B}$. Then, the inequalities

$$d(x, s) \leq d(x, x_0) + d(x_0, s) < \rho + d(x_0, s)$$

and

$$d(x, T \setminus \{s\}) \geq d(x_0, T \setminus \{s\}) - d(x, x_0) > d(x_0, T \setminus \{s\}) - \rho$$

imply

$$d(x, T \setminus \{s\}) - d(x, s) > d(x_0, T \setminus \{s\}) - d(x_0, s) - 2\rho = 0;$$

hence $x \in \psi_s(T)$. We thus have $x_0 + \rho\mathbf{B} \subset \psi_s(T) = F$, which is a contradiction with $x_0 \in \operatorname{bd} F$. Therefore $t_0 \in T$, and the proof is complete. $\square$

Corollary 11 *Let $s \in \mathbb{R}^n$ and $F \in \mathcal{V}(s)$. If $\{s\} \cup \operatorname{cl} L(F, s) \in \psi_s^{-1}(F)$, then $T^-(F, s)$ exists and coincides with $\{s\} \cup \operatorname{cl} L(F, s)$.*

We shall now present another sufficient condition for the existence of a smallest element of $\psi_s^{-1}(F)$, by combining the following proposition with the next lemma.

Proposition 12 *Let $s \in \mathbb{R}^n$, $T \in \mathcal{T}(s)$, and $t_0 \in T$ be such that the set $G_{t_0} := \psi_s(T) \cap \psi_{t_0}(T)$ is a facet of $\psi_s(T)$. Then, for every $T' \in \psi_s^{-1}(\psi_s(T))$, one has $t_0 \in T'$.*

PROOF. Let $T' \in \psi_s^{-1}(\psi_s(T))$. Since $s \in T'$, we will assume that $t_0 \neq s$, which implies that $s \notin G_{t_0}$. Let $u \in U := \operatorname{aff} G_{t_0} - \operatorname{aff} G_{t_0}$, and take $\overline{x} \in \operatorname{rint} G_{t_0}$. Take $\varepsilon > 0$ such that $\overline{x} + \varepsilon u \in G_{t_0}$. Since $d(\overline{x} + \varepsilon u, s) = d(\overline{x} + \varepsilon u, t_0)$, using the equality $d(\overline{x}, s) = d(\overline{x}, t_0)$ we deduce that $(t_0 - s)' u = 0$. Therefore, $t_0 - s$ is orthogonal to U. Observe that, by $G_{t_0} \subset \psi_s(T) = \psi_s(T')$, we have $d(\overline{x}, s) \leq d(\overline{x}, t)$ for every $t \in T'$.
Now, for $k \in \{1, 2, ...\}$, set $y_k := \left(1 + \frac{1}{k}\right)\overline{x} - \frac{1}{k}s$. One has $y_k \notin \psi_s(T)$, because $\overline{x}$ belongs to the intersection of G_{t_0}, a face of $\psi_s(T)$, with the segment having $s \in \psi_s(T) \setminus G_{t_0}$ and y_k as their endpoints. Therefore we have $d(y_k, t_k) < d(y_k, s)$ for some $t_k \in T'$. The sequence $\{t_k\}$ is bounded, because

$$d(t_k, \overline{x}) \leq d(t_k, y_k) + d(y_k, \overline{x}) < d(y_k, s) + d(y_k, \overline{x}) = \left(1 + \frac{2}{k}\right) d(\overline{x}, s) \leq 3d(\overline{x}, s). \tag{10}$$

Hence we can assume, w.l.o.g., that $\{t_k\}$ converges to some $\overline{t} \in T'$. The rest of the proof will be devoted to show that $\overline{t} = t_0$.
>From (10) and $y_k \to \overline{x}$ one gets $d(\overline{x}, \overline{t}) \leq d(\overline{x}, s)$, which implies that $d(\overline{x}, s) = d(\overline{x}, \overline{t})$. Take $\varepsilon > 0$ such that $\overline{x} + \varepsilon u \in G_{t_0}$. Since $d(\overline{x} + \varepsilon u, s) \leq d(\overline{x} + \varepsilon u, \overline{t})$, using the equality $d(\overline{x}, s) = d(\overline{x}, \overline{t})$ we deduce that $(\overline{t} - s)' u \leq 0$. This inequality must actually hold with the equal sign, because U is a linear subspace. Therefore, $\overline{t} - s$ is also orthogonal to U. Hence, as the dimension of U is $n - 1$, the point $\overline{t}$ belongs to the straight line determined by s and t_0. Since this line intersects the sphere centered at $\overline{x}$ with radius $d(\overline{x}, s)$ at

exactly two points, namely s and t_0, we have either $\overline{t} = s$ or $\overline{t} = t_0$.
For $k \in \{1, 2, ...\}$, define $S_k := \{(1-\lambda) y_k + \lambda x : x \in \operatorname{rint} G_{t_0},\ \lambda > 1\}$. Let $x \in \operatorname{rint} G_{t_0}$ and $\lambda > 1$. From the inequalities $d(x, s) \leq d(x, t_k)$ and $d(y_k, t_k) < d(y_k, s)$ (which are consequences of the choice of t_k), we deduce that

$$d((1-\lambda) y_k + \lambda x, t_k) > d((1-\lambda) y_k + \lambda x, s),$$

and hence $t_k \neq (1-\lambda) y_k + \lambda x$. This proves that $t_k \notin S_k$. On the other hand, $S_1 \subset S_k$, because for $x \in \operatorname{rint} G_{t_0}$ and $\lambda > 1$, setting $\mu := 1 + k(\lambda - 1)$ and $\widetilde{x} := \frac{\lambda}{1+k(\lambda-1)} x + \frac{(\lambda-1)(k-1)}{1+k(\lambda-1)} \overline{x}$, one has $\mu > 1$, $\widetilde{x} \in \operatorname{rint} G_{t_0}$ and $(1-\lambda) y_1 + \lambda x = (1-\mu) y_k + \mu \widetilde{x}$. Therefore $t_k \notin S_1$. Notice that $y_1 \notin \operatorname{aff} G_{t_0}$, because otherwise, as $\overline{x} \in \operatorname{aff} G_{t_0}$, we would have $s \in \psi_s(T) \cap \operatorname{aff} G_{t_0} = G_{t_0}$, which is a contradiction. Therefore, using that the dimension of $\operatorname{aff} G_{t_0}$ is $n-1$, we can see that the convex set S_1 (actually, each S_k) is open, because it is homeomorphic to an open set of the form $D \times]1, +\infty[$, where D is an open set in $\mathbb{R}^{n-1}$. This implies that $\overline{t} \notin S_1$, because $t_k \notin S_1$ and $t_k \to \overline{t}$. Since $s = -y_1 + 2\overline{x} \in S_1$, we conclude that $\overline{t} \neq s$, and hence $t_0 = \overline{t} \in T'$. $\square$

The next lemma allows us to obtain, from a given linear representation of $F \in \mathcal{V}(s)$, a set $T \in \psi_s^{-1}(F)$, by means of symmetries relative to the hyperplanes associated with the linear equalities in the linear representation.

Lemma 13 *Let $s \in \mathbb{R}^n$, and let $\sigma := \{c_k' x \leq d_k,\ k \in K\}$ be a representation of $F \in \mathcal{V}(s)$ such that s is Slater point of σ. If $c_k \neq \mathbf{0}$ for all k, the point t_k is the symmetric point of s with respect to $H_k := \{x \in \mathbb{R}^n : c_k' x = d_k\}$, and $T := \{s\} \cup \{t_k : k \in K\}$, then $V_T(s) = F$, that is,* $\operatorname{cl} T \in \psi_s^{-1}(F)$.

PROOF. Let σ and T be as in the statement. We define

$$\alpha_k := 2 \|c_k\|^{-2} (d_k - c_k' s).$$

We have $\alpha_k > 0$, because s is a Slater point of σ, and

$$t_k := \alpha_k c_k + s, \quad k \in K.$$

Straightforward calculations show that, for any $k \in K$,

$$\tfrac{\|t_k\|^2 - \|s\|^2}{2} = \alpha_k d_k,$$

so that

$$(t_k - s)' x \leq \tfrac{\|t_k\|^2 - \|s\|^2}{2} \ \Leftrightarrow\ c_k' x \leq d_k.$$

Thus, F is the Voronoi cell of T, and so that of $\operatorname{cl} T$, i.e., $\operatorname{cl} T \in \psi_s^{-1}(F)$. $\square$

In Example 6, since $\sigma := \{-x_2 \le 0\}$ is the unique minimal representation of F up to re-scaling, and the set T defined in Lemma 13 reduces to $\{(0,0)\}$ for $s := (0,0)$, we have $\operatorname{cl} T = \{(0,0)\} \notin \psi_{\mathbf{0}}^{-1}(F)$, so that the Slater condition is necessary in Lemma 13. Concerning Example 8, from Corollary 11 it follows that there exists a smallest element $T^{-}(\operatorname{cl}\mathbf{B}, \mathbf{0}) = \{(0,0)\} \cup 2\mathbf{S}^1$ of $\psi_{\mathbf{0}}^{-1}(\operatorname{cl}\mathbf{B})$, even though $\operatorname{cl}\mathbf{B}$ is not G-polyhedral. This example shows that from a smallest element of $\psi_s^{-1}(F)$ it is not always possible to get a minimal representation of F. It also shows the importance of the closedness assumption on the sets of sites T.

Theorem 14 *Let $s \in \mathbb{R}^n$, and let $\sigma := \{c_k' x \le d_k,\ k \in K\}$ be a minimal representation of $F \in \mathcal{V}(s)$, such that s is Slater point of σ. If $\dim F = n$, then there exists a smallest element $T^{-}(F,s)$ in $\psi_s^{-1}(F)$, namely, $T^{-}(F,s)$ is the closure of the set T defined in Lemma 13:*

$$T^{-}(F,s) = \operatorname{cl}\left[\{s\} \cup \left\{2\|c_k\|^{-2}(d_k - c_k' s)\, c_k + s : k \in K\right\}\right].$$

PROOF. Assume that σ, F and s are as in the statement. Then, by [5, Theorem 5.1 (III)], the set $\{x \in F : c_k' x = d_k\}$ is a facet of F for all $k \in K$. Let T be as in Lemma 13 and $T' \in \mathcal{T}(s)$. By Lemma 13, we have $\operatorname{cl} T \in \psi_s^{-1}(F)$. Since $T \subset T'$ by Proposition 12, we obtain $\operatorname{cl} T \subset T'$. Therefore $\operatorname{cl} T$ is the smallest element of $\psi_s^{-1}(F)$. $\square$

Example 15 *Consider the G-polyhedral set $F := \operatorname{conv}\{(k,k^2,0) : k \in \mathbb{Z}\}$ and $s := (0,1,0)$. We obtain two minimal representations σ_1 and σ_2 of F, by taking, for each $k \neq 0$, the two half-spaces whose boundaries contain the pair of adjacent extreme points $(k,k^2,0)$ and one of the two points $\left(0,1,\pm\frac{1}{2k}\right)$, in the case of σ_1, and one of the points $\left(0,1,\pm\frac{1}{2k+1}\right)$, in the case of σ_2. In other words, we take*

$$\sigma_1 := \left\{\begin{array}{l} (2k+1)\,x_1 - x_2 + 2k\left(k^2+k+1\right)x_3 \le k(k+1)\,, k \in \mathbb{Z} \\ (2k+1)\,x_1 - x_2 - 2k\left(k^2+k+1\right)x_3 \le k(k+1)\,, k \in \mathbb{Z} \end{array}\right\}$$

and

$$\sigma_2 := \left\{\begin{array}{l} (2k+1)\,x_1 - x_2 + (2k+1)\left(k^2+k+1\right)x_3 \le k(k+1)\,, k \in \mathbb{Z} \\ (2k+1)\,x_1 - x_2 - (2k+1)\left(k^2+k+1\right)x_3 \le k(k+1)\,, k \in \mathbb{Z} \end{array}\right\}.$$

Observe that s is Slater point for σ_1 and σ_2. Denote by T_1 and T_2 the sets obtained from σ_1 and σ_2, respectively, in the same way as T is obtained from σ in Lemma 13. One can easily check that these sets are closed; hence, by Lemma 13, one has $T_1, T_2 \in \psi_s^{-1}(F)$. Since $T_1 \cap T_2 = \{s\} \notin \psi_s^{-1}(F)$, there cannot be *a smallest element in $\psi_s^{-1}(F)$. Thus, the assumption on the dimension of F in Theorem 14 is not superfluous.*

Corollary 16 *Let $s \in \mathbb{R}^n$, and let $F \in \mathcal{V}(s)$ be polyhedral. Then, there exists a smallest element of $\psi_s^{-1}(F)$ if and only if $s \in \operatorname{int} F$.*

PROOF. Assume that $s \in \operatorname{int} F$. Since $\dim F = n$, there exists a minimal representation $\sigma := \{c_k' x \leq d_k,\ k \in K\}$ of F, with K finite, such that $\{x \in F : c_k' x = d_k\}$ is a facet of F for all $k \in K$. Then s is Slater point of σ, and Theorem 14 applies.
Now we assume that $s \in \operatorname{bd} F$ and there exists a smallest element $T^-(F, s)$ of $\psi_s^{-1}(F)$. Let $\sigma := \{c_k' x \leq d_k,\ k \in K\}$ be a minimal representation of F. Then K is finite, and the set $I := \{k \in K : c_k' s = d_k\}$ of active indices at s is non-empty. Let $J := K \backslash I$ be the set of inactive indices at s. Consider the following linear representations of F :

$$\sigma_1 := \left\{c_k' x \leq d_k + \frac{1}{m},\ k \in I, m \in \mathbb{N}; c_k' x \leq d_k,\ k \in J\right\}.$$

As s is a Slater point of σ_1, we can apply Lemma 13 to obtain a set

$$\begin{aligned} T_1 := \{s\} &\cup \left\{2\left\|c_k\right\|^{-2}\left(d_k + \tfrac{1}{m} - c_k' s\right) c_k + s, m \in \mathbb{N},\ k \in I\right\} \\ &\cup \left\{2\left\|c_k\right\|^{-2}\left(d_k - c_k' s\right) c_k + s,\ k \in J\right\}, \end{aligned}$$

which is closed (because $d_k - c_k' s = 0$ for all $k \in I$) and satisfies $T_1 \in \psi_s^{-1}(F)$ (by Lemma 13). Given any $k \in I$ and $m \in \mathbb{N}$, the element $2\left\|c_k\right\|^{-2}\left(d_k + \frac{1}{m} - c_k' s\right) c_k + s$ can be eliminated from T_1 without modifying the Voronoi cell, so it cannot belong to $T^-(F, s)$. Therefore

$$T^-(F, s) \subset T_2 := \{s\} \cup \left\{2\left\|c_k\right\|^{-2}\left(d_k - c_k' s\right) c_k + s : k \in J\right\},$$

which implies that $F \subsetneqq \psi_s(T_2) \subset \psi_s(T^-(F, s))$. This is a contradiction. $\square$

Theorems 7 and 14, or their respective Corollaries 9 and 16 in particular cases, are useful tools to solve the Voronoi inverse problem, as the next two examples illustrate.

Example 17 *Let $F := [-1, 1]^2$ and $s := (0, 0)$. >From Corollaries 9 (i) and 16, applied to the system $\{\pm x_1 \leq 1, \pm x_2 \leq 1\}$ (a minimal representation of F), we conclude that $T \in \psi_s^{-1}(F)$ if and only if T is closed, $\{(0,0), (2,0), (0,2), (-2,0), (0,-2)\} \subset T$ and $d(t, (\pm 1, \pm 1)) \geq \sqrt{2}$ for all $t \in T$.*

Example 18 *Let* $x_k := \left(\frac{1}{k}, \frac{1}{k^2}\right)$ *for* $k \in \mathbb{N}$,

$$F := \operatorname{cl}\operatorname{conv}\{x_k : k \in \mathbb{N}\} + \operatorname{cone}\{(-1,0),(2,3)\}.$$

and $s := (0,0) \in \operatorname{bd} F$*. By considering the line* l_k *joining* x_k *and* x_{k+1}*, we find that*

$$\sigma := \{(2k+1)\,x_1 - k\,(k+1)\,x_2 \leq 1,\ \ k \in \mathbb{N}\}$$

is a minimal representation of F*. Since* s *is a Slater point of* σ*, Theorem 14 implies the existence of the smallest element* $T^-(F,s)$ *of* $\psi_s^{-1}(F)$*, given by*

$$T^-(F,s) = \{s\} \cup \{t_k : k \in \mathbb{N}\},$$

where each t_k *is the symmetric point of* s *with respect to the line* l_k*.*
On the other hand, by Corollary 9 (i) applied to F*, we obtain*

$$T^+(F,s) = \left\{ t \in \operatorname{cone}\{(0,-1),(3,-2)\} : \frac{1}{k}t^1 + \frac{1}{k^2}t^2 \leq \frac{\|t\|^2}{2},\ \ k \in \mathbb{N} \right\}.$$

In this way we have solved the Voronoi inverse problem: $T \in \psi_s^{-1}(F)$ *if and only if* T *is closed and* $T^-(F,s) \subset T \subset T^+(F,s)$*.*

The next straightforward consequence of Theorem 14 extends to G-polyhedral sets a well-known property of polyhedral convex sets about uniqueness of minimal linear representations.

Corollary 19 *Each full dimensional G-polyhedral set has a unique minimal linear representation up to re-scaling.*

Corollary 20 *Let* $s \in \mathbb{R}^n$ *and let* $T \in \mathcal{T}(s)$ *be a finite set. Then,* $T^-(\psi_s(T), s) = T$ *if and only if* $\psi_s(T) \cap \psi_t(T)$ *is a facet of* $\psi_s(T)$ *for all* $t \in T \setminus \{s\}$*.*

PROOF. We just have to prove the "only if" part. The finiteness of T implies that $\psi_s(T)$ is a polyhedral convex set, and so a G-polyhedral set, while the existence of $T^-(\psi_s(T), s)$ entails that $s \in \operatorname{int}\psi_s(T)$ by Corollary 16. $\square$

The next corollary provides sufficient conditions for the existence of the smallest set of sites.

Corollary 21 *Let* $s \in \mathbb{R}^n$*,* $F \in \mathcal{V}(s)$ *and let*

$$T := \operatorname{cl}\left[\{s\} \cup \{t \in \mathbb{R}^n : \{x \in F : d(x,s) = d(x,t)\} \text{ is a facet of } F\}\right].$$

(i) *If the set* T *belongs to* $\psi_s^{-1}(F)$*, then* $T^-(F,s) = T$*.*
(ii) *If* F *is a G-polyhedral set and* $s \in \operatorname{int} F$*, then* $T^-(F,s) = T$*.*

PROOF. (i) is a direct consequence of Proposition 12.
(ii) We now assume that F is a G-polyhedral set and $s \in \operatorname{int} F$. Let $\sigma := \{c_k' x \leq d_k,\ k \in K\}$ be a minimal representation of F. Since s is a Slater point of σ and $\dim F = n$, the existence of $T^-(F,s)$ is assured by Theorem 14 and one has

$$T^-(F,s) = \operatorname{cl}\left[\{s\} \cup \left\{2\left\|c_k\right\|^{-2}(d_k - c_k' s)\, c_k + s : k \in K\right\}\right].$$

Now, for any $k \in K$ and $t = 2\left\|c_k\right\|^{-2}(d_k - c_k' s)\, c_k + s$, simple computations yield

$$\begin{pmatrix} t-s \\ \frac{\|t\|^2 - \|s\|^2}{2} \end{pmatrix} = 2\left\|c_k\right\|^{-2}(d_k - c_k' s)\begin{pmatrix} c_k \\ d_k \end{pmatrix},$$

and

$$\|x - s\| = \|x - t\| \Longleftrightarrow (t-s)' x = \tfrac{\|t\|^2 - \|s\|^2}{2}.$$

Hence

$$\{x \in F : d(x,s) = d(x,t)\} = \{x \in F : c_k' x = d_k\}. \tag{11}$$

Since F is full dimensional and σ is a minimal representation of it, [5, Theorem 5.1 (III)] yields that each inequality of σ determines a facet of F, i.e., $\{x \in F : c_k' x = d_k\}$ is a facet of F for all $k \in K$. The conclusion follows from (11). □

Both assumptions in Corollary 21 (ii) are necessary. On the one hand, taking $s := (0,0) \in \mathbb{R}^2$ and $F := \operatorname{cl} \mathbf{B}$, one has $s \in \operatorname{int} F$ and, by Corollary 11, the equality $T^-(F,s) = \{s\} \cup 2\mathbf{S}^1$ holds, while the exposed faces of F are not facets, so that the assumption that F is a G-polyhedral set is not superfluous. On the other hand, Example 15 shows a G-polyhedral set F with $s \in \operatorname{bd} F$ where $T^-(F,s) = T$ does not exist.

3.3 The Voronoi inverse problem for diagrams

Given a family of closed convex sets $\{F_i, i \in I\}$, *the Voronoi inverse problem for diagrams* consists in finding a (selection) mapping $\tau : I \to \mathbb{R}^n$ such that F_i is the Voronoi cell of $\tau(i)$ w.r.t. $\tau(I)$, i.e., $\psi_{\tau(i)}(\tau(I)) = F_i$, for all $i \in I$. This problem has a well-known geometric solution when $n = 2$ and I is finite (see, e.g., [8] and references therein). According to the previous results, if F_i is G-polyhedral and $\tau(i) \in \operatorname{int} F_i$, $i \in I$, and

$$\bigcup_{i \in I} T^-(F_i, s) \subset \tau(I) \subset \bigcap_{i \in I} T^+(F_i, s),$$

then τ solves the Voronoi inverse problem for diagrams. The formulation is simpler in the particular case that $I = \{1, ..., p\}$ because then, the selection problem consists in finding $(t_1, ..., t_p) \in (\mathbb{R}^n)^p$ such that $\psi_{t_i}(\{t_1, ..., t_p\}) = F_i$, $i = 1, ..., p$, and its solutions can be characterized as follows: $(t_1, ..., t_p) \in (\mathbb{R}^n)^p$ solves this inverse problem if and only if $t_i \in \operatorname{int} F_i, i = 1, ..., p$, and

$$\bigcup_{i=1}^{p} T^{-}(F_i, s) \subset \{t_1, ..., t_p\} \subset \bigcap_{i=1}^{p} T^{+}(F_i, s).$$

4 The set $\psi_s^{-1}(F)$ and linear representations of F

In this section, inspired by the proof of Theorem 7, we discuss some observations concerning the relationship of the set $\psi_s^{-1}(F)$ with the linear representations of F.

Let $\alpha_s : \mathbb{R}^n \setminus \{s\} \to (\mathbb{R}^n \setminus \{\mathbf{0}\}) \times \mathbb{R}$ be the mapping

$$\alpha_s(t) := \left(t - s, \frac{\|t\|^2 - \|s\|^2}{2}\right).$$

For every $T \in \mathcal{T}(s)$ one has

$$\psi_s(T) = \{x \in \mathbb{R}^n : a'x \leq b, (a, b) \in \alpha_s(T \setminus \{s\})\}. \tag{12}$$

So, given $F \in \mathcal{V}(s)$, each $T \in \psi_s^{-1}(F)$ induces a linear representation of F, whose index set is $\alpha_s(T \setminus \{s\})$.

Consider now the "inverse" mapping $\beta_s : (\mathbb{R}^n \setminus \{\mathbf{0}\}) \times \mathbb{R} \to \mathbb{R}^n$ defined as

$$\beta_s(a, b) := \frac{2(b - a's)}{\|a\|^2} a + s.$$

Then, $\beta_s \circ \alpha_s$ is the identity, while

$$(\alpha_s \circ \beta_s)(a, b) = \frac{2(b - a's)}{\|a\|^2}(a, b),$$

so that $\alpha_s \circ \beta_s$ is not the identity. The reason is that α_s is non-exhaustive; in fact,

$$\alpha_s(\mathbb{R}^n \setminus \{s\}) = \operatorname{gph}\left(\frac{\|\cdot + s\|^2 - \|s\|^2}{2}\right) \setminus \{(\mathbf{0}, 0)\}.$$

One can easily prove the following result:

Proposition 22 *Let $s \in \mathbb{R}^n$. The mapping $\alpha_s : \mathbb{R}^n \setminus \{s\} \to \operatorname{gph}\left(\frac{\|\cdot+s\|^2-\|s\|^2}{2}\right) \setminus \{(\mathbf{0},0)\}$ is a bijection; its inverse is the restriction of β_s to* $\operatorname{gph}\left(\frac{\|\cdot+s\|^2-\|s\|^2}{2}\right) \setminus \{(\mathbf{0},0)\}$.

The meaning of limiting ourselves to the set $\operatorname{gph}\left(\frac{\|\cdot+s\|^2-\|s\|^2}{2}\right) \setminus \{(\mathbf{0},0)\}$ is as follows. For $(a,b) \in (\mathbb{R}^n \setminus \{\mathbf{0}\}) \times \mathbb{R}$, with $a's < b$, defining $\left(\widetilde{a},\widetilde{b}\right) := (\alpha_s \circ \beta_s)(a,b)$ we get that the inequalities $a'x \leq b$ and $\widetilde{a}'x \leq \widetilde{b}$ are equivalent. The inequality $\widetilde{a}'x \leq \widetilde{b}$ can be interpreted as a normalized version of $a'x \leq b$, as it is the only equivalent inequality such that $\left(\widetilde{a},\widetilde{b}\right) \in \operatorname{gph}\left(\frac{\|\cdot+s\|^2-\|s\|^2}{2}\right)$.

For every set $D \subset (\mathbb{R}^n \setminus \{\mathbf{0}\}) \times \mathbb{R}$ such that $a's < b$ for all $(a,b) \in D$, we have

$$\{x \in \mathbb{R}^n : a'x \leq b, (a,b) \in D\} = V_{\beta_s(D)\cup\{s\}}(s). \tag{13}$$

Thus, if $F \in \mathcal{V}(s)$, for each linear representation of F where s is a Slater point, the data set D of coefficient vectors of this representation induces a set $T := \operatorname{cl}\beta_s(D) \cup \{s\} \in \psi_s^{-1}(F)$.

According to the previous remarks, there exists a bijective mapping from $\mathcal{T}(s)$ into the set of systems of "normalized" linear inequalities (i.e., those inequalities $a'x \leq b$ such that $a \neq \mathbf{0}$ and $(a,b) \in \operatorname{gph}\left(\frac{\|\cdot+s\|^2-\|s\|^2}{2}\right)$); by virtue of (12), for $T \in \mathcal{T}(s)$ the set $\alpha_s(T\setminus\{s\})$ defines such a system representing $\psi_s(T)$. Notice that s is a Slater point of those systems, since $b - a's = \|a\|^2/2 > 0$. Conversely, the closed convex set represented by the "normalized" linear system defined by the set $D \subset (\mathbb{R}^n \setminus \{\mathbf{0}\}) \times \mathbb{R}$ is precisely $\psi_s(\operatorname{cl}\beta_s(D) \cup \{s\})$. In summary, one has:

Theorem 23 *Let $s \in \mathbb{R}^n$, $F \in \mathcal{V}(s)$, and let $\mathcal{R}(F)$ be the set of all $D \subset \operatorname{gph}\left(\frac{\|\cdot+s\|^2-\|s\|^2}{2}\right) \setminus \{(\mathbf{0},0)\}$ such that $\{x \in \mathbb{R}^n : a'x \leq b, (a,b) \in D\} = F$. Then, the mappings $\psi_s^{-1}(F) \to \mathcal{R}(F)$ and $\mathcal{R}(F) \to \psi_s^{-1}(F)$ given by $T \mapsto \alpha_s(T\setminus\{s\})$ and $D \mapsto \operatorname{cl}\beta_s(D) \cup \{s\}$, respectively, are well-defined, bijective, and mutually inverse.*

5 A closure operator associated to the Voronoi mapping

We analyze a closure operator related to $T^+(\psi_s(T), s)$. The main result, Theorem 26, will enable us to show, in the next section, that if we allow for arbitrary (not necessarily closed) sets of sites, then for every $F \in \mathcal{V}(s)$ the collection of sets $\psi_s^{-1}(F)$ does not contain a smallest element $T^-(F,s)$.

We consider the mapping $\eta_s : \mathcal{T}(s) \to \mathcal{T}(s)$ defined by

$$\eta_s(T) := T^+(\psi_s(T), s).$$

By Theorem 7, one has

$$\begin{aligned}\eta_s(T) &= \left\{ t \in \mathbb{R}^n : \delta^*_{\psi_s(T)-s}(t-s) \leq \frac{\|t-s\|^2}{2} \right\} \\ &= \left\{ t \in \mathbb{R}^n : \left(t-s, \frac{\|t-s\|^2}{2} \right) \in C\left(\psi_s(T)-s\right) \right\}.\end{aligned}$$

Clearly, η_s is a closure operator, that is, it satisfies the following properties:

1) For every $T \in \mathcal{T}(s)$, one has $T \subset \eta_s(T)$.

2) η_s is monotone with respect to inclusion, that is, if $T, T' \in \mathcal{T}(s)$ and $T \subset T'$, then $\eta_s(T) \subset \eta_s(T')$.

3) η_s is idempotent, that is, $\eta_s^2 = \eta_s$.

To characterize the sets that are fixed with respect to the closure operator η_s (in other words, those sets $T \in \mathcal{T}(s)$ for which any enlargement provokes a shrinking of the cell), we need a lemma.

Lemma 24 *Let $T \subset \mathbb{R}^n$ be co-radiant, i.e., $\bigcup_{\lambda > 1} \lambda T \subset T$. If $\bar{t} \in T$ and $b \geq \frac{\|\bar{t}\|^2}{2}$, then*

$$(\bar{t}, b) \in \mathbb{R}_+ \left\{ \left(t, \frac{\|t\|^2}{2} \right), t \in T; \ \ (\mathbf{0}, 1) \right\}.$$

PROOF. The conclusion obviously holds if $\bar{t} = \mathbf{0}$. Suppose now that $\bar{t} \neq \mathbf{0}$. Since T is co-radiant, $\frac{2b}{\|\bar{t}\|^2}\bar{t} \in T$. Hence

$$(\bar{t}, b) = \frac{\|\bar{t}\|^2}{2b} \left(\frac{2b}{\|\bar{t}\|^2}\bar{t}, \frac{\left\| \frac{2b}{\|\bar{t}\|^2}\bar{t} \right\|^2}{2} \right) \in \mathbb{R}_+ \left\{ \left(t, \frac{\|t\|^2}{2} \right) : t \in T \right\}.$$

$\square$

We associate with each $T \in \mathcal{T}(s)$ the function $q_T : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by

$$q_T(x) := \begin{cases} \inf \left\{ \sum_{i=1}^{m} \lambda_i \|t_i\|^2 : t_i \in T, \lambda_i \geq 0, i = 1, ..., m; \right. \\ \qquad \left. \sum_{i=1}^{m} \lambda_i (1, t_i) = (1, x) \right\}, & x \in \operatorname{conv} T, \\ +\infty, & \text{otherwise.} \end{cases}$$

This function q_T is nothing else than the largest convex extension of $\|\cdot\|^2 |_T$.

A straightforward computation for the epigraph $\operatorname{epi} q_T$ of q_T in terms of the graph $\operatorname{gph} \|\cdot\|^2 |_T$ of $\|\cdot\|^2 |_T$ yields

$$\begin{aligned} \operatorname{epi} q_T &= \operatorname{conv} \operatorname{epi} \|\cdot\|^2 |_T = \operatorname{conv} \left(\operatorname{gph} \|\cdot\|^2 |_T + \mathbb{R}_+ (\mathbf{0}, 1) \right) \\ &= \operatorname{conv} \operatorname{gph} \|\cdot\|^2 |_T + \mathbb{R}_+ (\mathbf{0}, 1) \\ &= \operatorname{conv} \left\{ \left(t, \|t\|^2 \right) : t \in T \right\} + \mathbb{R}_+ (\mathbf{0}, 1) . \end{aligned} \tag{14}$$

If T is convex, $q_T(x) = \|x\|^2$ when $x \in T$, and $q_T(x) = +\infty$ otherwise.

Example 25 *Let $n = 1$, $T = \mathbb{R}_+$ and $s := 0$. Then, $\eta_s(T) = T$ and*

$$q_T(x) = \begin{cases} x^2, & x \geq 0, \\ +\infty, & x < 0. \end{cases}$$

It is easy to prove that every $T \in \mathcal{T}(s)$ satisfies the equality

$$\eta_s(T) = s + \eta_{\mathbf{0}}(T - s) ; \tag{15}$$

so the study of η_s reduces to that of $\eta_{\mathbf{0}}$.

Theorem 26 *Let $s \in \mathbb{R}^n$ and $T \in \mathcal{T}(s)$. Then $\eta_s(T) = T$ if and only if $T - s$ is co-radiant and satisfies*

$$\frac{q_{T-s}(t-s)}{\|t-s\|^2}(t-s) \in T - s \quad \textit{for all } t \in (\operatorname{conv} T) \setminus \{s\} .$$

PROOF. Using (15), it is enough to consider the case of $s = \mathbf{0}$. So we will prove that a set $T \in \mathcal{T}(\mathbf{0})$ satisfies $\eta_{\mathbf{0}}(T) = T$ if and only if it is co-radiant and satisfies

$$\frac{q_T(t)}{\|t\|^2} t \in T \quad \text{for all } t \in (\operatorname{conv} T) \setminus \{\mathbf{0}\} . \tag{16}$$

Assume first that $\eta_{\mathbf{0}}(T) = T$. Let $t \in T$ and $\lambda > 1$. Then, since $\eta_{\mathbf{0}}(T) = T$, we have $\left(t, \frac{\|t\|^2}{2}\right) \in C(\psi_{\mathbf{0}}(T))$; hence, every $x \in \psi_{\mathbf{0}}(T)$ satisfies $(\lambda t)' x \leq \lambda \frac{\|t\|^2}{2} \leq \frac{\|\lambda t\|^2}{2}$, that is, $\left(\lambda t, \frac{\|\lambda t\|^2}{2}\right) \in C(\psi_{\mathbf{0}}(T))$. Therefore $\lambda t \in \eta_{\mathbf{0}}(T) = T$, which shows that T is co-radiant. Let now $t \in (\operatorname{conv} T) \setminus \{\mathbf{0}\}$ and $t_i \in T$ and $\lambda_i \geq 0$ $(i = 1, ..., m)$ be such that $\sum_{i=1}^{m} \lambda_i = 1$ and $t = \sum_{i=1}^{m} \lambda_i t_i$. Since $\eta_{\mathbf{0}}(T) = T$, we have $\left(t_i, \frac{\|t_i\|^2}{2}\right) \in C(\psi_{\mathbf{0}}(T))$ for every i; hence, for every $x \in \psi_{\mathbf{0}}(T)$, we have $t'x = \sum_{i=1}^{m} \lambda_i t_i' x \leq \sum_{i=1}^{m} \lambda_i \frac{\|t_i\|^2}{2}$. Therefore $t'x \leq \frac{1}{2} q_T(t)$ or, equivalently, $\frac{q_T(t)}{\|t\|^2} t'x \leq \frac{\left\|\frac{q_T(t)}{\|t\|^2} t\right\|^2}{2}$, that is, $\left(\frac{q_T(t)}{\|t\|^2} t, \frac{\left\|\frac{q_T(t)}{\|t\|^2} t\right\|^2}{2}\right) \in C(\psi_{\mathbf{0}}(T))$. Hence $\frac{q_T(t)}{\|t\|^2} t \in \eta_{\mathbf{0}}(T) = T$.

Conversely, assume that T is co-radiant and satisfies (17). We will first prove that

$$\operatorname{cl} \operatorname{cone} \left\{ \left(t, \frac{\|t\|^2}{2}\right), t \in T; (\mathbf{0}, 1) \right\} = \mathbb{R}_+ \left\{ \left(t, \frac{\|t\|^2}{2}\right), t \in T; (\mathbf{0}, 1) \right\} \tag{17}$$

by showing that the set $\mathbb{R}_+ \left\{ \left(t, \frac{\|t\|^2}{2}\right), t \in T; (\mathbf{0}, 1) \right\}$ is convex and closed. To prove convexity, we will first show that every convex combination of elements of $\left\{ \left(t, \frac{\|t\|^2}{2}\right) : t \in T \right\}$ belongs to $\mathbb{R}_+ \left\{ \left(t, \frac{\|t\|^2}{2}\right), t \in T; (\mathbf{0}, 1) \right\}$. Let $t_i \in T$ and $\lambda_i \geq 0$ $(i = 1, ..., m)$ be such that $\sum_{i=1}^{m} \lambda_i = 1$, and define $t := \frac{q_T\left(\sum_{i=1}^{m} \lambda_i t_i\right)}{\left\|\sum_{i=1}^{m} \lambda_i t_i\right\|^2} \sum_{i=1}^{m} \lambda_i t_i$.

By (17), we have $t \in T$. From the equality

$$\left(t, \frac{\|t\|^2}{2}\right) = \frac{q_T\left(\sum_{i=1}^{m} \lambda_i t_i\right)}{\left\|\sum_{i=1}^{m} \lambda_i t_i\right\|^2} \left(\sum_{i=1}^{m} \lambda_i t_i, \frac{q_T\left(\sum_{i=1}^{m} \lambda_i t_i\right)}{2} \right)$$

it follows that

$$\left(\sum_{i=1}^{m} \lambda_i t_i, \frac{q_T\left(\sum_{i=1}^{m} \lambda_i t_i\right)}{2} \right) \in \mathbb{R}_+ \left\{ \left(t, \frac{\|t\|^2}{2}\right), t \in T \right\}.$$

Hence, given that $q_T\left(\sum_{i=1}^{m}\lambda_i t_i\right) \leq \sum_{i=1}^{m}\lambda_i q_T(t_i) = \sum_{i=1}^{m}\lambda_i \|t_i\|^2$, using Lemma 24 we can easily see that

$$\sum_{i=1}^{m}\lambda_i\left(t_i, \frac{\|t_i\|^2}{2}\right) \in \mathbb{R}_+\left\{\left(t, \frac{\|t\|^2}{2}\right), t \in T; (\mathbf{0},1)\right\}.$$

We have thus proved that every convex combination of elements of $\left\{\left(t, \frac{\|t\|^2}{2}\right) : t \in T\right\}$ belongs to $\mathbb{R}_+\left\{\left(t, \frac{\|t\|^2}{2}\right), t \in T; (\mathbf{0},1)\right\}$. Furthermore, using this fact and, again, Lemma 24, it is easy to see that every convex combination of elements of $\left\{\left(t, \frac{\|t\|^2}{2}\right), t \in T; (\mathbf{0},1)\right\}$ belongs to $\mathbb{R}_+\left\{\left(t, \frac{\|t\|^2}{2}\right), t \in T; (\mathbf{0},1)\right\}$. We thus conclude that the set $\mathbb{R}_+\left\{\left(t, \frac{\|t\|^2}{2}\right), t \in T; (\mathbf{0},1)\right\}$ is convex. To prove its closedness, since

$$\mathbb{R}_+\left\{\left(t, \frac{\|t\|^2}{2}\right), t \in T; (\mathbf{0},1)\right\} = \mathbb{R}_+\left\{\left(t, \frac{\|t\|^2}{2}\right) : t \in T\right\} \cup \mathbb{R}_+\left\{(\mathbf{0},1)\right\},$$

we only need to prove that $\mathbb{R}_+\left\{\left(t, \frac{\|t\|^2}{2}\right) : t \in T\right\}$ is closed. Let us consider sequences $\lambda_k \geq 0$ and $t_k \in T$ such that $\left(\lambda_k t_k, \lambda_k \frac{\|t_k\|^2}{2}\right)$ converges to some point (a,b). Then the sequence $t_k = \frac{\lambda_k\|t_k\|^2}{\|\lambda_k t_k\|^2}\lambda_k t_k$ converges to $\frac{2b}{\|a\|^2}a$, so this point belongs to T and, since $(a,b) = \frac{\|a\|^2}{2b}\left(\frac{2b}{\|a\|^2}a, \frac{\left\|\frac{2b}{\|a\|^2}a\right\|^2}{2}\right)$, we conclude that $(a,b) \in \mathbb{R}_+\left\{\left(t, \frac{\|t\|^2}{2}\right) : t \in T\right\}$. We have thus proved (17). We now need to check the inclusion $\eta_{\mathbf{0}}(T) \subset T$, since we already know that the opposite one holds. Let $t \in \eta_{\mathbf{0}}(T)$. Then, by (7) and (17), we have

$$\left(t, \frac{\|t\|^2}{2}\right) \in C\left(\psi_{\mathbf{0}}(T)\right) = \mathbb{R}_+\left\{\left(t', \frac{\|t'\|^2}{2}\right), t' \in T; (\mathbf{0},1)\right\}.$$

Hence $\left(t, \frac{\|t\|^2}{2}\right) = \lambda\left(t', \frac{\|t'\|^2}{2}\right)$ for some $\lambda \geq 0$ and $t' \in T$. We thus have $\lambda\|t'\|^2 = \|t\|^2 = \|\lambda t'\|^2 = \lambda^2\|t'\|^2$, and we conclude that either $t' = \mathbf{0}$, in which case $t = \mathbf{0} \in T$, or $\lambda = 1$, in which case $t = t' \in T$. $\square$

Notice that (17) necessarily holds whenever T is convex, as then $q_T(t-s) = \|t-s\|^2$ for all $t \in T$ (this is the case in Example 6 if we set $T := \{\mathbf{0}\} \times \mathbb{R}$, so that, as T is also co-radiant, one has $\eta_{\mathbf{0}}(T) = T$). Observe also that taking the closure in (17) is not superfluous (see Example 25).

Example 27 *Let $T := (\mathbb{R}^n \setminus \mathbf{B}) \cup \{\mathbf{0}\}$. Then T belongs to $\mathcal{T}(\mathbf{0})$ and is co-radiant. One can easily check that $q_T = \max\left\{\|\cdot\|, \|\cdot\|^2\right\}$, so condition (16)* *reduces to*

$$\left(\max\left\{\frac{1}{\|t\|},1\right\}\right)t\in T\ \forall t\in\mathbb{R}^n\backslash\{\mathbf{0}\}.$$

Since this condition clearly holds, one has

$$T=\eta_{\mathbf{0}}(T)=T^+(\psi_{\mathbf{0}}(T),\mathbf{0})=T^+\left(\frac{1}{2}\operatorname{cl}\mathbf{B},\mathbf{0}\right).$$

The next result shows that η_s transforms closed convex sets into convex cones.

Corollary 28 *Let $s\in\mathbb{R}^n$ and $T\in\mathcal{V}(s)$. Then $\eta_s(T)=s+\operatorname{cl}\mathbb{R}_+(T-s)$. In particular, if $T\in\mathcal{V}(\mathbf{0})$ is a cone, then $\eta_{\mathbf{0}}(T)=T$.*

PROOF. It is enough to show that, if $s:=\mathbf{0}$, then $\eta_s(T)=\operatorname{cl}\mathbb{R}_+T$. Recall that by [6, Proposition 18], the set $\psi_{\mathbf{0}}(T)$ is the normal cone to T at $\mathbf{0}$, which is equal to $(\mathbb{R}_+T)^\circ$; so, an application of Corollary 9 (ii) gives

$$\eta_{\mathbf{0}}(T)=T^+(\psi_{\mathbf{0}}(T),\mathbf{0})=T^+((\mathbb{R}_+T)^\circ,\mathbf{0})=(\mathbb{R}_+T)^{\circ\circ}=\operatorname{cl}\mathbb{R}_+T.$$

$\square$

6 Extension to non-closed sets of sites

In this final section we briefly discuss the extensions of the previous results to cells of arbitrary (possibly non-closed) sets of sites.

Denote by $\mathcal{U}(s):=\{T\subset\mathbb{R}^n:\ s\in T\}$ the class of subsets of $\mathbb{R}^n$ containing s, and let $\psi_s:\mathcal{U}(s)\to\mathcal{V}(s)$ be such that $\psi_s(T)=V_T(s)$ (i.e., ψ_s is the extension to non-closed sets of the mapping denoted in the same way that was introduced in Section 1).

Almost all the results of Sections 3 and 4 remain valid for arbitrary (not necessarily closed) sets of sites, e.g., Theorem 7, Corollary 9 and Theorem 26. The main difference of dealing with arbitrary, instead of closed, sets of sites is that $\psi_s^{-1}(F)$ never contains a smallest element in this new setting. We next show a counterpart of Theorem 14.

Proposition 29 *Let $\psi_s:\mathcal{U}(s)\to\mathcal{V}(s)$ be defined by $\psi_s(T)=V_T(s)$, and let $F\in\mathcal{V}(s)$ be different from $\mathbb{R}^n$. Then the following statements hold:*
(i) It does not exist a smallest element in $\psi_s^{-1}(F)$.
(ii) Let $\sigma:=\{c_k'x\leq d_k,\ k\in K\}$ be a minimal representation of F, with $\dim F=n$, such that s is Slater point of σ. Then, the set defined in Lemma 13 is a minimal element of $\psi_s^{-1}(F)$.

Proof: (i) Let s and F be as in the assumption, and assume that T_1 is the smallest element of $\psi_s^{-1}(F)$. Since $T^+(F,s) - s$ is co-radiant (Theorem 26), $T_2 - s := \bigcup_{\lambda>1} \lambda(T_1 - s) \subset T^+(F,s) - s$ and so $T_1 \subset \operatorname{cl} T_2 \subset T^+(F,s)$, which entails $\operatorname{cl} T_2 \in \psi_s^{-1}(F)$ by (2). Hence $T_2 \in \psi_s^{-1}(F)$ too, so $T_1 \subset T_2$. Since $F \neq \mathbb{R}^n$, we have $T_1 \neq \{s\}$. Let $t \in T_1 \setminus \{s\}$. As $T_1 \subset T_2$, in view of the definition of T_2, there exists $\alpha_1 \in (0,1)$ such that $s + \alpha_1(t-s) \in T_1$. The same argument, with t replaced by $s + \alpha_1(t-s) \in T_1$, shows the existence of $\alpha_2 \in (0, \alpha_1)$ such that $s + \alpha_2(t-s) \in T_1$. By repeating this procedure, we obtain a strictly decreasing sequence of positive numbers α_k such that $s + \alpha_k(t-s) \in T_1$. Clearly, the set obtained from T_1 by replacing the terms of the sequence $\{s + \alpha_k(t-s)\}$ by those of any of its subsequences belongs to $\psi_s^{-1}(F)$, which contradicts the minimality of T_1.

(ii) The argument is similar to that of Theorem 14 without taking the closure of the set defined in Lemma 13. □

In conclusion, when ψ_s is defined for not necessarily closed sets of sites, $\psi_s^{-1}(F)$ cannot be expressed by means of formula (2) except in the trivial case when $F := \mathbb{R}^n$.

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