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Research Article

Wiman and Arima Theorems for Quasiregular Mappings

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Wiman's theorem says that an entire holomorphic function of order less than 1/2 has a minimum modulus converging to ∞ along a sequence. Arima's theorem is a refinement of Wiman's theorem. Here we generalize both results to quasiregular mappings in the manifold setup. The so called fundamental frequency has an important role in this study.

1. Main Results

It follows from the Ahlfors theorem that an entire holomorphic function f of order ρ has no more than $[2\rho]$ distinct asymptotic curves where $[r]$ stands for the largest integer $\le r$. This theorem does not give any information if $\rho < 1/2$, This case is covered by two theorems: *if an entire holomorphic function f has order $\rho < 1/2$ then* $\limsup_{r\to\infty} \min_{|z|=r} |f(z)| = \infty$ (Wiman [1]) and *if f is an entire holomorphic function of order $\rho > 0$ and l is a number satisfying the conditions $0 < l \le 2\pi, l < \pi/\rho$, then there exists a sequence of circular arcs $\{|z| = r_k,\ \theta_k \le \arg z \le \theta_k + l\}, r_k \to \infty, 0 \le \theta_k < 2\pi$, along which $|f(z)|$ tends to ∞ uniformly with respect to* $\arg z$ (Arima [2]).

Below we prove generalizations of these theorems for quasiregular mappings for $n \ge 2$. The next two theorems are generalizations of the theorems of Wiman and of Arima for quasiregular mappings on manifolds.

Theorem 1.1. *Let $\mathcal{M}, \mathcal{N}$ be n-dimensional noncompact Riemannian manifolds without boundary. Assume that $h : \mathcal{M} \to (0,\infty)$ is a special exhaustion function of the manifold $\mathcal{M}$ and u is a nonnegative growth function on the manifold $\mathcal{N}$, which is a subsolution of* (3.4) *with the structure conditions* (3.2), (3.3) *and the structure constants $p = n$, ν_1, ν_2.*

Let $f : \mathcal{M} \to \mathcal{N}$ be a nonconstant quasiregular mapping. Suppose that the manifold $\mathcal{M}$ is such that

$$\int^{\infty} \lambda_n(\Sigma_h(t);1)dt = \infty. \tag{1.1}$$

If now

$$\liminf_{\tau\to\infty} \max_{h(m)=\tau} u(f(m)) \exp\left\{-C\int^{\tau} \lambda_n(\Sigma_h(t);1)dt\right\} = 0, \tag{1.2}$$

then

$$\limsup_{\tau\to\infty} \min_{h(m)=\tau} u(f(m)) = \infty. \tag{1.3}$$

Here

$$C = \left(n-1+n\left(\left(\frac{\nu_2}{\nu_1}\right)^2 K^2(f)-1\right)^{1/2}\right)^{-1} \tag{1.4}$$

is a constant, $K(f)$ is the maximal dilatation of f, $\Sigma_h(t)$ is an h-sphere in the manifold $\mathcal{M}$, $\lambda_n(U)$ is a fundamental frequency of an open subset $U \subset \Sigma_h(t)$, and $\lambda_n(\Sigma_h(t);1) = \inf \lambda_n(U)$, where the infimum is taken over all open sets $U \subset \Sigma_h(t)$ with $U \neq \Sigma_h(t)$. (See Sections 4 and 6.)

Theorem 1.2. *Let $\mathcal{M}, \mathcal{N}$ be n-dimensional noncompact Riemannian manifolds without boundary. Assume that $h : \mathcal{M} \to (0,\infty)$ is a special exhaustion function of the manifold $\mathcal{M}$ and u is a nonnegative growth function on the manifold $\mathcal{N}$, which is a subsolution of* (3.4) *with the structure conditions* (3.2), (3.3) *and the structure constants $p = n$, ν_1, ν_2.*

Let $f : \mathcal{M} \to \mathcal{N}$ be a quasiregular mapping and $M(\tau) = \max_{\Sigma_h(\tau)} u(f(m))$. If for some $\gamma > 0$ the mapping f satisfies the condition

$$\liminf_{\tau\to\infty} M(\tau+1) \exp\left\{-\gamma\int^{\tau} \lambda_n(\Sigma_h(t);1)dt\right\} = 0, \tag{1.5}$$

then for each $k = 1, 2, \ldots$ there exists an h-sphere $\Sigma_h(t_k)$ and an open set $U \subset \Sigma_h(t_k)$, for which

$$u(f)|_U \geq k, \qquad \lambda_n(U) < \frac{n\gamma}{C} \lambda_n(\Sigma_h(t_k);1). \tag{1.6}$$

The proofs of these results are based upon Phragmén-Lindelöf's and Ahlfors' theorems for differential forms of $\mathcal{WT}$-classes obtained in [3].

For n-harmonic functions on abstract cones, similar theorems were obtained in [4].

Our notation is as in [3, 5]. We assume that the results of [3] are known to the reader and we only recall some results on qr-mappings.

2. Quasiregular Mappings

Let $\mathcal{M}$ and $\mathcal{N}$ be Riemannian manifolds of dimension n. A continuous mapping $F : \mathcal{M} \to \mathcal{N}$ of the class $W^1_{n,\mathrm{loc}}(\mathcal{M})$ is called a quasiregular mapping if F satisfies

$$\left|F'(m)\right|^n \leq K J_F(m) \tag{2.1}$$

almost everywhere on $\mathcal{M}$. Here $F'(m) : T_m(\mathcal{M}) \to T_{F(m)}(\mathcal{N})$ is the formal derivative of $F(m)$, further, $|F'(m)| = \max_{|h|=1}|F'(m)h|$. We denote by $J_F(m)$ the Jacobian of F at the point $m \in \mathcal{M}$, that is, the determinant of $F'(m)$.

The best constant $K \geq 1$ in the inequality (2.1) is called the outer dilatation of F and denoted by $K_O(F)$. If F is quasiregular, then the least constant $K \geq 1$ for which we have

$$J_F(m) \leq K l\left(F'(m)\right)^n \tag{2.2}$$

almost everywhere on $\mathcal{M}$ is called the inner dilatation and denoted by $K_I(F)$. Here

$$l\left(F'(m)\right) = \min_{|h|=1}\left|F'(m)h\right|. \tag{2.3}$$

The quantity

$$K(F) = \max\{K_O(F), K_I(F)\} \tag{2.4}$$

is called the maximal dilatation of F and if $K(F) \leq K$, then the mapping F is called K-quasiregular.

If $F : \mathcal{M} \to \mathcal{N}$ is a quasiregular homeomorphism, then the mapping F is called quasiconformal. In this case, the inverse mapping F^{-1} is also quasiconformal in the domain $F(\mathcal{M}) \subset \mathcal{N}$ and $K(F^{-1}) = K(F)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be Riemannian manifolds of dimensions $\dim \mathcal{A} = k$ and $\dim \mathcal{B} = n - k$, $1 \leq k < n$, and with scalar products $\langle , \rangle_A, \langle , \rangle_B$, respectively. The Cartesian product $\mathcal{N} = \mathcal{A} \times \mathcal{B}$ has the natural structure of a Riemannian manifold with the scalar product

$$\langle , \rangle = \langle , \rangle_{\mathcal{A}} + \langle , \rangle_{\mathcal{B}}. \tag{2.5}$$

We denote by $\pi : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ and $\eta : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$ the natural projections of the manifold $\mathcal{N}$ onto submanifolds.

If $w_{\mathcal{A}}$ and $w_{\mathcal{B}}$ are volume forms on $\mathcal{A}$ and $\mathcal{B}$, respectively, then the differential form $w_{\mathcal{N}} = \pi^* w_{\mathcal{A}} \wedge \eta^* w_{\mathcal{B}}$ is a volume form on $\mathcal{N}$.

Theorem 2.1 (see [5]). *Let $F : \mathcal{M} \to \mathcal{N}$ be a quasiregular mapping and let $f = \pi \circ F : \mathcal{M} \to \mathcal{A}$. Then the differential form $f^* w_{\mathcal{A}}$ is of the class $\mathcal{WT}_2$ on $\mathcal{M}$ with the structure constants $p = n/k$, $\nu_1 = \nu_1(n, k, K_O)$, and $\nu_2 = \nu_2(n, k, K_O)$.*

Remark 2.2. The structure constants can be chosen to be

$$\nu_1^{-1} = \left(k + \frac{n-k}{\overline{c}^2}\right)^{-n/2} n^{n/2}\ K_O, \qquad \nu_2^{-1} = \underline{c}^{n-k}, \tag{2.6}$$

where $\overline{c} = \overline{c}(k,n,K_O)$ and $\underline{c} = \underline{c}(k,n,K_O)$ are, respectively, the greatest and smallest positive roots of the equation

$$\left(k\xi^2 + (n-k)\right)^{n/2} - n^{n/2} K_O \xi^k = 0. \tag{2.7}$$

3. Domains of Growth

Let $D \subset \mathbf{C}$ be an unbounded domain and let $w = f(z)$ be a holomorphic function continuous on the closure $\overline{D}$. The Phragmén-Lindelöf principle [6] traditionally refers to the alternatives of the following type:

(α) if $\operatorname{Re} f(z) \le 1$ everywhere on ∂D, then either $\operatorname{Re} f(z)$ grows with a certain rate as $z \to \infty$ or $\operatorname{Re} f(z) \le 1$ for all $z \in D$;

(β) if $|f(z)| \le 1$ on ∂D, then either $|f(z)|$ grows with a certain rate as $|z| \to \infty$ or $|f(z)| \le 1$ for all $z \in D$.

Here the rate of growth of the quantities $\operatorname{Re} f(z)$ and $|f(z)|$ depends on the "width" of the domain D near infinity.

It is not difficult to prove that these conditions are equivalent with the following conditions:

(α_1) if $\operatorname{Re} f(z) = 1$ on ∂D and $\operatorname{Re} f(z) \ge 1$ in D, then either $\operatorname{Re} f(z)$ grows with a certain rate as $z \to \infty$ or $f \equiv \text{const}$;

(β_1) if $|f(z)| = 1$ on ∂D and $|f(z)| \ge 1$ in D, then either $|f(z)|$ grows with a certain rate as $z \to \infty$ or $f \equiv \text{const}$.

Let D be an unbounded domain in $\mathbf{R}^n$ and let $f = (f_1, f_2, \ldots, f_n) : D \to \mathbf{R}^n$ be a quasiregular mapping. We assume that $f \in C^0(\overline{D})$. It is natural to consider the Phragmén-Lindelöf alternative under the following assumptions:

(a) $f_1(x)|_{\partial D} = 1$ and $f_1(x) \ge 1$ everywhere in D;

(b) $\sum_{i=1}^p f_i^2(x)|_{\partial D} = 1$ and $\sum_{i=1}^p f_i^2(x) \ge 1$ on D, $1 < p < n$;

(c) $|f(x)| = 1$ on ∂D and $|f(x)| \ge 1$ on D.

Several formulations of the Phragmén-Lindelöf theorem under various assumptions can be found in [7–11]. However, these results are mainly of qualitative character. Here we give a new approach to Phragmén-Lindelöf type theorems for quasiregular mappings, based on isoperimetry, that leads to almost sharp results. Our approach can be used to prove Phragmén-Lindelöf type results for quasiregular mappings of Riemannian manifolds.

Let $\mathcal{N}$ be an n-dimensional noncompact Riemannian C^2-manifold with piecewise smooth boundary $\partial\mathcal{N}$ (possibly empty). A function $u \in C^0(\overline{\mathcal{N}}) \cap W^1_{n,\mathrm{loc}}(\mathcal{N})$ is called a *growth function* with $\mathcal{N}$ as a *domain of growth* if (i) $u \ge 1$, (ii) $u \mid \partial\mathcal{N} = 1$ if $\partial\mathcal{N} \ne \emptyset$, and $\sup_{y \in \mathcal{N}} u(y) = +\infty$.

We consider a quasiregular mapping $f : \mathcal{M} \to \mathcal{N}$, $f \in C^0(\mathcal{M} \cup \partial M)$, where $\mathcal{M}$ is a noncompact Riemannian C^2-manifold, $\dim \mathcal{M} = n$, and $\partial\mathcal{M} \neq \emptyset$. We assume that $f(\partial\mathcal{M}) \subset \partial\mathcal{N}$. In what follows, we mean by the Phragmén-Lindelöf principle an alternative of the form: either the function $u(f(m))$ has a certain rate of growth in $\mathcal{M}$ or $f(m) \equiv \text{const}$.

By choosing the domain of growth $\mathcal{N}$ and the growth function u in a special way, we can obtain several formulations of Phragmén-Lindelöf theorems for quasiregular mappings. In view of the examples in [7], the best results are obtained if an n-harmonic function is chosen as a growth function. In the case (a), the domain of growth is $\mathcal{N} = \{y = (y_1, \dots, y_n) \in \mathbf{R}^n : y_1 \geq 0\}$ and as the function of growth, it is natural to choose $u(y) = y_1 + 1$; in the case (b), the domain $\mathcal{N}$ is the set $\{y = (y_1, \dots, y_n) \in \mathbf{R}^n : \sum_{i=1}^p y_i^2 \geq 1\}$, $1 < p < n$, and $u(y) = (\sum_{i=1}^p y_i^2)^{(n-p)/2(n-1)}$; in the case (c), the domain of growth is $\mathcal{N} = \{y \in \mathbf{R}^n : |y| > 1\}$ and $u(y) = \log|y| + 1$.

In the general case, we shall consider growth functions which are A-solutions of elliptic equations [12]. Namely, let $\mathcal{M}$ be a Riemannian manifold and let

$$A : T(\mathcal{M}) \longrightarrow T(\mathcal{M}) \tag{3.1}$$

be a mapping defined a.e. on the tangent bundle $T(\mathcal{M})$. Suppose that for a.e. $m \in \mathcal{M}$ the mapping A is continuous on the fiber T_m, that is, for a.e. $m \in \mathcal{M}$, the function $A(m, \cdot) : T_m \to T_m$ is defined and continuous; the mapping $m \mapsto A_m(X)$ is measurable for all measurable vector fields X (see [12]).

Suppose that for a.e. $m \in \mathcal{M}$ and for all $\xi \in T_m$, the inequalities

$$\nu_1 |\xi|^p \leq \langle \xi, A(m, \xi) \rangle, \tag{3.2}$$

$$|A(m, \xi)| \leq \nu_2 |\xi|^{p-1} \tag{3.3}$$

hold with $p > 1$ and for some constants $\nu_1, \nu_2 > 0$. It is clear that we have $\nu_1 \leq \nu_2$.

We consider the equation

$$\operatorname{div} A(m, \nabla f) = 0. \tag{3.4}$$

Solutions to (3.4) are understood in the weak sense, that is, A-solutions are $W^1_{p,\text{loc}}$-functions satisfying the integral identity

$$\int_{\mathcal{M}} \langle \nabla\theta, A(m, \nabla f) \rangle * \mathbb{1}_{\mathcal{M}} = 0 \tag{3.5}$$

for all $\theta \in W^1_p(\mathcal{M})$ with compact support in $\mathcal{M}$.

A function f in $W^1_{p,\text{loc}}(\mathcal{M})$ is an *A-subsolution* of (3.4) in $\mathcal{M}$ if

$$\operatorname{div} A(m, \nabla f) \geq 0 \tag{3.6}$$

weakly in $\mathcal{M}$, that is,

$$\int_{\mathcal{M}} \langle \nabla\theta, A(m, \nabla f)\rangle * \mathbb{1}_{\mathcal{M}} \leq 0, \tag{3.7}$$

whenever $\theta \in W_p^1(\mathcal{M})$, is nonnegative with compact support in $\mathcal{M}$.

A basic example of such an equation is the p-Laplace equation

$$\operatorname{div}\left(|\nabla f|^{p-2}\nabla f\right) = 0. \tag{3.8}$$

4. Exhaustion Functions

Below we introduce exhaustion and special exhaustion functions on Riemannian manifolds and give illustrating examples.

4.1. Exhaustion Functions of Boundary Sets

Let $h : \mathcal{M} \to (0, h_0)$, $0 < h_0 \leq \infty$, be a locally Lipschitz function such that

$$\operatorname{ess\,inf}_{Q} |\nabla h| > 0 \quad \forall Q \subset\subset \mathcal{M}. \tag{4.1}$$

For arbitrary $t \in (0, h_0)$, we denote by

$$B_h(t) = \{m \in \mathcal{M} : h(m) < t\}, \qquad \Sigma_h(t) = \{m \in \mathcal{M} : h(m) = t\} \tag{4.2}$$

the h-balls and h-spheres, respectively.

Let $h : \mathcal{M} \to \mathbf{R}$ be a locally Lipschitz function such that there exists a compact $K \subset \mathcal{M}$ with $|\nabla h(x)| > 0$ for a.e. $m \in \mathcal{M} \setminus K$. We say that the function h is an exhaustion function for a boundary set Ξ of $\mathcal{M}$ if for an arbitrary sequence of points $m_k \in \mathcal{M}$, $k = 1, 2, \ldots$, the function $h(m_k) \to h_0$ if and only if $m_k \to \xi$.

It is easy to see that this requirement is satisfied if and only if for an arbitrary increasing sequence $t_1 < t_2 < \cdots < h_0$, the sequence of the open sets $V_k = \{m \in \mathcal{M} : h(m) > t_k\}$ is a chain, defining a boundary set ξ. Thus the function h exhausts the boundary set ξ in the traditional sense of the word.

The function $h : \mathcal{M} \to (0, h_0)$ is called the exhaustion function of the manifold $\mathcal{M}$ if the following two conditions are satisfied:

(i) for all $t \in (0, h_0)$, the h-ball $\overline{B_h(t)}$ is compact;

(ii) for every sequence $t_1 < t_2 < \cdots < h_0$ with $\lim_{k\to\infty} t_k = h_0$, the sequence of h-balls $\{B_h(t_k)\}$ generates an exhaustion of $\mathcal{M}$, that is,

$$B_h(t_1) \subset B_h(t_2) \subset \cdots \subset B_h(t_k) \subset \cdots, \qquad \bigcup_k B_h(t_k) = \mathcal{M}. \tag{4.3}$$

Example 4.1. Let $\mathcal{M}$ be a Riemannian manifold. We set $h(m) = \operatorname{dist}(m, m_0)$ where $m_0 \in \mathcal{M}$ is a fixed point. Because $|\nabla h(m)| = 1$ almost everywhere on $\mathcal{M}$, the function h defines an exhaustion function of the manifold $\mathcal{M}$.

4.2. Special Exhaustion Functions

Let $\mathcal{M}$ be a noncompact Riemannian manifold with the boundary $\partial\mathcal{M}$ (possibly empty). Let A satisfy (3.2) and (3.3) and let $h : \mathcal{M} \to (0, h_0)$ be an exhaustion function, satisfying the following additional conditions:

(a_1) there is $h' > 0$ such that $h^{-1}((0, h'))$ is compact and h is a solution of (3.4) in the open set $K = h^{-1}((h', h_0))$;

(a_2) for a.e. $t_1, t_2 \in (h', h_0)$, $t_1 < t_2$,

$$\int_{\Sigma_h(t_2)} \left\langle \frac{\nabla h}{|\nabla h|}, A(x, \nabla h) \right\rangle d\mathcal{H}^{n-1} = \int_{\Sigma_h(t_1)} \left\langle \frac{\nabla h}{|\nabla h|}, A(x, \nabla h) \right\rangle d\mathcal{H}^{n-1}. \tag{4.4}$$

Here $d\mathcal{H}^{n-1}$ is the element of the $(n-1)$-dimensional Hausdorff measure on Σ_h. Exhaustion functions with these properties will be called *the special exhaustion functions of $\mathcal{M}$ with respect to A*. In most cases, the mapping A will be the p-Laplace operator (3.8) and, unless otherwise stated, A is the p-Laplace operator.

Since the unit vector $\nu = \nabla h / |\nabla h|$ is orthogonal to the h-sphere Σ_h, the condition (a_2) means that the flux of the vector field $A(m, \nabla h)$ through h-spheres $\Sigma_h(t)$ is constant.

In the following, we consider domains D in $\mathbf{R}^n$ as manifolds $\mathcal{M}$. However, the boundaries ∂D of D are allowed to be rather irregular. To handle this situation, we introduce (A, h)-transversality property for $\mathcal{M}$.

Let $h : \mathcal{M} \to (0, h_0)$ be a C^2-exhaustion function. We say that $\mathcal{M}$ satisfies the (A, h)-transversality property if for a.e. t_1, $t_2, h < t_1 < t_2 < h_0$, and for every $\varepsilon > 0$, there exists an open set

$$G = G_\varepsilon(t_1, t_2) \subset B_h(t_2) \setminus \overline{B}_h(t_1) \tag{4.5}$$

with piecewise regular boundary such that

$$\mathcal{H}^{n-1}(\Sigma_h(t_1) \cap \Sigma_h(t_2) \setminus \partial G) < \varepsilon, \tag{4.6}$$

$$\mathcal{H}^{n}\left(\left(B_h(t_2) / \overline{B}_h(t_1)\right) \setminus G\right) < \varepsilon, \tag{4.7}$$

$$\langle A(m, \nabla h(m), v) \rangle = 0, \tag{4.8}$$

where v is the unit inner normal to ∂G.

We say that $\mathcal{M}$ satisfies the h-transversality condition if $\mathcal{M}$ satisfies the (A, h)-transversality condition for the p-Laplace operator $A(m, \xi) = |\xi|^{p-2}\xi$. In this case, (4.8) reduces to

$$\langle \nabla h(m), v \rangle = 0. \tag{4.9}$$

Example 4.2. Let D be a bounded domain in $\mathbf{R}^2$ and let

$$\mathcal{M} = \left\{(x_1, x_2, x_3) \in \mathbf{R}^3 : (x_1, x_2) \in D, x_3 > 0\right\} \tag{4.10}$$

be a cylinder with base D. The function $h : (0,\infty) \to \mathbf{R}$, $h(x) = x_3$, is an exhaustion function for $\mathcal{M}$. Since every domain D in $\mathbf{R}^2$ can be approximated by smooth domains D' from inside, it is easy to see that for $0 < t_1 < t_2 < \infty$ the domain $G = D' \times (t_1, t_2)$ can be used as an approximating domain $G_\varepsilon(t_1, t_2)$. Note that the transversality condition (4.8) is automatically satisfied for the p-Laplace operator $A(m,\xi) = |\xi|^{p-2}\xi$.

Lemma 4.3. *Suppose that an exhaustion function $h \in C^2(\mathcal{M} \setminus K)$ satisfies* (3.4) *in $\mathcal{M} \setminus K$ and that the function $A(m,\xi)$ is continuously differentiable. If $\mathcal{M}$ satisfies the (A,h)-transversality condition, then h is a special exhaustion function on the manifold $\mathcal{M}$.*

Proof. It suffices to show (a_2). Let $h' < t_1 < t_2 < h_0$ and $\varepsilon > 0$. Choose an open set G as in the definition of the (A,h)-transversality condition. $|A(m,\nabla h(m))| \le M < \infty$ for every $m \in \mathcal{M}$, and (4.6)–(4.8) together with the Gauss formula imply for a.e. t_1, t_2

$$\begin{aligned}
&\left|\int_{\Sigma_h(t_2)} \left\langle \frac{\nabla h}{|\nabla h|}, A(m,\nabla h)\right\rangle d\mathcal{H}^{n-1} - \int_{\Sigma_h(t_1)} \left\langle \frac{\nabla h}{|\nabla h|}, A(m,\nabla h)\right\rangle d\mathcal{H}^{n-1}\right| \\
&\quad\le \left|\int_{\partial G\cup\Sigma_h(t_2)} \left\langle \frac{\nabla h}{|\nabla h|}, A(m,\nabla h)\right\rangle d\mathcal{H}^{n-1} - \int_{\partial G\cup\Sigma_h(t_1)} \left\langle \frac{\nabla h}{|\nabla h|}, A(m,\nabla h)\right\rangle d\mathcal{H}^{n-1}\right| + \varepsilon M \\
&\quad= \left|\int_{\partial G} \left\langle \frac{\nabla h}{|\nabla h|}, A(m,\nabla h)\right\rangle d\mathcal{H}^{n-1}\right| + \varepsilon M = \left|\int_{\partial G} \langle v, A(m,\nabla h)\rangle d\mathcal{H}^{n-1}\right| + \varepsilon M \\
&\quad= \left|\int_G \operatorname{div} A(m,\nabla h) d\mathcal{H}^{n}\right| + \varepsilon M = \varepsilon M.
\end{aligned} \tag{4.11}$$

Since $\varepsilon > 0$ is arbitrary, (a_2) follows. □

Example 4.4. Fix $1 \le n \le p$. Let $x_1, x_2, \ldots, x_n$ be an orthonormal system of coordinates in $\mathbf{R}^n, 1 \le n < p$. Let $D \subset \mathbf{R}^n$ be an unbounded domain with piecewise smooth boundary and let $\mathcal{B}$ be a $(p-n)$-dimensional compact Riemannian manifold with or without boundary. We consider the manifold $\mathcal{M} = D \times \mathcal{B}$.

We denote by $x \in D$, $b \in \mathcal{B}$, and $(x,b) \in \mathcal{M}$ the points of the corresponding manifolds. Let $\pi : D \times \mathcal{B} \to D$ and $\eta : D \times \mathcal{B} \to \mathcal{B}$ be the natural projections of the manifold $\mathcal{M}$.

Assume now that the function h is a function on the domain D satisfying the conditions (b_1), (b_2), and (3.8). We consider the function $h^* = h \circ \pi : \mathcal{M} \to (0,\infty)$.

We have

$$\begin{aligned}
\nabla h^* &= \nabla(h\circ\pi) = (\nabla_x h)\circ\pi, \\
\operatorname{div}\left(|\nabla h^*|^{p-2}\nabla h^*\right) &= \operatorname{div}\left(|\nabla(h\circ\pi)|^{p-2}\nabla(h\circ\pi)\right) \\
&= \operatorname{div}\left(|\nabla_x h|^{p-2}\circ\pi(\nabla_x h)\circ\pi\right) = \left(\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(|\nabla_x h|^{p-2}\frac{\partial h}{\partial x_i}\right)\right)\circ\pi.
\end{aligned} \tag{4.12}$$

Because h is a special exhaustion function of D, we have

$$\operatorname{div}\left(|\nabla h^*|^{p-2}\nabla h^*\right) = 0. \tag{4.13}$$

Let $(x,b) \in \partial\mathcal{M}$ be an arbitrary point where the boundary $\partial\mathcal{M}$ has a tangent hyperplane and let ν be a unit normal vector to $\partial\mathcal{M}$.

If $x \in \partial D$, then $\nu = \nu_1 + \nu_2$ where the vector $\nu_1 \in \mathbf{R}^k$ is orthogonal to ∂D and ν_2 is a vector from $T_b(\mathcal{B})$. Thus

$$\langle \nabla h^*, \nu\rangle = \langle (\nabla_x h) \circ \pi, \nu_1\rangle = 0, \tag{4.14}$$

because h is a special exhaustion function on D and satisfies the property (b_2) on ∂D. If $b \in \partial\mathcal{B}$, then the vector ν is orthogonal to $\partial\mathcal{B} \times \mathbf{R}^n$ and

$$\langle \nabla h^*, \nu\rangle = \langle (\nabla_x h) \circ \pi, \nu\rangle = 0, \tag{4.15}$$

because the vector $(\nabla_x h) \circ \pi$ is parallel to $\mathbf{R}^n$.

The other requirements for a special exhaustion function for the manifold $\mathcal{M}$ are easy to verify.

Therefore, *the function*

$$h^* = h^*(x,b) = h \circ \pi : \mathcal{M} \longrightarrow (0,\infty) \tag{4.16}$$

is a special exhaustion function on the manifold $\mathcal{M} = D \times \mathcal{B}$.

Example 4.5. We fix an integer k, $1 \leq k \leq n$, and set

$$d_k(x) = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}. \tag{4.17}$$

It is easy to see that $|\nabla d_k(x)| = 1$ everywhere in $\mathbf{R}^n \setminus \Sigma_0$, where $\Sigma_0 = \{x \in \mathbf{R}^n : d_k(x) = 0\}$. We shall call the set

$$B_k(t) = \{x \in \mathbf{R}^n : d_k(x) < t\} \tag{4.18}$$

a k-ball and the set

$$\Sigma_k(t) = \{x \in \mathbf{R}^n : d_k(x) = t\} \tag{4.19}$$

a k-sphere in $\mathbf{R}^n$.

We shall say that an unbounded domain $D \subset \mathbf{R}^n$ is k-admissible if for each $t > \inf_{x\in D} d_k(x)$, the set $D \cap B_k(t)$ has compact closure.

It is clear that every unbounded domain $D \subset \mathbf{R}^n$ is n-admissible. In the general case, the domain D is k-admissible if and only if the function $d_k(x)$ is an exhaustion function of D.

It is not difficult to see that if a domain $D \subset \mathbf{R}^n$ is k-admissible, then it is l-admissible for all $k < l < n$.

Fix $1 \le k < n$. Let Δ be a bounded domain in the $(n-k)$-plane $x_1 = \cdots = x_k = 0$ and let

$$D = \{x = (x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in \mathbf{R}^n : (x_{k+1}, \dots, x_n) \in \Delta\}. \tag{4.20}$$

The domain D is k-admissible. The k-spheres $\Sigma_k(t)$ are orthogonal to the boundary ∂D and therefore $\langle \nabla d_k, \nu \rangle = 0$ everywhere on the boundary. The function

$$h(x) = \begin{cases} \log d_k(x), & p = k, \\ d_k^{(p-k)/(p-1)}(x), & p \neq k, \end{cases} \tag{4.21}$$

satisfies (3.4). By Lemma 4.3, the function h is a special exhaustion function of the domain D. Therefore, the domain D has p-parabolic type for $p \ge k$ and p-hyperbolic type for $p < k$.

Example 4.6. Fix $1 \le k < n$. Let Δ be a bounded domain in the plane $x_1 = \cdots = x_k = 0$ with a (piecewise) smooth boundary and let

$$D = \{x = (x_1, \dots, x_n) \in \mathbf{R}^n : (x_{k+1}, \dots, x_n) \in \Delta\} = \mathbf{R}^{n-k} \times \Delta \tag{4.22}$$

be the cylinder domain with base Δ.

The domain D is k-admissible. The k-spheres $\Sigma_k(t)$ are orthogonal to the boundary ∂D and therefore $\langle \nabla d_k, \nu \rangle = 0$ everywhere on the boundary, where d_k is as in Example 4.5.

Let $h = \phi(d_k)$ where ϕ is a C^2-function with $\phi' \ge 0$. We have $\nabla h = \phi' \nabla d_k$ and since $|\nabla d_k| = 1$, we obtain

$$\begin{aligned} \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(|\nabla h|^{n-2} \frac{\partial h}{\partial x_i}\right) &= \sum_{i=1}^{k} \frac{\partial}{\partial x_i}\left((\phi')^{n-1} \frac{\partial d_k}{\partial x_i}\right) \\ &= (n-1)(\phi')^{n-2}\phi'' + \frac{k-1}{d_k}(\phi')^{n-1}. \end{aligned} \tag{4.23}$$

From the equation

$$(n-1)\phi'' + \frac{k-1}{d_k}\phi' = 0, \tag{4.24}$$

we conclude that *the function*

$$h(x) = (d_k(x))^{(n-k)/(n-1)} \tag{4.25}$$

satisfies (3.8) *in $D \setminus K$ and thus it is a special exhaustion function of the domain D.*

Example 4.7. Let (r,θ), where $r \ge 0$, $\theta \in S^{n-1}(1)$, be the spherical coordinates in $\mathbf{R}^n$. Let $U \subset S^{n-1}(1)$, $\partial U \ne \emptyset$, be an arbitrary domain with a piecewise smooth boundary on the unit sphere $S^{n-1}(1)$. We fix $0 \le r_1 < \infty$ and consider the domain

$$D = \{(r,\theta) \in \mathbf{R}^n : r_1 < r < \infty,\ \theta \in U\}. \tag{4.26}$$

As above it is easy to verify that the given domain is n-admissible and *the function*

$$h(|x|) = \log \frac{|x|}{r_1} \tag{4.27}$$

is a special exhaustion function of the domain D for $p = n$.

Example 4.8. Let $\mathcal{A}$ be a compact Riemannian manifold, $\dim \mathcal{A} = k$, with piecewise smooth boundary or without boundary. We consider the Cartesian product $\mathcal{M} = \mathcal{A} \times \mathbf{R}^n$, $n \ge 1$. We denote by $a \in \mathcal{A}$, $x \in \mathbf{R}^n$, and $(a,x) \in \mathcal{M}$ the points of the corresponding spaces. It is easy to see that the function

$$h(a,x) = \begin{cases} \log|x|, & p = n, \\ |x|^{(p-n)/(p-1)}, & p \ne n, \end{cases} \tag{4.28}$$

is a special exhaustion function for the manifold $\mathcal{M}$. Therefore, for $p \ge n$, the given manifold has p-parabolic type and for $p < n$, p-hyperbolic type.

Example 4.9. Let (r,θ), where $r \ge 0$, $\theta \in S^{n-1}(1)$, be the spherical coordinates in $\mathbf{R}^n$. Let $U \subset S^{n-1}(1)$ be an arbitrary domain on the unit sphere $S^{n-1}(1)$. We fix $0 \le r_1 < r_2 < \infty$ and consider the domain

$$D = \{(r,\theta) \in \mathbf{R}^n : r_1 < r < r_2,\ \theta \in U\} \tag{4.29}$$

with the metric

$$ds^2_{\mathcal{M}} = \alpha^2(r)dr^2 + \beta^2(r)dl^2_{\theta}, \tag{4.30}$$

where $\alpha(r), \beta(r) > 0$ are C^0-functions on $[r_1, r_2)$ and dl_θ is an element of length on $S^{n-1}(1)$.

The manifold $\mathcal{M} = (D, ds^2_{\mathcal{M}})$ is a warped Riemannian product. In the cases, $\alpha(r) \equiv 1$, $\beta(r) = 1$, and $U = S^{n-1}$ the manifold $\mathcal{M}$ is isometric to a cylinder in $\mathbf{R}^{n+1}$. In the cases, $\alpha(r) \equiv 1$, $\beta(r) = r$, $U = S^{n-1}$ the manifold $\mathcal{M}$ is a spherical annulus in $\mathbf{R}^n$.

The volume element in the metric (4.30) is given by the expression

$$d\sigma_{\mathcal{M}} = \alpha(r)\beta^{n-1}(r)dr\, dS^{n-1}(1). \tag{4.31}$$

If $\phi(r,\theta) \in C^1(D)$, then the length of the gradient $\nabla\phi$ in $\mathcal{M}$ takes the form

$$|\nabla\phi|^2 = \frac{1}{\alpha^2}(\phi_r')^2 + \frac{1}{\beta^2}|\nabla_\theta\phi|^2, \tag{4.32}$$

where $\nabla_\theta\phi$ is the gradient in the metric of the unit sphere $S^{n-1}(1)$.

For the special exhaustion function $h(r,\theta) \equiv h(r)$, (3.8) reduces to the following form:

$$\frac{d}{dr}\left(\left(\frac{1}{\alpha(r)}\right)^{p-1}(h_r'(r))^{p-1}\beta^{n-1}(r)\right) = 0. \tag{4.33}$$

Solutions of this equation are the functions

$$h(r) = C_1\int_{r_1}^{r}\frac{\alpha(t)}{\beta^{(n-1)/(p-1)}(t)}dt + C_2, \tag{4.34}$$

where C_1 and C_2 are constants.

Because the function h satisfies obviously the boundary condition (a_2) as well as the other conditions of special exhaustion functions listed in (4.2), we see that under the assumption

$$\int^{r_2}\frac{\alpha(t)}{\beta^{(n-1)/(p-1)}(t)}dt = \infty, \tag{4.35}$$

the function

$$h(r) = \int_{r_1}^{r}\frac{\alpha(t)}{\beta^{(n-1)/(p-1)}(t)}dt \tag{4.36}$$

is a special exhaustion function on the manifold $\mathcal{M}$.

Theorem 4.10. *Let $h : \mathcal{M} \to (0,h_0)$ be a special exhaustion function of a boundary set ξ of the manifold $\mathcal{M}$. Then*

(i) *if $h_0 = \infty$, the set ξ has p-parabolic type,*

(ii) *if $h_0 < \infty$, the set ξ has p-hyperbolic type.*

Proof. Choose $0 < t_1 < t_2 < h_0$ such that $K \subset B_h(t_1)$. We need to estimate the p-capacity of the condenser $(B_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M})$. We have

$$\operatorname{cap}_p\left(\overline{B}_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M}\right) = \frac{J}{(t_2-t_1)^{p-1}}, \tag{4.37}$$

where

$$J = \int_{\Sigma_h(t)}|\nabla h|^{p-1}d\mathcal{H}^{n-1}_{\mathcal{M}} \tag{4.38}$$

is a quantity independent of $t > h(K) = \sup\{h(m) : m \in K\}$. Indeed, for the variational problem [3, (2.9)], we choose the function φ_0, $\varphi_0(m) = 0$ for $m \in B_h(t_1)$,

$$\varphi_0(m) = \frac{h(m) - t_1}{t_2 - t_1}, \quad m \in B_h(t_2) \setminus B_h(t_1), \tag{4.39}$$

and $\varphi_0(m) = 1$ for $m \in \mathcal{M} \setminus B_h(t_2)$. Using the Kronrod-Federer formula [13, Theorem 3.2.22], we get

$$\begin{aligned}
\operatorname{cap}_p(B_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M}) &\le \int_{\mathcal{M}} \left|\nabla\varphi_0\right|^p * \mathbb{1}_{\mathcal{M}} \\
&\le \frac{1}{(t_2 - t_1)^p} \int_{t_1 < h(m) < t_2} |\nabla h(m)|^p * \mathbb{1}_{\mathcal{M}} \\
&= \int_{t_1}^{t_2} dt \int_{\Sigma_h(t)} |\nabla h(m)|^{p-1} d\mathcal{H}^{n-1}_{\mathcal{M}}.
\end{aligned} \tag{4.40}$$

Because the special exhaustion function satisfies (3.8) and the boundary condition (a_2), one obtains for arbitrary τ_1, τ_2, $h(K) < \tau_1 < \tau_2 < h_0$

$$\begin{aligned}
&\int_{\Sigma_h(t_2)} |\nabla h|^{p-1} d\mathcal{H}^{n-1}_{\mathcal{M}} - \int_{\Sigma_h(t_1)} |\nabla h|^{p-1} d\mathcal{H}^{n-1}_{\mathcal{M}} \\
&\quad = \int_{\Sigma_h(t_2)} |\nabla h|^{p-2} \langle \nabla h, \nu\rangle d\mathcal{H}^{n-1}_{\mathcal{M}} - \int_{\Sigma_h(t_1)} |\nabla h|^{p-2} \langle \nabla h, \nu\rangle d\mathcal{H}^{n-1}_{\mathcal{M}} \\
&\quad = \int_{t_1 < h(m) < t_2} \operatorname{div}_{\mathcal{M}}\left(|\nabla h|^{p-2} \nabla h\right) * \mathbb{1}_{\mathcal{M}} = 0.
\end{aligned} \tag{4.41}$$

Thus we have established the inequality

$$\operatorname{cap}_p(B_h(t_1), \mathcal{M} \setminus B_h(t_2); \mathcal{M}) \le \frac{J}{(t_2 - t_1)^{p-1}}. \tag{4.42}$$

By the conditions, imposed on the special exhaustion function, the function φ_0 is an extremal in the variational problem [3, (2.9)]. Such an extremal is unique and therefore the preceding inequality holds as an equality. This conclusion proves (4.37).

If $h_0 = \infty$, then letting $t_2 \to \infty$ in (4.37) we conclude the parabolicity of the type of ξ. Let $h_0 < \infty$. Consider an exhaustion $\{\mathcal{U}_k\}$ and choose $t_0 > 0$ such that the h-ball $B_h(t_0)$ contains the compact set K.

Set $t_k = \sup_{m \in \partial \mathcal{U}_k} h(m)$. Then for $t_k > t_0$, we have

$$\operatorname{cap}_p\left(\overline{\mathcal{U}}_{k_0}, \mathcal{U}_k; \mathcal{M}\right) \ge \operatorname{cap}_p(B_h(t_0), B_h(t_k); \mathcal{M}) = \frac{J}{(t_k - t_0)^{p-1}}, \tag{4.43}$$

and hence

$$\liminf_{k\to\infty} \operatorname{cap}_p\left(\overline{U}_{k_0}, \mathcal{U}_k; \mathcal{M}\right) \geq \frac{J}{(h_0 - t_0)^{p-1}} > 0, \tag{4.44}$$

and the boundary set ξ has p-hyperbolic type. □

5. Wiman Theorem

Now we will prove Theorem 1.1.

5.1. Fundamental Frequency

Let $U \subset \Sigma_h(\tau)$ be an open set. We need further the following quantity:

$$\lambda_p(U) = \inf \frac{\left(\int_U |\nabla h|^{-1} |\nabla_2 \varphi|^p \, d\mathcal{H}^{n-1}_{\mathcal{M}}\right)^{1/p}}{\left(\int_U |\nabla h|^{p-1} |\varphi|^p \, d\mathcal{H}^{n-1}_{\mathcal{M}}\right)^{1/p}}, \tag{5.1}$$

where the infimum is taken over all functions $\varphi \in W^1_p(U)$ with $\operatorname{supp}\varphi \subset U$ (By the definition, φ is a W^1_p-function on an open set U, if φ belongs to this class on every component of U.). Here $\nabla_2\varphi$ is the gradient of φ on the surface $\Sigma_h(\tau)$.

In the case $|\nabla h| \equiv 1$, this quantity is well-known and can be interpreted, in particular, as the best constant in the Poincaré inequality. Following [14], we shall call this quantity the fundamental frequency of the rigidly supported membrane U.

Observe a useful property of the fundamental frequency.

Lemma 5.1. *Let $U \subset \Sigma_h(\tau)$ be an open set and let U_i be the components of U, $i = 1, 2, \ldots$. Then*

$$\lambda_p(U) = \inf_i \lambda_p(U_i). \tag{5.2}$$

Proof. To prove this property, we fix arbitrary functions φ_i with $\operatorname{supp}\varphi_i \subset U_i$. Set $\varphi(m) = \varphi_i(m)$ for $m \in U_i$ and $\varphi = 0$ for $U \setminus (\bigcup_i U_i)$. Hence

$$\lambda_p^p(U_i) \int_{U_i} |\nabla h|^{p-1} |\varphi_i|^p \, d\mathcal{H}^{n-1} \leq \int_{U_i} |\nabla h|^{-1} |\nabla_2 \varphi_i|^p \, d\mathcal{H}^{n-1}. \tag{5.3}$$

Summation yields

$$\left(\inf_i \lambda_p^p(U_i)\right) \sum_i \int_{U_i} |\nabla h|^{p-1} |\varphi_i|^p \, d\mathcal{H}^{n-1} \leq \sum_i \int_{U_i} |\nabla h|^{-1} |\nabla_2 \varphi_i|^p \, d\mathcal{H}^{n-1} \tag{5.4}$$

and we obtain

$$\left(\inf_i \lambda_p^p(U_i)\right)\int_U |\nabla h|^{p-1}|\varphi|^p d\mathcal{H}^{n-1} \leq \int_U |\nabla h|^{-1}|\nabla_2\varphi|^p d\mathcal{H}^{n-1}. \tag{5.5}$$

This gives

$$\inf_i \lambda_p(U_i) \leq \lambda_p(U). \tag{5.6}$$

The reverse inequality is evident. Indeed, if U_i is a component of U, then evidently

$$\lambda_p(U) \leq \lambda_p(U_i) \tag{5.7}$$

and hence

$$\lambda_p(U) \leq \inf_i \lambda_p(U_i). \tag{5.8}$$

□

We also need the following statement.

Lemma 5.2. *Under the above assumptions for a.e. $\tau \in (0, h_0)$, we have*

$$\varepsilon(\tau; \mathcal{F}_B) \geq \frac{\lambda_p(\Sigma_h(\tau))}{c}, \tag{5.9}$$

where λ_p is the fundamental frequency of the membrane $\Sigma_h(\tau)$ defined by formula (5.1) *and*

$$c = c(\nu_1, \nu_2, p) = \begin{cases} c_1 & \text{for } p \leq 2, \\ c_2 & \text{for } p \geq 2, \end{cases} \tag{5.10}$$

where

$$\begin{gathered} c_1 = \sqrt{\nu_2^2 - \nu_1^2} + 2^{(2-p)/2}\ \nu_1 p^{-1}(p-1)^{(p-1)/p}, \\ c_2 = \sqrt{\nu_2^2 - \nu_1^2} + \nu_1\frac{p-1}{p}. \end{gathered} \tag{5.11}$$

For the proof, see Lemma 4.3 in [10].

We now use these estimates for proving Phragmén-Lindelöf type theorems for the solutions of quasilinear equations on manifolds.

Theorem 5.3. *Let $h : \mathcal{M} \to (0, \infty)$ be an exhaustion function. Suppose that the manifold $\mathcal{M}$ satisfies the condition*

$$\int^\infty \lambda_p(\Sigma_h(t))dt = \infty. \tag{5.12}$$

Let f be a continuous solution of (3.4) *with* (3.2) *and* (3.3) *on $\mathcal{M}$ such that*

$$\limsup_{m \to m_0} f(m) \leq 0, \quad \forall m_0 \in \partial\mathcal{M}. \tag{5.13}$$

Then either $f(m) \leq 0$ everywhere on $\mathcal{M}$ or

$$\liminf_{\tau \to \infty} \int_{\tau < h(m) < \tau+1} |\nabla h| |f(m)| |\nabla f(m)|^{p-1} * \mathbb{1} \exp\left\{ -c_3 \int^{\tau} \lambda_p(\Sigma_h(t))dt \right\} > 0, \tag{5.14}$$

$$\liminf_{\tau \to \infty} \int_{\tau < h(m) < \tau+1} |\nabla h(m)|^p |f(m)|^p * \mathbb{1} \exp\left\{ -c_3 \int^{\tau} \lambda_p(\Sigma_h(t))dt \right\} > 0. \tag{5.15}$$

In particular, if h is a special exhaustion function on $\mathcal{M}$, then

$$\liminf_{\tau \longrightarrow \infty} M(\tau + 1) \exp\left\{ -\frac{c_3}{p} \int^{\tau} \lambda_p(\Sigma_h(t))dt \right\} > 0. \tag{5.16}$$

Here

$$M(t) = \sup_{m \in \Sigma_h(t)} |f(m)| \tag{5.17}$$

and $c_3 = \nu_1 c^{-1}$ where c is the constant of Lemma 5.2.

Proof. We assume that at some point $m_1 \in \operatorname{int} \mathcal{M}$ we have $f(m_1) > 0$. We consider the set

$$\mathcal{O} = \{ m \in \mathcal{M} : f(m) > f(m_1) \}. \tag{5.18}$$

By, [3, Corollary 4.57] the set $\mathcal{O}$ is noncompact.

The function h is an exhaustion function on $\mathcal{O}$. Using the relation [3, 6.74] for the function $f(m) - f(m_1)$ on $\mathcal{O}$, we have

$$\liminf_{\tau \to \infty} \int_{\mathcal{O}(\tau)} |\nabla h| |f(m) - f(m_1)| |A(m, \nabla f)| * \mathbb{1} \exp\left\{ -\nu_1 \int_{\tau_0}^{\tau} \varepsilon(t; \mathcal{F}_{\mathcal{O}})dt \right\} > 0, \tag{5.19}$$

where $\mathcal{O}(\tau) = \{ m \in \mathcal{O} : \tau < h(m) < \tau + 1 \}$.

By Lemma 5.2, the following inequality holds

$$\varepsilon(t; \mathcal{F}_{\mathcal{O}}) \geq \frac{\lambda_p(\Sigma_h(t) \cap \mathcal{O})}{c}. \tag{5.20}$$

Because $\Sigma_h(t) \cap \mathcal{O} \subset \Sigma_h(t)$, it follows that $\lambda_p(\Sigma_h(t) \cap \mathcal{O}) \geq \lambda_p(\Sigma_h(t))$ and hence

$$\varepsilon(t; \mathcal{F}_{\mathcal{O}}) \geq \frac{\lambda_p(\Sigma_h(t))}{c}. \tag{5.21}$$

Thus using the requirement (3.3) for (3.4), we arrive at the estimate

$$\liminf_{\tau\to\infty}\int_{\mathcal{O}(\tau)}|\nabla h(m)|\left|f(m)-f(m_1)\right|\left|\nabla f(m)\right|^{p-1} * \mathbb{1}\exp\left\{-c_3\int^{\tau}\lambda_p(\Sigma_h(t))dt\right\}>0. \tag{5.22}$$

Further we observe that from the condition $f(m) > f(m_1) > 0$ on $\mathcal{O}$, it follows that

$$\begin{aligned}\int_{\mathcal{O}(\tau)}|\nabla h|\left|f(m)-f(m_1)\right|\left|\nabla f(m)\right|^{p-1} * \mathbb{1} &= \int_{\mathcal{O}(\tau)} f(m)|\nabla h|\left|\nabla f(m)\right|^{p-1} * \mathbb{1}\\ &\quad - f(m_1)\int_{\mathcal{O}(\tau)}|\nabla h|\ \left|\nabla f(m)\right|^{p-1} * \mathbb{1}\\ &\le \int_{\tau<h(m)<\tau+1}|\nabla h|\left|f(m)\right|\left|\nabla f(m)\right|^{p-1} * \mathbb{1}.\end{aligned} \tag{5.23}$$

From this relation, we arrive at (5.14).

The proof of (5.15) is carried out exactly in the same way by means of the inequality [3, 5.75].

In order to convince ourselves of the validity of (5.16), we observe that by the maximum principle we have

$$\int_{\tau<h(m)<\tau+1}|\nabla h(m)|^p\left|f(m)\right|^p * \mathbb{1} \le M^p(\tau+1)\int_{\tau<h(m)<\tau+1}|\nabla h(m)|^p * \mathbb{1}. \tag{5.24}$$

But h is a special exhaustion function and therefore by (4.37) we can write

$$\int_{\tau<h(m)<\tau+1}|\nabla h(m)|^p * \mathbb{1} = J, \tag{5.25}$$

where J is a number independent of τ.

The relation (5.15) implies then that (5.16) holds. □

Example 5.4. Let $\mathcal{A}$ be a compact Riemannian manifold with nonempty piecewise smooth boundary, $\dim\mathcal{A} = k \ge 1$, and let $\mathcal{M} = \mathcal{A}\times\mathbf{R}^n$, $n \ge 1$. Choosing as a special exhaustion function of $\mathcal{M}$ the function $h(a,x)$, defined in Example 4.8, we have

$$\Sigma_h(t) = \mathcal{A}\times S^{n-1}(t). \tag{5.26}$$

Then using the fact that $h(a,x)|_{\Sigma_h(t)} = t$, we find

$$|\nabla h(a,x)|_{\Sigma_h(t)} = h'(t) = \begin{cases} e^{-t} & \text{for } p=n\\ \dfrac{p-n}{p-1}t^{(1-n)/(p-n)} & \text{for } p\ne n.\end{cases} \tag{5.27}$$

Therefore, on the basis of (5.1) we get

$$\lambda_p(\Sigma_h(t)) = \frac{1}{h'(t)} \inf \frac{\left(\int_{\mathcal{A}\times S^{n-1}(t)} |\nabla_2\phi|^p d\mathcal{H}^{n-1}_{\mathcal{M}}\right)^{1/p}}{\left(\int_{\mathcal{A}\times \mathbf{R}^n} |\phi|^p d\mathcal{H}^{n-1}_{\mathcal{M}}\right)^{1/p}}. \tag{5.28}$$

Computation yields

$$\begin{aligned}
|\nabla_2\phi(a,x)|^2 &= |\nabla_{\mathcal{A}}\phi(a,x)|^2 + |\nabla_{S^{n-1}(t)}\phi(a,x)|^2 \\
&= |\nabla_{\mathcal{A}}\phi(a,x)|^2 + \frac{1}{t^2}\left|\nabla_{S^{n-1}(1)}\phi\left(a,\frac{x}{|x|}\right)\right|^2. \\
d\mathcal{H}^{n-1}_{\mathcal{M}} &= d\sigma_{\mathcal{A}}\, dS^{n-1}(t),
\end{aligned} \tag{5.29}$$

where $d\sigma_{\mathcal{A}}$ is an element of k-dimensional area on $\mathcal{A}$. Therefore,

$$\begin{aligned}
&\lambda_p(\Sigma_h(t)) \\
&= \frac{1}{h'(t)} \inf \frac{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(t)} \left(|\nabla_{\mathcal{A}}\phi(a,x)|^2 + |\nabla_{S^{n-1}(t)}\phi(a,x)|^2\right)^{p/2} dS^{n-1}(t)\right)^{1/p}}{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(t)} \phi^p(a,x) dS^{n-1}(t)\right)^{1/p}} \\
&= \frac{1}{h'(t)} \inf \frac{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} \left(|\nabla_{\mathcal{A}}\phi(a,x/|x|)|^2 + (1/t^2)|\nabla_{S^{n-1}(t)}\phi(a,x/|x|)|^2\right)^{p/2} dS^{n-1}(1)\right)^{1/p}}{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} \phi^p(a,x/|x|)\; dS^{n-1}(1)\right)^{1/p}}
\end{aligned} \tag{5.30}$$

and we obtain

$$\lambda_p(\Sigma_h(t)) = \frac{1}{h'(t)} \inf \frac{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} \left(|\nabla_{\mathcal{A}}\psi|^2 + (1/t^2)|\nabla_{S^{n-1}(1)}\psi|^2\right)^{p/2} dS^{n-1}(1)\right)^{1/p}}{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} \psi^p dS^{n-1}(1)\right)^{1/p}}, \tag{5.31}$$

where the infimum is taken over all functions $\psi = \psi(a,x)$ with

$$\psi(a,x) \in W^1_p\left(\mathcal{A}\times S^{n-1}(1)\right), \quad \psi(a,x)|_{a\in\partial\mathcal{A}} = 0, \quad \forall x \in S^{n-1}(1). \tag{5.32}$$

In the particular case $n = 1$, Theorem 5.3 has a particularly simple content. Here $h(x)$ is a function of one variable, and $\Sigma_h(t) = \mathcal{A}\times S^0(t)$ is isometric to $\Sigma_h(1)$. Therefore, $h'(t) \equiv 1$ and by (5.31) we have

$$\lambda_p(\Sigma_h(t)) \equiv \lambda_p(\Sigma_h(1)) \equiv \lambda_p(\mathcal{A}) \quad \forall t \in R^1. \tag{5.33}$$

In the same way, (5.16) can be written in the form

$$\liminf_{t\to\infty} \max_{|x|=t} |f(a,x)| \exp\left\{-\frac{c_3}{p}\, \lambda_n(\mathcal{A})\right\} > 0. \tag{5.34}$$

Let $n \geq 2$. We do not know of examples where the quantity (5.31) had been exactly computed. Some idea about the rate of growth of the quantity $M(\tau)$ in the Phragmén-Lindelöf alternative can be obtained from the following arguments. Simplifying the numerator of (5.31) by ignoring the second summand, we get the estimate

$$\lambda_p(\Sigma_h(t)) \geq \frac{1}{h'(t)} \inf_{\psi} \frac{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} |\nabla_{\mathcal{A}} \psi(a,x)|^p dS^{n-1}(1)\right)^{1/p}}{\left(\int_{\mathcal{A}} d\sigma_{\mathcal{A}} \int_{S^{n-1}(1)} \psi^p(a,x) dS^{n-1}(1)\right)^{1/p}}. \tag{5.35}$$

For each fixed $x \in S^{n-1}(1)$, the function $\psi(a,x)$ is finite on $\mathcal{A}$, because from the definition of the fundamental frequency it follows that

$$\left(\int_{\mathcal{A}} |\nabla_{\mathcal{A}} \psi(a,x)|^p \, d\sigma_{\mathcal{A}}\right)^{1/p} \geq \lambda_p(\mathcal{A}) \left(\int_{\mathcal{A}} \psi^p(a,x) d\sigma_{\mathcal{A}}\right)^{1/p}. \tag{5.36}$$

From this we get

$$\lambda_p(\Sigma_h(t)) \geq \frac{1}{h'(t)} \lambda_p(\mathcal{A}). \tag{5.37}$$

Thus

$$\int_{\tau_0}^{\tau} \lambda_p(\Sigma_h(r)) dr \geq \int_{\tau_0}^{\tau} \lambda_p(\mathcal{A}) \frac{dh(r)}{h'(r)} = \lambda_p(\mathcal{A}) \int_{\tau_0}^{\tau} r'(h) dh = \lambda_p(\mathcal{A})(r(\tau) - r(\tau_0)). \tag{5.38}$$

Here $r(h)$ is the inverse function of $h(r)$. Because

$$\max_{h(|x|)=\tau} |f(a,x)| \exp\left\{-\frac{c_3}{p}\lambda_p(\mathcal{A}) r(\tau)\right\} = \max_{|x|=r(\tau)} |f(a,x)| \exp\left\{-\frac{c_3}{p}\lambda_p(\mathcal{A}) r(\tau)\right\}, \tag{5.39}$$

the relation (5.16) can be written in the form (5.34).

Example 5.5. Let $U \subset S^{n-1}$ be an arbitrary domain with nonempty boundary. We consider a warped Riemannian product $\mathcal{M} = (r_1, r_2) \times U$ equipped with the metric (4.30) of the domain D. We now analyze Theorem 5.3 in this case.

The function $h(r)$, given by (4.36) under the requirement (4.35), is a special exhaustion function on $\mathcal{M}$. We compute the quantity $\lambda_p(\Sigma_h(\tau))$ as follows:

$$\begin{gathered}
|\nabla h(|x|)|_{\Sigma_h(\tau)} = h'(r(\tau)) = \frac{\alpha(r(\tau))}{\beta^{n-1}(r(\tau))}, \\
\left|\nabla_2 \phi\right|_{\Sigma_h(\tau)} = \frac{\left|\nabla_{S^{n-1}(1)}\phi\right|}{\beta(r(\tau))}, \\
d\mathcal{H}^{n-1}_{\mathcal{M}} = \beta^{n-1}(r(\tau))dS^{n-1}(1), \qquad r(\tau) = h^{-1}(\tau).
\end{gathered} \tag{5.40}$$

Therefore, observing that

$$\frac{1}{h'(r(\tau))} = r'(\tau), \tag{5.41}$$

we have

$$\begin{aligned}
\lambda_p(\Sigma_h(\tau)) &= \frac{1}{h'(r(\tau))} \inf_{\phi} \frac{\left(\int_{\Sigma_h(\tau)} \left|\nabla_2 \phi\right|^p d\mathcal{H}^{n-1}_{\mathcal{M}}\right)^{1/p}}{\left(\int_{\Sigma_h(\tau)} \phi^p d\mathcal{H}^{n-1}_{\mathcal{M}}\right)^{1/p}} \\
&= \frac{r'(\tau)}{\beta(r(\tau))} \inf \frac{\left(\int_U \left|\nabla_{S^{n-1}(1)}\phi\right|^p dS^{n-1}(1)\right)^{1/p}}{\left(\int_U \phi^p dS^{n-1}(1)\right)^{1/p}}.
\end{aligned} \tag{5.42}$$

Thus

$$\lambda_p(\Sigma_h(\tau)) = \frac{r'(\tau)}{\beta(r(\tau))}\lambda_p(U). \tag{5.43}$$

Further we get

$$\begin{gathered}
\int_{\tau_0}^{\tau} \lambda_h(\Sigma_h(\tau))d\tau = \lambda_p(U)\int_{r(\tau_0)}^{r(\tau)} \frac{dr}{\beta(r)}, \\
\max_{h(|x|)=\tau} \left|f(x)\right| \exp\left\{-\frac{c_3}{p}\lambda_p(U)\int^{r(\tau)} \frac{dr}{\beta(r)}\right\} = \max_{|x|=r(\tau)} \left|f(x)\right| \exp\left\{-\frac{c_3}{p}\lambda_p(U)\int^{r(\tau)} \frac{dr}{\beta(r)}\right\}.
\end{gathered} \tag{5.44}$$

Thus the relation (5.16) attains the form

$$\liminf_{r\to\infty} \max_{|x|=r} \left|f(x)\right| \exp\left\{-\frac{c_3}{p}\lambda_p(U)\int^{r} \frac{dr}{\beta(r)}\right\} > 0. \tag{5.45}$$

5.2. Proof of Theorem 1.1

We assume that

$$\limsup_{\tau\to\infty} \min_{m\in\Sigma_h(\tau)} u(f(m)) = K < \infty. \tag{5.46}$$

Consider the set

$$\mathcal{O} = \{m \in \mathcal{X} : u(f(m)) > qK\}, \quad q < 1. \tag{5.47}$$

It is clear that for a suitable choice of q, the set $\mathcal{O}$ is not empty.

By assumptions, the function u satisfies (3.4) with (3.2), (3.3) and structure constants $p = n$, ν_1, ν_2. Since f is quasiregular, by Lemma 14.38 of [12] the function $u(f(m))$ is a subsolution of another equation of the form (3.4) with structure constants $\nu_1' = \nu_1/K_O$, $\nu_2' = \nu_2 K_I$, where K_O and K_I are outer and inner dilatations of f. In view of the maximum principle for subsolutions, the set $\mathcal{O}$ does not have relatively compact components. Without restricting generality, we may assume that $\mathcal{O}$ is connected. Because for sufficiently large τ, the condition

$$\mathcal{O} \cap \Sigma_h(\tau) \neq \emptyset \tag{5.48}$$

holds; we see that

$$\lambda_n(\mathcal{O} \cap \Sigma_h(\tau)) \geq \lambda_n(\Sigma_h(\tau); 1). \tag{5.49}$$

Therefore, the condition (1.1) on the manifold $\mathcal{X}$ implies the following property:

$$\int^{\infty} \lambda_n(\mathcal{O} \cap \Sigma_h(\tau)) d\tau = \infty. \tag{5.50}$$

Observing that

$$\max_{m\in\Sigma_h(\tau)} u(f(m)) \geq \max_{m\in\Sigma_h(\tau)\cap\mathcal{O}} u(f(m)), \tag{5.51}$$

we see that by (1.2)

$$\liminf_{\tau\to\infty} \max_{\Sigma_h(\tau)\cap\mathcal{O}} u(f(m)) \exp\left\{-C\int^{\tau} \lambda_n(\mathcal{O} \cap \Sigma_h(t)) dt\right\} = 0 \tag{5.52}$$

with the constant C of Theorem 1.1.

It is easy to see that $C = c_3/n$. Using (5.16) with $p = n$ for the function $u(f(m))$ in the domain $\mathcal{O}$, we see that $u(f(m)) \equiv qK$ on $\mathcal{O}$. This contradicts with the definition of the domain $\mathcal{O}$.

Example 5.6. As the first corollary, we shall now prove a generalization of Wiman's theorem for the case of quasiregular mappings $f : \mathcal{M} \to \mathbf{R}^n$ where $\mathcal{M}$ is a warped Riemannian product.

For $0 \le r_1 < r_2 \le \infty$, let

$$D = \left\{ m = (r, \theta) \in \mathbf{R}^n : r_1 < r < r_2, \theta \in S^{n-1}(1) \right\} \tag{5.53}$$

be a ring domain in $\mathbf{R}^n$ and let $\mathcal{M} = (r_1, r_2) \times S^{n-1}(1)$ be an n-dimensional Riemannian manifold on D with the metric

$$ds^2_{\mathcal{M}} = \alpha^2(r) dr^2 + \beta^2(r) dl^2_{n-1}, \tag{5.54}$$

where $\alpha(r), \beta(r) > 0$ are continuously differentiable on $[r_1, r_2)$ and dl_{n-1} is an element of length on $S^{n-1}(1)$.

As we have proved in Example 4.9, under condition (4.35), the function

$$h(r) = \int_{r_1}^{r} \frac{\alpha(t)}{\beta(t)} dt \tag{5.55}$$

is a special exhaustion function on $\mathcal{M}$.

Let $f : \mathcal{M} \to \mathbf{R}^n$ be a quasiregular mapping. We set $u(y) = \log^+|y|$. This function is a subsolution of (3.4) with $p = n$ and also satisfies all the other requirements imposed on a growth function.

We find

$$\lambda_n\left(S^{n-1}(\tau); 1\right) = \frac{1}{\beta(r(\tau))} \lambda_n\left(S^{n-1}(1); 1\right) \tag{5.56}$$

and further

$$\lambda_n(\Sigma_h(\tau); 1) = \frac{\lambda_n(S^{n-1}(1); 1)}{\beta(r(\tau)) h'(r(\tau))}. \tag{5.57}$$

Therefore, the requirement (1.1) on the manifold will be fulfilled if

$$\int^{r_2} \frac{dr}{\beta(r)} = \infty \tag{5.58}$$

holds.

Because

$$\begin{aligned} &\max_{\Sigma_h(\tau)=\tau} \log^+ |f(r,\theta)| \exp\left\{ -C \int^{\tau} \lambda_n(\Sigma_h(t); 1) dt \right\} \\ &\quad \le \max_{r=h^{-1}(\tau)} \log^+ |f(r,\theta)| \exp\left\{ -C \lambda_n\left(S^{n-1}(1); 1\right) \int^{h^{-1}(\tau)} \frac{dr}{\beta(r)} \right\}, \end{aligned} \tag{5.59}$$

we see that, in view of (1.2), it suffices that

$$\liminf_{\tau\to r_2}\max_{\Sigma_h(\tau)}\log^+\left|f(r,\theta)\right|\exp\left\{-C\lambda_n\left(S^{n-1}(1);1\right)\int^{\tau}\frac{dt}{\beta(t)}\right\}=0. \tag{5.60}$$

In this way, we get the following corollary.

Corollary 5.7. *Let $f : \mathcal{M} \to \mathbf{R}^n$ be a nonconstant quasiregular mapping from the warped Riemannian product $\mathcal{M} = (r_1, r_2) \times S^{n-1}(1)$ and h a special exhaustion function of $\mathcal{M}$. If the manifold $\mathcal{M}$ has property* (5.58) *and the mapping f has property* (5.60), *then*

$$\limsup_{\tau\to r_2}\min_{\Sigma_h(\tau)}\left|f(r,\theta)\right|=\infty. \tag{5.61}$$

Example 5.8. Suppose that under the assumptions of Example 5.6, we have (in addition) $r_1 = 0$, $r_2 = \infty$, and the functions $\alpha(r) = \beta(r) \equiv 1$, that is, $\mathcal{M} = (0,\infty) \times S^{n-1}(1)$ with the metric $ds^2_{\mathcal{M}} = dr^2 + dl^2_{n-1}$ is an n-dimensional half-cylinder. As the special exhaustion function of the manifold $\mathcal{M}$, we can take $h \equiv r$. The condition (5.58) is obviously fulfilled for the manifold.

The condition (5.60) for the mapping f attains the form

$$\liminf_{r\to\infty}\max_{\theta\in S^{n-1}(1)}\log^+\left|f(r,\theta)\right|e^{-C\lambda_n(S^{n-1}(1);1)r}=0. \tag{5.62}$$

Corollary 5.9. *If $\mathcal{M} = (0,\infty) \times S^{n-1}(1)$ is a half-cylinder and $f : \mathcal{M} \to \mathbf{R}^n$ is a nonconstant quasiregular mapping satisfying* (5.62), *then*

$$\limsup_{r\to\infty}\min_{\theta\in S^{n-1}(1)}\left|f(r,\theta)\right|=\infty. \tag{5.63}$$

We assume that in Example 5.8 the quantities $r_1 = 0$, $r_2 = \infty$, and the functions $\alpha(r) \equiv 1$, $\beta(r) = r$, that is, the manifold is $\mathbf{R}^n$. As the special exhaustion function, we choose $h = \log|x|$. This function satisfies (3.6) *with $p = n$ and $\nu_1 = \nu_2 = 1$. The condition* (5.58) *for the manifold is obviously fulfilled.*

The condition (5.62) *attains the form*

$$\liminf_{r\to\infty}\max_{|x|=r}\log^+\left|f(x)\right|r^{-C'\lambda_n(S^{n-1}(1);1)}=0, \tag{5.64}$$

where

$$C'=\left(n-1+n\left(K^2(f)-1\right)^{1/2}\right)^{-1}. \tag{5.65}$$

We have the following corollary.

Corollary 5.10. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a nonconstant quasiregular mapping satisfying* (5.64). *Then*

$$\limsup_{r\to\infty}\min_{|x|=r}\left|f(x)\right|=\infty. \tag{5.66}$$

6. Asymptotic Tracts and Their Sizes

Wiman's theorem for the quasiregular mappings $f : \mathbf{R}^n \to \mathbf{R}^n$ asserts the existence of a sequence of spheres $S^{n-1}(r_k)$, $r_k \to \infty$, along which the mapping $f(x)$ tends to ∞. It is possible to further strengthen the theorem and to specify the sizes of the sets along which such a convergence takes place. For the formulation of this result it is convenient to use the language of asymptotic tracts discussed by MacLane [15].

6.1. Tracts

Let D be a domain in the complex plane C and let f be a holomorphic function on D. A collection of domains $\{\mathfrak{D}(s) : s > 0\}$ is called an *asymptotic tract* of f if

(a) each of the sets $\mathfrak{D}(s)$ is a component of the set

$$\{z \in D : |f(z)| > s > 0\}; \tag{6.1}$$

(b) for all $s_2 > s_1 > 0$, we have $\mathfrak{D}(s_2) \subset \mathfrak{D}(s_1)$ and $\bigcap_{s>0} \overline{\mathfrak{D}}(s) = \emptyset$.

Two asymptotic tracts $\{\mathfrak{D}'(s)\}$ and $\{\mathfrak{D}''(s)\}$ are considered to be different if for some $s > 0$ we have $\mathfrak{D}'(s) \cap \mathfrak{D}''(s) = \emptyset$.

Below we shall extend this notion to quasiregular mappings $f : \mathcal{M} \to \mathcal{N}$ of Riemannian manifolds. We study the existence of an asymptotic tract and its size.

Let $\mathcal{M}, \mathcal{N}$ be n-dimensional connected noncompact Riemannian manifolds and let $u = u(y)$ be a growth function on $\mathcal{N}$, which is a positive subsolution of (3.4) with structure constants $p = n$, ν_1, ν_2.

A family $\{\mathcal{M}(s)\}$ is called an asymptotic tract of a quasiregular mapping $f : \mathcal{M} \to \mathcal{N}$ if

(a) each of the sets $\{\mathcal{M}(s)\}$ is a component of the set

$$\{m \in \mathcal{M} : u(f(m)) > s > 0\}; \tag{6.2}$$

(b) for all $s_2 > s_1 > 0$, we have $\mathcal{M}(s_2) \subset \mathcal{M}(s_1)$ and $\bigcap_{s>0} \overline{\mathcal{M}}(s) = \emptyset$.

Let $f : \mathcal{M} \to \mathbf{R}^n$ be a quasiregular mapping having a point $a \in \mathbf{R}^n$ as a Picard exceptional value, that is, $f(m) \neq a$ and $f(m)$ attains on $\mathcal{M}$ all values of $B(a,r) \setminus \{a\}$ for some $r > 0$.

The set $\{\infty\} \cup \{a\}$ has n-capacity zero in $\mathbf{R}^n$ and there is a solution $g(y)$ in $\mathbf{R}^n \setminus \{a\}$ of (3.4) such that $g(y) \to \infty$ as $y \to a$ or $y \to \infty$ (cf. [12, Chapter 10, polar sets]). As the growth function on $\mathbf{R}^n \setminus \{a\}$, we choose the function $u(y) = \max(0, g(y))$. It is clear that this function is a subsolution of (3.4) in $\mathbf{R}^n \setminus \{a\}$.

The function $u(f(m))$ also is a subsolution of an equation of the form (3.4) on $\mathcal{M}$. Because the mapping $f(m)$ attains all values in the punctured ball $B(a,r)$, then among the components of the set

$$\{m \in \mathcal{M} : u(f(m)) > s\} \tag{6.3}$$

there exists at least one $\mathcal{M}(s)$ having a nonempty intersection with $f^{-1}(B(a,r))$. Then by the maximum principle for subsolutions, such a component cannot be relatively compact.

Letting $s \to \infty$, we find an asymptotic tract $\{\mathcal{M}(s)\}$, along which a quasiregular mapping tends to a Picard exceptional value $a \in \mathbf{R}^n$.

Because one can find in every asymptotic tract a curve Γ along which $u(f(m)) \to \infty$, we obtain the following generalization of Iversen's theorem [16].

Theorem 6.1. *Every Picard exceptional value of a quasiregular mapping $f : \mathcal{M} \to \mathbf{R}^n$ is an asymptotic value.*

The classical form of Iversen's theorem asserts that if f is an entire holomorphic function of the plane, then there exists a curve Γ tending to infinity such that

$$f(z) \longrightarrow \infty \quad \text{as } z \longrightarrow \infty \quad \text{on } \Gamma. \tag{6.4}$$

We prove a generalization of this theorem for quasiregular mappings $f : \mathcal{M} \to \mathcal{N}$ of Riemannian manifolds.

The following result holds.

Theorem 6.2. *Let $f : \mathcal{M} \to \mathcal{N}$ be a nonconstant quasiregular mapping between n-dimensional noncompact Riemannian manifolds without boundaries. If there exists a growth function u on $\mathcal{N}$ which is a positive subsolution of* (3.4) *with $p = n$ and on $\mathcal{M}$ a special exhaustion function, then the mapping f has at least one asymptotic tract and, in particular, at least one curve Γ on $\mathcal{M}$ along which $u(f(m)) \to \infty$.*

Proof. Let $h : \mathcal{M} \to (0,\infty)$ be a special exhaustion function of the manifold $\mathcal{M}$. Set

$$\liminf_{\tau \to \infty} \min_{h(m)=\tau} u\big(f(m)\big) = K. \tag{6.5}$$

If $K = \infty$, then $u(f(m))$ tends uniformly on $\mathcal{M}$ to ∞ for $h(m) \to \infty$. The asymptotic tract $\{\mathcal{M}(s)\}$ generates mutual inclusion of the components of the set $\{m \in \mathcal{M} : h(m) > s\}$.

Let $K < \infty$. For an arbitrary $s > K$, we consider the set

$$\mathcal{O}(s) = \big\{m \in \mathcal{M} : u\big(f(m)\big) > s\big\}. \tag{6.6}$$

Because $u(f(m))$ is a subsolution, the nonempty set $\mathcal{O}(s)$ does not have relatively compact components. By a standard argument, we choose for each $s > K$, as $\mathcal{M}(s)$, a component of the set $\mathcal{O}(s)$ having property (b) of the definition of an asymptotic tract. We now easily complete the proof for the theorem. □

6.2. Proof of Theorem 1.2

We fix a growth function u and a special exhaustion function h as in Section 4. Let $f : \mathcal{M} \to \mathcal{N}$ be a nonconstant quasiregular mapping. We set

$$M(\tau) = \max_{h(m)=\tau} u\big(f(m)\big). \tag{6.7}$$

Let K be the quantity defined in (6.5). The case $K = \infty$ is degenerate and has no interest in the present case.

Suppose now that $K < \infty$. For $s > K$, we consider the set $\mathcal{M}(s)$, defined in the proof of the preceding theorem. Define

$$\tau_0 = \tau_0(s) > \inf_{m \in \mathcal{M}(s)} h(m). \tag{6.8}$$

Because $u(f(m))$ is a subsolution of an equation of the form (3.4) on $\mathcal{M}$ by [3, Theorem 5.59], we have for an arbitrary $\tau > \tau_0$

$$\int_{B_h(\tau_0)\cap\mathcal{M}(s)} \left|\nabla u(f(m))\right|^n * \mathbb{1} \le \exp\left\{-\nu_1 \int_{\tau_0}^{\tau} \varepsilon(t)dt\right\} \int_{B_h(\tau)\cap\mathcal{M}(s)} \left|\nabla u(f(m))\right|^n * \mathbb{1}. \tag{6.9}$$

Using the inequality (4.5) of [10] for the quantity $\varepsilon(t)$, we get

$$\begin{aligned}\int_{B_h(\tau_0)\cap\mathcal{M}(s)} &\left|\nabla u(f(m))\right|^n * \mathbb{1} \\ &\le \exp\left\{-\frac{\nu_1}{c}\int_{\tau_0}^{\tau} \lambda_n(\Sigma_h(t)\cap\mathcal{M}(s))dt\right\} \int_{B_h(\tau)\cap\mathcal{M}(s)} \left|\nabla u(f(m))\right|^n * \mathbb{1},\end{aligned} \tag{6.10}$$

where

$$c = \sqrt{\nu_2^{-2} - \nu_1^{-2}} + \frac{n-1}{n}\nu_1. \tag{6.11}$$

By [3, 5.71], we have

$$\begin{aligned}\left(\frac{\nu_1}{\nu_2}\right)^n \int_{B_h(\tau)} \left|\nabla u(f(m))\right|^n * \mathbb{1} &\le n^n \int_{B_h(\tau+1)\setminus B_h(\tau)} |\nabla h|^n \left|u(f(m))\right|^n * \mathbb{1} \\ &\le n^n M^n(\tau+1)V(\tau),\end{aligned} \tag{6.12}$$

where

$$V(\tau) = \int_{B_h(\tau+1)\setminus B_h(\tau)} |\nabla_{\mathcal{M}} h|^n * \mathbb{1}. \tag{6.13}$$

But h is a special exhaustion function and as in the proof of (4.37) we get

$$V(\tau) \le J \equiv \text{const} \tag{6.14}$$

for all sufficiently large τ. Hence

$$\int_{B_h(\tau)} \left|\nabla u(f(m))\right|^n * \mathbb{1} \le J M^n(\tau+1) \tag{6.15}$$

and further

$$\int_{B_h(\tau_0)\cap \mathcal{M}(s)} |\nabla u(f(m))|^n * \mathbb{1} \le JM^n(\tau+1)\exp\left\{-C\int_{\tau_0}^{\tau} \lambda_n(\Sigma_h(t)\cap \mathcal{M}(s))dt\right\}, \tag{6.16}$$

where $C = \nu_1/c$ and c is defined in Lemma 5.2.

Under these circumstances, from the condition (1.5) for the growth of $M(\tau)$, it follows that for all $\varepsilon > 0$ and for all sufficiently large τ, we have

$$\int_{B_h(\tau_0)\cap \mathcal{M}(s)} |\nabla u(f(m))|^n * \mathbb{1} \le J\varepsilon \exp\left\{\int_{\tau_0}^{\tau} \left(n\gamma\lambda_n(\Sigma_h(t);1) - C\lambda_n(\Sigma_h(t)\cap \mathcal{M}(s))\right)dt\right\}. \tag{6.17}$$

If we assume that for all $\tau > \tau_0$

$$\int_{\tau_0}^{\tau} \left(n\gamma\lambda_n(\Sigma_h(t);1) - C\lambda_n(\Sigma_h(t)\cap \mathcal{M}(s))\right)dt \le 0, \tag{6.18}$$

then because $\varepsilon > 0$ was arbitrary, it would follow from (6.17) that $|\nabla u(f(m))| \equiv 0$ on $B_h(\tau_0)\cap \mathcal{M}(s)$ which is impossible.

Hence there exists $\tau = \tau(K) > \tau_0(K)$ for which

$$\lambda_n(\Sigma_h(\tau)\cap \mathcal{M}(s)) < \frac{n\gamma}{C}\lambda_n(\Sigma_h(\tau);1). \tag{6.19}$$

Letting $K \to \infty$, we see that $\tau_0 \to \infty$. Using each time the relation (6.17), we get Theorem 1.2.

In the formulation of the theorem, we used only a part of the information about the sizes of the sets $\mathcal{M}(s)$ which is contained in (6.17). In particular, the relation (6.17) to some extent characterizes also the linear measure of those $t > \tau_0$ for which the intersection of the sets $\mathcal{M}(s)$ with the h-spheres $\Sigma_h(t)$ is not too narrow.

We consider the case of warped Riemannian product $\mathcal{M} = (r_1, r_2)\times S^{n-1}(1)$ with the metric $ds^2_{\mathcal{M}}$ described in Example 5.6. Let h be a special exhaustion function of the manifold $\mathcal{M}$ of the type (4.36) with $p = n$, satisfying condition (4.35).

Here, as in Example 5.6,

$$\lambda_n(\Sigma_h(\tau);1) = \frac{\lambda_n(S^{n-1}(1);1)}{\beta(r(\tau))h'(r(\tau))}, \quad \lambda_n(U) = \frac{\lambda_n(U^*)}{\beta(r(\tau))h'(r(\tau))}, \tag{6.20}$$

where $r(\tau) = h^{-1}(\tau)$ and $U^* \subset S^{n-1}(1)$ is the image of the set U under the similarity mapping

$$x \longmapsto \frac{x}{\beta(r(\tau))} \tag{6.21}$$

of $\mathbf{R}^n$.

Let $f : \mathcal{M} \to \mathbf{R}^n$ be a nonconstant quasiregular mapping. We choose as a growth function u the function $u = \log^+|y|$. This function satisfies (3.6) with $p = n$ and $\nu_1 = \nu_2 = 1$. The condition (1.5) can be written as follows:

$$\liminf_{\tau \to r_2} \max_{r=\tau} \log^+ \left| f(r,\theta) \right| \exp\left\{ -\gamma \lambda_n \left(S^{n-1}(1); 1 \right) \int^r \frac{dt}{\beta(t)} \right\} = 0. \tag{6.22}$$

Hence we obtain

Corollary 6.3. *If a quasiregular mapping $f : \mathcal{M} \to \mathbf{R}^n$ has the property* (6.22) *for some $\gamma > 0$, then for each $k = 1, 2, \ldots$ there are spheres $S^{n-1}(t_k)$, $t_k \in (r_1, r_2)$, $t_k \to r_2$, and open sets $U \subset S^{n-1}(t_k)$ for which*

$$\left| f(m) \right| > k \quad \forall m \in U, \qquad \lambda_n(U) < \frac{n\gamma}{C'} \lambda_n \left(S^{n-1}(1); 1 \right), \tag{6.23}$$

where as above

$$C' = \left(n - 1 + n \left(K^2(f) - 1 \right)^{1/2} \right)^{-1}. \tag{6.24}$$

Corresponding estimates of the quantities $\lambda_n(U^*)$ and $\lambda_n(S^{n-1}(1); 1)$ were given in [7] in terms of the $(n-1)$-dimensional surface area and in terms of the best constant in the embedding theorem of the Sobolev space W_n^1 into the space C on open subsets of the sphere. This last constant can be estimated without difficulties in terms of the maximal radius of balls contained in the given subset.

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