Tuning Synchronization through mobility and limited vision

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Abstract: In this work we analyse the emergence of synchronization in a population of mobile Integrate-and-Fire oscillators with limited vision. By proposing novel interaction rules among oscillators we bridge phenomenology detected in a variety of previous models. In particular, we explore the effect that the effective asymmetry of interactions have on the non monotonic behaviour observed in the synchronization time of the population as a function of their velocities. We recover non linear features with the same origin as [1] but considering only geometrical interactions, and we study the scaling properties of the model as well as predict the values of the parameters where the different dynamical regimes take place.

I. INTRODUCTION

Synchronization is an emerging phenomenon in systems composed by dynamically interacting elementary units lacking any leader or hierarchy [4, 5]. This kind of collective behaviour appear in nature in a huge variety of contexts such as biology, ecology, climatology and sociology among others [6, 7]. For example, synchrony occurs in very different situations like metabolic processes in our cells or collective human actions based on collaboration. The versatility of this concept and the tools that the field of complex networks has given motivated numerous works addressed to characterize and predict this phenomenon. Reciprocally, synchronization in populations of units with oscillatory behaviour has been one of the more successful attempts to characterize the dynamical properties of time-dependant complex networks, which have been studied primarily from a static point of view.

The effects of changing interaction patterns on synchronization features has attracted attention from fields as diverse as chemotaxis [8], mobile ad hoc networks [9] and its implementation in wireless sensor networks [11] or genetics [10]. With this scope, a general framework of mobile oscillators performing random walks in a two dimensional space and interacting with the rest according some rules has been proposed [12]. The evolution of these systems has been approximated by linear dynamics. One paradigmatic example are the Kuramoto oscillators [13, 14], whose evolution after a short transient time is well described by a set of solvable linear equations.

In this work we consider another framework of dynamical interacting system, the so called Integrate and Fire Oscillators (IFO), in which the internal phases evolve continuously in time and there are instantaneous interactions between the units taking place in other time scale. There are several works addressed to characterize the emergence of synchronization in IFO populations considering different interaction rules. Mobility of the oscillators has been shown to be a necessary condition to reach synchronization in that models in which the network of interactions lies bellow the static percolation transition [15, 16]. Under these conditions, in some cases nonlinearity is observed in the emergence of synchronization through mobility (increase the velocity does not always favour synchronization), and hence a rich variety of models has been proposed.

Although the emergence of global synchronization is usually considered as a positive phenomenon on interactive dynamical systems, there are some examples, as it happens in the neuronal dynamics of the brain, where global synchronization is not a desired state [18]. Therefore, this non-linearity feature attracted interest since the prevention or favouring of synchronization by managing the dynamical properties of the system can be an useful tool in some contexts [19]. Actually, IFO have been used to model neural systems but other examples such as in the field of economics can be found [17].

The main objective of this work is to bridge the phenomenology observed for different interaction patterns in IFO populations an generally characterize the non monotonic behaviour in the emergence of synchronization through mobility. Our hypothesis is, given distinct models with different interaction rules display the same non monotonic behaviour in the emergence of synchronization, we can generalize the necessary conditions that must be satisfied for the occurrence of non-linearity. Furthermore, we want to show that the synchronization mechanisms proposed for modelling the evolution of these systems for a particular interaction rule [1] are generalizable, and that we can quantitatively characterize the non-trivial emergence of synchronization.

II. INTEGRATE AND FIRE OSCILLATORS MODEL

The basic model that will be used in this work consists in a population of N moving oscillators in a square of length L with either finite or periodic boundary conditions. All move with a fixed velocity modulus V and are initially given random orientations $\theta_i \in [0, 2\pi]$, $i = 1, ..., N$. We associate to each oscillator an internal phase $\phi_i \in (0, 1)$ which increases uniformly in time until a maximum of 1 is reached. At this point, the oscillator is reoriented in a random direction, its phase is reset to 0 and a firing event occurs. Upon such an event at time t, the firing oscillator influences other oscillators (which we call its neighbours) updating their phases by a factor ϵ:

$$\phi_i(t) = 1 \Rightarrow \begin{cases} \phi_i(t^+) = 0 \\ \theta_i(t^+) \in U[0, 2\pi] \\ \phi_{\text{neigh}}(t^+) = (1+\epsilon)\phi_{\text{neigh}}(t) \end{cases}$$

A firing event may push a neighbour phase to reach the maximum and hence another firing event occurs. The whole succession of fires takes place in frozen time, which is restarted when no more phases can possibly be updated.

We consider that the system is synchronized when a succession of N firing events takes place, or equivalently when all the oscillators have exactly the same internal phase. We characterize the time that the system needs to reach the synchronization state with a discrete time T_{sync} defined as the number of cycles completed by a given reference oscillator. Notice that T_{sync} describes entirely the dynamics of the model because the synchronized state cannot be altered once reached, being an attractor.

For this general model, one can choose different interaction rules that define how the affected neighbours by a firing oscillator are chosen. This interaction structure plays a fundamental role in the dynamics of IFO systems [4]. Is worth to note that we are attending extreme situations with minimal interaction rules, where mobility is a necessary condition for the emergence of synchronization because the network of interactions are bellow the static percolation transition. Moreover, different choices for the boundary conditions can be set, although our experiments show little variation of the relevant aspects of each considered model (see Appendix VIII B).

In [1], what we call a *topological model*, in which each oscillator influences only its nearest neighbour during a firing event, is considered. Periodic boundary conditions are taken. In the cited paper the reorientations are performed by the influenced oscillators instead of the firing ones. This difference is not relevant in the scope of this work as we discuss in Appendix VIII A, so we consider reorientations of the firing oscillators for the sake of clarity in comparing the different models. The results of [1] show that motion is effectively a necessary condition for reaching synchronization and that there is a surprising non monotonic dependence of T_{sync} with the velocity V of the oscillators. Specifically, three dynamical regimes are observed: For small velocities, synchronization arises mainly due to the synchronization and recursive merging of well defined local clusters of oscillators, which is a slow process leading to large values of T_{sync}; For fast velocities, no local coherent structure can be found and all oscillators interact randomly with each-other, leading to a global emergence of synchronization in a rapid and effective way. These two regimes are well known and studied in depth in previous works [4]. The novelty of this model, however, is that a new regime is observed for intermediate velocities where the interplay of the two synchronization mechanisms leads to frustration, and the system is not able to reach the synchronized state.

The local or global nature of the emergence of synchronization can be characterized by the mixing parameter χ which is the averaged fraction of distinct neighbours influenced by an oscillator per unit of time:

$$\chi = \frac{\sum_{i=1}^{N} n_i}{N^2 T_{\text{sync}}}. \tag{1}$$

where n_i is the number of different oscillators influenced by oscillator i during the synchronization process. High values of χ indicate fast changes of neighbours or equivalently global connectivity and vice versa.

This non monotonic behaviour in the emergence of synchronization is not observed if we consider instead a *geometrical model* like the one proposed in [3]. In this case, each oscillator influences all the oscillators lying closer than a certain interaction range distance R. Periodic boundary conditions are taken in this model as well. The results show that T_{sync} decreases monotonously with the velocity, so we only observe the two earlier synchronization regimes without the intermediate regime. Therefore, the faster are the changes of neighbours (or equivalently the higher is the mixing), the faster the system globally synchronizes.

In both models, the motion of the oscillators allows the emergence of the synchronized state. However, the differences of the interaction rules lead to important differences in the underlying time-dependant structure of interactions in each model, and hence we observe distinct relationships between dynamics and the emergence of global phenomena such as synchronization.

In a later work [2], an interesting addition to the moving IFO model was added by implementing another notion of geometric interaction that we call the *robots model*: Each oscillator has a limited cone of vision of radius R and angle α oriented in the direction of its motion. When an oscillator reaches maximum phase and fires, influences all oscillators that have the firing oscillator located inside their cone. Additionally, in this model no reorientations are performed during the firing events and the boundary conditions are finite: When an oscillator reaches a boundary, it reorients itself in a random orientation chosen from the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ with respect the boundary normal. In addition, in frozen time there is only a single update for oscillator. The effects of boundary conditions and number of allowed updates are shown to not cause significant differences in the emergence of synchronization through mobility as we discuss in Appendix VIII B. Most strikingly, this model displays again a non monotonic behaviour in the emergence of the synchronization through mobility. The results are similar to the ones obtained in the topological model and one can again identify the same three dynamical regimes, so that there is also a region where synchronization is prevented.

Philosophically, this model lies among the two earlier studied models, yet it displays the non monotonic behaviour using exclusively geometric interactions, which are interesting from the technological point of view since the election of the neighbours is instantaneous, there is no need of comparing the positions of the oscillators around you. This has even allowed the realization of experiments with robots as can be seen in [2]. Moreover, as we discussed in the Introduction, there are interest in detecting which properties can prevent the emergence of synchronization by tuning the dynamical changes on the networks. Therefore, the recovering of a non monotonic behaviour is an important feature of this model and indicates that the mechanisms behind the no-sync zone detected in [1] are by no means particular to the topological model.

Inspired by [1], here we defend that the non monotonic behaviour emerges as a result of what we call an *effective asymmetry* of the interaction pattern, as seen by the fact that the geometrical model is unable to recover this non monotonic behaviour [3]. The effective asymmetry is the consequence of the interplay among two effects: The *non-reciprocity* (in frozen time, the interactions between oscillators are not reciprocal) and the *persistence* (in static conditions, each oscillator interact only with a particular subgroup of neighbours).

In the following sections we thus interpolate between the topological and the robots model and justify that the interaction pattern must satisfy effective asymmetry for the existence of the non monotonic behaviour. Given that the robots model can have interesting applications, we conclude our work by studying in detail this model and its dependences with the relevant parameters ϵ and R for the representative case of $\alpha = \pi$ that correctly captures the main properties of the model. Furthermore, we perform a quantitative analysis allowing us to predict the position of the non-sync zone using similar arguments as in [1] and thus validating the importance of effective asymmetry for the emergence of non-trivial synchronization behaviours in populations of moving oscillators.

The work is organized as follows: In section III we consider the topological model but introducing the notion of the cone of vision in order to study the effects of persistence on a non-reciprocal notion of interaction. In section IV we consider the geometrical model introducing again the notion of the cone in order to study the effects of reciprocity. In sections V and VI we analyse in depth the robots model and we quantitatively characterize the non monotonic behaviour observed. In Appendix VIII A and VIII B we discuss the effects of secondary features of the model that don't cause significant changes in the emergence of synchronization through mobility. In Appendix VIII C and VIII D we study the properties of the two time estimators used in section VI to quantitatively characterize the synchronization mechanisms of the robots model.

III. PERSISTENCE BREAKING. TOPOLOGICAL CONE OF VISION.

Consider again the topological model but including the notion of the cone of vision, that is select for each oscillator its nearest neighbour inside an infinit cone of angle α oriented along the motion direction. Notice that for $\alpha = 2\pi$ this model is exactly the topological one. The interactions of this model are non-reciprocal because the direction of the fires depends on the instant spatial distribution of oscillators.

In Fig.1 we show the T_{sync} dependence on the velocity V of the oscillators for different angles of interaction α. Notice that in the fast velocity regime the synchronization state is reached quickly for every angle. This is due we are under *fast switching* conditions regardless of α: Each oscillator needs short time to interact with the rest of oscillators, so the connectivity is global and the synchronization emerges quickly because the whole system is synchronizing at once. It's worth noting that in this regime the global nature of the interactions are due to the mobility of the oscillators so the influence of the angle is negligible and no information about persistence can be obtained in this regime. This statement is supported by the mixing dependence on V shown in Fig.2: In the fast regime, χ is high and doesn't depend on α.

For small angles ($\alpha < \pi$) there are long range interactions (the first oscillator inside the cone can be located far away from the firing one) and hence, as we are reorienting the firing oscillators, we are breaking the persistence and we can have almost fast switching conditions due long range interactions regardless of mobility although not as effective as the mobility-induced fast-switching regime due to geometric constraints. Therefore, in Fig.1, for small angles, we observe quick synchronization for all velocity and hence monotonic behaviour is obtained. Supporting this argument, in Fig.2 we observe that, in the slow regime, χ is significantly greater than 0 for small angles as consequence of the long range interactions.

By increasing the angle α we observe in Fig.1 a transition from a monotonic behaviour towards a non-monotonic behaviour where the non-sync region is obtained. For large α, the range of the interactions decrease so the oscillators interact only with their immediate surroundings in the slow regime, enforcing the persistence. In Fig.2 we observe that, for large angles, χ is significantly greater than 0 only in the fast regime. Therefore, it seems clear that persistence is a necessary condition to obtain non monotonic behaviour in the emergence of synchronization.

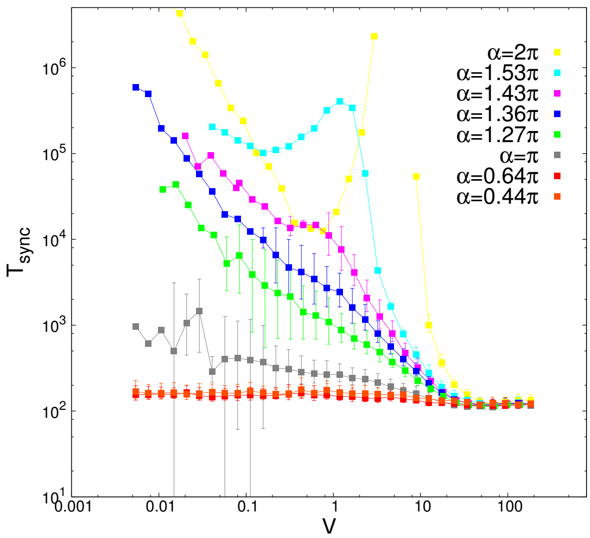

FIG. 1: Topological Cone of Vision. T_{sync} vs V for different α with: L=200, N=20 and ϵ=0.1. We observe quick synchronization for small angles due to fast switching conditions through mobility for large V and through long range interactions for small V. We also observe a transition to non monotonic behaviour by increasing α. This is due for large α persistence is satisfied. Here and in the following figures, error bars correspond to one standard deviation and are only shown for those points which, during the simulations, its calculation has not diverged.

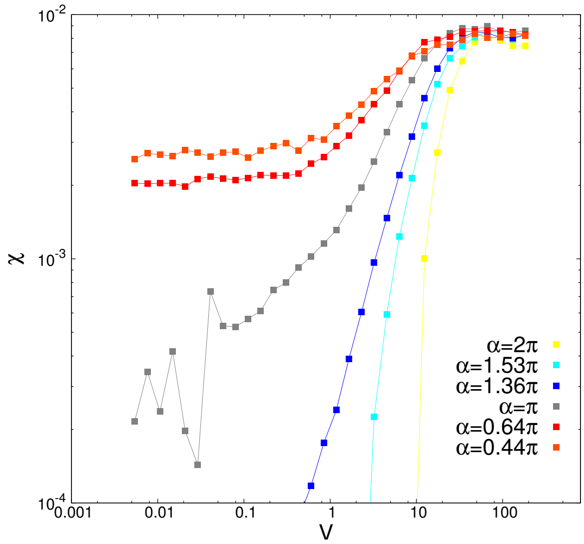

FIG. 2: Topological Cone of Vision. χ vs V for different α with: L = 200, N = 20 and ϵ=0.1. For large V, we observe high χ regardless of α. So the influence of α is negligible and fast switching conditions are always satisfied through mobility. For small V, we have higher values of χ by decreasing α and hence we increment the ratio of neighbour changes and fast switching conditions can be satisfied even for low velocities through long range interactions, in detriment of the persistence.

IV. TUNING RECIPROCITY. GEOMETRICAL CONE OF VISION.

Now we want to study the effects of reciprocity. Consider the geometrical model but with the notion of the cone of vision, that is select for each oscillator all the neighbours inside a cone of radius R and angle α oriented along the motion direction. For $\alpha = 2\pi$, the interactions are totally reciprocal. By decreasing α we are introducing non-reciprocity in the interaction pattern.

Here we fix the average number of neighbours $\bar{k}$ to 1 in order to compare with the previous results:

$$\bar{k} = \frac{\alpha R^2(N-1)}{2L^2} = 1. \tag{2}$$

In Fig.3 we show the T_{sync} dependence on the mobility. Again, in the fast regime we have fast switching and hence quick synchronization through mobility regardless of the angle. For small angles, fast switching conditions are satisfied even in the slow regime through long range interactions as in the case of the topological cone.

Moreover, we observe a monotonic behaviour for every angle (even for the small ones in which the interactions are non-reciprocal). This is because there are reorientations. The direction of the interactions are randomly fixed with the direction of the motion and don't depend on the spatial distribution of oscillators. By decreasing α we increase the range of interactions and if we reorient the firing oscillators, the interactions take place in all directions even for small angles α, so that there are not spatial correlations between consecutive fires and the underlying network of interactions evolves locally in a isotropic way. Therefore, while decreasing α we are also breaking persistence and the effective asymmetry is not satisfied.

Notice that in the fast velocity regime, the global synchronization mechanism is more effective for large angles. We do not properly understand this phenomenon which could be studied in future work.

V. RESTORING PERSISTENCE BY STOPPING REORIENTATIONS. THE ROBOTS MODEL

In the former section we tried to study the effects of introducing non-reciprocity in the interactions but at the same time we broke the persistence due the reorientations.

Is therefore necessary to study another model which clearly should be the robots model, where no reorientations are performed. In this case we fix the direction of the interactions randomly with the motion, which persists unalterable during a succession of fires only finished when the oscillator reach the borders. This ensures persistence regardless of the angle. For large angles the in-

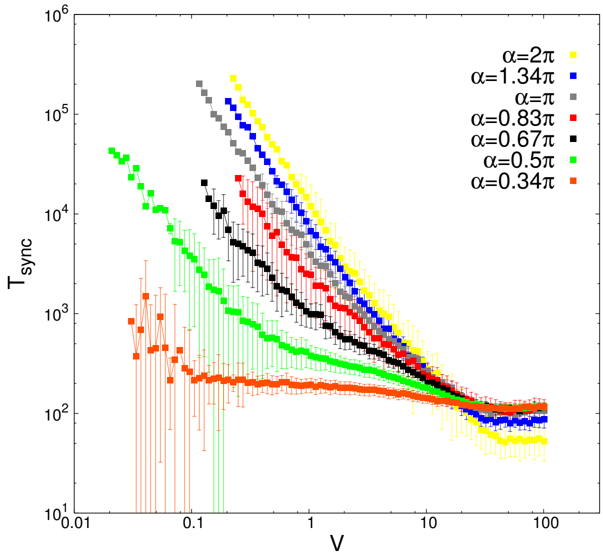

FIG. 3: Geometrical Cone of Vision. $T_{\rm sync}$ vs V for different α with: $L = 200$, $N = 20$, $\epsilon = 0.1$ and $\bar{k} = 1$. The behaviour in the emergence of synchronization is always monotonic regardless of α. This is due the interactions are in any case effectively asymmetric due to the reorientations: If we decrease α we increment the non-reciprocity and decrement the persistence and vice versa.

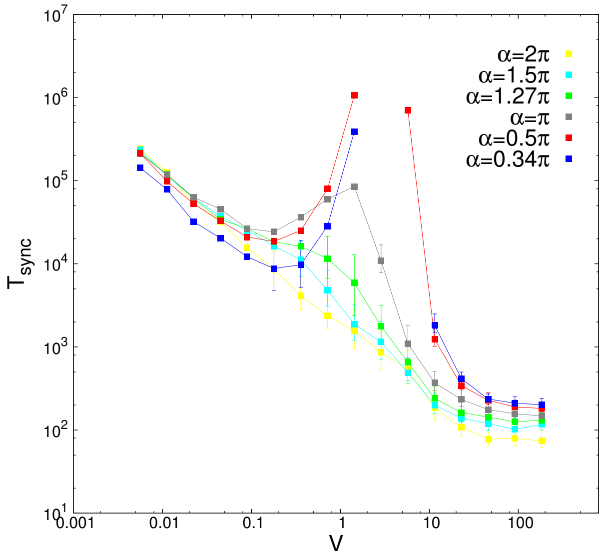

FIG. 4: Robots Model. $T_{\rm sync}$ vs V for different α with: $L = 200$, $N = 20$, ϵ=0.1 and $\bar{k}$=1. We observe non monotonic behaviour for small α. This is due the interactions are effectively asymmetric given non-reciprocity is satisfied and there are no reorientations so that persistence is also satisfied.

teractions are still reciprocal and, again, by decreasing α we are reinforcing the non-reciprocity. So for small angles this interaction pattern is effectively asymmetric. We fix again $\bar{k} = 1$ in order to compare with the other models.

Confirming our hypothesis, in Fig.4 we observe how the non monotonic behaviour in the emergence of sync is recovered by decreasing α.

With all, we conclude that enforcing the effective asymmetry of the interactions leads to non monotonic behaviour in the emergence of synchronization through mobility. The next step is to test whether the synchronization mechanisms for the different dynamic regimes proposed in [1] are generalizable to any model displaying non monotonic behaviour or not.

Other interesting observation of this results is that the synchronization mechanism in the slow regime is more effective for smaller angles. In the following section we characterize this mechanism but we still don't have a proper explanation for this phenomenon.

VI. LOCALIZING THE GEOMETRICAL PEAK. TUNING SYNCHRONIZATION THROUGH MOBILITY AND LIMITED VISION.

The aim of this section is the understanding and modelling of synchronization in the slow velocity regime in order to give a quantitative explanation to the appearance of the peak for the robots model. The synchronization in the fast regime is due the fast switching (global sync). Our hypothesis is that there is a different mechanism of synchronization for slow velocities analogous to the one proposed for the topological model (local sync).

Following the same procedure as in [1] (see [20]), we want to compare the time-scale that governs the changes of the connected component configurations in the network of interactions (the normalized neighbour time $\tau_{\rm neigh}/N$) with the characteristic synchronization time of the most common configurations taking place (the local synchronization time $\tau_{\rm sync}$). We want to determine a regime of velocities where this two time estimators are comparable, and see if this regime coincides with the local sync region as it happens for the topological model. In such case, we justify the existence of a local synchronization mechanism.

We focus on the case $\alpha = \pi$. This is a significant case that correctly captures the general behaviour of the robots model.

On one hand, let's consider the neighbour time τ_{neigh} as the average time an oscillator remains inside the cone of vision of another oscillator. We calculated both analytically and computationally this estimator. In the simulations we computed the neighbour time as the average number of consecutive interactions between two oscillators. This is the most natural way to compute τ_{neigh} given the definition of $T_{\rm sync}$. The analytical calculation of this estimator for the case $\alpha = \pi$ is shown in Appendix VIII C and we obtained the following expression:

$$\tau_{neigh} = \frac{3R}{\pi V}. \quad (3)$$

The dependences on V and R are correctly captured as we verified by comparing with the computational results.

Consider the normalization N of the neighbour time estimator τ_{neigh}/N. It is an estimator of the time-scale governing the connectivity changes of the network (if τ_{neigh} is the time an oscillator needs to change its neighbour, on average, every τ_{neigh}/N there is a change in the network of interactions).

On the other hand, we calculated computationally the local synchronization time estimator τ_{sync}: We consider random static networks connected with the cone of vision and we compute the average time that the non-isolated oscillators need to synchronize with its static neighbours. In Appendix VIII D we discuss the properties of this estimator.

Finally, to compare τ_{sync} with τ_{neigh}/N as a function of V, N and R, let's consider the control parameter

$$\eta = \frac{\tau_{\text{sync}}}{\tau_{\text{neigh}}/N} = \frac{\pi V N \tau_{\text{sync}}}{3R}. \tag{4}$$

In Fig.5 we show that the slow velocity regime of effective synchronization is characterized by $\eta \sim 1$ for every value of the relevant system parameters ϵ and $\bar{k}$. Therefore, in the slow regime the two considered estimators are comparable, so that oscillators locally synchronize in small connected components before they break, and hence there is an effective local mechanism that give rise to synchronized clusters before global synchronization is reached.

Moreover, if $R < V$ the oscillators should mostly change their neighbour in every consecutive interaction and fast switching through mobility must work. Confirming this idea, the beginning of the fast velocity regime of effective synchronization for each case is characterized by such η that $R = V$ ($\eta = \frac{\pi}{3} N \tau_{sync}$).

VII. CONCLUSIONS AND FUTURE WORK

The main result of this work is that the idea introduced in [1] for the topological model, which points to effective asymmetry as the necessary condition to obtain non monotonic behaviour in the emergence of synchronization through mobility, is generalizable to models with geometric interaction rules. This generalization has been done in the following way: In section III we have shown that, for a non-reciprocal interaction rule, if we allow long range interactions in detriment of persistence, we lose the non monotonic behaviour in the emergence of synchronization; In section IV we have shown that we can't obtain non monotonic behaviour with the geometrical cone with reorientations because if we enforce non-reciprocity by decreasing the angle we also break the persistence and vice versa; Finally, in section V, we have shown that for the robots model, in which we stop reorientations, we can reinforce the non-reciprocity of interactions by decreasing α without breaking persistence and hence satisfy effective asymmetry and obtain non monotonic behaviour.

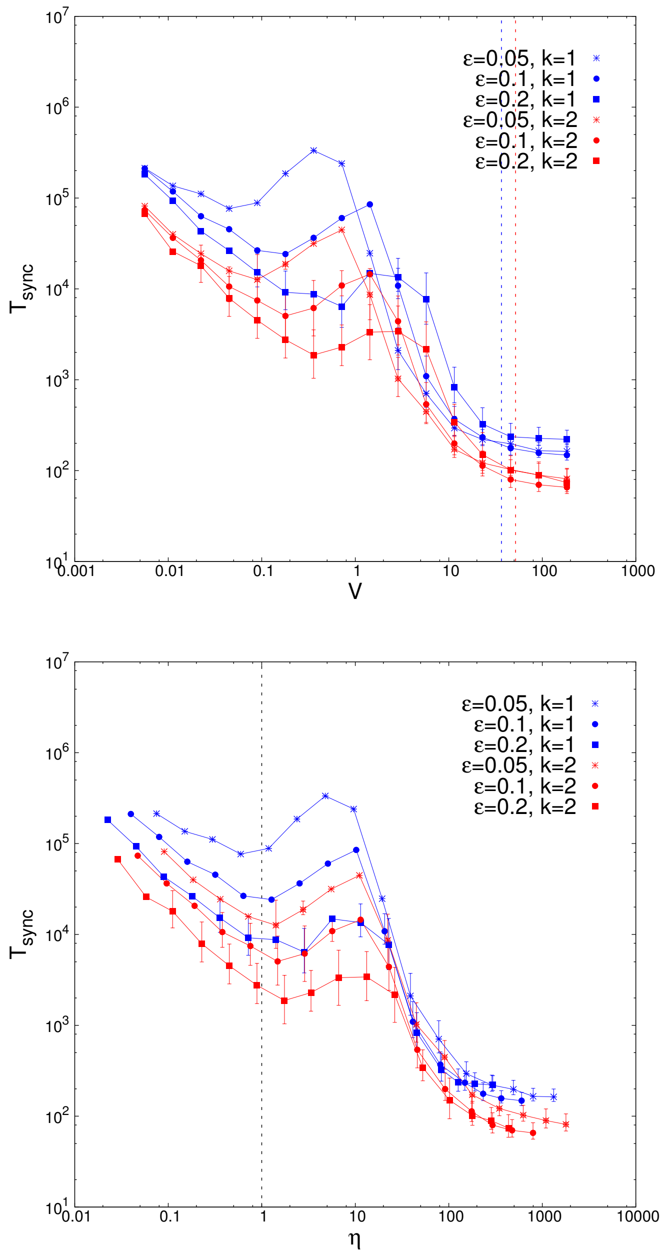

FIG. 5: ROBOTS MODEL. Collapse of the peaks through η for different ϵ and $\bar{k}$ with: $\alpha = \pi$, $N = 20$ and $L = 200$. The reason for the collapse is that the slow regime of synchronization can be characterized with $\eta \sim 1$. The beginning of the fast regime of effective synchronization can be characterized with $R = V$ (vertical colour lines in upper figure), or equivalently with $\eta = \pi N \tau_{sync}/3$.

Moreover, given that we obtained a non monotonic behaviour for the robots model analogous to the topological one, in section VI we checked whether the synchronization mechanisms proposed for the topological model can be also generalized or not. We found that for the robots model there is also a local synchronization mechanism for the slow regime. Furthermore, we have shown a way to characterize this slow regime of synchronization through the control parameter η (see expression (4) and Fig.5). This control parameter allows us to work in any of the three dynamical regimes whatever the relevant parameters of the system may be, in particular allows us to prevent synchronization.

With all, the principal aim of this work has been the generalization of the results presented in [1] by introducing significant changes in the interaction rules in IFO populations. Obviously, other significant changes could be introduced. Nevertheless, the generalization done for the robots model has interesting applications since this model has clear interest from the technological point of view.

In all the results, the density of oscillators has been fixed ($N = 20$ and $L = 200$) because the features of the different models are shown to be correctly reflected and the simulation times are not excessively high. One interesting future work could be the study of the emergence of synchronization if we consider large populations of oscillators and check if the results presented here are still valid.

Other secondary phenomena could be also studied in future work. For example, we have neither given an explanation to the effects of the angle in the fast switching regime through high velocity for the geometrical cone of vision (see section IV), nor to the better effectiveness of the slow regime sync mechanism for smaller angles in the robots model (see section V).

VIII. APPENDIX

A. Reorient who Fires or who Receives

In the cases of the topological and the geometrical cone with reorientations, we reoriented the firing oscillators in every firing event. In the model presented in [1] the reorientations are performed on the neighbours of the firing oscillator (the oscillators who receive). This difference has shown to be negligible in the behaviour of the emergence of synchronization: The dependence of T_{sync} on V is exactly the same but we observe slightly lower values of T_{sync} if we reorient the oscillator who receives. That is, the monotonic or non monotonic behaviour is obtained in the same way but the height of the peak in non monotonic behaviour conditions are slightly lower if we reorient who receives. The mixing dependences on velocity are similar as well.

B. Boundary Conditions and number of allowed Updates

Periodic and finite boundary conditions are shown to not modify the behaviour in the emergence of synchronization through mobility for any of the models considered. However, with periodic boundary conditions some secondary effects have been detected. In particular, for the robots model with reorientations and periodic boundary conditions, we observe an anomalous behaviour: We observe an increment of T_{sync} by decreasing α with $\bar{k}$ fixed and low velocity. This is due for $R > L/2$ (or equivalently $\theta < 8/N$), the oscillators can have the same neighbour with two opposite orientations and this favours the synchronization process.

In the robots model we only allow one update per oscillator in every succession of firing events in frozen time. This has been done in order to compare to the results of [2]. This variation of the model causes an homogeneous increment of T_{sync} for all V. Nevertheless, the behaviour of the emergence of synchronization through mobility is exactly the same, with the only difference that in general the system needs more time to synchronize.

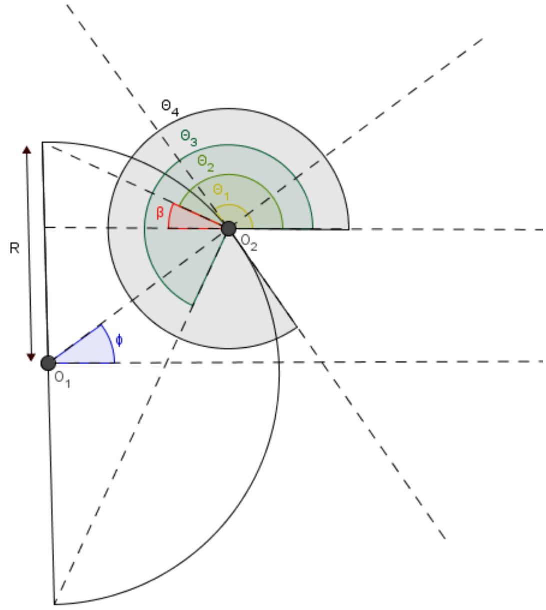

FIG. 6: APPENDIX C. Representation of the integral variables: Φ is the inicial angular coordinate of oscillator O_2; angles θ_1, θ_2, θ_3, θ_4 are the extremes of integration and β is an instrumental variable in the calculation.

C. Analytical calculation of the Neighbour Time.

In this Appendix we perform the analytical calculation of the estimator τ_{neigh} for $\alpha = \pi$, although the procedure can be generalized for an arbitrary α.

Consider the representation of Fig.6: The oscillator with the cone O_1 is located in the origin oriented with velocity $v_1 = V\hat{i}$ and the oscillator O_2 is located in some point of the semicircle (at distance R and angle $\Phi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ from O_1). Given both oscillators have the same modulus velocity V, the positions considered for O_2 are the only ones from which this oscillator can enter inside

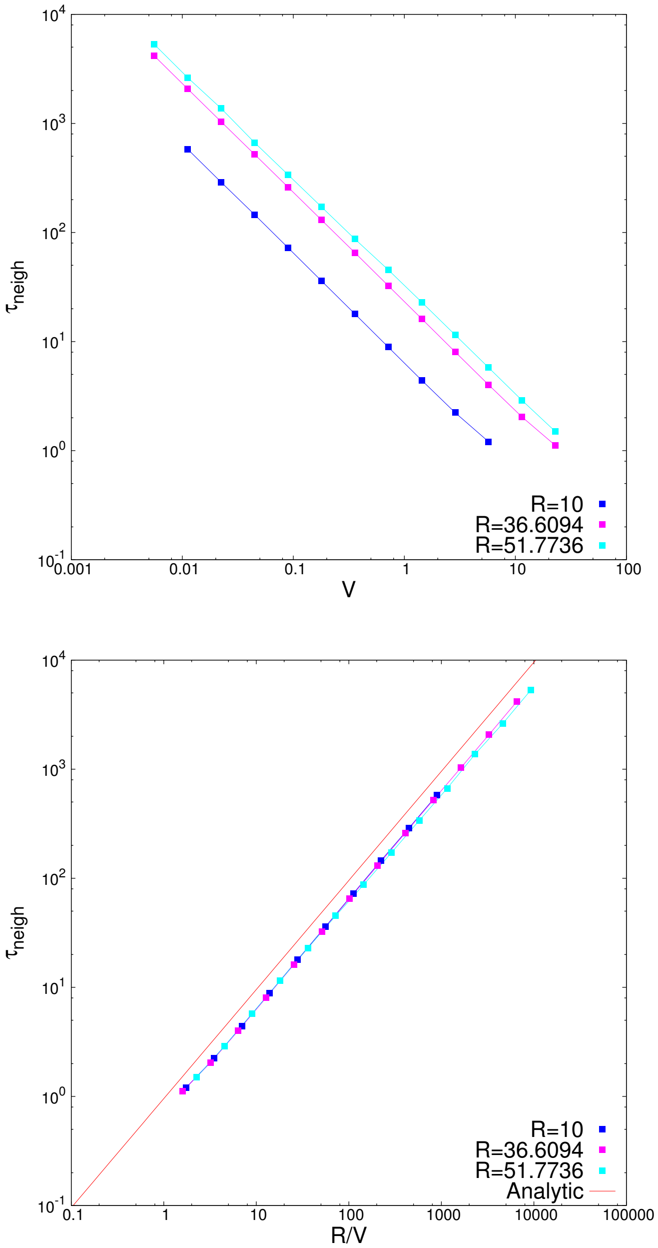

FIG. 7: APPENDIX C. Simulated values of τ_{neigh} vs R/V with $\alpha = \pi$. We observe the collapse of the estimator as we expect from expression (24). The discrepancy with the analytic results is due in the simulations the notion of the time is discrete and in the analytic method the time is continuous.

the cone. Let γ be the random orientation of O_2 and $v_2 = V(\cos(\gamma)\hat{i} + \sin(\gamma)\hat{j})$ its velocity. Hence the relative velocity v is

$$v = v_2 - v_1 = V((\cos(\gamma) - 1)\hat{i} + \sin(\gamma)\hat{j}). \quad (5)$$

Consider the horizontal relative distance $x(t)$ and the vertical relative distance $y(t)$ where t is the time:

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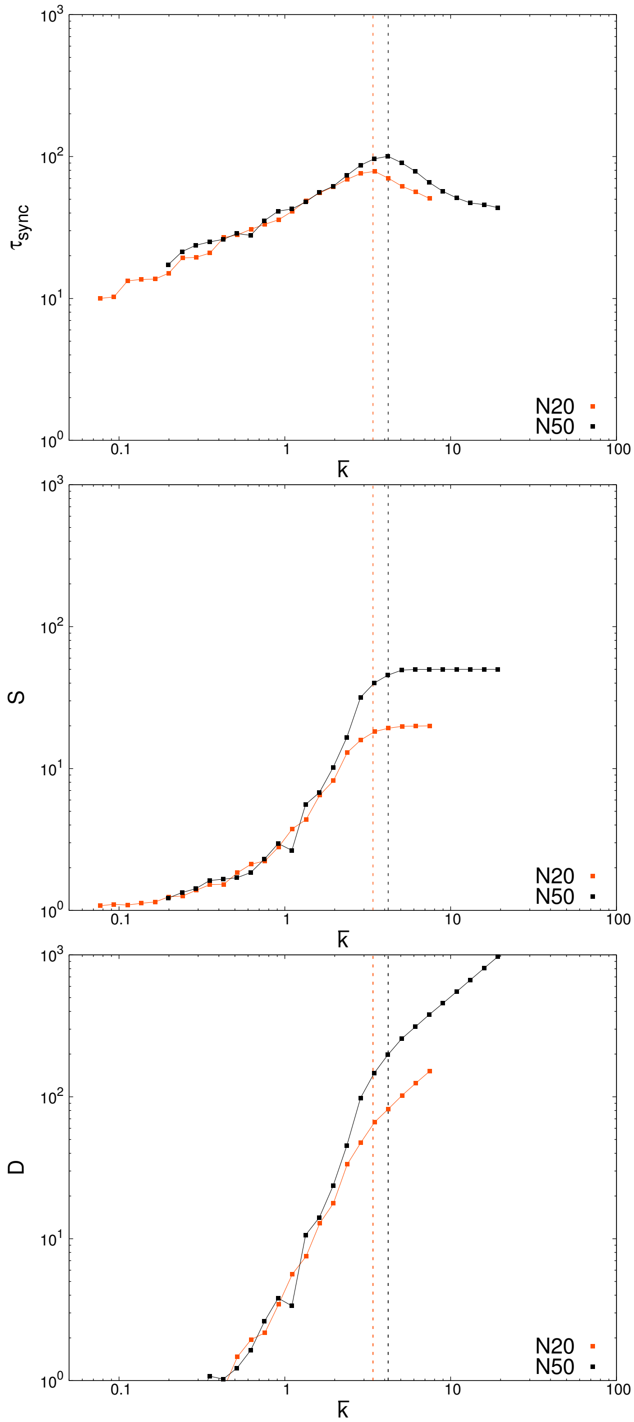

FIG. 8: APPENDIX D. τ_{sync}, S, D vs $\bar{k}$ respectively with $\alpha = \pi$, $L = 200$ and $\epsilon = 0.02$ (we vary the parameter R). We observe a non monotonic dependence of τ_{sync} because, for low $\bar{k}$, we are increasing the average size of the connected components S and the number of links D by increasing $\bar{k}$, and, for high $\bar{k}$, we keep increasing the number of links D but the size of the components S remains constant by increasing $\bar{k}$, because the whole network is connected which favours synchronization. Vertical lines indicate the position of the peak and the transition between the two regimes.

$$x(t) = R\cos(\Phi) + tV(\cos(\gamma) - 1), \tag{6}$$

$$y(t) = R\sin(\Phi) + tV\sin(\gamma). \tag{7}$$

Consider the variable

$$\theta = \tan^{-1}(\frac{\sin(\gamma)}{cos(\gamma) - 1}) \tag{8}$$

which is the polar coordinate of the relative velocity. Notice that, with the notation introduced in Fig.6, O_2 enter inside the cone if $\theta \in (\theta_1(\Phi), \theta_4(\Phi))$. Therefore, the integral we must solve to average τ_{neigh} is

$$\tau_{neigh} = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\Phi \frac{1}{\pi}\int_{\theta_1(\Phi)}^{\theta_4(\Phi)} d\theta T(\theta, \Phi) \tag{9}$$

where $T(\theta, \Phi)$ is the time that O_2 needs to leave the cone.

Notice that if $\theta \in (\theta_1(\Phi), \theta_2(\Phi))$, then $T(\theta, \Phi) \equiv T_1$ satisfies

$$x(T_1)^2 + y(T_1)^2 = R^2 \tag{10}$$

and $T_1 > 0$.

If $\theta \in (\theta_2(\Phi), \theta_3(\Phi))$, then $T(\theta, \Phi) \equiv T_2$ satisfies

$$x(T_2) = 0. \tag{11}$$

And finally if $\theta \in (\theta_3(\Phi), \theta_4(\Phi))$, then $T(\theta, \Phi) \equiv T_3$ satisfies

$$x(T_3)^2 + y(T_3)^2 = R^2 \tag{12}$$

and $T_3 > 0$.

Therefore, the integral (9) becomes

$$\tau_{neigh} = \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\Phi \frac{1}{\pi}\int_{\theta_1(\Phi)}^{\theta_2(\Phi)} d\theta T_1 +$$

$$\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\Phi \frac{1}{\pi}\int_{\theta_2(\Phi)}^{\theta_3(\Phi)} d\theta T_2 + \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\Phi \frac{1}{\pi}\int_{\theta_3(\Phi)}^{\theta_4(\Phi)} d\theta T_3 \tag{13}$$

Now we need to determine θ_1, θ_2, θ_3 and θ_4. Consider

$$\beta = \tan^{-1}(\frac{1 - \sin(|\Phi|)}{\cos(\Phi)}). \tag{14}$$

We have

$$\theta_1 = \Phi + \frac{\pi}{2}, \tag{15}$$

$$\theta_2 = \pi - \beta, \tag{16}$$

$$\theta_3 = \theta_2 + \frac{\pi}{2} = \frac{3\pi}{2} - \beta, \tag{17}$$

$$\theta_4 = \Phi + \frac{3\pi}{2}. \tag{18}$$

Moreover, from(8),

$$\cos(\gamma) - 1 = \frac{\sin(\gamma)}{\tan(\theta)}, \tag{19}$$

$$\sin(\gamma) = \frac{-2\tan(\theta)}{\tan^2(\theta) + 1}. \tag{20}$$

Now substitute expressions (19) and (20) in (10), (11) and (12) and isolate T_i discarding the null solutions. We obtain

$$T_1 = \frac{R(\sin(\Phi)\tan(\theta) + \cos(\Phi))}{V}, \tag{21}$$

$$T_2 = \frac{R\cos(\Phi)(\tan^2(\theta) + 1)}{2V}, \tag{22}$$

$$T_3 = \frac{R(\sin(\Phi)tan(\theta) + \cos(\Phi))}{V}, \tag{23}$$

Finally substitute expressions (21), (22) and (23) in the integral (13). The resulting integral is solvable analytically and we obtain the final expression

$$\tau_{neigh} = \frac{3R}{\pi V}. \tag{24}$$

Expression (24) correctly matches with the results for this estimator in the simulations as can be seen in Fig.7.

D. Local Synchronization Time Estimator Analysis

The estimator τ_{sync} depend on the average number of neighbours $\bar{k}$. In Fig.8 we observe that the dependence is not monotonic. For small $\bar{k}$, the estimator increases by increasing $\bar{k}$ and for large $\bar{k}$ the estimator decreases.

We studied the average size S and the average number of links D of the connected components appearing in the static network connected with the cone of vision. As can be seen in Fig.8, for small $\bar{k}$, we increase the size of the components S and the number of links D by increasing $\bar{k}$, so τ_{sync} becomes larger. For large $\bar{k}$, the size of the components S remains constant because a giant connected component appears and we only increase the number of links D by increasing $\bar{k}$. This favours synchronization of the connected component in this regime and τ_{sync} decreases.

The transition between the two regimes (the peak in the figure τ_{sync} vs $\bar{k}$) depend on N (for larger populations the size of the components is still growing for larger $\bar{k}$).

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