A SURVEY ON ASSIGNMENT MARKETS

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Abstract. The assignment game is a two-sided market, say buyers and sellers, where demand and supply are unitary and utility is transferable by means of prices. This survey is structured in three parts: a first part, from the introduction of the assignment game by Shapley and Shubik (1972) until the publication of the book of Roth and Sotomayor (1990), focused on the notion of core; the subsequent investigations that broaden the scope to other notions of solution for these markets; and its extensions to assignment markets with multiple sides or multiple partnership. These extended two-sided assignment markets, that allow for multiple partnership, better represent the situation in a labour market or an auction.

1. **The preliminaries.** When the marriage problem was introduced by Gale and Shapley in "College Admission and the Stability of Marriage" (1962), its counterpart in which utility can be transferred from agents on one side of the market to agents on the opposite side by means of prices or salaries had already been considered in the economic literature. In the 1953 edition of the book of von Neumann and Morgenstern, "Theory of Games and Economic Behavior", several instances of two or three-person markets are considered as examples of coalitional games, and a description of their *solutions*[1] is included. There, the general case with several buyers and sellers is also formulated (see sections 61 to 64).

One of the first formulations of the assignment problem related to a situation in economics (assigning industrial plants to locations), is given by Koopmans and Beckmann (1957) in an Econometrica paper. Their aim is to match two sets of an equal number n of objects by making up pairs consisting of one object from each set. For each of the n^2 possible pairs a value is given and the problem is to find a matching where the sum of values of the matched pairs is as high as possible.

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[1]In von Neumann-Morgenstern (1953), the term solution refers to what is now known as von Neumann-Morgenstern solution or stable set.

They make use of a remark made by von Neumann (1953) based in a theorem due to Birkhoff (1946) to show that an optimal assignment can be obtained by solving a linear program. Then, by considering the associated dual linear program, they introduce a system of rents on the locations that sustain the optimal assignment. This notion of equilibrium prices is formally defined in Gale (1960), who also proves that equilibrium prices exist for any assignment problem.

We now write these preliminar contributions with a unified notation. Like in Gale (1960), and also, some years later, in Shapley and Shubik (1972), let us present the assignment problem as a model for a house market.

Let M be a finite set of m buyers and M' a finite set of m' sellers, these sets being disjoint. Each buyer $i \in M$ is willing to buy at most one house and each seller $j \in M'$ has exactly one house on sale. Assume $h_{ij} \geq 0$ is how much buyer $i \in M$ values the house of seller $j \in M'$ and $c_j \geq 0$ is the reservation value of this seller, meaning he will not sell his house for a lower price. Then, whenever $h_{ij} \geq c_j$, there is room to agree on some price $h_{ij} \geq p \geq c_j$ and the join profit of this trade is $(h_{ij} - p) + (p - c_j)$. As a consequence, we consider the *valuation matrix* $A = (a_{ij})_{(i,j)\in M\times M'}$ where $a_{ij} = \max\{h_{ij} - c_j, 0\}$ for all $i \in M, j \in M'$.

The problem of assigning buyers to sellers in such a way that the sum of values of the matched pairs is as high as possible is a well known operations research problem, namely the assignment problem, and reduces to solving an integer linear program:

$$\begin{array}{ll} \max & \sum_{i\in M}\sum_{j\in M'} a_{ij}\mu_{ij} \\ \text{s.t.} & \sum_{i\in M}\mu_{ij} \leq 1, \text{ for all } j \in M', \\ & \sum_{j\in M'}\mu_{ij} \leq 1, \text{ for all } i \in M, \\ & \mu_{ij} \in \{0,1\} \text{ for all } (i,j) \in M\times M'. \end{array} \tag{1}$$

In this linear program, $\mu = (\mu_{ij})_{(i,j)\in M\times M'}$ represents a *feasible assignment (or matching)* of buyers to sellers such that if $\mu_{ij} = 1$, then buyer i buys the house of seller j and if $\mu_{ij} = 0$ she does not. We assume without loss of generality that matrix A is square (same number of buyers and sellers) and hence (1) can be rewritten with equality constrains: the linear program on the left of (2) is the relaxation of the previous integer linear program (1) and a solution can be obtained by solving the linear program on the right of (2). Notice that the matrices $(\mu_{ij})_{(i,j)\in M\times M'}$ in the feasible set of the linear program on the right of (2) are doubly stochastic matrices:

$$\begin{array}{ll} \max & \sum_{i\in M}\sum_{j\in M'} a_{ij}\mu_{ij} \\ \text{s.t.} & \sum_{i\in M}\mu_{ij} \leq 1, \text{ for all } j \in M', \\ & \sum_{j\in M'}\mu_{ij} \leq 1, \text{ for all } i \in M, \\ & \mu_{ij} \geq 0 \text{ for all } (i,j) \in M\times M', \end{array} \qquad \begin{array}{ll} \max & \sum_{i\in M}\sum_{j\in M'} a_{ij}\mu_{ij} \\ \text{s.t.} & \sum_{i\in M}\mu_{ij} = 1, \text{ for all } j \in M', \\ & \sum_{j\in M'}\mu_{ij} = 1, \text{ for all } i \in M, \\ & \mu_{ij} \geq 0 \text{ for all } (i,j) \in M\times M'. \end{array} \tag{2}$$

As a consequence, Birkhoff's theorem guarantees the existence of a solution to (2) with $\mu_{ij} \in \{0,1\}$ for all $(i,j) \in M\times M'$, and hence this is also a solution to the assignment problem (1).

In the sequel, we identify the feasible assignment μ of (1) with the set of bijections between a subset of M and a subset of M', in such a way that when $\mu_{ij} = 1$ we write $(i,j) \in \mu$, or equivalently $\mu(i) = j$ or $\mu^{-1}(j) = i$. Moreover, we denote by $\mathcal{M}(M, M')$ the set of matchings between M and M', and by $\mathcal{M}_A(M, M')$ those which are optimal with respect to a given valuation matrix A: $\mu \in \mathcal{M}_A(M, M')$ if $\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}$ for all $\mu' \in \mathcal{M}(M, M')$.

The above relation of the assignment problem with a linear program, allows to consider the linear program that is dual to the first program in (2):

$$\begin{array}{ll} \min & \sum_{i\in M} u_i + \sum_{j\in M'} v_j \\ \text{s.t.} & u_i + v_j \geq a_{ij} \text{ for all } (i,j)\in M\times M' \\ & u_i \geq 0 \text{ for all } i\in M, \\ & v_j \geq 0 \text{ for all } j \in M'. \end{array} \tag{3}$$

Gale (1960) characterizes the competitive equilibrium prices by means of the optimal solutions of this dual program. To see that, we assume without loss of generality $m = m'$ and null reservation values, $c_j = 0$ for all $j \in M'$, and introduce the necessary definitions.

Given a vector of non-negative prices $p \in \mathbb{R}^{M'}$, the *demand* set of buyer $i \in M$ at prices p is

$$D_i(p) = \{j \in M' \mid a_{ij} - p_j = \max_{k\in M'}\{a_{ik} - p_k\}\}.$$

Then, a pair (p, μ) formed by a vector of prices and a matching is a *competitive equilibrium* if $\mu(i) \in D_i(p)$ for all $i \in M$ and $p_j = 0$ whenever $j \in M'$ is unmatched by μ. In this case we say that p is an equilibrium price vector. Gale's Theorem 5.6 shows that for any solution (u, v) of the dual program (3), the prices $p_j = v_j$ for all $j \in M'$ are equilibrium prices.

2. **From Shapley and Shubik to Roth and Sotomayor.** The first rigorous presentation of this two-sided market as a coalitional game, the *assignment game*, can be found in Shapley and Shubik (1972). To give a complete description of the game, we must determine the amount each coalition $S \subseteq M \cup M'$ can attain on its own, which implies to solve an assignment problem with the valuation matrix restricted to rows and columns of agents in S:

$$w_A(S) = \max_{\mu\in\mathcal{M}(S\cap M, S\cap M')} \sum_{(i,j)\in\mu} a_{ij}.$$

An important property of this coalitional function is that players on the same side of the market appear as *substitutes* in coalitional terms, while agents on opposite sides appear as *complements* (see Shapley, 1962): given two agents i, j not contained in $S \subseteq M \cup M'$,

$$\begin{array}{rcl} w_A(S\cup\{i,j\}) - w_A(S\cup\{i\}) & \leq & w_A(S\cup\{j\}) - w_A(S) \text{ if } i,j\in M \text{ or } i,j\in M', \\ w_A(S\cup\{i,j\}) - w_A(S\cup\{i\}) & \geq & w_A(S\cup\{j\}) - w_A(S) \text{ if } i\in M \text{ and } j \in M'. \end{array}$$

The paper of Shapley and Shubik focuses on the *core*[2] of the game. They show that the core of the assignment game is non-empty, since it coincides with the set of solutions of the dual linear program (3). This fact reveals that: a) only core constraints related to individual coalitions and mixed-pair coalitions (those formed by a buyer and a seller) are relevant to describe the core of the assignment game, and b) efficiency together with core constraints for mixed-pairs and individuals implies there are no side-payments in the core other than the price a buyer pays a seller to acquire his house.

[2]The core of a coalitional game (N, w) is the set of payoff vectors $x \in \mathbb{R}^N$ such that $\sum_{i\in N} x_i = w(N)$ and $\sum_{i\in S} x_i \geq w(S)$, for all $S \subseteq N$.

To summarize, $(u,v)\in\mathbb{R}^M\times\mathbb{R}^{M'}$ is in the core $C(w_A)$ of the assignment game $(M\cup M', w_A)$ if and only if there exists a matching $\mu\in\mathcal{M}(M,M')$ such that

$$\begin{array}{l} u_i+v_j=a_{ij} \text{ for } (i,j)\in\mu, \\ u_i=0 \text{ if } i \text{ unmatched by } \mu,\ v_j=0 \text{ if } j \text{ unmatched by } \mu \\ u_i+v_j\geq a_{ij} \text{ for } (i,j)\in M\times M',\ u_i\geq 0 \text{ for all } i\in M,\ v_j\geq 0 \text{ for all } j\in M'. \end{array} \quad (4)$$

Notice that any matching μ supporting a core allocation must be optimal, and moreover any optimal matching can play this role.

In the book of Roth and Sotomayor, the payoff vectors that satisfy (4) are named *(pairwise)-stable payoff vectors*, since there is no buyer-seller pair (i,j) that are not matched together by μ but both could improve their payoff by breaking their partnerships by μ and being matched one to each other to share a_{ij}. The coincidence of the core and the set of pairwise-stable allocations is the first similarity between the assignment game and the two-sided matching problems without money.

Up to this point, we can state the following result.

Theorem 2.1. *For the assignment game, the four sets below coincide:*

- *The core.*
- *The set of dual solutions to the assignment problem.*
- *The set of competitive equilibrium payoff vectors.*
- *The set of pairwise-stable payoff vectors.*

The second main contribution in the paper of Shapley and Shubik (1972) is the study of the structure of the core of the assignment game. If we consider on the core elements the partial order defined by one side of the market, for instance $(u,v)\leq_M(u',v')$ if and only if $u_i\leq u'_i$ for all $i\in M$, it results that the core has the structure of a *complete lattice* with respect to this order. Indeed, Shapley and Shubik prove that given two core elements (u,v) and (u',v'), the join

$$(u,v)\vee(u',v')=\big((\max\{u_i,u'_i\})_{i\in M},(\min\{v_j,v'_j\})_{j\in M'}\big)$$

and the meet

$$(u,v)\wedge(u',v')=\big((\min\{u_i,u'_i\})_{i\in M},(\max\{v_j,v'_j\})_{j\in M'}\big)$$

also belong to the core.

A consequence of this lattice structure of the core is the existence of two special extreme core points. In one of them, $(\overline{u}^A,\underline{v}^A)$ each buyer maximizes her payoff in the core, while each seller minimizes his. This core element is related to the minimum competitive equilibrium price vector. In the other one, $(\underline{u}^A,\overline{v}^A)$, each seller maximizes his core payoff while buyers get their minimum one, and this is related to the maximum competitive equilibrium price vector. Moreover, these are the two more distant points inside the core. What is remarkable is that all agents on the same side of the market, despite being competing for the best deal, obtain their maximum core payoff in the same core element.

The reader will notice that the lattice structure of the core and the existence of the *buyers-optimal and the sellers-optimal core allocations* resembles the lattice structure of the set of stable matchings in the marriage problem and the existence of the men-optimal and women-optimal matchings.

The results of Shapley and Shubik are the starting point of Chapter 8 in Roth and Sotomayor (1990). Besides that, the chapter covers the main contributions on the assignment game before that date. We will mention some of them which analyze in depth the properties of the buyers-optimal and sellers-optimal core allocations.

Lemma 8.4 in Roth and Sotomayor (1990) provides a neat proof of a result presented, independently, in Demange (1982) and Leonard (1983). It says that the maximum core payoff of an agent, be it a buyer or a seller, is her/his marginal contribution to the grand coalition. That is, given an assignment game $(M \cup M', w_A)$,

$$\overline{u}_i^A = w_A(M \cup M') - w_A((M \setminus \{i\}) \cup M'), \text{ for all } i \in M, \tag{5}$$

$$\overline{v}_j^A = w_A(M \cup M') - w_A(M \cup (M' \setminus \{j\})), \text{ for all } j \in M'. \tag{6}$$

Notice that, if μ is an optimal matching for this market, then the maximum core payoff of buyer $i \in M$, when matched to $j = \mu(i)$, can be written also in this way:

$$\overline{u}_i^A = a_{ij} - \left[w_A((M \setminus \{i\}) \cup M') - w_A((M \setminus \{i\}) \cup (M' \setminus \{j\})) \right].$$

Hence, the price paid by i (assume without loss of generality that the reservation values c_j, for $j \in M'$, are null) is

$$p_j = w_A((M \setminus \{i\}) \cup M') - w_A((M \setminus \{i\}) \cup (M' \setminus \{j\})), \tag{7}$$

which does not depend on the valuations of buyer i.

As a consequence of this fact, a non-cooperative approach to the assignment game can be done. Consider any mechanism that, from the valuations revealed by the buyers, implements the buyers-optimal core allocation (or the minimum competitive prices). Because of (7), buyers do not have *incentives* to misrepresent their true valuations.

Later on in this survey an auction mechanism that produces the buyers-optimal core allocation will be reviewed. By the moment, let us collect the existence result of Shapley and Shubik (1972), the computational result of Demange (1982) and the property regarding incentives of Leonard (1983). A similar statement could be written regarding the sellers-optimal core allocation.

Theorem 2.2. *There exists an element in the core of the assignment game where each buyer receives her marginal contribution to the grand coalition.*

In any mechanism that, from the valuation matrix, implements the buyers-optimal core allocation, truth telling is a dominant strategy for each buyer.

A similar result in the marriage market states that any mechanism that yields the men-optimal stable matching (in terms of the stated preferences) makes it a dominant strategy for each men to state his true preferences.

Another question that was first studied by Mo (1988) in the assignment game, and later also consider by Roth and Sotomayor (1990) for the marriage market, is concerned with the effect on the core of changing the market by introducing a new agent.

Let $(M \cup M', w_A)$ be an assignment game and assume a new buyer i^* enters the market. The new game will be $((M \cup \{i^*\}) \cup M', w_{A'})$ where $a'_{ij} = a_{ij}$ for all $i \in M$ and $j \in M'$. Then,

$$\overline{u}_i^A \geq \overline{u}_i^{A'} \text{ and } \underline{u}_i^A \geq \underline{u}_i^{A'} \text{ for all } i \in M,$$
$$\overline{v}_j^A \leq \overline{v}_j^{A'} \text{ and } \underline{v}_j^A \leq \underline{v}_j^{A'} \text{ for all } j \in M'.$$

This means that each of the previously existing buyers is (weakly) worse off in the market with the new entrant and each of the sellers is (weakly) better off in this new market.

In Mo (1988), more precise conclusions are drawn regarding the changes in the core when the market faces a new entrant. If the new buyer i^* gets matched by some optimal assignment of the new market, then there exists a non-empty set of agents such that for all buyer i in this set $\overline{u}_i^{A'} = \underline{u}_i^A$; and for all seller j in this set $\underline{v}_j^{A'} = \overline{v}_j^A$. That is, all buyers in this set are so punished by the presence of the new entrant that their best payoff in the core of the new market equals their worst payoff in the core of the original market.

As the reader that reaches this point will realize, the first studies on the assignment market focus on the core and in particular on the two extreme core elements that are optimal for one side of the market. However, in most instances, it seems unfair to solve the situation by allocating the utility of an optimal matching by means of one of these two extreme points that favours one sector and damages the opposite. A first attempt to propose a not so extreme solution is due to Thompson (1980). He defines the *fair division point* of an assignment market $(M \cup M', w_A)$, as the midpoint between the buyers-optimal core allocation and the sellers-optimal core allocation:

$$\tau(w_A) = \frac{1}{2}(\overline{u}^A, \underline{v}^A) + \frac{1}{2}(\underline{u}^A, \overline{v}^A). \tag{8}$$

Some monotonicity properties of this single-valued solution are analyzed years later in Núñez and Rafels (2002b).

Thompson's fair division point can be quite easily computed, just by solving three linear programs. The first linear program aims to obtain an optimal matching. Then, solving a second linear program we obtain the buyers-optimal core allocation,

$$\begin{array}{ll} \max & \sum_{i\in M} u_i \\ & \sum_{i\in M} u_i + \sum_{j\in M'} v_j = w_A(M\cup M') \\ & u_i + v_j \geq a_{ij}, \text{ for all } (i,j)\in M\times M' \\ & u_i \geq 0, \text{ for all } i\in M \\ & v_j \geq 0, \text{ for all } j \in M', \end{array} \tag{9}$$

and a similar program yields the sellers optimal core allocation.

Notice that Thompson's fair division point may still lie on the boundary of the core for some instances of assignment markets, and in that case there is a pair of agents who are paid their minimum joint core payoff.

Rochford (1984) characterizes certain points in the interior of the core of the assignment game as fixed points of a rebargaining process. Given an optimal matching μ and a core element $x = (u, v)$ of an assignment game $(M \cup M', w_A)$, each pair of optimally matched partners $(i, j) \in \mu$ engage in a Nash bargaining problem, using as their threats the maximum they could obtain if matched to a different partner and paying him/her according to x. If it results that the solution to this two-person Nash bargaining problem is (u_i, v_j), for each pair $(i, j) \in \mu$, we say that $x = (u, v)$ is a fixed point for this procedure. The set of fixed points of this procedure is named the set of *symmetrically pairwise-bargained* (SPB) allocations and it is proved to coincide with the intersection of the core and the kernel.[3] Making use of Tarski's fixed-point theorem, Roth and Sotomayor (1988) observe that the set of SPB allocations inherits the lattice structure of the core and hence there is also a SPB allocation that is optimal for each side of the market.

[3]The kernel of a coalitional game is a solution concept introduced by Davis and Maschler (1965).

Some years later, Driessen (1998) shows that in fact the set of SPB allocations of an assignment game $(M \cup M', w_A)$ coincides with the kernel of the game. As a consequence, a geometric characterization of the core elements of the kernel for arbitrary coalitional games, due to Mashler et al. (1979), can be applied to the assignment game. Under the assumption that A is square, for each $(u,v) \in C(w_A)$ and (i,j) in an optimal matching μ, let us denote by $\delta_{ij}(u,v)$ the maximum amount that i can transfer to j without getting out of the core, that is,

$$\begin{aligned}\delta_{ij}(u,v) &= \max\{\varepsilon \geq 0 \mid (u - \varepsilon e^{\{i\}}, v + \varepsilon e^{\{j\}}) \in C(w_A)\} \text{ and}\\ \delta_{ji}(u,v) &= \max\{\varepsilon \geq 0 \mid (u + \varepsilon e^{\{i\}}, v - \varepsilon e^{\{j\}}) \in C(w_A)\},\end{aligned}$$

where $e^{\{k\}} \in \mathbb{R}^M$, for $k \in M \cup M'$, satisfies $e^{\{k\}}_r = 0$ for all $r \neq k$ and $e^{\{k\}}_k = 1$. Then, a payoff vector $(u,v) \in C(w_A)$ is a SPB allocation (or a kernel allocation) if and only if, given an optimal matching $\mu \in \mathcal{M}_A(M,M')$, the maximum that a buyer can transfer to her assigned seller without getting out of the core equals the maximum that the seller can transfer to the buyer without getting outside the core:

$$\delta_{ij}(u,v) = \delta_{ji}(u,v) \text{ for all } (i,j) \in \mu. \tag{10}$$

Geometrically, condition (10) says that each SPB allocation bisects certain core segments. From the point of view of computation, condition (10) can be stated as follows.

Theorem 2.3. *Let $(M \cup M', w_A)$ be a square assignment game, μ an optimal matching and $(u,v) \in C(w_A)$. The core allocation (u,v) is a SPB allocation (or a kernel element) if and only if, for all $(i,j) \in \mu$,*

$$\delta_{ij}(u,v) = \delta_{ji}(u,v), \text{ where} \tag{11}$$

$\delta_{ij}(u,v) = \min_{k \in M' \setminus \{j\}}\{u_i, a_{ik} - u_i - v_k\}$, $\delta_{ji}(u,v) = \min_{k \in M \setminus \{i\}}\{v_j, a_{kj} - u_k - v_j\}$.

We now illustrate with a simple example what has been developed in this section. Let $M = \{1,2\}$ be the set of buyers, $M' = \{1', 2'\}$ the set of sellers and the valuation matrix A:

	1'	2'
1	**6**	5
2	2	**4**

. The core is defined by

$$\begin{aligned}&u_1 + v_1 = 6\\ &u_1 + v_2 \geq 5\\ &u_2 + v_1 \geq 2\\ &u_2 + v_2 = 4\\ &u_i \geq 0,\ v_j \geq 0.\end{aligned} \tag{12}$$

and the shadowed region in Figure 1 represents the projection of the core in the two-dimensional space of the buyers' payoffs. The reader will check that in the buyers-optimal core allocation $(6,4;0,0)$ each buyer receives her marginal contribution, while in the sellers-optimal core allocation $(1,0;5,4)$, each seller does.

The core elements on the piece-wise linear curve $B - C - D - G$ are those that satisfy the bisection property $\delta_{11'}(u,v) = \delta_{1'1}(u,v)$. Those on the piece-wise linear curve $A - C - E - F$ satisfy the other bisection property: $\delta_{22'}(u,v) = \delta_{2'2}(u,v)$. Hence, the segment $C - D$ is the set of SPB allocations.

Over the period 1972-1990, some extensions of the assignment game, that allow for more than two sides or for multiple partnership, began to be considered. We consign these extensions to Section 4 in this survey.

A generalization of the assignment game (with unitary demands and supplies) was carried out by Demange and Gale (1985). These authors consider a two-sided assignment game where matchings are one-to-one but the utility functions of the

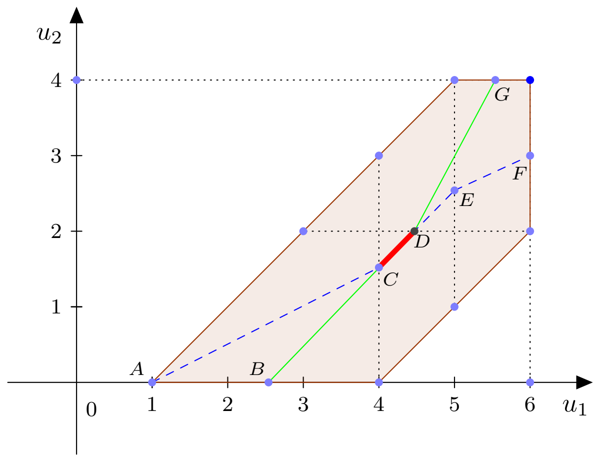

FIGURE 1. The core and the kernel of a 2x2 assignment game

agents are not linear in money. This model is also developed in Chapter 9 of the book of Roth and Sotomayor (1990) and although it has strong similarities with the classical Shapley and Shubik assignment game (the set of stable payoff vectors is non-empty and coincides with the core and a core element optimal for each side of the market exists), there are surprising differences, such as the fact that the core may not be a convex set.

Another generalization developed in the period before 1990 is due to Kaneko (1982) by introducing assignment games with non-transferable utilities. In this setting the non-emptyness of the core and the relationship with the competitive prices are also proved to hold.

In the next section we cotinue describing the advances in the study of the two-sided assignment game with unitary supplies and demands, and transferable utility.

3. **Further contributions on the assignment game after Roth and Sotomayor.** We organize the contributions on the Shapley and Shubik assignment game that appear after 1990 in different blocks: first we present further properties of the core mainly related with the property of exactness, secondly the question regarding the existence of von Neumann-Morgenstern stable sets for the assignment game is answered, in third place we review contributions related to single-valued solutions such as the Shapley value and the nucleolus and finally some contributions based on an axiomatic approach to solutions.

3.1. **More about the core.** From Figure 2, the reader will realize that the core of a 2x2 assignment game, when projected to the space of payoffs to one side of the market, has a quite particular shape: it is a *45-degree lattice*. Quint (1991) proves that this also holds for markets with more agents on each side and in fact this property gives a *geometric characterization of the core* of the assignment game. Given any non-empty 45-degree lattice in $\mathbb{R}^m$, that is,

$$L = \left\{ u \in \mathbb{R}^m \;\middle|\; \begin{array}{l} u_i - u_k \geq d_{ik} \text{ for all } i,k \in \{1,\dots,m\}, i \neq k \\ b_i \leq u_i \leq e_i \text{ for all } i \in \{1,\dots,m\} \end{array} \right\}$$

where $d_{ik}, b_i, e_i \in \mathbb{R}$, $b_i \geq 0$ and $e_i \geq 0$, there exists a non-negative matrix A with m rows and either m or $m+1$ columns, such that the lattice L is the core of the assignment game defined by the valuation matrix A, that is $L = C(w_A)$.

For $m = 2$, the lattice L determines either a unique valuation matrix A with two rows and two columns such that $C(w_A) = L$, or a unique valuation matrix with two rows and three columns such that $C(w_A) = L$. But with more agents the same 45-degree lattice may represent the core of several valuation matrices of the same dimension. The reader will easily check that the two following matrices

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{y} \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \tag{13}$$

define markets with the same core, since $C(w_A) = C(w_B) = \{(t,t,t;1-t,1-t,1-t) \mid 0 \leq t \leq 1\}$.

Quint (1991) poses the question of which is the maximum matrix $\overline{A}$ among those with $C(w_A) = L$, given a 45-degree lattice L. Clearly, this maximality requieres that no matrix entry can be raised without modifying the core, that is, for each $(i,j) \in M \times M'$ there must be a core element $(u,v) \in C(w_{\overline{A}})$ with $u_i + v_j = \overline{a}_{ij}$. And this is a weaker form of exactness.[4]

Solymosi and Raghavan (2001) characterize, in terms of the valuation matrix, the exactness of an assignment game. A square assignment game $(M \cup M', w_A)$, with an optimal matching placed on the main diagonal, is exact if and only if matrix A has:

- **a dominant diagonal**: $a_{ii} \geq a_{ij}$ and $a_{ii} \geq a_{ji}$, for all $i,j \in M$, and
- **a doubly dominant diagonal**: $a_{ij} + a_{kk} \geq a_{ij} + a_{jk}$, for all $i,j,k \in M$.

Of course, when an assignment game is exact, its valuation matrix is maximum among those leading to the same core. However exactness is a suficient but not a necessary condition for that, just exactness for mixed-pair coalitions (*buyer-seller exactness*) is needed. The reader will check that the assignment game in (12) is not exact, since it has not a dominant diagonal, but no matrix entry can be raised without modifying the core.

Núñez and Rafels (2002a) show that given any assignment game $(M \cup M', w_A)$ there exists a unique matrix $\overline{A}$ that is buyer-seller exact and gives rise to the same core, $C(w_A) = C(w_{\overline{A}})$. Under the assumption that A is square and μ an optimal matching, for all $(i,j) \in M \times M'$, the entry in this matrix $\overline{A}$ is given by

$$\overline{a}_{ij} = a_{i\mu(i)} + a_{\mu^{-1}(j)j} + w_A(M \cup M' \setminus \{\mu^{-1}(j), \mu(i)\}) - w_A(M \cup M').$$

Moreover, an assignment game $(M \cup M', w_A)$ is buyer-seller exact if and only if A has a doubly dominant diagonal. For instance, the buyer-seller exact representative of matrices A and B in (13) is the matrix with entries $\overline{a}_{ij} = 1$ for all $i,j = 1,2,3$. Such a matrix is known as a *glove market* matrix.

Given any assignment game $(M \cup M', w_A)$, the entries of the buyer-seller exact representative $\overline{A}$ can be obtained by solving a combinatorial optimization problem for each $(i,j) \in M \times M'$, but also by means of an iterative procedure that computes $\overline{A}$ in $\mathcal{O}(m^4)$ time (see Núñez and Solymosi, 2014).

The analysis of all valuation matrices that define assignment games with the same core is continued in Martínez de Albéniz et al. (2011) to find that they form a join

[4] A coalitional game (N, v) is *exact* if for all $S \subseteq N$ there exists $x \in C(w_A)$ such that $\sum_{i \in S} x_i = v(S)$.

semilattice with a unique maximal element (which is $\overline{A}$) and finitely many minimal elements. To reach that, the coincidence of the cores of two assignment games with the same player set is characterized: two square assignment games $(M \cup M', w_A)$ and $(M \cup M', w_B)$ have the same core if and only if

$$w_A((M \cup M') \setminus \{i,j\}) = w_B((M \cup M') \setminus \{i,j\}) \text{ for all } (i,j) \in M \times M'. \quad (14)$$

Note that matrices A and B in (13) satisfy this condition.

The buyer-seller exact representative of an assignment game has proved to be a useful tool in several studies such as the characterization of the dimension of the core in Núñez and Rafels (2008), the representation of an assignment market in terms of glove markets in Núñez and Rafels (2009), and also the computation of its extreme core allocations.

The first results on the *extreme core points* of the assignment game $(M \cup M', w_A)$ are due to Balinski and Gale (1985). There, given any core element $(u,v) \in C(w_A)$, its associated tight graph is constructed with set of nodes $M \cup M'$ and edges $(i,j) \in M \times M'$ whenever $u_i + v_j = a_{ij}$. Then, (u,v) is an extreme core point, $(u,v) \in \mathrm{ext}(C(w_A))$, if and only if each component of its tight graph has an agent with null payoff.

Some years later, Hamers et al. (2002) show that every extreme core allocation of an assignment game is a *marginal worth vector*.[5] But not all marginal worth vectors of an assignment game are extreme core points, only those that satisfy the core constraints. This provides a way to compute the extreme core allocations: compute all marginal worth vectors and select those that belong to the core. However, this procedure is very costly since you need the worth of all coalitions. A characterization of the extreme core allocations in Núñez and Rafels (2003) is interesting from a theoretical point of view.

A simple procedure to compute the extreme core points of an assignment game is developed in Izquierdo et al. (2007). Given $(M \cup M', w_A)$, we recursively define a payoff vector $x^{\sigma,A} \in \mathbb{R}^M \times \mathbb{R}^{M'}$, named a *max-payoff vector*, for each possible order σ on the player set in the following way: $x^{\sigma,A}_{\sigma(1)} = 0$ and

$$x^{\sigma,A}_{\sigma(k)} = \begin{cases} \max_{\substack{\sigma(l)<\sigma(k) \\ \sigma(l) \in M'}} \{0, a_{\sigma(k)\sigma(l)} - x^{\sigma,A}_{\sigma(l)}\} & \text{if } \sigma(k) \in M, \\ \max_{\substack{\sigma(l)<\sigma(k) \\ \sigma(l) \in M'}} \{0, a_{\sigma(l)\sigma(k)} - x^{\sigma,A}_{\sigma(l)}\} & \text{if } \sigma(k) \in M'. \end{cases} \quad (15)$$

It is proved that each extreme core allocation of the assignment game is a max-payoff vector. The converse statement does not hold in general, since max-payoff vectors satisfy all core inequalities but may fail to be efficient. Only when the assignment game is exact, that is, has a square valuation matrix A with a dominant diagonal and a doubly dominant diagonal, the set of max-payoff vectors coincides with the set of extreme core allocations. However, even when the assignment game is not exact, we have a more efficient method to compute the extreme core points: obtain all max-payoff vectors and select those that are efficient. Notice that to do that you do not need the worth of all coalitions, only the valuation matrix.

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[5] Given a coalitional game (N,v) and any order σ on the player set N, the marginal worth vector $m^{\sigma,v} \in \mathbb{R}^N$ pays each agent his/her contribution to the set of predecessors acording to the order σ. That is, $m^{\sigma,v}_{\sigma(1)} = v(\{\sigma(1)\})$ and, for all $i \in \{2,\ldots,n\}$,

$$m^{\sigma,v}_{\sigma(i)} = v(\{\sigma(1),\ldots,\sigma(i)\}) - v(\{\sigma(1),\ldots,\sigma(i-1)\}).$$

As an exercise, take the order $\sigma = (1, 2', 1', 2)$ on the player set of the assignment game in (12). The associated max-payoff vector is $x_1^{\sigma,A} = 0$, $x_{2'}^{\sigma,A} = \max\{0, 5 - 0\} = 5$, $x_{1'}^{\sigma,A} = \max\{0, 6 - 0\} = 6$ and $x_2^{\sigma,A} = \max\{0, 2 - 6, 4 - 5\} = 0$. Then $x^{\sigma,A} = (0, 0, 6, 5)$ is not efficient, which is not surprising since A has not a dominant diagonal. Some other orders, take for instance $\sigma' = (2, 2', 1, 1')$, lead to an extreme core point.

Clearly, for assignment matrices with special properties the core and its extreme core points may be obtained in a still simpler way. For instance, Martínez de Albéniz and Rafels (2014) study those assignment games with a valuation matrix that has the inverse Monge property (each 2×2 submatrix has an optimal matching on its main diagonal).

The core has been the most studied notion of solution not only for assignment games but also for arbitrary coalitional games. However, the first solution proposed in the theory was not the core but the notion of stable set of von Neumann and Morgenstern.

3.2. **von Neumann-Morgenstern stable sets.** In their book *Theory of Games and Economic Behavior*, von Neumann and Morgenstern (1944) put forward a notion of solution for coalitional games that later on was known with the name of *stable set*.

The definition of stable sets relies on a *binary relation of domination* between imputations.[6] An imputation y dominates another imputation x if there exists a non-empty coalition S such that each player $k \in S$ is paid more in y than in x and moreover the allocation y is feasible for S since $\sum_{k\in S} y_k$ does not exceed the worth of coalition S. Making use of this binary relation that allows for the comparison between imputations, a stable set is defined as a set of imputations that is internally stable (two imputations in the set do not dominate one another) and externaly stable (any imputation outside the set is dominated by some imputation in the set).

It is immediate to check that the core of a coalitional game, whenever non-empty, is the set of undominated imputations. Indeed, if a core element x is dominated by an imputation y through a coalition S, from $y_k > x_k$ for all $k \in S$ and $\sum_{k\in S} y_k \leq v(S)$ we obtain $\sum_{k\in S} x_k < v(S)$, which contradicts x being in the core. As a consequence, the core of a game is always internally stable. The problem is that it may not be externaly stable and, in that case, if we reject a payoff vector as a solution for the game because it is outside the core we must take into acount that its dominating imputation may also be outside the core and hence no better than the first one.

When the core of a game is a stable set, it is the only one. Besides, we know from Lucas (1968) that a coalitional game may have no stable sets,

In the case of the assignment game, the above dominance relation is determined simply by the mixed-pair coalitions. Given an assignment game $(M \cup M', w_A)$ and two imputations (u, v) and (u', v'), we say $(u', v')\, dom(u, v)$ if there is a pair $(i, j) \in M \times M'$ such that $u'_i > u_i$, $v'_j > v_j$ and $u'_i + v'_j \leq a_{ij}$. Solymosi and Raghavan (2001) prove that *the core of the (square) assignment game is a stable set if and only if its valuation matrix has a dominant diagonal.*

The existence of stable sets for the assignment game was an open problem since Shapley and Shubik (1972). It seems that after the paper on the core they planned

[6] An imputation $x \in \mathbb{R}^M \times \mathbb{R}^{M'}$ is an efficient and individually rational payoff vector, that is, $\sum_{k\in M\cup M'} x_k = w_A(M \cup M')$ and $x_k \geq 0$ for all $k \in M \cup M'$.

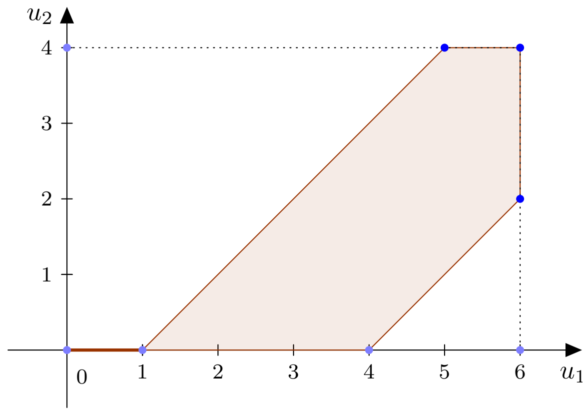


FIGURE 2. A stable set for a 2x2 assignment game

a second paper on the stable sets of the assignment game. In some personal notes of Shapley and also in Shubik (1984) it is conjectured that the union of the core of the assignment game with the core of some specific subgames, those that are compatible with a given optimal matching, form a stable set. This conjecture has been recently proved in Núñez and Rafels (2013).

Given an assignment game $(M \cup M', w_A)$ and an optimal matching $\mu \in \mathcal{M}_A(M, M')$, a subgame where a subset of buyers $I \subseteq M$ and a subset of sellers $J \subseteq M'$ have been excluded is μ-compatible if the restriction of μ to the submatrix $A_{(M \setminus I) \times (M' \setminus J)}$ is an optimal matching for the submarket and moreover $I \times J$ contains no pair whose members are matched together by μ and with positive value.

In the assignment game in (12), the compatible subgames are those in which either $I = \{1\}$, or $I = \{2\}$ or $J = \{2'\}$ are eliminated. The core of the compatible subgame $(\{1,2\} \cup \{1'\}, w_{A_{|\{1,2\} \times \{1'\}}})$ is the interval $[(0,0;6),(4,0;2)]$ and, once we complete these payoff vectors with a payoff of 4 to the excluded player $2'$, we get the *extended core* of the compatible subgame:

$$\hat{C}(w_{A_{|\{1,2\} \times \{1'\}}}) = [(0,0;6,4),(4,0;2,4)].$$

The union of this segment and the core of the game forms a stable set of the assignment game in (12). In this example, the cores of the remaining compatible subgames are not relevant since they correspond to segments which are already included in the core of the initial assignment game.

Figure 2 illustrates the stable set we have described for the market in (12). The reader can deduce from the picture that this stable set also has a lattice structure with respect to the usual order defined on the payoff vectors to one side of the market, the two extreme points of the lattice being (6,4;0,0) and (0,0,;6,4).

Given any optimal matching μ for the assignment game $(M \cup M', w_A)$, the stable set $V^\mu(w_A)$ defined as the union of the cores of all μ-compatible subgames excludes third-party payments. This means that, as it also happens in the core, the only side-payments that take place are those between a buyer and her optimally matched seller by means of the price paid for the object: for all $(u,v) \in V^\mu(w_A)$, $u_i + v_j = a_{ij}$ for all $(i,j) \in \mu$ and the payoff to players that are unmatched by μ is zero. In fact, $V^\mu(w_A)$ is the unique stable set of $(M \cup M', w_A)$ with this property.

The interpretation that can be given to the imputations in the set $V^\mu(w_A)$ is as follows. Assume agents on an assignment market have agreed on an optimal matching μ and there is a buyer such that her withdrawal from the market leaves her optimally matched seller unmatched in the submarket. Then, this buyer may take with her the whole value of their partnership and the agents that remain may agree on a core allocation of the submarket that results.

We collect in the next theorem the main findings regarding von Neumann-Morgenstern stability for the assignment game.

Theorem 3.1. *Given an assignment game $(M \cup M', w_A)$ with as many buyers as sellers,*

- *the core is stable if and only if A has a dominant diagonal,*
- *for each optimal matching μ, the union of the (extended) cores of all μ-compatible subgames is the unique stable set that excludes third-party payments with respect to μ.*
- *any stable set of the assignment game has the structure of a complete lattice.*

Both the core and the stable sets of the assignment game, usually contain infinitely many allocations. But to reach a negotiated solution in such a market, it is necessary to select or agree on one single allocation under some criteria of fairness.

3.3. **Single-valued solutions.** A first example of single-valued solution is *Thomson's fair division rule*, that we have already mentioned in Section 2. Recall that this solution always selects a core element (since it is the midpoint of the buyers-optimal and the sellers-optimal core allocations), it is easily computed and satisfies some desirable monotonicity properties (see Núñez and Rafels 2002b).

The best-known single-valued solutions for coalitional games are probably the Shapley value and the nucleolus. We briefly review in this section how these two solutions apply to the assignment game.

The *Shapley value* (Shapley, 1953) is the average of all the marginal worth vectors. This means that, according to this value, each agent receives an average of all her contributions to the coalitions she takes part of. The Shapley value is always efficient, that is, it exactly allocates the worth of the grand coalition. However, when applied to an assignment game $(M \cup M', w_A)$, it may happen that the values of an optimally matched pair do not add up to the valuation of this pair, that is it may be $\Phi_i(w_A) + \Phi_j(w_A) \neq a_{ij}$ for some $(i, j) \in \mu$ and some optimal matching μ. Similarly, consider a non-square assignment game and assume without loss of generality that seller $j \in M'$ is unmatched by an optimal matching μ. Unless the j-th column in the valuation matrix A is null, seller j will have some positive contribution $(w_A(\{i, j\}) - w_A(\{i\}) = a_{ij})$ and hence its Shapley value will be positive. These arguments show that the Shapley value of an assignment game may not select a core allocation.

Hoffmann and Sudhölter (2007) prove that when a square assignment game is exact (its valuation matrix has a dominant diagonal and a doubly dominant diagonal), then the Shapley value is in the core. However, this is not a necessary condition.

Apart from not being a core selection, another drawback of the Shapley value when applied to the assignment game is that, as far as we know, the worth of all coalitions is needed for its computation.

Compared to the Shapley value, the *nucleolus* (Schmeidler, 1969) of a coalitional game with a non-empty core always selects a core allocation. The nucleolus is sustained by an idea of fairness similar to Rawls criterium but extended not only to individuals but also to coalitions. As it is the case of other solution concepts for assignment games previously reviewed in this survey, when applied to the assignment game the definition of the nucleolus is substantially simplified, since only individual and mixed-pair coalitions need to be considered.

Given any assignment game $(M \cup M', w_A)$, we associate with each imputation (u, v) a vector $\theta(u, v)$ formed by all excesses[7] $-u_i$, for all $i \in M$, $-v_j$ for all $j \in M'$ and $a_{ij} - u_i - v_j$, for all $(i, j) \in M \times M'$, non-increasingly ordered, that is, $\theta_l(u, v) \geq \theta_{l+1}(u, v)$ for all $l \in \{1, 2, \ldots, mm'+m+m'-1\}$. Note that excesses of core elements are always non-positive at any core coalition, and the more negative an excess is, the more satisfied are the members of the coalition with the proposed imputation since they are being paid high above the worth of the coalition. The nucleolus is the imputation $\eta(w_A)$ that minimizes the non-increasingly ordered vector of excess with respect to the lexicographic order $\leq_L$:

$$\theta(\eta(w_A)) \leq_L \theta(u, v), \quad \text{for all other imputation } (u, v).$$

Since the core of the assignment game is non-empty, the nucleolus always lies on the core.

The nucleolus of any coalitional game can be computed by successively solving a sequence of linear programs (see Maschler et al., 1979). A specific, and more eficient, *algorithm to compute the nucleolus* of an assignment game is due to Solymosi and Raghavan (1994). But even with this algorithm, the computation of the nucleolus is not easy.

In the general class of coalitional games, two games may have the same core and different nucleolus. But it is proved in Núñez (2004) that two assignment games with the same core have the same nucleolus. This suggests that the nucleolus of an assignment game can be uniquely determined by its position inside the core. Indeed, Llerena and Núñez (2011) provide a *geometric characterization* of the nucleolus of an square assignment game that we now describe.

Let $(M \cup M', w_A)$ be an assignment game with square valuation matrix A. For each coalition $R \subseteq M$ or $R \subseteq M'$, the incidence vector $e^R \in \mathbb{R}^m$ is such that $e^R_k = 1$ if $k \in R$ and $e^R_k = 0$ if $k \notin R$. Now, for each $S \subseteq M$ and $T \subseteq M'$, $S, T \neq \emptyset$, we define the maximum transfer that, departing from the core element (u, v), can be made from agents in S to agents in T without getting outside the core:

$$\delta_{S,T}(u, v) = \max\{\varepsilon \geq 0 \mid (u - \varepsilon e^S, v + \varepsilon e^T) \in C(w_A)\}.$$

Notice that $\delta_{S,T} = 0$ if S and T do not correspond one to another by some optimal matching.

Then, the nucleolus $\eta(w_A)$ of a square assignment game $(M \cup M', w_A)$ is the only core element (u, v) that, given an optimal matching $\mu \in \mathcal{M}_A(M, M')$, satisfies the bisection property with respect to all coalition $\emptyset \neq S \subseteq M$, that is, $(u, v) = \eta(w_A)$ if and only if, for all $\emptyset \neq S \subseteq M$ and $\mu \in \mathcal{M}_A(M, M')$,

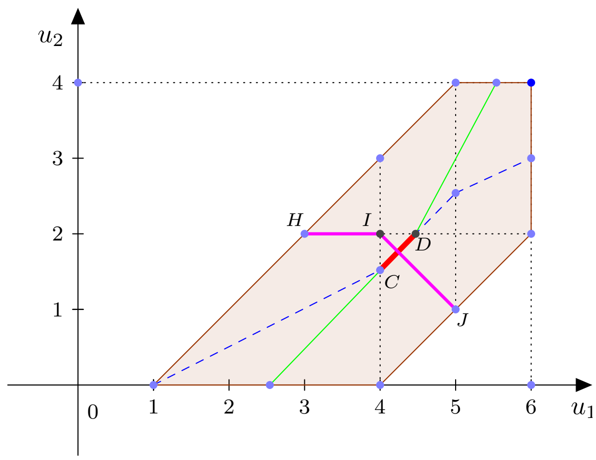
$$(u, v) \in C(w_A) \text{ and } \delta_{S,\mu(S)}(u, v) = \delta_{\mu(S),S}(u, v).$$

This bisection property of the nucleolus of the assignment game stregthens the bisection property that characterizes the core elements of the kenel, where the above equations are only required for individual coalitions $S = \{i\} \subseteq M$.

To compute the nucleolus of the assignment game in (12), we only need to add to the two bisection equations satisfied by the kernel, the one corresponding to

[7]The excess of an imputation x at a given coalition S is the difference between the worth of coalition S and the amount that x allocates to S, $\sum_{i \in S} x_i$.

coalition $S = \{1, 2\}$, that is, $\delta_{\{1,2\},\{1',2'\}}(u, v) = \delta_{\{1',2'\},\{1,2\}}(u, v)$, that we draw as the piece-wise linear curve $H - I - J$ in next figure. We obtain in this way that the nucleolus is $\eta(w_A) = (4.25, 1.75; 1.75, 2.25)$, since it is the intersection of all the bisection curves which is attained at the intersection of the segments $C - D$ and $I - J$.



This geometric characterization of the nucleolus has been proved useful in Martínez de Albéniz et al. (2013a) to provide formulas for the nucleolus of a 2×2 assignment game, and also in Martínez de Albéniz et al. (2013b) to develope a procedure of computation for square assignment games with more agents on each side.

When several notions of solutions are available for a cooperative situation, we often compare the properties of each solution in order to choose the one that better represents our idea of fairness. For assignment games, this axiomatic approach to the study of solutions has not been completed yet. To the best of our knowledge, only the core and the nucleolus have been axiomatically characterized.

3.4. **Axiomatic approach.** The first axiomatic approach to a solution on the domain of assignment games is due to Sasaki (1995). This paper provides and axiomatic characterization of the core:

- The core of the assignment game is the only solution that satisfies *consistency, continuity, couple rationality, individual rationality, Pareto optimality and weak pairwise monotonicity.*

When comparing different axiomatic approaches, the reader must check very carefully which is the domain, what is understood as a solution and finally which is the precise definition of the axioms. In Sasaki (1995) a solution is a correspondence that assigns to each assignment problem a subset of feasible allocations, where a feasible allocation consists of a matching and a payoff vector that allocates to the agents the total value of this matching. Some of the axioms of this characterization of the core do not need further explanation, but let us make some remarks on weak pairwise monotonicity and consistency. A solution is *weak pairwise monotonic* if given an element of the solution, whenever the value of a pair in the valuation matrix is increased (the value of the other pairs remaining the same), there is an element in the solution of the new game where the joint payoff to the two agents in this pair has not decreased.

Raughly speaking, *consistency* of a solution says that if an allocation is recommended by the solution, and if some agents decide to leave the game with their payoffs, then a new allocation obtained by restricting the original allocation to members of such a reduced game is also recommended by the solution. The different ways in which the reduced game can be defined gives rise to the different notions of consistency. In Sasaki (1995) the reduced game is simply the subgame (this notion of consistency is known as *projection consistency*), and moreover consistency is only required there for certain subgames.

Another notion of reduced game that has been widely used to characterize solutions of coalitional games, such as the core and the nucleolus, is the *Davis and Maschler reduced game* (see Davis and Maschler, 1965). The problem is that if we depart from an assignment game and we make the Davis and Maschler reduced game at a given allocation and coalition, this reduced game may not be on the domain of assignment games, not even be superadditive.[8]

This drawback is overcome by a definition of reduced game due to Owen (1992) which fits very well for assignment games and is closely related to the Davis and Maschler definition. This reduced game is defined on the domain of assignment games (or markets) with reservation values. An assignment market with reservation values consists on a finite set of buyers M, a finite set of sellers M' ($M \cap M' = \emptyset$), a valuation matrix $A = (a_{ij})_{(i,j)\in M\times M'}$ and a vector of reservation values for each side of the market, $p \in \mathbb{R}^M$ and $q \in \mathbb{R}^{M'}$. We denote such a market by $\gamma = (M, M', A, p, q)$. The reservation value p_i of buyer $i \in M$ represents what this buyer gets if not matched with any seller (analogously for sellers). Hence, given any coalition $S \subseteq M \cup M'$, the worth of this coalition is

$$w_\gamma(S) = \max_{\mu\in\mathcal{M}(M,M')} \left\{ \sum_{(i,j)\in\mu} a_{ij} + \sum_{i\notin supp(\mu)} p_i + \sum_{j\notin supp(\mu)} q_j \right\},$$

where $supp(\mu)$ is the set of agents that form part of some pair in μ.

Obviously, the class of assignment games with reservation values contains the classical assignment games defined by Shapley and Shubik, just taking $p_i = 0$ for all $i \in M$ and $q_j = 0$ for all $j \in M'$. Moreover, an assignment game with reservation values $(M \cup M', w_\gamma)$, where $\gamma = (M, M', A, p, q)$, is strategically equivalent[9] to a Shapley and Shubik assignment game, just taking the same sets of agents and the valuaton matrix $\tilde{A}$ defined by $\tilde{a}_{ij} = max\{0, a_{ij} - p_i - q_j\}$. An advantage of working on the class of assignment games with reservation values is that it is closed by strategic equivalence.

Owen defines a notion of reduction for assignment markets (and games) with reservation values. Given such a market $\gamma = (M, M', A, p, q)$, a payoff vector $x = (u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ and a coalition $S \subseteq M \cup M'$, the *derived market* at x and S is $\gamma^{S,x} = (S \cap M, S \cap M', A_{|(S\cap M)\times(S\cap M')}, p^{S,x}, q^{S,x})$ where the new valuation matrix is the restriction of A to rows and columns corresponding to agents in coalition S

[8] A game (N, v) is superadditive if for any disjoint coalitions S, $T \subseteq N$, it holds $v(S \cup T) \geq v(S) + v(T)$.

[9] Two games (N, v) and (N, w) are strategically equivalent if there exists $\alpha > 0$ and a vector $d \in \mathbb{R}^N$ such that $w(S) = \alpha v(S) + \sum_{i\in S} d_i$ for all coalition $S \subseteq N$.

and the new reservation values are

$$
\begin{aligned}
p_i^{S,x} &= \max_{j \in M' \setminus S} \{p_i, a_{ij} - v_j\} \text{ for all } i \in M, \\
q_j^{S,x} &= \max_{i \in M \setminus S} \{q_j, a_{ij} - u_i\} \text{ for all } j \in M'.
\end{aligned}
$$

The main result in Owen (1992) states that if $x = (u, v) \in C(w_\gamma)$, then the derived game at x and $S \subseteq N$ is the superadditive cover[10] of the Davis and Maschler reduced game at x and S. Moreover, the core of the derived game and of the Davis and Maschler reduced game coincide, which suggests that the derived game is an appropriate notion of reduction for the assignment games with reservation values. Hence, when analyzing solutions we may ask for consistency with respect to the derived game.

On the domain of assignment games with reservation values, two axiomatic characterizations are provided by Toda (2003, 2005). The first one adapts to assignment games with reservation values the axiomatic characterization of the core of coalitional games in Peleg (1986), replacing Davis and Maschler consistency by consistency with respect to Owen's derived game.

- On the domain of assignment games with reservation values, the core is the only solution that satisfies *individual rationality, Pareto optimality, consistency (w.r.t. Owen's derived game) and superadditivity.*

It is important to remark that, in the above axiomatization, as well as in those that follow in this section, a solution is considered to be a correspondence that assigns to any assignment game a set of payoff vectors (u, v) that are feasible by some matching, that is to say, there exists a matching μ such that $u_i + v_j = a_{ij}$ for all $(i, j) \in \mu$, $u_i = 0$ if i is unmatched by μ, and $v_j = 0$ if j is unmatched by μ.

The axiomatic characterization of the core in Toda (2005) relies on the projection consistency notion instead of the derived consistency.

- On the domain of assignment games with reservation values, the core is the only solution that satisfies *Pareto optimality, projection consistency, pairwise monotonicity and individual monotonicity.*

Individual monotonicity of the core means that if the reservation value of an agent increases, the other data of the market remaining the same, there is a core element in the new market in which this agent is paid at least as much as in the old one. Similarly, pairwise monotonicity states that if a matrix entry is increased, there is a core element of the new market in which both involved agents receive at least as much as in a core element of the old market. Another characterization of the core is obtained just by replacing individual monotonicity by *population monotonicity*, which raughly speaking says that if a new agent enters the market none of the agents on the opposite side is worse off and none of the agents on his/her own side is better off.

Toda (2005) also gives an axiomatization of the core on the domain of Shapley and Shubik assignment games, that is when reservation values are null.

- On the domain of assignment games with null reservation values, the core is the only solution that satisfies *Pareto optimality, projection consistency, pairwise monotonicity and population monotonicity.*

[10]The superadditive cover of a game (N, v) is the minimal game (N, w) such that $w(S) \geq v(S)$ for all $S \subseteq N$ and w is superadditive.

Apart from the core, another solution concept that has been axiomatically characterized is the nucleolus. On the domain of assignment games with reservation values, Llerena et al. (2014) combine derived consistency with a sort of symmetry property between the two sides of the market.

- On the domain of assignment games with reservation values, the nucleolus is the only solution that satisfies *consistency with respect to Owen's derived game and symmetry of maximum complaints of the two sides.*

Symmetry of maximum complaints of the two sides only requires that at each solution outcome (u, v) of a market with as many buyers as sellers, the most dissatisfied buyer has the same complaint as the most dissatisfied seller,

$$\max_{i\in M}\{p_i - u_i\} = \max_{j\in M'}\{q_j - v_j\},$$

where we interpret the dissatisfaction of an agent with a given outcome as the difference between his reservation value and the amount that this outcome allocates to him.

4. **Extensions of the assignment market: multiple sides and multiple partners.** The model of the assignment game, as defined by Shapley and Shubik, relies on several economic assumptions: there are two sides in the market, utility is identified with money, side-payments are allowed and the supply of each seller and the demand of each buyer are unitary. We have already discussed how some of these assumptions have been relaxed in the literature: the assignment game with different utility functions of Demange and Gale (1985) and the assignment game with non-transferable utility of Kaneko (1982). We now consider two different extensions that follow from relaxing the two-sided nature of the market and the unitary demand and supply of the agents. It results that some of the nice properties and structure of solutions do not hold when we consider these generalized assignment markets.

4.1. **Multi-sided assignment markets.** Let us consider a market γ with m different sides or sectors ($m \geq 2$). Each sector consists of a finite set of agents, $N_1, N_2, \ldots, N_m$ and in this market value is only generated by the cooperation of exactly one agent from each sector. If $i_1 \in N_1$, $i_2 \in N_2$, ..., $i_m \in N_m$, then $a_{i_1 i_2 \ldots i_m}$ is a non-negative real number that represents the value that can be attained by the m-tuple $(i_1, i_2, \ldots, i_m)$. The market γ is thus described by $(N_1, N_2, \ldots, N_m; A)$ where A is an m-dimensional non-negative matrix. A matching in this market is a subset of m-tuples in $N_1 \times N_2 \times \cdots \times N_m$, where each agent belongs to at most one m-tuple. Then, the worth of an arbitrary coalition is the maximum possible value that it can attain by a matching among its agents.

As in the two-sided case, the only coalitions that are relevant to describe the core of a multi-sided assignment game are the individual coalitions and those corresponding to m-tuples: given an optimal matching μ for γ, a payoff vector $x \in \mathbb{R}^{N_1} \times \cdots \times \mathbb{R}^{N_m}$ belongs to the core $C(\gamma)$ if and only if $x \geq 0$, $\sum_{t=1}^{m} x_{i_t} \geq a_{i_1 \ldots i_m}$ for all $(i_1, i_2, \ldots, i_m) \in N_1 \times N_2 \times \cdots \times N_m$, $\sum_{t=1}^{m} x_{i_t} = a_{i_1 \ldots i_m}$ whenever $(i_1, i_2, \ldots, i_m) \in \mu$ and $x_k = 0$ if k is single by μ. In other words, the core consists of those individually rational and efficient allocations that cannot be blocked by any m-tuple.

A main difference with the two-sided case is that multi-sided assignment games may have an empty core (see Kaneko and Wooders, 1982). For three-sided assignment games with exactly two agents on each side, Lucas (1995) characterizes the non-emptiness of the core in terms of two inequalities: given a $2 \times 2 \times 2$ assignment game with an optimal matching $\mu = \{(1,1,1),(2,2,2)\}$, the core is non-empty if and only if

$$\begin{array}{rcl} a_{111} + 2a_{222} & \geq & a_{122} + a_{212} + a_{221} \\ 2a_{111} + a_{222} & \geq & a_{112} + a_{121} + a_{211}. \end{array} \tag{16}$$

As an application, let us consider a market with three sides and two agents on each side, $N_1 = \{i_1, i_2\}$, $N_2 = \{j_1, j_2\}$ y $N_3 = \{k_1, k_2\}$. The three-dimensional valuation matrix is

$$\begin{array}{c} i_1 \\ i_2 \end{array} \quad \underset{k_1}{\overset{\begin{array}{cc} j_1 & j_2 \end{array}}{\begin{pmatrix} \mathbf{3} & 2 \\ 4 & 0 \end{pmatrix}}} \quad \underset{k_2}{\overset{\begin{array}{cc} j_1 & j_2 \end{array}}{\begin{pmatrix} \alpha & 0 \\ 0 & \mathbf{2} \end{pmatrix}}}$$

For $0 \leq \alpha \leq 5$, an optimal matching for this market is $\mu = \{(i_1, j_1, k_1), (i_2, j_2, k_2)\}$ with an optimal value of 5. From Lucas' conditions, such a market has a non-empty core if and only if $0 \leq \alpha \leq 2$. But even when it is non-empty, the core of a multi-sided assignment game loses some of the properties that hold for the two-sided case. For instance, in the above example with $0 \leq \alpha \leq 2$, there is no core allocation where the first agent on the second sector, j_1, receives his contribution to the grand coalition, which is 3. However, Sherstyuk (1999) proves that, if the core of a multi-sided assignment game is non-empty, then agents in the same sector maximize their core payoff at a same core element. Sufficient conditions on the valuation matrix that guarantee that all agents attain their contribution to the grand coalition are also provided.

There are in the literature few studies on multi-sided assignment games. Tejada and Rafels (2010) extend to this setting the symmetrically pairwise-bargained allocations of Rochford (1984) and Tejada (2010) shows that, when the core of a multi-sided assignment market is non-empty, then, as in the two-sided case, it coincides with the set of competitive equilibrium payoff vectors.

Instead of focusing on the general multi-sided case, several authors have studied particular subclasses of multi-sided assignment games where the existence of core allocations can be guaranteed. See for instance Quint (1991) , Stuart (1997), Sherstyuk (1999) and Tejada (2012). Most of these subclasses, except that of Sherstyuk (1999) have a common feature: the valuation of an m-tuple is obtained by addition of some parameters attached either to each individual in this m-tuple or to each pair involved in this m-tuple.

For instance, in the supplier-firm-buyer game of Stuart (1997) each supplier-firm pair (i,j) has a valuation $b^1_{ij} \geq 0$ and each firm-buyer pair (j,k) has a valuation $b^2_{jk} \geq 0$. Then, the value of the triplet (i,j,k) in the corresponding three-sided assignment game is $a_{ijk} = b^1_{ij} + b^2_{jk}$ and the core is always non-empty.

On the other hand, Tejada (2013) assumes that agents in sector $k \in \{1, 2, \ldots, m-1\}$ are sellers of good of type k, while the sector N_m is composed of buyers, each of them demanding an $(m-1)$-tuple of goods of different type. Each seller i_k of sector k has a reservation value of $c^k_{i_k} \geq 0$ and each buyer i_m has a willingness to pay of $w_{i_m} \geq 0$, that meaning that he/she values equally any $(m-1)$-tuple of sellers. Then, the value of an m-tuple $(i_1, i_2, \ldots, i_m)$ is $a_{i_1 i_2 \ldots i_m} = \max\{0, w_{i_m} - c^1_{i_1} - c^2_{i_2} - \cdots - c^{m-1}_{i_{m-1}}\}$. This model is no more than the multi-sided generalization of Böhm-Bawerk horse markets. The core is proved to be non-empty and moreover core allocations can be easily obtained from the core elements of an interesting convex game played by the sectors. Some properties of the nucleolus and the center of gravity of the core of this game are shown in (Tejada and Núñez, 2012).

4.2. **Assignment markets with multiple partnership.** We now consider two-sided markets where at least some agents may establish more than one partnership. Let M be a finite set of buyers and M' a finite set of sellers, M and M' disjoint. Let $Q = \{1, 2, \ldots, q\}$ be the types of indivisible goods present in the market. Each seller $j \in M'$ owns $s_j^k \in \mathbb{Z}_+$ goods of type $k \in \{1, 2, \ldots, q\}$, with $s_j = \sum_{k \in Q} s_j^k \geq 1$ and $\mathbb{Z}_+$ the set of non-negative integers. Each buyer $i \in M$ wants to buy up to $r_i \in \mathbb{Z}_+ \setminus \{0\}$ goods and valuates in $a_{ik} \geq 0$ one good of type $k \in Q$. Hence, such a market is defined by (M, M', Q, A, r, s). Throughout this section we assume that buyers' utilities are additively separable.

One can distinguish in the literature two different models dealing with such a situation, depending on whether we allow that a same buyer-seller pair trades in total more than one unit or not. In this second case the different partnerships must be established with different partners. This fact affects the definition of a feasible matching in each setting and, as a consequence, the associated coalitional game, since the worth of a coalition S of buyers and sellers is defined as the maximum value that can be obtained by matching buyers in S to objects belonging to sellers in S.

However, when all agents on one of the sides have unitary capacity ($r_i = 1$ for all $i \in M$ or $s_j = \sum_{k \in Q} s_j^k = 1$ for all $j \in M'$), there is no room for the above differentiation. Usually, the assignment markets where agents on one side have unitary capacity are named many-to-one assignment markets. Precisely, the first extension of the assignment game that appeared in the literature was of this sort.

Many-to-one assignment markets.

Kaneko (1976) studies a two-sided market with q different types of goods, where buyers have unitary capacity, $r_i = 1$ for all $i \in M$, while each seller may have several units of each type of good. In Kaneko's model, each seller has a reservation value for each good he possesses but, after normalization, his market is equivalent to a many-to-one assignment market as defined above. A matching is a set of buyer-type pairs such that each buyer appears in exactly[11] one pair and each type of object k in at most $\sum_{j \in M'} s_j^k$ pairs. The matrix A represents the value that each buyer $i \in M$ assigns to each type of object $k \in Q$.

To this assignment market, one can associate a one-to-one assignment game, simply by considering each unit of good as a different seller. Then, from each core element of this one-to-one assignment game it is obtained a competitive equilibrium payoff vector of the many-to-one market.

The notion of competitive equilibrium in this market is the usual one: a vector of prices $p \in \mathbb{R}^Q$ and a matching μ form a competitive equilibrium if a) for each buyer $i \in M$, the object $\mu(i)$ is in her demand set, given prices p,

$$a_{i\mu(i)} - p_{\mu(i)} \geq a_{ik} - p_k, \text{ for all } k \in Q,$$

[11]In this survey we intend to present in a unified way different markets that have appeared in the literature with slight differences on the notion of matching, which has implications for instance in the way the authors define the competitive equilibrium. To overcome that, we will assume in this section that the set of objects Q contains as many dummy objects as necessary, so that we can assume that in a matching all buyers complete their capacity and, moreover, the demand of a buyer at any given vector of prices is non-empty. Obviously, each buyer will value at zero any dummy object and the price of such an object will also be null.

and b) the price of a good is null if some unit of this good remains unassigned by μ.

Moreover, all competitive equilibrium payoff vectors are core elements of the many-to-one assignment game. The converse implication does not hold: there are core elements that do not correspond to a competitive equilibrium. To summarize:

- In a many-to-one assignment market where each buyer has unitary capacity and each seller some units of each type of good, the set of competitive equilibrium payoff vectors is non-empty and (strictly) included in the core.

A different many-to-one assignment market is considered in the final section of Sotomayor (2002), where the unitary capacity is placed on the sellers' side. This is a job market where M is the set of firms (buyers), M' is the set of heterogeneous workers (sellers), each firm can hire up to $r_i \geq 1$ workers but each worker can work for at most one firm ($s_j = 1$ for all $j \in M'$). Here, a competitive equilibrium is a pair formed by a matching μ between firms and workers and a vector of non-negative prices (salaries), $p \in \mathbb{R}^{M'}$, such that non-assigned workers have null price and, given these prices, each firm gets a group of workers in its demand set: for all $i \in M$ and all subset of workers R of cardinality r_i,

$$\sum_{j \in \mu(i)} a_{ij} - p_j \geq \sum_{j \in R} a_{ij} - p_j. \qquad (17)$$

- In a many-to-one market where firms (buyers) may hire several workers (sellers) and each worker can work for only one firm, the core coincides with the set of competitive equilibrium payoff vectors and it is non-empty.
- The set of core payoff vectors for the sellers, or equivalently the set of competitive equilibrium price vectors, has a lattice structure under the partial order of $\mathbb{R}^{M'}$.
- There exists a core element where all buyers attain simultaneously their maximum core payoff and all sellers their minimum one. And another core element where all sellers attain simultaneously their maximum core payoff and all buyers their minimum one.

Regarding the differences with the one-to-one assignment game, Sotomayor (2002) provides examples that show that, in her many-to-one market, the opposition of interests between firms and workers along the core is no longer true and firms can manipulate the mechanism which produces the best core payoff for them.

The non-emptyness of the core of this many-to-one market is derived from the general case where also workers may work for several firms, as we will show later on.

Many-to-many assignment games where each buyer-seller pair can trade at most one unit.

This market model has been developed by M. Sotomayor along the last years. It is introduced in Sotomayor (1992) with the name of *multiple-partner assignment games*. Agents[12] in this market belong to two disjoint finite sets, a set of buyers M and a set of sellers M'. Buyers $i \in M$ are interested in sets of objects owned by different sellers and the quota r_i of a buyer is the number of different objects she is allowed to acquire. Each seller $j \in M'$ owns a lot of s_j identical objects. As usual, a non-negative matrix A represents the value generated by each buyer-seller pair.

[12] In Sotomayor (1992) agents are interpreted as firms and workers in a labor market.

We assume that the set M contains as many copies of a dummy buyer i_0 (and M' as many copies of a dummy seller j_0) as necessary. Then a matching is a set of buyer-seller different pairs such that each non-dummy agent belongs to as many pairs as her/his quota indicates. A *feasible outcome* for this market is formed by a matching and a way to split the value of each matched pair between its partners: $((u_{ij})_{(i,j)\in\mu}, (v_{ij})_{(i,j)\in\mu}; \mu)$ with

$$\begin{aligned} &u_{ij} + v_{ij} = a_{ij},\ u_{ij} \geq 0,\ v_{ij} \geq 0 \text{ for all } (i,j) \in \mu, \text{ and} \\ &u_{ij_0} = u_{i_0 j} = u_{i_0 j_0} = 0,\ v_{ij_0} = v_{i_0 j} = v_{i_0 j_0} = 0, \end{aligned} \tag{18}$$

in case these payoffs are defined. A *pairwise stable* outcome is a feasible outcome that satisfies

$$u_{ij} + v_{lk} \geq a_{ik} \text{ whenever } (i,j) \in \mu, ; (l.k) \in \mu \text{ and } (i,k) \notin \mu. \tag{19}$$

The interpretation is that in a pairwise-stable outcome there is no buyer and seller who are not matched to each other but the sum of the payoffs each of them receive because of some of their partnerships is less than what they would earn by trading with each other.

Notice that from each feasible outcome we can deduce a payoff vector $(u, v) \in \mathbb{R}^M \times \mathbb{R}^{M'}$ where $u_i = \sum_{j\in\mu(i)} u_{ij}$ for all $i \in M$ and $v_j = \sum_{i\in\mu(i)} v_{ij}$ for all $j \in M'$.

In Sotomayor (1992), the existence of pairwise-stable outcomes is proved in two different ways: by means of linear programing and by considering an associated one-to-one assignment game with as many copies of each agent in $M \cup M'$ as his/her quota indicates. From each core element of this one-to-one assignment game, a pairwise-stable outcome of the many-to-many market (M, M', A, r, s) is obtained. Next we summarize the main facts regarding this model.

- In a many-to-many assignment game where a matching does not admit repeated pairs, the set of pairwise-stable outcomes is non-empty.
- Pairwise stable payoff vectors are core allocations, but not all core allocation can be obtained from a pairwise-stable allocation.
- There exists an optimal pairwise-stable outcome for each side of the market, which is the worst for the opposite side.
- The set of pairwise-stable outcomes is endowed with a lattice structure under two convenient partial order relations (Sotomayor, 1999).
- The set of competitive equilibria (p, μ) is non-empty. Each competitive equilibrium gives a pairwise-stable outcome but the converse implication does not hold in general. However, the buyers-optimal competitive equilibrium outcome equals the buyers-optimal pairwise stable outcome, and similarly for sellers (Sotomayor, 2007)

The notion of competitive equilibrium in Sotomayor (2007) is analogous to that in (17). It is also proved there that the set of competitive equilibrium price vectors inherits the lattice structure of the set of pairwise-stable allocations. On the contrary, the core of this many-to-many assignment game loses the oposition of interest between the two sides of the market since it does not preserve the lattice structure: the best core allocation for agents on one side is not the worst core allocation for agents on the opposite side. Indeed, the worst core allocation for a given side of the market may not exist. Nevertheless, the existence of a core element that is optimal for each side of the market still remains an open problem.

To better understand this model and some of the results stated above, let us consider a market with two buyers $M = \{1, 2\}$ with capacities $r_1 = r_2 = 2$ and two sellers $M' = \{1', 2'\}$ with capacities $s_1 = 1$ and $s_2 = 2$. We denote by 0' the dummy seller and the valuation matrix A is

$$\begin{array}{c|c|c|} & 1' & 2' \\ \hline 1 & 4 & 2 \\ \hline 2 & 3 & 4 \\ \hline \end{array} \qquad (20)$$

There is only one optimal matching, $\mu = \{(1,1'), (1,2'), (2,2'), (2,0')\}$ and hence the worth of the grand coalition is 10. Note that this matching, together with the payoffs

$$(u_{11}, u_{12}, u_{22}, u_{20}; v_{11}, v_{12}, v_{22}, v_{20}) = (0,0,0,0;4,2,4,0)$$

forms a pairwise-stable outcome, since $u_{2k} + v_{11} \geq a_{21} = 3$ for $k \in \{2, 0\}$. However, it does not correspond to a competitive equilibrium since seller 2 would receive a different price for his two identical objects. Instead, the pairwise-stable outcome $(u'_{11}, u'_{12}, u'_{22}, u'_{20}; v'_{11}, v'_{12}, v'_{22}, v'_{20}) = (0,0,2,0;4,2,2,0)$ corresponds to the equilibrium prices $p_1 = 4, p_2 = 2$. In both cases, the corresponding payoff vectors $(U_1, U_2; V_1, V_2) = (0,0,4,6)$ and $(U'_1, U'_2; V'_1, V'_2) = (0,2,4,4)$ are core allocations.

Pérez-Castrillo and Sotomayor (2014) study the manipulability of *competitive equilibrium rules*, which are rules that select a unique competitive equilibrium payoff vector for each valuation matrix of the many-to-many assignment markets we are considering.[13] They prove that, when more than one competitive equilibrium price vector exist, every competitive equilibrium rule is manipulable. This means that there exists an agent that, by misrepresenting his/her true valuation receives, under the selected competitive equilibrium rule, a higher payoff than the one he/she would receive by revealing the true valuation. Even more, different to the one-to-one case (see Theorem 2.2), examples are provided that show that the buyers-optimal competitive equilibrium rule can be manipulated by both buyers and sellers. However, if a buyer $i \in M$ has capacity $r_i = 1$, then she cannot manipulate the buyers-optimal competitive equilibrium rule (and analogously for sellers with unitary capacity).

Another contribution to this model of many-to-many markets is Fagebaume et al. (2010), where it is shown that the maximum of any pair of pairwise-stable payoff vectors for the buyers (respectively, sellers) is pairwise-stable but the minimum need not be. Therefore, the set of pairwise-stable payoff vectors for the buyers (respectively, sellers) is not a lattice, but it has a maximum element. Recall that by pairwise-stable payoff vector for the buyers we mean the vector formed by the payoff to each buyer obtained from a related pairwise-stable outcome, by addition of the profits this buyer gets in each partnership.

Many-to-many assignment markets where each buyer-seller pair can trade several units.

Consider now a market similar to the one above, with the only difference that now agents are allowed to trade more than one object with a same agent on the opposite side of the market. Let us assume, to begin with, that each seller $j \in M'$ owns s_j identical objects. Then, a matching is a set of buyer-seller pairs where each buyer $i \in M$ belongs to at most r_i pairs and each seller $j \in M'$ belongs to at most s_j pairs. The problem of finding a matching (within the agents capacities) that maximizes the sum of values of the matched pairs is known in Operations

[13]In this paragraph, to let both buyers and sellers act strategically by revealing her/his valuations, do not assume the reservation values of sellers to be normalized at zero.

Research as the *transportation problem* and it is solved by means of the following linear program:

$$\begin{array}{ll} \max & \sum_{M\times M'} a_{ij}\mu_{ij} \\ \text{where} & \sum_{j\in W}\mu_{ij}\le r_i, \text{ for all } i\in M, \\ & \sum_{i\in F}\mu_{ij}\le s_j, \text{ for all } j\in M', \\ & \mu_{ij}\ge 0, \text{ for all } (i,j)\in M\times M', \end{array} \tag{21}$$

since it is well-known that whenever all r_i and s_j are integer, then the linear program has an integer solution $(\mu_{ij})_{(i,j)\in M\times M'}$. As usual, in this many-to-many assignment game, the worth of a coalition of agents is the addition of the values of all the pairs in an optimal matching among buyers and sellers in this coalition.

Consider the same market as in (20) but assume now that each buyer-seller pair is allowed to trade more than one unit. An optimal matching is now $\mu = \{(1,1'),(1,0'),(2,2'),(2,2')\}$ and the worth of the gran coalition raises to 12.

This market game is first consider in Thompson (1980) and Crawford and Knoer (1981). The non-emptiness of the core of this game is shown by Sánchez-Soriano (2001) and Sotomayor (2002). To see that, just consider the dual linear program

$$\begin{array}{ll} \min & \sum_{i\in M} r_i y_i + \sum_{j\in M'} s_j z_j \\ \text{where} & y_i + z_j \ge a_{ij}, \text{ for all } (i,j)\in M\times M', \\ & y_i\ge 0, z_j\ge 0, \text{ for all } (i,j)\in M\times M', \end{array}$$

and check that if (y,z) is a solution of this dual program, then the payoff vector (u,v) belongs to the core, where $u_i = r_i y_i$ for all $i\in M$ and $v_j = s_j z_j$ for all $j\in M'$. However, unlike the Shapley and Shubik assignment game, there are core allocations that cannot be obtained from any solution of the dual program. In those core allocations that come from a solution of the dual transportation problem, sellers do not discriminate buyers, that is, each seller sells all its objects at the same price and, similarly, each buyer receives the same profit from each object she buys.

Let us summarize the main results, pointing out similarities and differences with the Shapley and Shubik assignment game:

- The core is non-empty but strictly greater than the set of dual solutions.
- The opposition of interests of buyers and sellers along the core is not necessarily true, and the core is not a lattice.
- The existence of an optimal core element for each side of the market is an open problem.
- The pairwise-egalitarian solution, a single-valued solution for these markets where each agent is paid one half of all the partnerships in which she/he takes part, is analyzed in Sánchez-Soriano (2003, 2006).

Sotomayor (2013) presents the *time-sharing assignment game*, a continuous matching extension of the above model: each participant owns a fixed amount of units of a divisible good (labor time), which is to be distributed among the partnerships he forms in any way he likes and exchanged for money. For a partnership to be active both members must contribute the same positive amount of labor and each agent should receive equal individual payoff per each unit of labor time he contributes to the partnership. For this model, different solution concepts are studied and correlated.

Another many-to-many assignment market is considered in Jaume et al. (2012). The difference is that sellers in this market own a given number of indivisible units of potentially many different goods.[14] As in Sotomayor's (2002) model, buyers value each good and want to buy at most an exogenously fixed number of units. The notion of competitive equilibrium in this paper is slightly different: given a vector of prices of the objects, each buyer substracts the corresponding price from her valuation of each object, identifies which objects provide the maximum difference or surplus, and then demands as many of these objects as the capacity of the buyer allows. This definition of demand is defined by Sotomayor (2012) as non-discriminatory, since each buyer gets the same utility from all objects in her demand set. Then, as usual, a competitive equilibrium consists of a price vector and a matching such that each buyer gets a set of objects in her demand set.

The authors in Jaume et al. (2012) show, using well-known duality theorems of linear programming, that each market has at least a competitive equilibrium and that the set of equilibrium price vectors has a lattice structure that does not translate to the set of utilities of buyers (or sellers) that can be attained at equilibrium. For this same market, the notion of setwise stability, different from the core notion, is introduced in Massó and Neme (2014) and some limit results for replicated markets are proved:

- The set of setwise stable payoffs includes the non-empty set of competitive equilibrium payoffs and it is contained in the core.
- The set of setwise stable payoffs of a two-fold replicated market already coincides with the set of competitive equilibrium payoffs.
- The sequence of cores of replicated markets converges to the set of competitive equilibrium payoffs when the number of replicas tends to infinity.

Other notions of group stability for this market are considered in Arribillaga et al. (2014), together with a notion of competitive equilibrium also studied in Sotomoyor (2007, 2013), where, given prices, each buyer demands those packages that, within her capacity, maximize her aggregate surplus. Then, demand is discriminatory since a buyer may obtain different utility from different objects in a demanded package.

In all the assignment markets we have considered in this survey, we have assumed that the valuation of each buyer is additively separable. Let us mention, to conclude this section, the *package assignment game* of Bikhchandani and Ostroy (2002) in which agents trade packages of several objects and buyers' values are non-additive. Optimal assignments can still be formulated as a linear programming problem and relationships between different notions of equilibrium prices and the core are established. Nevertheless, unlike the two-sided models we have reviewed before, equilibrium prices and core allocations may not exist.

5. **A final remark on auction markets.** An assignment market with only one seller is the setting of an auction, either a single-object auction or a multi-item auction, depending on the number of objects on sale by the seller.

A Shapley and Shubik (one-to-one) assignment market with only one seller is analyzed in Chapter 7 of Roth and Sotomayor (1990). The only seller has a single object on sale, each buyer desires to buy this object and we can assume without loss of generality that the valuations of the m buyers are ordered in this way, $a_{11} \geq a_{21} \geq \cdots \geq a_{m1}$. Assume also that the seller has null reservation value for his object. In a core allocation, an optimal matching assigns the object to buyer 1 at a price $a_{21} \leq p \leq a_{11}$. Notice then that in the buyers-optimal core allocation

[14] The case with only one seller with several heterogeneous goods had been studied in Camiña (2006).

the buyer with the highest valuation wins the object and pays the second highest value which is the minimum core price, that being the outcome of an auction well known as *second-price auction*. It was already known from Vickrey (1961) that in the second-price sealed-bid auction it is a dominant strategy for each buyer to state her true valuation. However, coalitions of bidders have more strategic possibilities by forming bidder rings.

In the multi-item generalization of the second-price auction, there are several objects in the market and each buyer is interested in acquiring at most one item and submits a sealed-bid listing her valuation of all the items. The auctioneer then allocates the items in accordance with the minimum equilibrium price and the resulting payoff is the buyers-optimal core allocation of the corresponding Shapley and Shubik assignment game. As we have seen in Section 2, under this mechanism no buyer has incentives to misrepresent her list of valuations.

Demange et al. (1986) provide a dynamic multi-item auction mechanism that yields the minimum competitive price in a finite number of steps. This mechanism is a version of the Hungarian algorithm and a generalization of the English auction. At every step, given the price vector announced by the auctioneer, the buyers indicate their demand set. If it is possible to assign each item to a buyer who demands it, the auction stops. If no such assignment exists, there exists a minimal over-demanded set, that is, an over-demanded set with the property that none of its proper subsets is over-demanded. The auctioneer then locates such a minimal over- demanded set and raises the price of each item in the set by one unit. This auction mechanism is extended in Sotomayor (2009) to the multiple partners assignment game introduced in Sotomayor (1992), to also obtain the minimum competitive equilibrium payoff vector.

The sellers-optimal core allocation of the one-to-one assignment game has also been implemented in Pérez-Castrillo and Sotomayor (2002), not by means of an auction but of a selling and buying procedure in which both buyers and sellers act strategically. In this mechanism, sellers simultaneously fix prices first and then buyers sequentially decide which object to buy and report their indiferences (along with the previous buyers' indifferences) to the following buyer. Reporting truthfully the indiferences is a dominant strategy for the buyers and when they do so, the subgame perfect equilibria outcomes correspond to the maximum equilibrium price vector.

The extension of the second-price auction to the case in which one seller, 0, has some units of several heterogeneous goods to sell and valuations of buyers may not be additively separable is usually known as the *Vickrey-Clarke-Gloves mechanism* (see Ausubel and Milgrom, 2002, 2004). This is also the setting of the package assignment game (Bikchandani and Ostroy, 2002), in the case there is only one seller. Assume Q is the set of different types of goods and $w \in \mathbb{R}^Q$ the vector of goods the seller has on offer. Each buyer $i \in M$ reports a valuation function $v_i : \mathbb{R}^Q_+ \longrightarrow \mathbb{R}$ that denotes buyers i's value for each non-negative vector $x_i \in \mathbb{R}^Q$. Then the auctioneer chooses an allocation $x^* = (x_i^*)_{i \in M}$ such that x^* maximizes $\sum_{i \in M} v_i(x_i)$ subject to $\sum_{i \in M} x_i \leq w$, and the price p_i each buyer pays is such that $v_i(x_i^*) - p_i$ is the contribution of buyer i to the market. Then, truthfull reporting is a dominant strategy for each bidder in the VCG mechanism.

It is proved in Ausubel and Milgrom (2002) that when the Vickrey payoff vector (the vector of contributions of the buyers) is in the core of the one-seller package assignment game, then it is the buyers-optimal core allocation. Otherwise, there is no core allocation where all buyers simultaneously maximize their payoff. Moreover, if the coalitional function is buyers-submodular, $w(S \cup \{0\}) - w(S) \leq w(T \cup \{0\}) - w(T)$ for all $T \subseteq S \subseteq M$, then the Vickrey payoff vector is a core allocation.

Several imperfections of the VCG mechanism for this market are shown in Ausubel and Milgrom (2005a) and better procedures for real-world auction design are proposed in Ausubel and Milgrom (2005b).

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