This is a previous version of the article published in RACSAM. 2015, 109(2): 717-730. doi:10.1007/s13398-014-0211-5

On Gruenhage spaces, separating σ-isolated families, and their relatives

F. García*, M. A. Melguizo

Departamento de Matemática Aplicada
Escuela Politécnica Superior
Universidad de Alicante
03080 San Vicente del Raspeig
Alicante, Spain

Abstract

We prove that, given a topological space X, the following conditions are equivalent. (α) X is a Gruenhage space. (β) X has a countable cover by sets of small local diameter (property SLD) by $\mathcal{F}\cap\mathcal{G}$ sets. (γ) X has a separating σ-isolated family $\mathcal{M}\subset\mathcal{F}\cap\mathcal{G}$. (δ) X has a one-to-one continuous map into a metric space which has a σ-isolated base of $\mathcal{F}\cap\mathcal{G}$ sets.

Besides, we provide an example which shows

$$\text{Fragmentability} \not\Rightarrow \text{ property SLD } \not\Rightarrow \text{ the space to be Gruenhage.}$$

Keywords: Gruenhage space, σ-isolated family, countable cover by sets of small local diameter, fragmentability, descriptive compact
2000 MSC: 54H99, 54E99, 46B26, 46A99

1. Introduction

1.1. Main Results

The notion of Gruenhage space is based on that of a σ-distributively point-finite open cover. The latter was introduced by Gruenhage [4] in order to solve a problem stated in [32]. Gruenhage spaces have already been studied in [3, 21, 27, 28, 29, 30, 31].

*corresponding author
Email addresses: Fernando.gc@ua.es (F. García), ma.mp@ua.es (M. A. Melguizo)

The main result of this work is Theorem 1.1, which has been stated briefly in the first paragraph of the abstract. Before stating it in full, we have to fix some terminology.

Given a set A, we will denote by $\#A$ its cardinality. A family $\mathcal{A}$ of subsets of a set X is said to be *separating* (resp. T_2*-separating*), if given distinct x, $y \in X$ there exists $A \in \mathcal{A}$ such that $\#\{x,y\} \cap A = 1$ (resp. disjoint A, $B \in \mathcal{A}$ such that $x \in A$ and $y \in B$). As usual, we will denote by $\mathcal{F}$ (resp. by $\mathcal{G}$) the family of closed (resp. open) subsets of a topological space, by $\mathcal{F} \cap \mathcal{G}$ the family of sets which are the intersection of an open and a closed set, by $(\mathcal{F} \cap \mathcal{G})_\sigma$ the family of sets which are countable unions of elements of $\mathcal{F} \cap \mathcal{G}$, and by Borel$(X, \tau)$ the Borel σ-algebra for the topology τ on X. Given two families of sets $\mathcal{A}$ and $\mathcal{B}$, by $\mathcal{A} \subset \mathcal{B}$ it is understood that every set in $\mathcal{A}$ also belongs to $\mathcal{B}$. If d is a metric on X, we will denote by τ_d the topology on X generated by d. In what follows, the reader will find some notions which have not been introduced yet. These are defined in the next subsection.

Theorem 1.1. *The following are equivalent for a topological space (X, τ).*

(i) X is a Gruenhage space.

(ii) X has property SLD by $\mathcal{F} \cap \mathcal{G}$ sets (the decomposition may also be taken finite and disjoint).

(iii) For some metric d on X, τ_d has a τ-σ-isolated network $\mathcal{N} \subset \mathcal{F} \cap \mathcal{G}$.

(iv) There is a separating σ-isolated family $\mathcal{M} \subset \mathcal{F} \cap \mathcal{G}$ (the family may also be taken T_2-separating).

(v) X has property d-SLD for some metric d and $\tau_d \subset (\mathcal{F} \cap \mathcal{G})_\sigma$. Hence Borel$(X, \tau_d) \subset$ Borel(X, τ).

(vi) There is a metric space (Z, d) and a one-to-one map $\Phi : X \to Z$ which has a σ-isolated base $\mathcal{B} \subset \mathcal{F} \cap \mathcal{G}$.

It is worth noting that the notions which appear in the former result are closely related to LUR renorming [16, 17, 18]. Property SLD was introduced and studied by Jayne, Namioka, and Rogers [10], the notion of σ-isolated family was introduced by Hansell [6], and the class of maps with a σ-isolated function base was studied by Hansell [8]. More connections between General Topology and Functional Analysis can be found in [13].

Our characterization had its genesis in the works of Stegall [31] and Ribarska [27]. In the latter it is proved that any Gruenhage space has property

SLD. In addition, it is proved that if d is a metric on a regular topological space (X,τ), X has property d-SLD, and either if d is lower semicontinuous or if τ_d is finer than τ, then X is a Gruenhage space. After a deep inspection of the proofs of these results, we state assertions (ii) and (iii) in Theorem 1.1. Assertion (iv) provides a simpler version of [31, Proposition 7.4]. Also, in (iv), we obtain a T_2-separation assumption which does not imply metrizability; in contrast with Rosenthal-type theorems [2, 5, 15, 24]. Assertion (v) provides information about the relationship between the topologies τ and τ_d. In particular, an inclusion between the corresponding Borel σ-algebras is obtained. Let us note that both σ-algebras coincide when we consider descriptive compacta, or when we consider Banach spaces which have property JNR. Regarding the latter, we find an open problem [19, Problem 1.7]. Finally, assertion (vi) is related to the study of measurable selectors of upper semi-continuous multivalued functions [18].

Next, let us focus on two classes of topological spaces which are directly related to Gruenhage compacta. These are the class of descriptive compacta and that of fragmentable spaces. Descriptive compacta were introduced by Hansell [6] and comprehensively studied in [20], and connections with LUR renorming theory can be found in [23]. The first proof of the fact that any descriptive compact is Gruenhage compact is based on the characterization of descriptive compacta in terms of property SLD due to Oncina and Raja in [20]. On the other hand, the notion of fragmentable space was introduced by Jayne and Rogers in [11] in connection with Borel selectors of multivalued functions. In general, fragmentable compact spaces are not particularly well behaved from the point of view of renorming. In [25] Ribarska stated an internal characterization for fragmentable spaces, by means of which she proved that any Gruenhage space is fragmentable.

The following result is already known [20, 23, 25] and shows the behaviour of descriptive compacta, Gruenhage spaces and fragmentable ones.

Proposition 1.2. *Let us consider the following properties for a topological space X.*

(i) X is a descriptive compact.

(ii) X is a Gruenhage space.

(iii) X is fragmentable.

Then we have (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii), and no implication can be reversed. When X is a compact space and every collection of open sets admits a

σ-isolated refinement, then the three properties above are equivalent.

In the former result we have not made any mention of property SLD. However, we have the following implications

$$\text{fragmentability} \Leftarrow \text{Gruenhage space} \Rightarrow \text{property SLD}.$$

The next example, together with the fact that $[0,c]$ is not a Gruenhage space [26], helps us clarify the relationship between Gruenhage spaces, spaces which have property SLD, and fragmentable spaces. As usually, c denotes the cardinality of the continuum and c^+ the cardinal successor of c.

Example 1.3. $[0,c^+]$ is fragmentable but does not have property SLD, whereas $[0,c]$ is fragmentable and has property SLD.

Let us recall that it is unknown if there exists any σ-fragmentable Banach space which does not have property JNR. However, Example 1.3 shows that those two notions are not equivalent out of the frame of Banach spaces.

Finally, it seems natural to state the following problem regarding property SLD and fragmentability.

Problem 1.4. *Let (X,τ) be a topological space which has property SLD. Is (X,τ) necessarily fragmentable?*

The plan of the paper is as follows. In the next subsection we have compiled the definitions of most of the notions which appear in the work. Moreover, we give a proof of Proposition 1.2. In Section 2 we study properties of metrics associated with separating σ-isolated families. In Proposition 2.4 we prove that those metrics provide property SLD. In Proposition 2.7 we show that if the separating σ-isolated family is contained in $\mathcal{F}\cap\mathcal{G}$, then the metric associated also fragments the space. These propositions are used in several proofs of the work, for instance in the proof of Theorem 1.1 in Section 3. Besides, in Section 3, we state and prove Corollary 3.3. This allows us to claim that in a Gruenhage space it is not restrictive to assume that the metric which fragments it generates a topology finer than that generated by the metric which provides property SLD. Remark 3.4 yields extra information about the Stegall characterization and the definition of Gruenhage space. The main goal of Section 4 is to prove Example 1.3. Nevertheless, before that proof, some results concerning spaces which have property SLD without $\mathcal{F}\cap\mathcal{G}$ sets are provided (Proposition 4.1 and Corollaries 4.2 and 4.3).

1.2. Notation, definitions, and previous results

Next, let us introduce some notions needed in the development of this work.

All spaces considered in this paper are assumed to be Hausdorff. Given a set X and a family $\mathcal{A}$ of subsets of X, we denote by $ord(x, \mathcal{A})$ the cardinality of $\{A \in \mathcal{A}\colon x \in A\}$, by ω the first infinite ordinal and by n any positive integer.

A topological space (X, τ) is called a *Gruenhage space* if there exists an open cover $\mathcal{U}$ of X which is *σ-distributively point finite separating.* That is to say, $\mathcal{U} = \cup_{n<\omega}\mathcal{U}_n$ so that, for each pair of distinct $x, y \in X$, there exists $n < \omega$ such that the subfamily $\mathcal{U}_n$ separates x and y, and also either $ord(x, \mathcal{U}_n)$ or $ord(y, \mathcal{U}_n)$ is finite. A compact space K is called a *Gruenhage compact* if K is compact and a Gruenhage space.

Let us introduce now the notion of property SLD. Given a metric d on a set X and a d-bounded set $A \subset X$, we will denote by d-diam(A) the diameter of the set A respect to the metric d. Let d be a metric on a topological space (X, τ), it is said that (X, τ) has a countable cover by sets of small local d-diameter (*X has property d-SLD*) if for every $\varepsilon > 0$, there exists a countable set N_ε such that there exists a decomposition $X = \cup_{n\in N_\varepsilon} X_n^\varepsilon$ in such a way that, for every $n \in N_\varepsilon$ and every $x \in X_n^\varepsilon$, there is a τ-open set U which contains x and verifies that d-diam$(U \cap X_n^\varepsilon) < \varepsilon$. Moreover, we say that *X has property d-SLD by $\mathcal{F} \cap \mathcal{G}$ sets* if X has property d-SLD and each $X_n^\varepsilon \in \mathcal{F} \cap \mathcal{G}$. In addition, we say that X *has property d-SLD by a finite decomposition* if X has property d-SLD and each N_ε is finite. Finally, we say that *X has property d-SLD by a disjoint decomposition* if X has property d-SLD and $X_n^\varepsilon \cap X_m^\varepsilon = \emptyset$ for different n, m and every $\varepsilon > 0$. We say that *X has property SLD* if X has property d-SLD for some metric d on X. Moreover, we say that *X has property SLD by $\mathcal{F} \cap \mathcal{G}$ sets* (and/or *by a finite decomposition*, and/or *by a disjoint decomposition*) if X has property d-SLD by $\mathcal{F} \cap \mathcal{G}$ sets (and/or by a finite decomposition, and/or by a disjoint decomposition) for some metric d on X. Property SLD is equivalent to property $P(d, \tau)$ introduced by Raja in [22] and used in [9]. We will denote by weak, the weak topology of a Banach space. A *Banach space X has property JNR* if (X, weak) has property SLD respect to the metric given by its norm.

Given a family $\mathcal{A}$ of subsets of a set X, we denote by $\cup\mathcal{A}$ the set obtained as the union of elements of the family $\mathcal{A}$. Given a sequence of families of subsets, $\{\mathcal{A}_n\}_{n<\omega}$, we will denote by $\cup_{n<\omega}\mathcal{A}_n$ the family formed by the union of all of the families $\mathcal{A}_n$. A *family* $\mathcal{A}$ of subsets of a topological space X is said to be *isolated*, if for every $A \in \mathcal{A}$ there exists an open set U such that $U \supseteq A$ and $U \cap \widetilde{A} = \emptyset$ for every $\widetilde{A} \in \mathcal{A} \setminus \{A\}$; equivalently, $A \cap \overline{\bigcup \mathcal{A} \setminus \{A\}} = \emptyset$. A *family* $\mathcal{A}$ is said to be σ*-isolated* if it can be expressed as $\mathcal{A} = \cup_{n<\omega} \mathcal{A}_n$ and each subfamily $\mathcal{A}_n$ is isolated. In fact, when $\mathcal{A}$ is a σ-isolated family, we denote by $\mathcal{A}_n$ those isolated subfamilies of $\mathcal{A}$ corresponding to a decomposition of $\mathcal{A}$, i. e., $\mathcal{A} = \cup_{n<\omega} \mathcal{A}_n$.

Let us fix topological spaces X and Y, and a map $\Phi : X \to Y$. We say that a *family* $\mathcal{B}$ of subsets of X *is a function base for* Φ if, whenever V is open in Y, then $\Phi^{-1}(V)$ is union of sets of $\mathcal{B}$.

A family $\mathcal{N}$ of subsets of a topological space (X, τ) is said to be a *network for* τ if every open set is a union of sets from $\mathcal{N}$. A compact space K is called *descriptive compact* if it has a σ-isolated network.

Let d be a metric on a *topological space* (X, τ), X is called d*-fragmentable* (or fragmented by d), if for each nonempty $A \subset X$ and $\varepsilon > 0$ there is some (non empty) relatively open set in A with d-diameter less than ε. A *topological space* (X, τ) is *fragmentable* if there exists a metric d on X such that (X, τ) is d-fragmentable. A *topological space* (X, τ) is said to be σ*-fragmented by* d, if given $\varepsilon > 0$ there exists a countable decomposition $X = \cup_{n<\omega} X_n$ such that for every n, each $A \subset X_n$ has a (non empty) relatively open set of d-diameter less than ε. A topological space (X, τ) is said to be σ*-fragmentable* if (X, τ) is σ-fragmented for some metric d on X. A *Banach space* X is said to be σ*-fragmentable* if (X, weak) is σ-fragmented by the metric provided by the norm.

We complete this subsection by giving a proof of Proposition 1.2.

Proof of Proposition 1.2. In the introduction we have given the corresponding references for the proofs of the implications (i)⇒(ii)⇒(iii). However, these implications can also be obtained as consequence of Corollary 2.9, Theorem 1.1, and Corollary 3.3.

Now assume that X is a fragmentable compact space, and that every collection of open sets admits a σ-isolated refinement. Let d be a fragmenting metric on the compact X. By [25, Corollary 1.11] (or [1, Theorem 5.1.12]), it is not restrictive to suppose that τ_d is finer than the initial topology. Now we apply [6, Theorem 1.10] and X admits a σ-isolated network. □

2. Metrics associated with separating σ-isolated families

The aim of this section is to construct metrics associated with separating σ-isolated families and to study some of their properties. At the beginning, it will be shown that those metrics provide property SLD. Later on, it will be proved that, when the separating σ-isolated families are subfamilies of $\mathcal{F} \cap \mathcal{G}$, the associated metrics also fragment the space. These results will be used in the subsequent sections.

Let us begin with a lemma without proof which will allow us to provide the first definition of the section.

Lemma 2.1. *Let (X, τ) be a topological space, $\mathcal{M} = \cup_{n<\omega} \mathcal{M}_n$ be a separating family of subsets of X such that each $\mathcal{M}_n$ is a disjoint cover of X. Let us define $d : X \times X \to \mathbb{R}$ by*

$$d(x, y) := 1/n_0 \text{ where } n_0 = \min\{n \colon \exists M \in \mathcal{M}_n \text{ s.t. } \#\{x, y\} \cap M = 1\}.$$

Then d is a metric on X and $\mathcal{M}$ is a subbase for τ_d. In particular, the open ball $B(x, 1/n)$ coincides with the set $\cap\{M \in \mathcal{M}_i \colon x \in M \text{ and } i \leq n\}$, for every $x \in X$ and $n < \omega$.

Definition 2.2. The metric d in the former lemma is called the metric generated by the family $\mathcal{M}$.

Next, we introduce the notion of metric associated with a separating σ-isolated family.

Definition 2.3. Let (X, τ) be a topological space and d a metric on X. It is said that d is a metric associated with a separating σ-isolated family, if there exists a separating σ-isolated family $\mathcal{M}$ such that d is a metric generated by the family

$$\bigcup_{n<\omega} \mathcal{M}_n \cup \{X \setminus \cup \mathcal{M}_n\}.$$

In this situation d is said to be the metric associated with $\mathcal{M}$.

Let us show how this kind of metrics provide property SLD.

Proposition 2.4. *Let (X, τ) be a topological space and d a metric on X associated with a separating σ-isolated family. Then X has property d-SLD by a finite and disjoint decomposition.*

Proof. Let $\mathcal{M} = \cup_n \mathcal{M}_n$ be a separating σ-isolated family and d a metric associated with $\mathcal{M}$. For any $n < \omega$ and $M \in \mathcal{M}_n$, we choose an open set $U_M \supset M$ such that $U_M \cap (\cup \mathcal{M}_n \setminus \{M\}) = \emptyset$. In this way, we define the family $\mathcal{U}_n := \{U_M \colon M \in \mathcal{M}_n\}$ for every $n \geq 1$, and finally, $\mathcal{U} := \cup_{n<\omega} \mathcal{U}_n$.

In order to obtain property d-SLD we define, for every $n < \omega$, $X_1^n := \cup\mathcal{M}_n$ and $X_2^n := X \setminus \cup\mathcal{M}_n$. The necessary decomposition of X will be obtained after taking finite intersections. Namely, let us fix $\varepsilon > 0$ and n_0 such that $1/n_0 < \varepsilon$. We define

$$X_{i_1,i_2,\ldots,i_{n_0}} := X_{i_1}^1 \cap \ldots \cap X_{i_{n_0}}^{n_0}, \text{ where } i_j \in \{1,2\}.$$

Then

$$X = \bigcup_{\substack{1\leq i_j\leq 2\\ 1\leq j\leq n_0}} X_{i_1,i_2,\ldots,i_{n_0}},$$

is a finite and pairwise disjoint union. Moreover, let us fix $x \in X_{i_1,i_2,\ldots,i_{n_0}}$. For every $l \in \{1,\ldots,n_0\}$, we will define the set V_l in the following way. If $x \in X_1^l$, we define $V_l := U \in \mathcal{G}$, where $U \in \mathcal{U}_n$ and $x \in U$. On the contrary, if $x \in X_2^l$, we define $V_l := X \in \mathcal{G}$. In both situations, the elements of each set $V_l \cap X_{i_j}^l$ are not separated by the cover $\mathcal{M}_l \cup \{X \setminus \cup\mathcal{M}_l\}$. Now, we define $V := \cap_{1\leq l\leq n_0} V_l \in \mathcal{G}$. Then, $x \in V$ and

$$d\text{-diam}(V \cap X_{i_1,i_2,\ldots,i_{n_0}}) < 1/n_0 < \varepsilon.$$

□

Next, we state a lemma which will be used in several places of this paper.

Lemma 2.5. *Let (X,τ) be a topological space and $\mathcal{M}$ an isolated subfamily of $\mathcal{F}\cap\mathcal{G}$. Then $\cup\mathcal{M} \in \mathcal{F}\cap\mathcal{G}$.*

As a consequence, a σ-isolated family $\mathcal{M}$ is contained in $\mathcal{F}\cap\mathcal{G}$ if, and only if, $\cup\mathcal{M}_n \in \mathcal{F}\cap\mathcal{G}$, for every $n < \omega$.

Proof. We refer the reader to [6, Lemma 3.3]. In fact, we apply that lemma to the special case $\mathcal{B}_0$. □

Now, we will study some properties of those metrics which are associated with separating σ-isolated families contained in $\mathcal{F}\cap\mathcal{G}$. Let us define them.

Definition 2.6. Let (X,τ) be a topological space and d a metric on X. It is said that d is a metric $\mathcal{F}\cap\mathcal{G}$ associated with a separating σ-isolated family, if there exists a separating σ-isolated family, $\mathcal{M}$, such that for every $n < \omega$, $\cup\mathcal{M}_n = F_n \cap G_n$ (for some $F_n \in \mathcal{F}$ and $G_n \in \mathcal{G}$) and d is a metric generated by the family

$$\bigcup_{n<\omega} \mathcal{M}_n \cup \{F_n \setminus G_n, X \setminus F_n\}.$$

In this situation d is said to be the metric $\mathcal{F}\cap\mathcal{G}$ associated with $\mathcal{M}$.

Proposition 2.7. *Let (X, τ) be a topological space and d a metric on X which is $\mathcal{F} \cap \mathcal{G}$ associated with a separating σ-isolated family. Then the following statements hold.*

(i) X has property d-SLD by $\mathcal{F} \cap \mathcal{G}$ sets and by a finite and disjoint decomposition.

(ii) X is fragmented by d.

Proof. Let d be a metric $\mathcal{F} \cap \mathcal{G}$ associated with the separating σ-isolated family $\mathcal{M} = \cup \mathcal{M}_n$. Given $n \geq 1$, we fix sets $F_n \in \mathcal{F}$ and $G_n \in \mathcal{G}$ such that $\cup \mathcal{M}_n = F_n \cap G_n$ in such a way that d is a metric generated by

$$\bigcup_{n<\omega} \mathcal{M}_n \cup \{F_n \setminus G_n, X \setminus F_n\}.$$

Now, for any $n \geq 1$ and $M \in \mathcal{M}_n$, we choose an open set $U_M \supset M$ such that $U_M \cap (\cup \mathcal{M}_n \setminus \{M\}) = \emptyset$. In this way, for every $n \geq 1$, we define the family $\mathcal{U}_n := \{U_M \colon M \in \mathcal{M}_n\}$, and finally $\mathcal{U} := \cup_{n \geq 1} \mathcal{U}_n$.

Let us prove part (i). For every $n \geq 1$, we define

$$X_1^n := X \setminus F_n \in \mathcal{G},\ X_2^n := \cup \mathcal{M}_n \in \mathcal{F} \cap \mathcal{G}, \text{ and } X_3^n := F_n \setminus G_n \in \mathcal{F}.$$

Taking finite intersections, we will obtain a decomposition of X which will provide property d-SLD. Namely, we fix $\varepsilon > 0$ and choose n_0 such that $1/n_0 < \varepsilon$. Next, we define

$$X_{i_1, i_2, \ldots, i_{n_0}} := X_{i_1}^1 \cap \ldots \cap X_{i_{n_0}}^{n_0} \in \mathcal{F} \cap \mathcal{G}, \text{ where } i_j \in \{1, 2, 3\}.$$

Then $X = \bigcup_{\substack{1 \leq i_j \leq 3 \\ 1 \leq j \leq n_0}} X_{i_1, i_2, \ldots, i_{n_0}}$, and it is a finite and pairwise disjoint union. Moreover, let us fix $x \in X_{i_1, i_2, \ldots, i_{n_0}}$. For every $l \in \{1, \ldots, n_0\}$, we will define a set V_l in the following way. If $x \in X_1^l$, we define $V_l := X \setminus F_l \in \mathcal{G}$. However, if $x \in X_2^l$, we define $V_l := G_l \cap U \in \mathcal{G}$, where $U \in \mathcal{U}_n$ and $x \in U$. Finally, if $x \in X_3^l$, we define $V_l := X \in \mathcal{G}$. In all cases, the elements of each set $V_l \cap X_{i_j}^l$ are not separated by the cover $\mathcal{M}_l \cup \{F_l \setminus G_l, X \setminus F_l\}$. Now, we define $V := \cap_{1 \leq l \leq n_0} V_l \in \mathcal{G}$. Then $x \in V$ and d-diam$(V \cap X_{i_1, i_2, \ldots, i_{n_0}}) < 1/n_0 < \varepsilon$.

The proof of statement (ii) is similar to the previous one. Let us fix now a subset $A \subset X$ and $\varepsilon > 0$. Let $n_0 < \omega$ be such that $1/n_0 < \varepsilon$. We define the open set V_1 in this way. If $A \not\subset F_1$, then $V_1 := X \setminus F_1$. Instead, if $A \cap F_1 \cap G_1 \neq \emptyset$, we choose some $U \in \mathcal{U}_1$ such that $A \cap U \neq \emptyset$ and define

$V_1 := G_1 \cap U$. Finally, if $A \subset F_1 \setminus G_1$, then we define $V_1 := X$. If $1 < l \leq n_0$, then we fix $A_{l-1} := A \cap V_1 \cap \ldots V_{l-1}$, and define the corresponding open set V_l in the analogous procedure. Thus, if $A_{l-1} \not\subset F_l$ then $V_l := X \setminus F_l$. If $A_{l-1} \subset F_l \cap G_l$, we choose some $U \in \mathcal{U}_l$ such that $A_{l-1} \cap U \neq \emptyset$ and define $V_l := G_l \cap U$. Finally, if $A_{l-1} \subset F_l \setminus G_l$, then we define $V_l := X$. Therefore, we have obtained a finite family of open sets $\{V_l\}_{l=1}^{n_0}$, such that $V := \cap_{l=1}^{n_0} V_l \in \tau$, $V \cap A \neq \emptyset$, and d-diam$(V \cap A) < 1/n_0 < \varepsilon$. □

In Propositions 2.4 and 2.7, we are not able to relate the topology τ_d with the initial topology τ of the space X. Nevertheless, in this inquiry line, we state the following result.

Proposition 2.8. *Let (X, τ) be a topological space and d a metric associated (or $\mathcal{F} \cap \mathcal{G}$ associated) with a separating σ-isolated family $\mathcal{M}$. Assume that $\mathcal{M}$ is a network for some topology $\widetilde{\tau}$ on X. Then τ_d is finer than $\widetilde{\tau}$.*

Proof. We will only study the case of a metric $\mathcal{F} \cap \mathcal{G}$ associated with a separating σ-isolated family since, in the other case, the proof is exactly the same. Given a separating σ-isolated family $\mathcal{M} \subset \mathcal{F} \cap \mathcal{G}$ on X, we consider, for every $n < \omega$, $F_n \in \mathcal{F}$ and $G_n \in \mathcal{G}$ such that $\cup \mathcal{M}_n = F_n \cap G_n$ in such a way that d is the metric generated by

$$\bigcup_{n<\omega} \mathcal{M}_n \cup \{F_n \setminus G_n, X \setminus F_n\}.$$

The conclusion is a consequence of the fact that $\mathcal{M}$ is contained in a subbase for τ_d. Namely, we fix $x \in X$ and $U \in \widetilde{\tau}$ such that $x \in U$. Then there exists $N \in \mathcal{M}_n$ such that $x \in N \subset U$. Now, for every $i < n$, we take $M_i \in \mathcal{M}_i \cup \{F_i \setminus G_i, X \setminus F_i\}$ such that $x \in M_i$. In conclusion,

$$x \in B_d(x, 1/n) = \cap_{i<n} M_i \cap N \subset U.$$

□

Next result is a direct consequence of the former propositions. Besides, it is related to Proposition 1.2 of the introduction.

Corollary 2.9. *Let (X, τ) be a regular topological space with a σ-isolated network. Then there exists a metric d on X such that the following statements hold* *(i) X has property d-SLD by $\mathcal{F}\cap\mathcal{G}$ sets and by a finite and disjoint decomposition.*

(ii) X is fragmented by d.

(iii) τ_d is finer than τ.

Proof. Let us consider a σ-isolated network $\mathcal{N}$ on X. By Lemma 2.5 and the regularity hypothesis, it is not restrictive to assume that $\mathcal{N}\subset\mathcal{F}\cap\mathcal{G}$; see [20] for more details. Then, for every $n<\omega$, there exist $F_n\in\mathcal{F}$ and $G_n\in\mathcal{G}$ such that $\cup\mathcal{N}_n=F_n\cap G_n$. Let d be a metric generated by

$$\bigcup_{n<\omega}\mathcal{N}_n\cup\{F_n\setminus G_n, X\setminus F_n\}.$$

Assertions (i) and (ii) are consequence of Proposition 2.7 and assertion (iii) of Proposition 2.8. □

3. Gruenhage spaces

The aim of this section is to prove Theorem 1.1 and to establish some of its consequences. Let us begin this section by showing some useful results. The first one is an equivalence in [17, Proposition 2]. Here, we will provide a direct proof in order to have a self-contained work.

Proposition 3.1. *Let (X,τ) be a topological space and d a metric on X. Then X has property d-SLD if, and only if, τ_d has a τ-σ-isolated network.*

Proof. Let us assume that X has property d-SLD. Then, for every $m\geq 1$, we split $X=\cup_{i\geq 1}X_i^m$ in such a way that for every $x\in X_i^m$, there exists an open set U containing x such that $d\text{-diam}(U\cap X_i^m)<1/m$. Now, let $\mathcal{B}=\cup_{l\geq 1}\mathcal{B}^l$ be a base of τ_d such that each subfamily $\mathcal{B}^l$ is discrete. For any m, i, and $B\in\mathcal{B}$, set $U_B^{m,i}:=\text{int}_{X_i^m}(B\cap X_i^m)$, i.e., the interior of the set $B\cap X_i^m$ in the induced topology in X_i^m. Now for any l, m, and i, we define the isolated family

$$\mathcal{N}^{l,m,i}:=\{U_B^{m,i}:B\in\mathcal{B}^l\}.$$

Finally, we claim that the family $\mathcal{N}:=\cup_{l,m,i}\mathcal{N}^{l,m,i}$, is a network for τ_d. Let us check it. We fix arbitrary $x\in X$ and $r>0$. To shorten notation, we will denote by B_d the open ball with radius r and centre x. Let $l\geq 1$ and $B_0\in\mathcal{B}^l$ such that $x\in B_0\subset B_d$. It is not restrictive to assume that there exists some

$0 < \hat{r} < r$ such that $B_0 = B(x, \hat{r})$, otherwise we could follow the argument with a subset $B(x, \hat{r})$ of B_0. Let m_0 be such that $0 < 1/m_0 < \hat{r}$. Consider the decomposition $X = \cup_{i \geq 1} X_i^{m_0}$ given by property d-SLD for $\varepsilon = 1/m_0$. Now, we fix $i_0 \geq 1$ such that $x \in X_i^{m_0}$. Then there exists $U \in \tau$ verifying

$$x \in U \cap X_{i_0}^{m_0} \subset B_0 \subset B_d. \tag{1}$$

Thus $x \in U_{B_0}^{m_0,i_0} = \text{int}_{X_{i_0}^{m_0}}(B_0 \cap X_{i_0}^{m_0})$. In fact $U_{B_0}^{m_0,i_0} \subset B_d$, because if $y \in U_{B_0}^{m_0,i_0}$, there exists $V \in \tau$ such that $y \in V \cap X_{i_0}^{m_0} \subset B_0 \cap X_{i_0}^{m_0} \subset B_d$. Let us note that in the last inclusion we have used (1).

For the reverse implication, we assume that τ_d has a σ-isolated network $\mathcal{N}$. If we fix $\varepsilon > 0$, then the sets $X_n^\varepsilon := \cup\{N \in \mathcal{N}_n : d\text{-diam } N < \varepsilon\}$ provide the suitable decomposition of X for property d-SLD. □

Now, we focus our efforts in the proof of Theorem 1.1. For that reason we begin to manage property SLD by $\mathcal{F} \cap \mathcal{G}$ sets.

Proposition 3.2. *Let (X, τ) be a topological space and d a metric on X, the following are equivalent.*

(i) (X, τ) has property d-SLD by $\mathcal{F} \cap \mathcal{G}$ sets.

(ii) τ_d has a τ-σ-isolated network $\mathcal{N} \subset \mathcal{F} \cap \mathcal{G}$.

(iii) (X, τ) has property d-SLD and $\tau_d \subset (\mathcal{F} \cap \mathcal{G})_\sigma$.

Proof. The implication (i)⇒(ii) follows from the only if part of the proof of the former proposition. In this case, we have also to apply that each X_i^m is a $\mathcal{F} \cap \mathcal{G}$-set. Thus, since each $U_B^{m,i}$ can be written as $U_B^{m,i} = U \cap X_i^m$ for some $U \in \tau$, then $U_B^{m,i} \in \mathcal{F} \cap \mathcal{G}$.

The implication (ii)⇒(i) follows from the if part of the proof of the former proposition. In this case we have also to apply that each X_n^ε is a isolated union of $\mathcal{F} \cap \mathcal{G}$ sets. Applying Lemma 2.5, X_n^ε is again a $\mathcal{F} \cap \mathcal{G}$ set.

In the implication (ii)⇒(iii), the first assertion follows from the former proposition. For the second one we fix a network $\mathcal{N} \subset \mathcal{F} \cap \mathcal{G}$ for τ_d such that $\mathcal{N} = \cup_{n \geq 1} \mathcal{N}_n$ and each $\mathcal{N}_n$ is τ-isolated. Hence, given $U \in \tau_d$, there exists $\mathcal{M} \subset \mathcal{N}$ such that $U = \cup \mathcal{M}$. For every n, we define the family $\mathcal{M}_n := \mathcal{M} \cap \mathcal{N}_n$. Hence $U = \cup_{n \geq 1} \cup \mathcal{M}_n$. Applying Lemma 2.5 again, we conclude that each set $\cup \mathcal{M}_n$ is a $\mathcal{F} \cap \mathcal{G}$ set, which finishes the proof of this implication.

Finally, we will prove (iii)⇒(ii). Let us fix a base $\mathcal{B} = \cup_{n<\omega}\mathcal{B}_n$ for τ_d such that each subfamily $\mathcal{B}_n$ is d-discrete. By hypothesis, each $B \in \mathcal{B}$ can be decomposed as $B = \cup_{i<\omega}U_B^i$ in such a way that each $U_B^i \in \mathcal{F} \cap \mathcal{G}$. Next, we define

$$\mathcal{B}_{n,i} := \{U_B^i, B \in \mathcal{B}_n\}.$$

Then $\cup_{n,i<\omega}\mathcal{B}_{n,i} \subset \mathcal{F} \cap \mathcal{G}$ is a network for τ_d. To obtain a σ-isolated network for τ_d, we have to refine the former one. By Proposition 3.1, there exists a network for τ_d, $\mathcal{N} = \cup_{n<\omega}\mathcal{N}_n$, such that each subfamily $\mathcal{N}_n$ is τ-isolated. For every $N \in \mathcal{N}_n$, we define

$$\widetilde{N} := \overline{N} \setminus \overline{\cup(\mathcal{N}_n \setminus \{N\})}.$$

Since each family $\mathcal{N}_n$ is isolated, then each family

$$\widetilde{\mathcal{N}_n} := \{\widetilde{N} \colon N \in \mathcal{N}_n\},$$

is also isolated. Therefore, each family

$$\widetilde{\mathcal{N}}_{n,k,i} := \{\widetilde{N} \cap U_B^i; N \in \mathcal{N}_n,\, B \in \mathcal{B}_k\} \subset \mathcal{F} \cap \mathcal{G},$$

is also τ-isolated. Moreover, the family $\bigcup_{n,i,k} \widetilde{\mathcal{N}}_{n,k,i}$ is a network for τ_d (because $\cup_{n,i}\mathcal{B}_{n,i}$ is a network for τ_d). □

Before showing Theorem 1.1, we state an interesting consequence of the former result.

Corollary 3.3. *Let (X,τ) be a topological space and d a metric on X such that X has property d-SLD by $\mathcal{F} \cap \mathcal{G}$ sets. Then there exists another metric d_1 on X which fragments X and such that τ_{d_1} is finer than τ_d.*

Proof. Let us consider the σ-isolated network $\mathcal{N} \subset \mathcal{F} \cap \mathcal{G}$ for τ_d given by Proposition 3.2 assertion (ii). Then, for every $n < \omega$, there exist $F_n \in \mathcal{F}$ and $G_n \in \mathcal{G}$ such that $\cup\mathcal{N}_n = F_n \cap G_n$. Let d_1 be a metric generated by

$$\bigcup_{n<\omega} \mathcal{N}_n \cup \{F_n \setminus G_n, X \setminus F_n\}.$$

Then Propositions 2.7 and 2.8 apply. □

The former result, together with [19, Theorem 1.3], provides a direct proof of the known fact that each Banach space having property JNR is fragmentable by a metric which generates a finer topology than that given by the initial norm. The first proof of this fact is based on a game which characterizes fragmentability [14].

Proof of Theorem 1.1. The equivalence of assertions (ii), (iii), and (v) follows from Proposition 3.2. Let us note that equivalence (i)⇔(iv) is essentially stated in [27]. However, we provide a complete and self contained proof.

(i)⇒(ii) Let $\mathcal{U} = \cup_{n\geq 1}\mathcal{U}_n$ be a σ-distributively point finite separating open cover on X. For all integers $n \geq 1$ and $p \geq 0$, we define the set

$$C_{n,p} := \{x \in \cup\mathcal{U}_n;\ ord(x, \mathcal{U}_n) = p\} \in \mathcal{F} \cap \mathcal{G},$$

and consider the family $\mathcal{U}_{n,p}$ of open sets consisting of intersections of p distinct elements of $\mathcal{U}_n$. Let us note that the sequence of families $(\mathcal{U}_{n,p})_{n,p}$ satisfies the same properties as the original sequence $(\mathcal{U}_n)_n$. Now we are in the situation of [31, Proposition 7.4]. Let us relabel the sub indexes in order to simplify the notation. We will denote the sets $C_{n,p}$ by C_n and the families $\mathcal{U}_{n,p}$ by $\mathcal{U}_n$. For every $n \geq 1$, we fix $F_n \in \mathcal{F}$ and $G_n \in \mathcal{G}$ such that $C_n = F_n \cap G_n$. Then $\mathcal{M} := \cup_{n<\omega}\{U \cap C_n; U \in \mathcal{U}_n\}$ is a separating σ-isolated family by $\mathcal{F} \cap \mathcal{G}$ sets. Let d be a metric generated by

$$\bigcup_{n<\omega} \mathcal{M}_n \cup \{F_n \setminus G_n, X \setminus F_n\}.$$

By Proposition 2.7, X has property d-SLD by $\mathcal{F} \cap \mathcal{G}$ sets and by a finite and disjoint decomposition.

The implication (ii)⇒(iii) follows from Proposition 3.2.

(iii)⇒(iv) The network $\mathcal{N}$ given in (iii) is a T_2-separating family and it inherits this separation property from τ_d.

(iv)⇒(i) Let us fix now a separating family $\mathcal{N} \subset \mathcal{F} \cap \mathcal{G}$ such that $\mathcal{N} = \cup_{n<\omega}\mathcal{N}_n$ and each subfamily $\mathcal{N}_n$ is isolated. We will obtain a σ-distributively separating open cover. Since $\mathcal{N}$ is σ-isolated, given $n \geq 1$ and $N \in \mathcal{N}_n$, there exist G_N and $F_N \in \mathcal{F}$ in such a way that $N = F_N \cap G_N$ and $G_N \cap M = \emptyset$, $\forall M \in \mathcal{N}_n \setminus \{N\}$. For every $n \geq 1$, Lemma 2.5 yields that the set $C_n := \cup\mathcal{N}_n = F_n \cap G_n$, for some $F_n \in \mathcal{F}$ and $G_n \in \mathcal{G}$. Besides, we define a family of open sets

$$\mathcal{V}_n := \{G_N; N \in \mathcal{N}_n\} \cup \{G_N \cap F_n^c; N \in \mathcal{N}_n\}.$$

Now, we will check that the family $\mathcal{V} := \cup_{n<\omega}\mathcal{V}_n$ is a σ-distributively separating cover. Let x, $y \in X$, then there exists $n_0 \in \omega$ and $N \in \mathcal{N}_{n_0}$ such that $\#\{x, y\} \cap N = 1$. It is not restrictive to assume that $x \in N$ and, in this case, we will prove that $ord(x, \mathcal{V}_{n_0}) = 1$. Since $x \notin G_M$ for every $M \in \mathcal{N}_{n_0} \setminus \{N\}$, then either $x \in G_N$ or $x \in G_N \cap F_{n_0}^c$. But $x \notin G_N \cap F_{n_0}^c$ because $x \in C_{n_0} \subset F_{n_0}$.

The implications (ii)⇒(v)⇒(iii) follow from Proposition 3.2.

(iii)⇒(vi) Let us fix the metric d of (iii) and consider the identity map $I_d : (X, \tau) \to (X, d)$. Then, the network given by assertion (iii) is just the desired σ-isolated base for the identity map.

(vi)⇒(iv) Let us check that $\mathcal{B}$ is separating. For this purpose we fix distinct x, $y \in X$. Then there exists $\varepsilon > 0$ such that $\Phi(y) \notin B(\Phi(x), \varepsilon)$. Then $y \notin \Phi^{-1}(B(\Phi(x), \varepsilon)) = \cup\mathcal{B}_1$, for some $\mathcal{B}_1 \subset \mathcal{B}$. □

The former proof provides the following extra information regarding Gruenhage spaces.

Remark 3.4.

(1) It is not restrictive to assume in [31, Proposition 7.4] that the families involved are T_2-separating.

(2) Every Gruenhage space X has an open cover $\mathcal{U} = \cup_{n<\omega}\mathcal{U}_n$, such that if $x, y \in X$, then there exists n_0 verifying that

(a) x and y are separated by the subfamily $\mathcal{U}_{n_0}$;

(b) $ord(x, \mathcal{U}_{n_0}) = 1$.

Next, a consequence of Theorem 1.1. As usual, by $\tau_{\text{pointwise}}$ we denote the pointwise convergence topology.

Corollary 3.5. *Let (X, τ) be a topological space, K a compact space, and $\Phi : (X, \tau) \to (C(K), \tau_{pointwise})$ a one-to-one continuous map which has a σ-isolated base. Then X is a Gruenhage space.*

Proof. By [7, Theorem 1], the map Φ has a σ-isolated base $\mathcal{B} \subset \mathcal{F} \cap \mathcal{G}$ with respect the norm topology of $C(K)$. Now assertion (vi) of Theorem 1.1 applies. □

4. Property SLD and fragmentability

Characterizations for spaces having property SLD can be found in [17]. We begin this section by providing two results in this aim. The first one checks that $\mathcal{F} \cap \mathcal{G}$ sets are suitably preserved to get Theorem 1.1.

Proposition 4.1. *The following are equivalent for a topological space (X, τ).*

(i) X has property d-SLD (the decomposition may also be taken finite and disjoint).

(ii) There exists a separating σ-isolated family on X (the family may also be taken T_2-separating).

(iii) There exists a metric space (Z, d) and a one-to-one map $\Phi : X \to Z$ which has a σ-isolated base.

Proof. (i)⇒(ii) follows from Proposition 3.1 and implication (ii)⇒(i) from Proposition 2.4. For the proof of the implication (ii)⇒(iii) we consider a separating σ-isolated family $\mathcal{M}$ and a metric d generated by the family $\cup_n \mathcal{M}_n \cup \{X \setminus \cup \mathcal{M}_n\}$. Then

$$\mathcal{B} := \{\cap_{i \in I} M_i \colon \; M_i \in \mathcal{M}_i \cup \{X \setminus \cup \mathcal{M}_i\},\, i \in I < \omega\},$$

is a base for the identity map between (X, τ) and (X, d). Finally, let us check (iii)⇒(ii). We assume that $\Phi : (X, \tau) \to (Z, d)$ is a one-to-one map with a σ-isolated base $\mathcal{B}$. Next, we check that $\mathcal{B}$ is separating. If we fix distinct x, $y \in X$, then $\Phi(x) \neq \Phi(y)$. Thus, there exists $\varepsilon > 0$ such that $\Phi(y) \notin B(\Phi(x); \varepsilon)$. Hence, $y \notin \Phi^{-1}(B(\Phi(x); \varepsilon))$ and there exists an element $B \in \mathcal{B}$ such that $y \notin B$ but $x \in B$. □

Let us note that the following result is a consequence of Proposition 3.1, however it is more suitable to state it here because of the topic of this section.

Corollary 4.2. *Let (X, τ) be a topological space and d a metric on X such that X has property d-SLD. Then, there exists another metric d_1 on X such that X has property d_1-SLD by a finite and disjoint decomposition and, in addition, τ_{d_1} is finer than τ_d.*

Proof. We consider the σ-isolated network $\mathcal{N}$ for τ_d provided by Proposition 3.1. Let d_1 be a metric generated by

$$\bigcup_{n<\omega} \mathcal{N}_n \cup \{X \setminus \cup \mathcal{N}_n\}.$$

Then Propositions 2.4 and 2.8 apply. □

Corollary 4.3. *Let (X, τ) be a topological space, (Y, d) a metric space, and $\Phi : (X, \tau) \to (Y, d)$ a one-to-one Baire map. Then X has property SLD.*

Proof. By [18, Corollary 2.4], the map Φ has a σ-isolated function base. □

Now we will prove Example 1.3 stated in the introduction. Its proof will use the following result

Lemma 4.4. *Let X be a topological space which has property SLD. Then there exists a collection of sets pairwise disjoint, $\{X_i\}_{i<c}$, such that $X = \cup_{i<c} X_i$ and $\{x\}$ is a G_δ set of X_i, for every $x \in X_i$.*

Proof. For every $m < \omega$, we consider the decomposition $X = \cup_{k<\omega} X_k^m$ given by condition SLD for $\varepsilon = 1/m$. By Corollary 4.2, it is not restrictive to assume that each previous decomposition is disjoint. Now we define, for every element of the Baire space $\bar{k} = (k_1, k_2, \dots) \in \omega^\omega$, the set $X_{\bar{k}} = \cap_{i<\omega} X_{k_i}^i$. It is clear that each element of $X_{\bar{k}}$ is a G_δ set of $X_{\bar{k}}$. □

Proof of Example 1.3. It is easy to check that the topological space $[0, \gamma]$ is fragmentable by the discrete metric for every ordinal γ. In fact, if we fix $A \subset [0, \gamma]$, we can consider $\alpha = \min A$. If $\alpha > 0$ then $\{\alpha\} = (\beta, \alpha + 1) \cap A$ for each $\beta \in X$, $\beta < \alpha$. If $\alpha = 0$, then $\{0\} = [0, 1) \cap A$.

Next, we will show that $[0, c^+]$ does not have property SLD. We will do it by contradiction. For this purpose, we suppose that X has property SLD. Applying Lemma 4.4, we can decompose $[0, c^+] = \cup_{i<c} X_i$ in such a way that for every $x \in X_i$, $\{x\}$ is a G_δ set of X_i. Now, we fix the stationary set

$$Z := \{\alpha \in [0, c^+] : \text{cofinality}(\alpha) > \omega\}.$$

If $\alpha \in Z \cap X_i$, then there exists $\varphi(\alpha) < \alpha$ such that

$$(\varphi(\alpha), \alpha] \cap X_i = \{\alpha\}.$$

We now apply [12, Fodor lemma] to the map $\varphi : Z \to [0, c^+]$. Then there exists a stationary subset $Z' \subset Z$ such that the corresponding restriction is constant, i. e., $\varphi|_{Z'} \equiv \gamma \in [0, c^+]$. Cardinality of the set $\{z \in Z'; z > \gamma\}$ is bigger than c^+. Thus, there exists $i < c$ such that the cardinality of the set $Z \cap X_i$ is bigger than c. As a consequence, we can take α and β both belonging to $Z \cap X_i$, satisfying $\alpha < \beta$ and

$$\alpha \in (\gamma, \beta] \cap X_i = \{\beta\},$$

which is a contradiction.

Finally, let us check now that $[0, c]$ has property SLD. Indeed, let X be a topological space of cardinality at most continuum. Then there exists a one-to-one map $g : X \to \mathbb{R}$. Let us define the metric d on X by $d(x, y) := |g(x) - g(y)|$, for every x, $y \in X$. Then X has property d-SLD. In fact, for every $\varepsilon > 0$, X may be decomposed into countably many sets of d-diameter less than ε. □

Acknowledgements

We would like to thank José Orihuela his interest and suggestions on this material, Antonio Avilés his help with Example 1.3, and Alberto Escapa for valuable discussions.

This work has been partially supported by the Generalitat Valenciana project GV/2014/072 and Universidad de Alicante project GRE11-08.

References

[1] Fabian M. J.: Gateaux differentiability of convex functions and topology. John Wiley & Sons, New York (1997)

[2] Fabian M. J., Habala P., Hájek P., Montesinos V., Pelant J., Zizler V.: Functional Analysis and Infinite-Dimensional Geometry. Springer Verlag, New York (2001)

[3] Fabian M. J., Montesinos V., Zizler V.: Gul'ko, descriptive, and Gruenhage compact spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math., RACSAM 104(2), 201–220 (2010)

[4] Gruenhage G.: A note on Gul'ko compact spaces. Proc. Amer. Math. Soc. 100, 371–376 (1987)

[5] Habala P., Hájek P., Zizler V.: Introduction to Banach spaces. Matfyzpress, Prague (1996)

[6] Hansell R. W.: Descriptives sets and the topology of nonseparable Banach spaces. Serdica Math. J. 27, 1–66 (2001)

[7] Hansell R. W.: Absolute Souslin-$\mathcal{F}$-spaces and other weak-invariants for the norm topology. Topology Appl. 111, 151–160 (2001)

[8] Hansell R. W.: Descriptive topology. In: Husek M. and Van Mill J. (eds.) Recent Progress in General Topology, chap. 8. Elsevier Science B. V., Amsterdam (1992)

[9] Haydon R.: Locally uniformly convex norms in Banach spaces and their duals. J. Funct. Anal. 254, 2023–2039 (2008)

[10] Jayne J. E., Namioka I., Rogers C. A.: σ-Fragmentable Banach spaces. Mathematika 39(2), 197–215 (1992)

[11] Jayne J. E., Rogers C. A.: Borel selectors for upper semicontinuous set-valued maps. Acta Math. 155, 41–79 (1985)

[12] Jech T.: Set Theory. Springer-Verlag, Berlin (2002)

[13] Kakol J., Kubis W., López-Pellicer M.: Descriptive Topology in Selected Topics of Functional Analysis. Springer Science, New York (2011)

[14] Kenderov P., Moors W.: Fragmentability and sigma-fragmentability of Banach spaces. J. London Math. Soc. 60(2), 203–223 (1999)

[15] Miscenko A.: Spaces with a point-countable base. Soviet. Math. Dock. 3, 855–858 (1962)

[16] Moltó A., Orihuela J., Troyanski S., Valdivia M.: Locally uniformly rotund renorming and fragmentability. Proceedings of the London Mathematical Society, 75, 619–640 (1997)

[17] Moltó A., Orihuela J., Troyanski S., Valdivia M.: On weakly locally uniformly rotund Banach spaces. J. Funct. Anal. 163, 252–271 (1999)

[18] Moltó A., Orihuela J., Troyanski S., Valdivia M.: A Nonlinear Transfer Technique for Renorming. Lecture Notes in Mathematics 1951, Springer-Verlag, Berlin (2009)

[19] Oncina L.: The JNR property and the Borel structure of a Banach space. Serdica Math. J. 26, 13–32 (2000)

[20] Oncina L., Raja M.: Descriptive compact spaces and renorming. Studia Math. 165(1), 39–52 (2004)

[21] Orihuela J., Smith R.J., Troyanski S.: Strictly convex norms and topology. Proc. London Math. Soc. 104(3), 197–222 (2011)

[22] Raja M.: Kadec norms and Borel sets in a Banach space. Studia Math. 136(1), 1–16 (1999)

[23] Raja M.: Weak* locally uniformly rotunds norms and descriptive compact spaces. Journal of Funct. Anal. 197, 1–13 (2003)

[24] Rosenthal H. P.: The heredity problem for weakly compactly generated Banach spaces. Compositio Math. 28, 83–111 (1974)

[25] Ribarska N. K.: Internal characterization of fragmentable spaces. Mathematika 34, 243–257 (1987)

[26] Ribarska N. K.: A Radon-Nikodym compact which is not a Gruenhage space. Comptes rendus de l'Academie bulgare des Sciences 41(7), 9–11 (1988)

[27] Ribarska N. K.: On having a countable cover by sets of small local diameter. Studia Math. 140(2), 99–116 (2000)

[28] Smith R. J.: Gruenhage compacta and strictly convex dual norms. J. Math. Anal. Appl. 350, 745–750 (2009)

[29] Smith, R. J.: Strictly convex norms, G_δ-diagonals and non-Gruenhage spaces. Proc. Amer. Math. Soc. 140, 3117–3125 (2012)

[30] Smith, R. J., Troyanski, S.: Renormings of C(K) spaces. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math., RACSAM 104(2), 375–412 (2010)

[31] Stegall C.: The topology of certain spaces of measures. Topology Appl. 41, 73–112 (1991)

[32] Talagrand M.: Espaces de Banach faiblement $\mathcal{K}$-analytiques. Ann. of Math. 110, 407–438 (1979)