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A Natural Extension of the Classical Envelope Theorem in Vector Differential Programming

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Abstract

The aim of this paper is to extend the classical envelope theorem from scalar to vector differential programming. The obtained result allows us to measure the quantitative behaviour of a certain set of optimal values (not necessarily a singleton) characterized to become minimum when the objective function is composed with a positive function, according to changes of any of the parameters which appear in the constraints. We show that the sensitivity of the program depends on a Lagrange multiplier and its sensitivity.

Key words: Envelope Theorem; Set-valued Map; Tangential Regularity; Contingent or Bouligand Derivative; Clarke Derivative.

1 Introduction

The "classical" envelope theorem is a corollary of the Kuhn-Tucker theorem which characterizes the rate of change of the optimal value of a problem with respect to variations on some of its parameters. It was firstly introduced into economic theory by Hotelling [1] in 1932. Since the work of Samuelson [3], in 1947, and Viner [2], in 1952, the envelope theorem has become a standard tool in modern economic analysis. Many of the central results in competitive consumer and producer theory are applications of the envelope theorem. The famous lemmas of Hotelling, Shephard, and Roy are directly deducted from it. Over the years, several extensions of the traditional envelope theorems have emerged, as a response to the different necessities that have arisen. Among the most important authors who have contributed to this task, we can highlight Samuelson ([3, page 34]), who provided in 1947 the first proof of the envelope theorem for the generic class of differentiable unconstrained optimization problems; Afriat [3], who provided in 1971 a proof of the envelope theorem for the class of differentiable constrained optimization problems; Epstein [5], who in 1978 derived an envelope expression for a general parameter in optimal control problems; Caputo [6], who covered in 1996 static games with locally differentiable Nash equilibria; and Rincon-Zapatero and Santos [7], who in 2009 extended the classical C^1 envelope theorem to infinite horizon stochastic dynamic programming; additionally we can cite some others important authors such as Silberberg [8, 9], Rockafellar [10], Benveniste and Scheinkman [11], and so on.

Another significant step was taken in 1998 by Balbás, Fernández and Jiménez-Guerra [12], who extended the classical result to the field of vector programming in a quite general context of arbitrary Banach spaces. In this work, by applying a selection in the efficient set, two versions of the envelope theorem for differentiable and convex programs were stated. In the paper the authors used the so-called T-optimal solutions, concept successfully utilized in many other works of sensitivity analysis [13–22]. These solutions are characterized to become minimum when the

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objective function is composed with a positive function, T, and under weak requirements are dense in the efficient set.

The objective of this paper is to extend the former approach for differential programs even further, by eluding the aforementioned selection through the introduction of set-valued derivatives in the study. Then, the obtained result will allow us to measure the quantitative behaviour of certain sets of optima, no necessarily singleton, according to changes on some of the parameters of the problem. The study will be accomplished by using two criteria of regularity: derivability and tangential regularity. Thus three derivatives will be involved, the contingent, adjacent, and circatangent derivatives. Another goal of this work is that the obtained result extends the classical envelope theorem from scalar to vector optimization, leaving the first case as a particular instance of the second. This fact not always happen as can be seen in [18], in which an envelope theorem for vector convex programs with inequality constraints was formulated, but the classical scalar case is not exactly included as a particular instance of it.

The article is organized as follows. Section 2 introduces notation, basic concepts, and some results which will be used throughout the paper. In Section 3 we state and prove the main results of the paper, Theorems 3.10 and 3.11. In addition, we provide Example 3.13 which illustrates the sensitivity analysis done.

2 Notation and preliminaries

2.1 Definition of the problem

Let X, P, Y, Z, and W be five Banach spaces such that Y, Z, and W are ordered vector spaces, and W is also a Banach lattice. Let Y_+, Z_+, and W_+ denote the positive cones of Y, Z, and W, respectively. Moreover, assume that Y_+ and Z_+ are closed, Y_+ is also pointed, and the order of W verifies the infimum axiom. Let $T : Y \to W$ be a positive (i.e., $T(Y_+\backslash\{0\}) \subset W_+\backslash\{0\}$) linear and continuous surjective map such that Ker T has a topological supplement, Y_T. For example, when Y is a Hilbert space, the orthogonal complement of Ker T can be chosen as Y_T. Let $\widehat{T}$ denote the restriction of T to Y_T and π the natural projection from Y onto Ker T. It follows from the open mapping theorem, [23, Theorem 2.11], that the inverse operator $\widehat{T}^{-1}$is continuous. Moreover, let us consider an open and convex set $V \subset P$, an open set $D \subset X$, and two continuously Fréchet differentiable maps defined $f : D \subset X \to Y$ and $g : D \times V \subset X \times P \to Z$. Fixed $x_0 \in D$, we define the map $g[x_0] : V \subset P \to Z$ by $g[x_0](p) := g(x_0, p)$, for every $p \in V$. Similarly on the other variable, fixed $p_0 \in V$, we define the map $g[p_0] : D \subset X \to Z$ by an analogous way $g[p_0](x) := g(x, p_0)$, for every $x \in D$.

Let us denote by (1_p) the following differentiable optimization program:

$$\left.\begin{array}{c} \text{Min} \;\; f(x) \\ x \in D,\; g(x,p) = 0 \end{array}\right\} (1_p)$$

with $p \in V$. We adopt here the concept of T-optimal solution introduced in [12]. We say that $x_p \in D$ is a T-optimal solution of (1_p) if $Tf(x_p) \leq Tf(x)$ for every $x \in D$ such that $g(x,p) = 0$. Note that every T-optimal solution of (1_p) is an optimal solution of (1_p), i.e. $f(x_p) - f(x) \notin Y_+\backslash\{0\}$ for every $x \in D$ such that $g(x,p) = 0$. We say that a T-optimal solution x_p of (1_p) is regular, if $g[x_p]$ is Fréchet differentiable at p, the corresponding Fréchet differential $dg[x_p]_p$ is surjective, and Ker $dg[x_p]_p$ has a topological supplement S_{x_p}.

Throughout the paper, $\mathfrak{L}(X,Y)$ denotes the space of all linear and continuous maps from the Banach space X into the Banach space Y endowed with the usual norm. For short, the composition of two maps R and S will be represented by SR instead of $S \circ R$. Let us fix $p \in V$, a non negative operator $L_p \in \mathfrak{L}(Z,W)$ (i.e., $L_p(Z_+) \subset W_+$), and a T-optimal solution $x_p \in D$ of (1_p). Following again [12], it is said that L_p is a Lagrange T-multiplier of (1_p) associated to x_p if $Tdf_{x_p} = -L_p dg[p]_{x_p}$.

2.2 Some useful tools to manage set-valued maps

Now, we recall some of the basic concepts of set-valued analysis which will be useful in the current work (for further information see for instance the book [24]).

Let $A \subset X$ be a nonempty set and $x \in \overline{A}$. The Bouligand or contingent cone $T_A(x)$ is defined by

$$T_A(x) = \{v \in X\colon \liminf_{h \to 0+} \frac{d(A, x + hv)}{h} = 0\}.$$

Therefore, $v \in T_A(x)$ if and only if there exist two sequences, $\{h_n\}_{n=1}^{\infty} \subset R_+ \backslash \{0\}$ converging to 0 and $\{v_n\}_{n=1}^{\infty} \subset X$ converging to v, such that $x + h_n v_n \in A$ for all $n \in \mathbb{N}$. The intermediate or adjacent cone $T_A^{\flat}(x)$ is defined by

$$T_A^{\flat}(x) = \{v \in X\colon \lim_{h \to 0+} \frac{d(A, x + hv)}{h} = 0\}.$$

Therefore, $v \in T_A^{\flat}(x)$ if and only if for every sequence $\{h_n\}_{n=1}^{\infty} \subset R_+ \backslash \{0\}$ converging to 0 there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset X$ converging to v such $x + h_n v_n \in A$ for all $n \in \mathbb{N}$. Finally, the Clarke or circatangent cone $C_A(x)$ is defined by

$$C_A(x) = \{v \in X\colon \lim_{h \to 0+ \ \hat{x} \underset{\hat{x} \in A}{\to} x} \frac{d(A, \hat{x} + hv)}{h} = 0\}.$$

Therefore, $v \in C_A(x)$ if and only if for every two sequences , $\{h_n\}_{n=1}^{\infty} \subset R_+ \setminus \{0\}$ converging to 0 and $\{x_n\}_{n=1}^{\infty} \subset A$ converging to x, there exists a sequence $\{v_n\}_{n=1}^{\infty} \subset X$ converging to v such that $x_n + h_n v_n \in A$ for all $n \in \mathbb{N}$. The following inclusions are fulfilled: $C_A(x) \subset T_A^{\flat}(x) \subset T_A(x)$.

Let $F : A \rightrightarrows Y$ be a set-valued map and $(x, y) \in \mathrm{Graph}(F) = \{(x, y) \in X \times Y \mid y \in F(x)\}$. The Bouligand or contingent derivative $DF(x, y)$ of F at (x, y) is the set-valued map from X to Y defined by $\mathrm{Graph}(DF(x, y)) = T_{\mathrm{Graph}(F)}(x, y)$, the adjacent derivative $D^{\flat}F(x, y)$ of F at (x, y) is the set-valued map from X to Y defined by $\mathrm{Graph}(D^{\flat}F(x, y)) = T^{\flat}_{\mathrm{Graph}(F)}(x, y)$, and the Clarke derivative or circaderivative $CF(x, y)$ of F at (x, y) is the set-valued map from X to Y defined by $\mathrm{Graph}(CF(x, y)) = C_{\mathrm{Graph}(F)}(x, y)$.

We say that F is derivable at $(x, y) \in \mathrm{Graph}(F)$ if $DF(x, y) = D^{\flat}F(x, y)$. If F is single-valued and Fréchet differentiable at x then F is derivable at $(x, F(x))$ and $DF(x, F(x))(u) = dF_x(u)$ for every $u \in X$. We say that F is tangentially regular at $(x, y) \in \mathrm{Graph}(F)$ if $DF(x, y) = CF(x, y)$. If F is single-valued and continuously differentiable at x then F is tangentially regular at $(x, F(x))$ and $CF(x, F(x))(u) = dF_x(u)$ for every $u \in X$.

We will devote the last part of this subsection to remind two properties on regularity of set-valued maps. These properties will be useful in the proof of Theorem 3.10 in Section 3.

Throughout this subsection $\Sigma : V \subset P \rightrightarrows \mathfrak{L}(P, Y)$ denotes a set-valued map, $(p_0, G_0) \in \mathrm{Graph}\Sigma$, and $\check{\Sigma} : V \subset P \rightrightarrows Y$ is the set-valued map defined by $\check{\Sigma}(p) := \Sigma(p)(p)$, for every $p \in V$.

If Σ is a single-valued and Fréchet differentiable map at p_0, then $\check{\Sigma}$ is also Fréchet differentiable at p_0 and

$$\check{\Sigma}'(p_0, q) = \Sigma'(p_0, q)(p_0) + G_0(q),$$

for every $q \in P$, [13, Lemma 11]. Nevertheless, this fact does not remain true for derivable or tangentially regular set-valued maps.

Being Σ derivable, a necessary and sufficient condition to guarantee derivability of $\check{\Sigma}$ is that Σ fulfils Property $\mathfrak{R}$, [19, Theorem 6]. Here we recall that property.

Definition 2.1. [19, Definition 5]. *We say that the set-valued map $\Sigma : V \subset P \rightrightarrows \mathfrak{L}(P, Y)$ satisfies property $\mathfrak{R}$ at $(p_0, G_0) \in$ GraphΣ when:*

Given three sequences $\{p_n\}_{n=1}^{\infty} \subset P$, $\{h_n\}_{n=1}^{\infty} \subset \mathbb{R}_+ \backslash \{0\}$, and $\{G_n\}_{n=1}^{\infty} \subset \mathfrak{L}(P, Y)$ such that:

a.1) $\{p_n\}_{n=1}^{\infty}$ is convergent and $\{h_n\}_{n=1}^{\infty}$ converges to 0, *a.2)* $G_n \in \Sigma(p_0 + h_n p_n)$ *for every* $n \in \mathbb{N}$ *and the sequence*

$$\left\{ \frac{G_n(p_0 + h_n p_n) - G_0(p_0)}{h_n} \right\}_{n=1}^{\infty} \text{ is convergent.}$$

Then, there exist two sequences, $\{\bar{p}_n\}_{n=1}^{\infty} \subset P$ *and* $\{\bar{G}_n\}_{n=1}^{\infty} \subset \mathfrak{L}(P, Y)$ *such that:*

b.1) $\lim_{n\to\infty} \bar{p}_n = \lim_{n\to\infty} p_n$,

b.2) $\bar{G}_n \in \Sigma(p_0 + h_n \bar{p}_n)$ *for every* $n \in \mathbb{N}$ *and*

$$\lim_{n\to\infty} \frac{\bar{G}_n(p_0 + h_n \bar{p}_n) - G_0(p_0)}{h_n} = \lim_{n\to\infty} \frac{G_n(p_0 + h_n p_n) - G_0(p_0)}{h_n},$$

b.3) the sequence

$$\left\{ \frac{\bar{G}_n - G_0}{h_n} \right\}_{n=1}^{\infty} \text{ is convergent in } \mathfrak{L}(P, Y).$$

Nonetheless, [22, Example 3.1] shows that Property $\mathfrak{R}$ is not sufficient to assure the tangential regularity of $\check{\Sigma}$ even when Σ is tangentially regular. To guarantee tangential regularity of $\check{\Sigma}$, the set-valued map Σ must also to verify an additional property of regularity called $\mathfrak{S}$. Here we remember it.

Definition 2.2. [19, Definition 3.2]. *We say that the set-valued map* $\Sigma : V \subset P \rightrightarrows \mathfrak{L}(P, Y)$ *satisfies property* $\mathfrak{S}$ *at* $(p_0, G_0) \in Graph\Sigma$ *when:*

Given two sequences $\{p_n\}_{n=1}^{\infty} \subset P$ *and* $\{R_n\}_{n=1}^{\infty} \subset \mathfrak{L}(P, Y)$ *such that:*

a.1) $\{p_n\}_{n=1}^{\infty}$ *converges to* p_0 *and* $R_n \in \Sigma(p_n)$ *for every* $n \in \mathbb{N}$,

a.2) $\{R_n(p_n)\}_{n=1}^{\infty}$ *converges to* $G_0(p_0)$.

Then, there exists a sequence $\{\bar{R}_n\}_{n=1}^{\infty} \subset \mathfrak{L}(P, Y)$ *such that:*

b.1) $\bar{R}_n \in \Sigma(p_n)$ *and* $\bar{R}_n(p_n) = R_n(p_n)$ *for every* $n \in \mathbb{N}$,

b.2) $\{\bar{R}_n\}_{n=1}^{\infty}$ *converges to* G_0.

Finally, [22, Theorem 3.1] shows that if a set-valued map Σ is tangentially regular and satisfies Properties $\mathfrak{R}$ and $\mathfrak{S}$ at (p_0, G_0), then $\check{\Sigma}$ is also tangentially regular at $(p_0, G_0(p_0))$ and

$$C\check{\Sigma}(p_0, G_0(p_0))(q) = C\Sigma(p_0, G_0)(q)(p_0) + G_0(q), \tag{1}$$

for every $q \in P$. [22, Example 3.2] shows that tangential regularity of Σ and $\check{\Sigma}$ do not imply Σ to enjoy Property $\mathfrak{S}$ nor (1) be satisfied.

3 Sensitive Analysis

Let us begin this section by introducing some necessary ingredients in order to do the sensitivity analysis of the problem (1_p) introduced in Subsection 2.1.

Definition 3.1. *Let us fix* $p \in V$ *and a* T*-optimal regular solution* $x_p \in D$ *of* (1_p)*. We say that:*

(i) the map $\mathcal{G}_{x_p} \in \mathfrak{L}(P, Y)$ *is a Lagrange premultiplier of* (1_p) *associated to* x_p *if*

(a) $T\mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1} dg[p]_{x_p} = -T df_{x_p}$,

(b) $\mathcal{G}_{x_p}(Ker\ dg[x_p]_p) \subset Ker\ T$,

(c) $\pi\mathcal{G}_{x_p}(p) = \pi f(x_p)$,

(ii) the map $G_{x_p} \in \mathfrak{L}(Z, Y)$ *is a Lagrange multiplier of* (1_p) *associated to* x_p *if* $G_{x_p} dg[x_p]_p$ *is a Lagrange premultiplier of* (1_p) *associated to* x_p.

Condition $(i.a)$ is the analogous to the condition which defines the notion of Lagrange T-multiplier introduced in [12] and commented at the end of Subsection 2.1.

Our next step is to ensure that the former premultipliers and multipliers there exist. In the following proof, and throughout the remain of the work, we will denote by $\lambda \cdot x$ or $x \cdot \lambda$ the canonical product of the scalar λ and the vector x.

From now on, we fix a continuously Fréchet differentiable map $\beta : V \subset P \to P^*$ such that $\beta(q)(q) = 1$ for every $q \in V$. The existence of such a β is guaranteed. Indeed, since V is open and convex and $0 \notin V$, [23, Theorem 3.4] provides a $p^* \in P^*$ such that $p^*(0) < p^*(q)$ for every $q \in V$. Consider $\beta(q) := p^*/p^*(q)$ for all $q \in V$ and we get the required β.

Proposition 3.2. *Let us fix* $p \in V$ *and* x_p *a* T*-optimal regular solution of* (1_p)*. The following statements hold.*

(i) There exists $\mathcal{G}_{x_p} \in \mathfrak{L}(P, Y)$ *a Lagrange premultiplier of* (1_p) *associated to* x_p.

(ii) If $dg[x_p]_p(p) \neq 0$*, then there exists* $G_{x_p} \in \mathfrak{L}(Z, Y)$*, a Lagrange multiplier of* (1_p) *associated to* x_p.

Proof. Statement (i). From [14, Theorem 2] there exists $L_p \in \mathfrak{L}(Z, W)$ such that

$$L_p dg[p]_{x_p} = -T df_{x_p}.$$

Consider

$$\mathcal{G}_{x_p}(q) := \widehat{T}^{-1} L_p dg[x_p]_p(q) + \pi f(x_p) \cdot \beta(p)(q), \forall q \in P.$$

Let us check that $\mathcal{G}_{x_p}$ is a Lagrange premultiplier associated to x_p.
Condition (a): Since given any $z \in Z$,

$$\begin{aligned} T\mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1}(z) &= T\widehat{T}^{-1} L_p dg[x_p]_p dg[x_p]_p|_{S_{x_p}}^{-1}(z) + T(\pi f(x_p)) \cdot \beta(p)(dg[x_p]_p|_{S_{x_p}}^{-1}(z)) \\ &= L_p(z) + 0_Y = L_p(z), \end{aligned}$$

we obtain that

$$T\mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1} dg[p]_{x_p} = L_p dg[p]_{x_p} = -T df_{x_p}.$$

Condition (b): For any $q \in \text{Ker } dg[x_p]_p$, we have that

$$\mathcal{G}_{x_p}(q) = \widehat{T}^{-1} L_p dg[x_p]_p(q) + \pi f(x_p) \cdot \beta(p)(q) = 0_Y + \pi f(x_p) \cdot \beta(p)(q) \in \text{Ker } T.$$

Condition (c): $\pi\mathcal{G}_{x_p}(p) = \pi\widehat{T}^{-1}(L_p(dg[x_p]_p(p))) + \pi f(x_p) \cdot \beta(p)(p) = 0_Y + \pi f(x_p)$.

Statement (ii). Let us fix $\mathcal{G}_{x_p}$ a Lagrange premultiplier of (1_p) associated to x_p and $z_0 \in Z^*$ such that $z_0(dg[x_p]_p(p)) \neq 0$. Decompose $p = p' + p'' \in \text{Ker } dg[x_p]_p \oplus S_{x_p}$ and consider

$$G_{x_p}(z) := \mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1}(z) + \mathcal{G}_{x_p}(p') \cdot \frac{z_0(z)}{z_0(dg[x_p]_p(p))}, \forall z \in Z.$$

Let us check that G_{x_p} is a Lagrange multiplier of (1_p) associated to x_p, or equivalently, that

$G_{x_p} dg[x_p]_p$ is a Lagrange premultiplier associated to x_p.
Condition (a): Since $T(\mathcal{G}_{x_p}(p')) = 0_W$, fixed any $x \in X$, we have

$$
\begin{aligned}
TG_{x_p} dg[x_p]_p dg[x_p]_p|_{S_{x_p}}^{-1} dg[p]_{x_p}(x) &= TG_{x_p} dg[p]_{x_p}(x) \\
= T\mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1} dg[p]_{x_p}(x) &+ T(\mathcal{G}_{x_p}(p')) \cdot \frac{z_0(dg[p]_{x_p}(x))}{z_0(dg[x_p]_p(p))} \\
= T\mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1} dg[p]_{x_p}(x) &= -Tdf_{x_p}(x).
\end{aligned}
$$

Condition (b): By definition $G_{x_p} dg[x_p]_p(\text{Ker } dg[x_p]_p) = 0_Y$.
Condition (c): Taking into account that

$$
\begin{aligned}
G_{x_p}(dg[x_p]_p(p)) &= \mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1}(dg[x_p]_p(p' + p'')) + \mathcal{G}_{x_p}(p') \cdot \frac{z_0(dg[x_p]_p(p))}{z_0(dg[x_p]_p(p))} \\
&= \mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1}(dg[x_p]_p(p')) + \mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1}(dg[x_p]_p(p'')) + \mathcal{G}_{x_p}(p') \\
&= 0 + \mathcal{G}_{x_p}(p'') + \mathcal{G}_{x_p}(p') = \mathcal{G}_{x_p}(p),
\end{aligned}
$$

we obtain that $\pi G_{x_p} dg[x_p]_p(p) = \pi \mathcal{G}_{x_p}(p) = \pi f(x_p)$.

$\square$

Now come into play the set-valued maps which we will derive using the tools introduced in Subsection 2.2.

Definition 3.3. *Regarding the program (1_p), we consider the following set-valued maps:*

(i) The T-perturbation map of (1_p), defined as

$$
\begin{aligned}
&\Phi : V \subset P \rightrightarrows Y \\
&\quad p \rightrightarrows \Phi(p) := \{f(x_p) \colon x_p \text{ is a } T\text{-optimal regular solution of } (1_p)\}.
\end{aligned}
$$

(ii) The T-dual perturbation map of (1_p), defined as

$$
\begin{aligned}
&\Psi : V \subset P \rightrightarrows \mathfrak{L}(Z, Y) \\
&\quad p \rightrightarrows \Psi(p) := \{\mathcal{G}_{x_p} \in \mathfrak{L}(P, Y) \colon \text{ it is a Lagrange premultiplier of } (1_p) \\
&\qquad\qquad\qquad \text{associated to a } T\text{-optimal regular solution } x_p \text{ of } (1_p)\}.
\end{aligned}
$$

Throughout this section we assume the following assumption.

Hypothesis 3.4. *There exists a Fréchet differentiable selection $\gamma : V \to D$, $\gamma(p) = x_p$, where x_p is a T-optimal regular solution of program (1_p).*

The following result shows that the composition $T\Psi$ is, in fact, a single-valued map on V.

Proposition 3.5. *Let us fix $p \in V$, x_p a T-optimal regular solution of (1_p), and $\mathcal{G}_{x_p}$ a Lagrange premultiplier of (1_p) associated to x_p. The following statements hold.*

(i) $T\mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1} dg[x_p]_p = T\mathcal{G}_{x_p}$.

(ii) If $\bar{\mathcal{G}}_{\bar{x}_p}$ is a Lagrange premultiplier of (1_p) associated to any T-optimal regular solution of (1_p), $\bar{x}_p$, then $T\mathcal{G}_{x_p} = T\bar{\mathcal{G}}_{\bar{x}_p}$.

Proof. Statement (i). Let us fix any $q \in P$ and decompose $q = q' + q'' \in \text{Ker } dg[x_p]_p \oplus S_{x_p} = P$. Taking into account that

$$
dg[x_p]_p|_{S_{x_p}}^{-1} dg[x_p]_p(q) = q'',
$$

we get that

$T\mathcal{G}_{x_p}(q) = T\mathcal{G}_{x_p}(q') + T\mathcal{G}_{x_p}(q'') = 0 + T\mathcal{G}_{x_p}(q'') = T\mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1} dg[x_p]_p(q)$.

Statement (ii). Define the map $S : V \subset P \to W$ by $S(p) := Tf(x_p)$, where each $x_p \in D$ is a T-optimal solution of (1_p). Taking $L_p := T\mathcal{G}_{x_p} dg[z_p]_p|_{S_{x_p}}^{-1}$ in [12, Theorem 7], we get that

$$dS_p(q) = T\mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1} dg[x_p]_p(q),\ \forall q \in P.$$

Hence, the uniqueness of Fréchet differential yields

$$T\mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1} dg[x_p]_p = T\bar{\mathcal{G}}_{\bar{x}_p} dg[\bar{x}_p]_p|_{S_{\bar{x}_p}}^{-1} dg[\bar{x}_p]_p,$$

and Statement (i) leads to

$$T\mathcal{G}_{x_p} = T\bar{\mathcal{G}}_{\bar{x}_p}.$$

□

The next notion we introduce will be an useful tool in the proof of Theorem 3.10.

Definition 3.6. *Let us fix $p \in V$, a T-optimal regular solution $x_p \in D$ of (1_p), and a Lagrange premultiplier $\mathcal{G}_{x_p}$ of (1_p) associated to x_p. We define the canonical reduction of $\mathcal{G}_{x_p}$ as the map defined by*

$$\mathfrak{B}[\mathcal{G}_{x_p}](q) := \widehat{T}^{-1}(T(\mathcal{G}_{x_p}(q))) + \pi\mathcal{G}_{x_p}(p) \cdot \beta(p)(q),\ \forall q \in P. \tag{2}$$

Remark 3.7. *Condition (ii) of the former proposition allows us to claim that $\mathfrak{B}[\mathcal{G}^1_{x_p}] = \mathfrak{B}[\mathcal{G}^2_{x_p}]$ for any two Lagrange T-premultipliers $\mathcal{G}^1_{x_p}$ and $\mathcal{G}^2_{x_p}$ of (1_p) associated to the same x_p.*

Proposition 3.8. *Let us fix $p \in V$, a T-optimal regular solution x_p of (1_p), and a Lagrange premultiplier $\mathcal{G}_{x_p}$ of (1_p) associated to x_p. Then $\mathfrak{B}[\mathcal{G}_{x_p}]$ is also a Lagrange premultiplier of (1_p) associated to x_p.*

Proof. Condition (a) Since $\pi f(x_{x_p}) = \pi\mathcal{G}_{x_p}(p)$,

$$T\mathfrak{B}[\mathcal{G}_{x_p}] = T\widehat{T}^{-1}T\mathcal{G}_{x_p} + T(\pi f(x_{x_p})) \cdot \beta(p) = T\mathcal{G}_{x_p} + 0,$$

and therefore

$$T\mathfrak{B}[\mathcal{G}_{x_p}] dg[x_p]_p|_{S_{x_p}}^{-1} dg[p]_{x_p} = T\mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1} dg[p]_{x_p} = -T df_{x_p}.$$

Condition (b) For any $q \in \operatorname{Ker} g[x_p]_p$, we have that $\mathcal{G}_{x_p}(q) \in \operatorname{Ker} T$, and so

$$\mathfrak{B}[\mathcal{G}_{x_p}](q) = \widehat{T}^{-1}(T(\mathcal{G}_{x_p}(q))) + \pi f(x_{x_p}) \cdot \beta(p)(q) = \pi f(x_{x_p}) \cdot \beta(p)(q) \in \operatorname{Ker} T.$$

Condition (c) Since $\pi\widehat{T}^{-1} = 0$ and $\beta(p)(p) = 1$, we have

$$\pi\mathfrak{B}[\mathcal{G}_{x_p}](p) = \pi\widehat{T}^{-1}(T(\mathcal{G}_{x_p}(q))) + \pi f(x_{x_p}) \cdot \beta(p)(p) = \pi f(x_p).$$

□

Proposition 3.9. *Let us fix $p \in V$, a T-optimal regular solution x_p of (1_p), and a Lagrange premultiplier $\mathcal{G}_{x_p}$ of (1_p) associated to x_p. Consider the vector space*

$$\mathfrak{J}_p = \{R \in \mathfrak{L}(P, Ker\ T) \colon R(p) = 0\}.$$

The following statements hold:

(i) If $R \in \mathfrak{J}_p$, then $\mathcal{G}_{x_p} + R$ is a Lagrange premultiplier of (1_p) associated to x_p.

(ii) If $\mathcal{G}^*_{x_p}$ is another Lagrange premultiplier of (1_p) associated to x_p, then $\mathcal{G}_{x_p} - \mathcal{G}^*_{x_p} \in \mathfrak{J}_p$.

Proof. Statement (i)
Condition (a) Taking into account that $T\mathcal{G}_{x_p} = T(\mathcal{G}_{x_p} + R)$, we get that

$$T(\mathcal{G}_{x_p} + R)dg[x_p]_p|_{S_{x_p}}^{-1} dg[p]_{x_p} = T\mathcal{G}_{x_p} dg[x_p]_p|_{S_{x_p}}^{-1} dg[p]_{x_p} = -Tdf_{x_p}.$$

Condition (b) Since $\mathcal{G}_{x_p}$ is a Lagrange premultiplier associated to x_p, $\mathcal{G}_{x_p}(\text{Ker } dg[x_p]_p) \subset \text{Ker } T$. Moreover, by definition of R, $R(\text{Ker } dg[x_p]_p) \subset \text{Ker } T$. Therefore $(\mathcal{G}_{x_p} + R)(\text{Ker } dg[x_p]_p) \subset \text{Ker } T$.
Condition (c) $\pi(\mathcal{G}_{x_p} + R)(p) = \pi\mathcal{G}_{x_p}(p) + \pi R(p) = \pi f(x_p) + 0_Y$.
Statement (ii).
Proposition 3.5 (ii) yields $T\mathcal{G}_{x_p} = T\mathcal{G}^*_{x_p}$, or equivalently that

$$(\mathcal{G}_{x_p} - \mathcal{G}^*_{x_p})(P) \subset \text{Ker } T.$$

On the other hand, $\pi\mathcal{G}_{x_p}(p) = \pi\mathcal{G}^*_{x_p}(p) = \pi f(x_p)$. In addition, $T(\mathcal{G}_{x_p}(p)) = T(\mathcal{G}^*_{x_p}(p))$ which yields $\widehat{T}^{-1}T(\mathcal{G}_{x_p}(p)) = \widehat{T}^{-1}T(\mathcal{G}^*_{x_p}(p))$. Consequently

$$(\mathcal{G}_{x_p} - \mathcal{G}^*_{x_p})(p) = \widehat{T}^{-1}T(\mathcal{G}_{x_p}(p)) - \widehat{T}^{-1}T(\mathcal{G}^*_{x_p}(p)) + \pi\mathcal{G}_{x_p}(p) - \pi\mathcal{G}^*_{x_p}(p) = 0.$$

$\square$

The above proposition shows that the set of all the Lagrange premultipliers associated to a T-optimal regular solution is an affine space. In particular, if we denote by $\mathfrak{M}_{x_p}$ the set of all the Lagrange premultipliers of (1_p) associated to x_p, then it can be decomposed as

$$\mathfrak{M}_{x_p} = \mathfrak{R}[\mathcal{G}_{x_p}] + \mathfrak{J}_p,$$

for any $\mathcal{G}_{x_p}$ Lagrange premultiplier associated to x_p. Moreover, we can write

$$\Psi(p) = \cup\{\mathfrak{M}_{x_p} : x_p \text{ is a } T-\text{optimal regular solution of } (1_p)\}.$$

The following theorem is a cornerstone of our research.

Theorem 3.10. *Let us fix $(p_0, \mathcal{G}_{x_{p_0}}) \in$ Graph Ψ and define the set-valued map $\check{\Psi} : V \subset P \rightrightarrows Y$ by $\check{\Psi}(p) := \Psi(p)(p)$ for every $p \in V$. If Ψ is derivable (respectively tangentially regular) at $(p_0, \mathcal{G}_{x_{p_0}})$ and $T\Psi$ is Fréchet differentiable at $p_0 \in V$, then $\check{\Psi}$ is derivable (respectively tangentially regular) at $(p_0, \mathcal{G}_{x_{p_0}}(p_0))$ and*

$$D\check{\Psi}(p_0, \mathcal{G}_{x_{p_0}}(p_0))(p) = D\Psi(p_0, \mathcal{G}_{x_{p_0}})(p)(p_0) + \mathcal{G}_{x_{p_0}}(p), \ \forall p \in P. \tag{3}$$

Proof. In order to simplify the expressions involved in this proof, we will do the following abuse of notation. Given $y \in Y$ and $p^* \in P^*$, sometimes we will write $y \cdot p^*$ to denote de element of $\mathfrak{L}(P, Y)$ defined by $(y \cdot p^*)(p) := y \cdot p^*(p)$, $\forall p \in P$, where the last $\cdot$ denotes de multiplication of a vector and a scalar.

The proof is divided in two parts. The first is devoted to the case of Ψ derivable, and the second one to the case of Ψ tangentially regular.

Part I. Let us assume that Ψ is derivable at $(p_0, \mathcal{G}_{x_{p_0}})$. By [19, Theorem 6] we have just to prove that Ψ has Property $\mathfrak{R}$ at $(p_0, \mathcal{G}_{x_{p_0}})$. For that purpose we fix three sequences: $\{p_n\}_{n=1}^{\infty} \subset V$, $\{h_n\}_{n=1}^{\infty} \subset \mathbb{R}_+$, and $\{\mathcal{G}_n\}_{n=1}^{\infty} \subset \mathfrak{L}(P, Y)$ such that $\{h_n\}_{n=1}^{\infty}$ converges to 0, $\{p_n\}_{n=1}^{\infty}$ is convergent, $\mathcal{G}_n \in \Psi(p_0 + h_n p_n)$ for all $n \in \mathbb{N}$, and the sequence

$$\left\{ \frac{\mathcal{G}_n(p_0 + h_n p_n) - \mathcal{G}_{x_{p_0}}(p_0)}{h_n} \right\}_{n=1}^{\infty} \tag{4}$$

converges. Let $x_{p_0+h_np_n}$ be the T-optimal solution of $(1_{p_0+h_np_n})$ associated to $\mathcal{G}_n$. Consider now the sequence $\{\bar{\mathcal{G}}_n\}_{n=1}^{\infty} \subset \mathfrak{L}(P,Y)$, defined as

$$\bar{\mathcal{G}}_n(p) := \mathfrak{B}[\mathcal{G}_n](p) + (\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(p) - (\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(p_0 + h_np_n) \cdot \beta(p_0 + h_np_n)(p),$$

for every $p \in P$ and $n \in \mathbb{N}$. We will check that $\bar{\mathcal{G}}_n \in \Psi(p_0 + h_np_n)$, $\bar{\mathcal{G}}_n(p_0 + h_np_n) = \mathcal{G}_n(p_0 + h_np_n)$ for every $n \in \mathbb{N}$, and

$$\left\{ \frac{\bar{\mathcal{G}}_n - \mathcal{G}_{x_{p_0}}}{h_n} \right\}_{n=1}^{\infty}$$

converges.

By Proposition 3.8, each $\mathfrak{B}[\mathcal{G}_n]$ is a Lagrange premultiplier associated to $x_{p_0+h_np_n}$. Then, by Proposition 3.9 (i), it is enough to show that the map R defined by

$$R(p) := (\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(p) - (\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(p_0 + h_np_n) \cdot \beta(p_0 + h_np_n)(p),\ \forall p \in P,$$

belongs to $\mathfrak{I}_{p_0+h_np_n}$. Since $\mathcal{G}_{x_{p_0}}$ and $\mathfrak{B}[\mathcal{G}_{x_{p_0}}]$ are Lagrange T-premultipliers, Proposition 3.5 (ii) yields

$$(\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(P) \subset \operatorname{Ker} T,$$

which implies that $R \in \mathfrak{L}(P, \operatorname{Ker} T)$. The condition $R(p_0 + h_np_n) = 0$ is immediate. Therefore, $\bar{\mathcal{G}}_n \in \Psi(p_0 + h_np_n)$ for all $n \in \mathbb{N}$.

On the other side we have that

$$\begin{aligned} \bar{\mathcal{G}}_n(p_0 + h_np_n) &= \mathfrak{B}[\mathcal{G}_n](p_0 + h_np_n) = \widehat{T}^{-1}T\mathcal{G}_n(p_0 + h_np_n) + \pi f(x_{p_0+h_np_n}) = \\ &= \widehat{T}^{-1}T\mathcal{G}_n(p_0 + h_np_n) + \pi\mathcal{G}_n(p_0 + h_np_n) = \mathcal{G}_n(p_0 + h_np_n) \end{aligned}$$

for every $n \in \mathbb{N}$. Finally, let us analyse the convergence of

$$\left\{ \frac{\bar{\mathcal{G}}_n - \mathcal{G}_{x_{p_0}}}{h_n} \right\}_{n=1}^{\infty}.$$

Let us fix $n \in \mathbb{N}$, we have

$$\begin{aligned} \frac{\bar{\mathcal{G}}_n - \mathcal{G}_{x_{p_0}}}{h_n} &= \frac{\mathfrak{B}[\mathcal{G}_n] - \mathfrak{B}[\mathcal{G}_{x_{p_0}}]}{h_n} - \frac{\left(\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}]\right)(p_0 + h_np_n) \cdot \beta(p_0 + h_np_n)}{h_n} \\ &= \frac{\mathfrak{B}[\mathcal{G}_n] - \mathfrak{B}[\mathcal{G}_{x_{p_0}}]}{h_n} - \frac{\left(\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}]\right)(p_0) \cdot \beta(p_0 + h_np_n)}{h_n} \\ &\quad - \left(\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}]\right)(p_n) \cdot \beta(p_0 + h_np_n). \end{aligned}$$

Using $\mathcal{G}_{x_{p_0}}(p_0) = \mathfrak{B}[\mathcal{G}_{x_{p_0}}](p_0)$, the former expression can be written as

$$\frac{\mathfrak{B}[\mathcal{G}_n] - \mathfrak{B}[\mathcal{G}_{x_{p_0}}]}{h_n} - \left(\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}]\right)(p_n) \cdot \beta\left(p_0 + h_np_n\right).$$

Let us note that the sequence

$$\left\{\left(\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}]\right)(p_n) \cdot \beta\left(p_0 + h_np_n\right)\right\}_{n=1}^{\infty}$$

converges to $\left(\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}]\right)(u)\cdot\beta(p_0)$, where u is the limit of $\{p_n\}_{n=1}^{\infty}$. Hence, the sequence

$$\left\{ \frac{\bar{\mathcal{G}}_n - \mathcal{G}_{x_{p_0}}}{h_n} \right\}_{n=1}^{\infty}$$

converges if, and only if, the sequence

$$\left\{\frac{\mathfrak{B}[\mathcal{G}_n]-\mathfrak{B}[\mathcal{G}_{x_{p_0}}]}{h_n}\right\}_{n=1}^{\infty}$$

converges. Let us check that the last one does. Indeed, fixed $n \in \mathbb{N}$ and $p \in P$, we have that

$$\frac{\mathfrak{B}[\mathcal{G}_n]-\mathfrak{B}[\mathcal{G}_{x_{p_0}}]}{h_n}=\widehat{T}^{-1}\left(\frac{T\mathcal{G}_n-T\mathcal{G}_{x_{p_0}}}{h_n}\right)+\frac{\pi f(x_{p_0+h_np_n})\cdot\beta(p_0+h_np_n)-\pi f(x_{p_0})\cdot\beta(p_0)}{h_n}.$$

Adding and subtracting $\pi f(x_{p_0+h_np_n})\cdot\beta(p_0)$, the former line can be expressed as

$$\frac{\mathfrak{B}[\mathcal{G}_n]-\mathfrak{B}[\mathcal{G}_{x_{p_0}}]}{h_n}=\widehat{T}^{-1}\left(\frac{T\mathcal{G}_n-T\mathcal{G}_{x_{p_0}}}{h_n}\right)+\pi f(x_{p_0+h_np_n})\cdot\frac{\beta(p_0+h_np_n)-\beta(p_0)}{h_n} \tag{5}$$

$$+\frac{\pi f(x_{p_0+h_np_n})-\pi f(x_{p_0})}{h_n}\cdot\beta(p_0) \tag{6}$$

Let us study separately the convergence of each of the three terms of the right-hand side of the former equality.
First term. Since $T\Psi$ is differentiable, the sequence

$$\left\{\frac{T\mathcal{G}_n-T\mathcal{G}_{x_{p_0}}}{h_n}\right\}_{n=1}^{\infty}$$

converges, and therefore, from the continuity of $\widehat{T}^{-1}$, the sequence

$$\left\{\widehat{T}^{-1}\left(\frac{T\mathcal{G}_n-T\mathcal{G}_{x_{p_0}}}{h_n}\right)\right\}_{n=1}^{\infty} \tag{7}$$

converges, too.

Second term. Since $\pi f(x_{p_0+h_np_n})$ converges to $\pi f(x_{p_0})$ and $\dfrac{\beta(p_0+h_np_n)-\beta(p_0)}{h_n}$ converges to $d\beta_{p_0}(u)$, then

$$\left\{\pi f(x_{p_0+h_np_n})\cdot\frac{\beta(p_0+h_np_n)-\beta(p_0)}{h_n}\right\}_{n=1}^{\infty}$$

converges too.

Third term. We will express it in a more suitable way. Indeed, since

$$\frac{\mathfrak{B}[\mathcal{G}_n](p_0+h_np_n)-\mathfrak{B}[\mathcal{G}_{x_{p_0}}](p_0)}{h_n}=$$

$$\widehat{T}^{-1}\left(\frac{T\mathcal{G}_n(p_0+h_np_n)-T\mathcal{G}_{x_{p_0}}(p_0)}{h_n}\right)+\frac{\pi f(x_{p_0+h_np_n})-\pi f(x_{p_0})}{h_n}$$

$$=\widehat{T}^{-1}\left(\frac{T\mathcal{G}_n-T\mathcal{G}_{x_{p_0}}}{h_n}\right)(p_0)+T\mathcal{G}_n(p_n)+\frac{\pi f(x_{p_0+h_np_n})-\pi f(x_{p_0})}{h_n},$$

for all $n \in \mathbb{N}$, we get that

$$\frac{\pi f(x_{p_0+h_np_n})-\pi f(x_{p_0})}{h_n}=$$

$$\frac{\mathfrak{B}[\mathcal{G}_n](p_0+h_np_n)-\mathfrak{B}[\mathcal{G}_{x_{p_0}}](p_0)}{h_n}-\widehat{T}^{-1}\left(\frac{T\mathcal{G}_n-T\mathcal{G}_{x_{p_0}}}{h_n}\right)(p_0)-T\mathcal{G}_n(p_n), \tag{8}$$

for all $n \in \mathbb{N}$. Now, since

$$\mathcal{G}_n(p_0 + h_n p_n) = \mathfrak{B}[\mathcal{G}_n](p_0 + h_n p_n)$$

for all $n \in \mathbb{N}$, and $\mathcal{G}_{x_{p_0}}(p_0) = \mathfrak{B}[\mathcal{G}_{x_{p_0}}](p_0)$, the convergence of (4) yields the convergence of

$$\left\{ \frac{\mathfrak{B}[\mathcal{G}_n](p_0 + h_n p_n) - \mathfrak{B}[\mathcal{G}_{x_{p_0}}](p_0)}{h_n} \right\}_{n=1}^{\infty}. \tag{9}$$

Consequently, the convergence of $\{T\mathcal{G}_n(p_n)\}_{n=1}^{\infty}$ to $T\mathcal{G}_{x_{p_0}}(u)$, jointly with the convergence of (7) yields that the sequence

$$\left\{ \frac{\pi f(x_{p_0+h_n p_n}) - \pi f(x_{p_0})}{h_n} \right\}_{n=1}^{\infty}$$

converges, and therefore that the sequence

$$\left\{ \frac{\pi f(x_{p_0+h_n p_n}) - \pi f(x_{p_0})}{h_n} \cdot \beta(p_0) \right\}_{n=1}^{\infty}$$

converges too.
Hence, the sequence

$$\left\{ \frac{\mathfrak{B}[\mathcal{G}_n] - \mathfrak{B}[\mathcal{G}_{x_{p_0}}]}{h_n} \right\}_{n=1}^{\infty}$$

is convergent.

Part II. Let us prove the theorem now when Ψ is tangentially regular at $(p_0, \mathcal{G}_{x_{p_0}})$. Since Ψ satisfies property $\mathfrak{B}$ at $(p_0, \mathcal{G}_{x_{p_0}})$, by using Theorem 3.1 of [22], we have just to prove that Ψ satisfies property $\mathfrak{S}$ at $(p_0, \mathcal{G}_{x_{p_0}})$.
Let $\{a_n\}_{n=1}^{\infty} \subset V$ and $\{\mathcal{R}_n\}_{n=1}^{\infty} \subset \mathfrak{L}(P, Y)$ be two sequences such that $\{a_n\}_{n=1}^{\infty}$ converges to p_0, $\mathcal{R}_n \in \Psi(a_n)$ for every $n \in \mathbb{N}$, and $\{\mathcal{R}_n(a_n)\}_{n=1}^{\infty}$ converges to $\mathcal{G}_{x_{p_0}}(p_0)$.
Let x_{a_n} be the T-optimal solution of (1_{a_n}) associated to $\mathcal{R}_n$, and consider, as above, $\mathfrak{B}[\mathcal{R}_n]$ the Lagrange multiplier of (1_{a_n}) associated to x_{a_n} defined as

$$\mathfrak{B}[\mathcal{R}_n](p) := \widehat{T}^{-1} T\mathcal{R}_n(p) + \pi f(x_{a_n}) \cdot \beta(a_n)(p), \tag{10}$$

for every $p \in P$ and $n \in \mathbb{N}$.
Consider now the sequence $\{\bar{\mathcal{R}}_n\}_{n=1}^{\infty} \subset \mathfrak{L}(P, Y)$, defined as

$$\bar{\mathcal{R}}_n(p) := \mathfrak{B}[\mathcal{R}_n](p) + (\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(p) - (\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(a_n) \cdot \beta(a_n)(p), \tag{11}$$

for every $p \in P$ and $n \in \mathbb{N}$.
We will check that $\bar{\mathcal{R}}_n(a_n) = \mathcal{R}_n(a_n)$, $\bar{\mathcal{R}}_n \in \Psi(a_n)$ for every $n \in \mathbb{N}$, and that the limit $\lim_{n\to\infty} \bar{\mathcal{R}}_n = \mathcal{G}_{x_{p_0}}$.
Indeed, since $\mathcal{R}_n$ is a Lagrange premultiplier associated to x_{a_n}, from Definition 3.1 we get that

$$\pi f(x_{a_n}) = \pi \mathcal{R}_n(a_n),$$

and thus,

$$\begin{aligned} \bar{\mathcal{R}}_n(a_n) &= \mathfrak{B}[\mathcal{R}_n](a_n) = \widehat{T}^{-1} T\mathcal{R}_n(a_n) + \pi f(x_{a_n}) = \\ &= \widehat{T}^{-1}\, T\mathcal{R}_n(a_n) + \pi \mathcal{R}_n(a_n) = \mathcal{R}_n(a_n), \end{aligned} \tag{12}$$

for every $n \in \mathbb{N}$.
Moreover, since

$$\bar{\mathcal{R}}_n(p) - \mathfrak{B}[\mathcal{R}_n](p) = (\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(p) - (\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(a_n) \cdot \beta(a_n)(p),\ \forall p \in P, \tag{13}$$

then $\bar{\mathcal{R}}_n - \mathfrak{B}[\mathcal{R}_n] \in \mathfrak{J}_{a_n}$. Now, Proposition 3.9 (i) yields that $\bar{\mathcal{R}}_n$ is a Lagrange multiplier of (1_{a_n}) associated to x_{a_n}, and therefore, $\bar{\mathcal{R}}_n \in \Psi(a_n)$ for all $n \in \mathbb{N}$.

Finally, let us check that

$$\lim_{n\to\infty} \bar{\mathcal{R}}_n = \mathcal{G}_{x_{p_0}}.$$

Indeed, from (11) we have that

$$\lim_{n\to\infty} \bar{\mathcal{R}}_n = \mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}] + \lim_{n\to\infty} \mathfrak{B}[\mathcal{R}_n] - \lim_{n\to\infty} (\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(a_n) \cdot \beta(a_n). \tag{14}$$

Let us consider separately the limits of the right-hand side of (14).

On one hand we have that

$$\lim_{n\to\infty} \mathfrak{B}[\mathcal{R}_n] = \lim_{n\to\infty} \widehat{T}^{-1} T\mathcal{R}_n - \lim_{n\to\infty} \pi f(x_{a_n}) \cdot \beta(a_n). \tag{15}$$

Since $T\Psi$ is Fréchet differentiable at p_0, and therefore continuous, the continuity of $\widehat{T}^{-1}$ directly yields that

$$\lim_{n\to\infty} \widehat{T}^{-1} T\mathcal{R}_n = \widehat{T}^{-1} T\mathcal{G}_{x_{p_0}}.$$

Furthermore, since $T\mathcal{R}_n = T\mathfrak{B}[\mathcal{R}_n]$, taking into account that both $\mathcal{R}_n$ and $\mathfrak{B}[\mathcal{R}_n]$ are associated solutions to x_{a_n}, we have that $\mathcal{R}_n(a_n) = \mathfrak{B}[\mathcal{R}_n](a_n)$ for every $n \in \mathbb{N}$, and therefore, we get that $\{\mathfrak{B}[\mathcal{R}_n](a_n)\}_{n=1}^{\infty}$ converges to $\mathcal{G}_{x_{p_0}}(p_0)$. Likewise, since $T\mathcal{G}_{x_{p_0}} = T\mathfrak{B}[\mathcal{G}_{x_{p_0}}]$, $\mathcal{G}_{x_{p_0}}(p_0) = \mathfrak{B}[\mathcal{G}_{x_{p_0}}](p_0)$. Hence $\{\mathfrak{B}[\mathcal{R}_n](a_n)\}_{n=1}^{\infty}$ converges to $\mathfrak{B}[\mathcal{G}_{x_{p_0}}](p_0)$, and therefore since

$$\begin{aligned}
&\left\|\mathfrak{B}[\mathcal{R}_n](a_n) - \mathfrak{B}[\mathcal{G}_{x_{p_0}}](p_0)\right\| = \\
&\quad = \left\|\widehat{T}^{-1} T\mathcal{R}_n(a_n) + \pi f(x_{a_n}) \cdot \beta(a_n)(a_n) - (\widehat{T}^{-1} T\mathcal{G}_{x_{p_0}}(p_0) + \pi f(x_{p_0}) \cdot \beta(p_0)(p_0))\right\| \geq \\
&\qquad \geq \left| \left\|\widehat{T}^{-1} T\mathcal{R}_n(a_n) - \widehat{T}^{-1} T\mathcal{G}_{x_{p_0}}(b_0)\right\| - \left\|\pi f(x_{a_n}) - \pi f(x_{p_0})\right\| \right|,
\end{aligned}$$

and

$$\lim_{n\to\infty} \left\|\widehat{T}^{-1}\ T\mathcal{R}_n(a_n) - \widehat{T}^{-1}\ T\mathcal{G}_{x_{p_0}}(p_0)\right\| = 0,$$

we obtain that

$$\lim_{n\to\infty} \pi f(x_{a_n}) = \pi f(x_{p_0}).$$

Thus

$$\lim_{n\to\infty} \pi f(x_{a_n}) \cdot \beta(a_n) = \pi f(x_{p_0}) \cdot \beta(p_0),$$

and hence, from (15) we have

$$\lim_{n\to\infty} \mathfrak{B}[\mathcal{R}_n] = \widehat{T}^{-1}\ T\mathcal{G}_{x_{p_0}} + \pi f(x_{p_0}) \cdot \beta(p_0) = \mathfrak{B}[\mathcal{G}_{x_{p_0}}].$$

On the other side,

$$\lim_{n\to\infty} (\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(a_n) \cdot \beta(a_n) = (\mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}])(p_0) \cdot \beta(p_0) = 0.$$

Therefore, from (14), we obtain that

$$\lim_{n\to\infty} \bar{\mathcal{R}}_n = \mathcal{G}_{x_{p_0}} - \mathfrak{B}[\mathcal{G}_{x_{p_0}}] + \mathfrak{B}[\mathcal{G}_{x_{p_0}}] = \mathcal{G}_{x_{p_0}}.$$

$\square$

Theorem 3.11. *Consider $p_0 \in V$, x_{p_0} a T-optimal regular solution of (1_{p_0}) and $G_{x_{p_0}}$ a Lagrange multiplier of (1_{p_0}) associated to x_{p_0}. If Ψ is derivable (respectively tangentially regular) at $(p_0, G_{x_{p_0}} dg[x_{p_0}]_{p_0})$ and $T\Psi$ is Fréchet differentiable (respectively continuously Fréchet differentiable) at p_0, then Φ is derivable (respectively tangentially regular) at $(p_0, f(x_{p_0}))$ and*

$$D\Phi(p_0, f(x_{p_0}))(p) = G_{x_{p_0}} dg[x_{p_0}]_{p_0}(p) + \pi D\Psi(p_0, G_{x_{p_0}} dg[x_{p_0}]_{p_0})(p)(p_0), \tag{16}$$

for every $p \in P$.

Proof. Consider $\check{\Psi}(p) := \Psi(p)(p)$ for every $p \in V$. Since $\pi\Phi(p) = \pi\Psi(p)(p)$ for every $p \in V$ we have that

$$\Phi(p) = \widehat{T}^{-1}T\Phi(p) + \pi\Phi(p) = \widehat{T}^{-1}T\Phi(p) + \pi\check{\Psi}(p) \tag{17}$$

for every $p \in V$.
First, [12, Theorem 6] yields that $T\Phi$ is Fréchet differentiable at p_0 and

$$[T\Phi]'(p_0, p) = TG_{x_{p_0}} dg[x_{p_0}]_{p_0}(p),$$

for every $p \in P$. Thus,

$$[\widehat{T}^{-1}T\Phi]'(p_0, p) = \widehat{T}^{-1}TG_{x_{p_0}} dg[x_{p_0}]_{p_0}(p)$$

for every $p \in P$.
On the other hand, Theorem 3.10 yields that $\check{\Psi}$ is derivable (respectively tangentially regular) at $(p_0, G_{x_{p_0}} g'_{x_{p_0}}(p_0, p_0))$ and

$$D\check{\Psi}(p_0, G_{x_{p_0}} dg[x_{p_0}]_{p_0}(p_0))(p) = D\Psi(p_0, G_{x_{p_0}} dg[x_{p_0}]_{p_0})(p)(p_0) + G_{x_{p_0}} dg[x_{p_0}]_{p_0}(p), \tag{18}$$

for every $p \in P$. Now, by [13, Lemma 9], we have that $T\check{\Psi}$ is Fréchet differentiable (respectively continuously Fréchet differentiable) at p_0, and then, by [19, Theorem 8] (respectively [21, Theorem 2.4]), we obtain that $\pi\check{\Psi}$ is derivable (respectively tangentially regular) at $(p_0, \pi G_{x_{p_0}}(p_0))$ and

$$D(\pi\check{\Psi})(p_0, \pi G_{x_{p_0}} dg[x_{p_0}]_{p_0}(p_0))(p) = \pi D\Psi(p_0, G_{x_{p_0}} dg[x_{p_0}]_{p_0})(p)(p_0) + \pi G_{x_{p_0}} dg[x_{p_0}]_{p_0}(p),$$

for every $p \in P$. By applying [24, Propositions 5.1.2. and 5.2.2] to (17) we get that Φ is derivable (respectively tangentially regular) at $(p_0, f(x_{p_0}))$ and

$$\begin{aligned} D\Phi(p_0, f(x_{p_0}))(p) &= \widehat{T}^{-1}TG_{x_{p_0}} dg[x_{p_0}]_{p_0}(p) + \pi D\Psi(p_0, G_{x_{p_0}} dg[x_{p_0}]_{p_0})(p)(p_0) + \pi G_{x_{p_0}} dg[x_{p_0}]_{p_0}(p) \\ &= G_{x_{p_0}} dg[x_{p_0}]_{p_0}(p) + \pi D\Psi(p_0, G_{x_{p_0}} dg[x_{p_0}]_{p_0})(p)(p_0) \end{aligned}$$

for every $p \in P$. $\square$

Remark 3.12. *Note that the classical envelope theorem is included in Theorem 3.11 as a particular instance of it. Indeed, taking $Z := W := \mathbb{R}$ and T as the identity map, the set-valued map Υ becomes a conventional Fréchet differentiable (respectively continuously Fréchet differentiable) point-to-point map, and then, Theorem 3.11 takes the form*

$$d\Upsilon_{b_0}(u) = D\Upsilon(b_0, f(x_{b_0}))(u) = G_{x_{p_0}} dg[x_{p_0}]_{p_0}(u), \quad \forall u \in \mathrm{I\!R},$$

since Ker $T = \{0\}$, and the contingent (respectively circatangent) derivative and the Fréchet differential coincide. Consequently, our approach extends the classical result from scalar to vector optimization by means of the contingent (respectively circatangent) derivative, providing a set-valued extension of this.

The following example shows how Theorem 3.11 works.

Example 3.13. *Let us define* $\mu(\{n\}) := e^{-n}$ $\forall n \in \mathbb{N} = \{0, 1, 2, \dots\}$ *and the Hilbert space* $L^2(\mu) := \{(\lambda_n)_n \in \mathbb{R}^{\mathbb{N}} \colon \sum_{n=0}^{+\infty} \lambda_n^2 e^{-n} < +\infty\}$.

Let us consider $X := L^2(\mu)$, $Y := \mathbb{R}^3$, $Z := W := \mathbb{R}$, $V := (\frac{9}{10}, \frac{11}{10}) \subset \mathbb{R}$, $T := (1, 1, \sqrt{2})$ $D = \{(u_n)_n \in X : -\frac{\pi}{2} < \left[\sum_n u_{3n} e^{-3n}\right]^3 < \frac{\pi}{2}\}$, *and the problem*

$$\begin{cases} Min \ (-\sum_n u_{3n+2} e^{-3n-2} + \sum_n \sqrt{2} u_{3n+1} e^{-3n-1} - \sum_n u_{3n} e^{-3n}, \\ \quad -\sum_n u_{3n+2} e^{-3n-2} - \sum_n \sqrt{2} u_{3n+1} e^{-3n-1} - \sum_n u_{3n} e^{-3n}, \\ \quad \sum_n \sqrt{2} u_{3n+2} e^{-3n-2} - \sum_n \sqrt{2} u_{3n} e^{-3n}); \\ \tan\left(\left[\sum_n u_{3n} e^{-3n}\right]^3\right) + 1/2 - \sin(\pi p^2/3) = 0, \ (u_n)_n \in D. \end{cases}$$

Solving Problem (1_p) we obtain the T-optimal solution set-valued map

$$\begin{aligned} \Phi(p) \quad = \quad & \{(-\sqrt[3]{\arctan(-1/2 + \sin(\pi p/3))} + \sqrt{2}\mu - \lambda, -\sqrt[3]{\arctan(-1/2 + \sin(\pi p/3))} - \sqrt{2}\mu - \lambda, \\ & -\sqrt{2}\sqrt[3]{\arctan(-1/2 + \sin(\pi p/3))} + \sqrt{2}\lambda) : \lambda, \mu \in \mathbb{R}\}. \end{aligned}$$

For short, we will denote $a = \dfrac{\sqrt{3} - 1}{2}$ and $b = \dfrac{-\pi/18}{\sqrt[3]{\arctan^2 a}\,(1 + a^2)}$. Let us study the sensitivity of (1_p) at $p = 1$, $x_1 = (\sqrt[3]{\arctan a}, 0, 0, \cdots)$, and so

$$f(x_1) = \left(-\sqrt[3]{\arctan a}, -\sqrt[3]{\arctan a}, -\sqrt{2}\sqrt[3]{\arctan a}\right).$$

We first analyse the sensitivity by calculating

$$D\Phi(1, f(x_1))(u) = \left\{\left(bu + \sqrt{2}\mu - \lambda, bu - \sqrt{2}\mu - \lambda, \sqrt{2}bu + \lambda\right) : \lambda, \mu \in \mathrm{I\!R}\right\}. \qquad (19)$$

Let us now apply Theorem 3.11 to verify (16). Since Ker T is the linear space generated by $\{(-\sqrt{2}, 0, 1), (-1, 1, 0)\}$, we have $G_{x_1}(u) = \left(-\frac{4}{3\xi(1)}u, -\frac{4}{3\xi(1)}u, -\frac{4\sqrt{2}}{3\xi(1)}u\right)$, and the T-optimal dual solution set-valued map of (1_p) is

$$\Psi\,(p) = \left\{\left(\frac{4\pi\cos(\pi p/3)}{9\xi(p)} + \frac{\sqrt{2}\mu - \lambda}{p}, \frac{4\pi\cos(\pi p/3)}{9\xi(p)} - \frac{\sqrt{2}\mu + \lambda}{p}, \frac{4\sqrt{2}\pi\cos(\pi p/3)}{9\xi(p)} + \frac{\sqrt{2}\lambda}{p}\right) : \lambda, \mu \in \mathrm{I\!R}\right\},$$

where $\xi(p) := \sqrt[3]{\arctan^2[-1/2 + \sin(\pi p/3)]}[-9 + 4\sin(\pi p/3) + 4\cos^2(\pi p/3)]$ and $p \in V$. Thus, taking into account that $dg[x_1]_1 = -\dfrac{\pi}{6}$, we have that

$$\begin{aligned} D\Psi\,(1, G_{x_1} dg[x_1]_1)\,(u)(1) = \{(& -\frac{\pi^2}{81}\frac{\arctan(a)(72\sqrt{3} - 93) + 8 - 2\sqrt{3}}{\arctan^{5/3}(a)(\sqrt{3} - 4)^3}u - \sqrt{2}\mu + \lambda, \\ & -\frac{\pi^2}{81}\frac{\arctan(a)(72\sqrt{3} - 93) + 8 - 2\sqrt{3}}{\arctan^{5/3}(a)(\sqrt{3} - 4)^3}u + \sqrt{2}\mu + \lambda, \\ & -\frac{\sqrt{2}\pi^2}{81}\frac{\arctan(a)(72\sqrt{3} - 93) + 8 - 2\sqrt{3}}{\arctan^{5/3}(a)(\sqrt{3} - 4)^3}u - \sqrt{2}\lambda) \colon \lambda, \mu \in \mathrm{I\!R}\}, \end{aligned}$$

for every $u \in \mathrm{I\!R}$. Hence

$$\pi D\Psi\,(1, G_{x_1} dg[x_1]_1)\,(u)(1) = \{(-\sqrt{2}\mu + \lambda, \sqrt{2}\mu + \lambda, -\sqrt{2}\lambda) \colon \lambda, \mu \in \mathrm{I\!R}\}.$$

Finally, we obtain that

$$\begin{aligned} & G_{x_1} dg[x_1]_1 + \pi D\Psi\left(1, G_{x_1} dg[x_1]_1\right) = \\ = \quad & \left\{\left(bu, bu, \sqrt{2}bu\right) + (-\sqrt{2}\mu + \lambda, \sqrt{2}\mu + \lambda, -\sqrt{2}\lambda) \colon \lambda, \mu \in \mathbb{R}\right\} = \\ = \quad & D\Phi(1, f(x_1))(u), \end{aligned} \tag{20}$$

for every $u \in \mathbb{R}$, as Theorem 3.11 states.

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