Series concatenation of 2D convolutional codes

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***Abstract*—In this paper we study two-dimensional (2D) convolutional codes which are obtained from series concatenation of two 2D convolutional codes. In this preliminary work we confine ourselves to dealing with finite-support 2D convolutional codes and make use of the so-called Fornasini-Marchesini input-state-output (ISO) model representations. In particular, we show that the series concatenation of two 2D convolutional codes is again a 2D convolutional code and we explicitly compute an ISO representation of the code. Within these ISO representations we study when the structural properties of reachability and observability of the two given ISO representations carry over to the resulting 2D convolutional code.**

I. Introduction

A Serial Concatenated Convolutional Code (SCCC) is based on the application of two convolutional coding techniques twice on the data input, first on the direct data sequence and second on the interleaved one [1]. This technique results in improving the probability of error (decreasing exponentially with code length), while decoding complexity increases only polynomially. Note that turbo codes are instances of codes that comprise a concatenation of two (or more) convolutional encoders. In fact, a thorough study of SCCC started after the discovery of turbo codes in 1993. Despite the fact that they are, in general, more involved that many block codes, these codes have not found widespread commercial use in several communication standards.

Concatenated convolutional codes have traditionally been investigated by means of generator matrices. In [2] the first analysis of two kinds of concatenated convolutional codes using linear systems theory was proposed. In this paper the authors derived conditions for the minimality of an input-state-output representation of the concatenated code. Furthermore, working over an arbitrary field (not necessarily binary), they presented a lower bound on the free distance of the models of concatenation in terms of the free distance of the outer and inner codes.

In this preliminary work we address the problem of concatenation of two-dimensional (2D) convolutional codes. 2D convolutional codes are a natural generalization of standard (1D) convolutional codes. These codes have a practical potential in applications as they are very suitable to encode data recorded in two dimensions, e.g., pictures, storage media, wireless applications, etc. However, in comparison to 1D convolutional codes, little research has been done in the area of 2D convolutional codes [5], [3], [11]. In this first approach we confine ourselves to finite-support 2D convolutional codes and make use of the so-called Fornasini-Marchesini input-state-output (ISO) model representations. First we show that the series concatenation of two 2D convolutional codes results in another 2D convolutional code and we explicitly compute an ISO representation. Then, we investigate under which conditions fundamental properties such as modally observability and modally/locally reachability of ISO representations of two 2D convolutional codes carry over after serial concatenation. In fact, we show that while the interconnection of two modally observable 2D systems is also modally observable, the same does not happen for the properties of reachability.

II. Preliminaries

Denote by $\mathbb{F}[z_1, z_2]$ the ring of polynomials in two indeterminates with coefficients in $\mathbb{F}$, by $\mathbb{F}(z_1 z_2)$ the field of fractions of $\mathbb{F}[z_1, z_2]$ and by $\mathbb{F}[[z_1, z_2]]$ the ring of formal powers series in two indeterminates with coefficients in $\mathbb{F}$.

In this section we start by giving some preliminaries on matrices over $\mathbb{F}[z_1, z_2]$.

Definition II.1. A matrix $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$, with $n \geq k$ is,

1) *unimodular* if $n = k$ and $\det(G(z_1, z_2)) \in \mathbb{F}\backslash\{0\}$;
2) *right factor prime* (rFP) if for every factorization
$$G(z_1, z_2) = \overline{G}(z_1, z_2)T(z_1, z_2),$$
with $\overline{G}(z_1, z_2)$ in $\mathbb{F}[z_1, z_2]^{n\times k}$ and $T(z_1, z_2)$ in $\mathbb{F}[z_1, z_2]^{k\times k}$, $T(z_1, z_2)$ is unimodular;
3) *right zero prime* (rZP) if the ideal generated by the $k \times k$ minors of $G(z_1, z_2)$ is $\mathbb{F}[z_1, z_2]$.

A matrix is left factor prime (ℓFP) / left zero prime (ℓZP) if its transpose is rFP / rZP, respectively. When we consider polynomial matrices in one indeterminate, the notions 2) and 3) of the above definition are equivalent. However this is not the case for polynomial matrices in two indeterminates. In fact, zero primeness implies factor primeness, but the contrary does not happen. The following lemmas give characterizations of right factor primeness and right zero primeness that will be needed later.

Lemma II.2 ([7], [9]). *Let* $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$, *with* $n \geq k$. *Then the following are equivalent:*

1) $G(z_1, z_2)$ *is right factor prime;*
2) *for all* $\hat{u}(z_1, z_2)$ *in* $\mathbb{F}(z_1, z_2)^k$, $G(z_1, z_2)\hat{u}(z_1, z_2)$ *in* $\mathbb{F}[z_1, z_2]^n$ *implies* $\hat{u}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^k$.
3) *the* $k \times k$ *minors of* $G(z_1, z_2)$ *have no common factor.*

Lemma II.3 ([7], [9]). *Let* $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$, *with* $n \geq k$. *Then the following are equivalent:*

1) $G(z_1, z_2)$ *is right zero prime;*
2) $G(z_1, z_2)$ *admits a polynomial left inverse.*

It is well known that given a full column rank polynomial matrix $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$, there exists a square polynomial matrix $V(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{k\times k}$ and a rFP matrix $\bar{G}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$ such that

$$G(z_1, z_2) = \bar{G}(z_1, z_2)V(z_1, z_2).$$

The following lemma will be needed in the sequel. Let $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$, $H(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{(n-k)\times n}$, $n > k$, c_i the ith column of $H(z_1, z_2)$ and r_j the jth row of $G(z_1, z_2)$. We say that the full size minor of $H(z_1, z_2)$ constituted by the columns $c_{i_1}, \ldots, c_{i_{n-k}}$ and the full size minor of $G(z_1, z_2)$ constituted by the rows $r_{j_1}, \ldots, r_{j_k}$ are corresponding maximal order minors of $H(z_1, z_2)$ and $G(z_1, z_2)$, if

$$\{i_1, ..., i_{n-k}\} \cup \{j_1, ..., j_k\} = \{1, \ldots, n\}$$

and $\{i_1, ..., i_{n-k}\} \cap \{j_1, ..., j_k\} = \emptyset$.

Lemma II.4 ([5]). *Let* $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$ *and* $H(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{(n-k)\times n}$ *be a* rFP *and a* ℓFP *matrices, respectively, such that* $H(z_1, z_2)G(z_1, z_2) = 0$. *Then the corresponding maximal order minors of* $H(z_1, z_2)$ *and* $G(z_1, z_2)$ *are equal, modulo a unit of the ring* $\mathbb{F}[z_1, z_2]$.

Next we give some preliminaries on 2D linear systems, which we will use to construct 2D finite support convolutional codes. In particular we consider the Fornasini-Marchesini state space model representation of 2D systems [4]. In this model a first quarter plane 2D linear system, denoted by

$$\Sigma = (A_1, A_2, B_1, B_2, C, D),$$

is given by the updating equations

$$\begin{aligned} x(i+1, j+1) &= A_1 x(i, j+1) + A_2 x(i+1, j) \\ &\quad + B_1 u(i, j+1) + B_2 u(i+1, j) \qquad (1) \\ y(i, j) &= Cx(i, j) + Du(i, j), \end{aligned}$$

where $A_1, A_2 \in \mathbb{F}^{\delta\times\delta}$, $B_1, B_2 \in \mathbb{F}^{\delta\times k}$, $C \in \mathbb{F}^{(n-k)\times\delta}$, $D \in \mathbb{F}^{(n-k)\times k}$, $\delta, n, k \in \mathbb{N}$, $n > k$ and with past finite support of the input and of the state and zero initial conditions (i.e., $u(i, j) = x(i, j) = 0$ for $i < 0$ or $j < 0$ and $x(0, 0) = 0$). We say that $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ has dimension δ, local state $x(i, j)$, input $u(i, j)$ and output $y(i, j)$, at (i, j).

The input, state and output 2D sequences (trajectories), $\{u(i, j)\}_{(i,j)\in\mathbb{N}^2}$, $\{x(i, j)\}_{(i,j)\in\mathbb{N}^2}$, $\{y(i, j)\}_{(i,j)\in\mathbb{N}^2}$, respectively, can be represented as formal power series,

$$\hat{u}(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} u(i, j)z_1^i z_2^j \in \mathbb{F}[[z_1, z_2]]^k$$

$$\hat{x}(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} x(i, j)z_1^i z_2^j \in \mathbb{F}[[z_1, z_2]]^\delta$$

$$\hat{y}(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} y(i, j)z_1^i z_2^j \in \mathbb{F}[[z_1, z_2]]^{n-k}$$

In the sequel we shall use the sequence and the corresponding series interchangeably. Given an input trajectory $\hat{u}(z_1, z_2)$ with corresponding state $\hat{x}(z_1, z_2)$ and output $\hat{y}(z_1, z_2)$ trajectories obtained from (1), the triple $(\hat{x}(z_1, z_2), \hat{u}(z_1, z_2), \hat{y}(z_1, z_2))$ is called an input-state-output trajectory of $\Sigma = (A_1, A_2, B_1, B_2, C, D)$. The set of input-state-output trajectories of Σ is given by

$$\ker_{\mathbb{F}[[z_1,z_2]]} X(z_1, z_2) =$$

$$= \left\{\hat{r}(z_1, z_2) \in \mathbb{F}[[z_1, z_2]]^{n+\delta} \mid X(z_1, z_2)\hat{r}(z_1, z_2) = 0\right\}$$

where

$$X(z_1, z_2) = \begin{bmatrix} I_\delta - A_1 z_1 - A_2 z_2 & -B_1 z_1 - B_2 z_2 & 0 \\ -C & -D & I_{n-k} \end{bmatrix}. \quad (2)$$

Next we present the (local and modal) reachability and observability properties of such systems.

Definition II.5 ([4]). Let $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ be a system with dimension δ.

1) A state $\bar{x} \in \mathbb{F}^\delta$ is *reachable* if there exists an input-state-output trajectory $((x(i, j), u(i, j), y(i, j)))_{(i,j)\in\mathbb{N}^2}$ of Σ, and a pair $(i_1, j_1) \in \mathbb{N}^2$ such that $\bar{x} = x(i_1, j_1)$. The system Σ is *locally reachable* if the set of all reachable states is $\mathbb{F}^\delta$, i.e., if and only if the reachability matrix

$$\mathcal{R} = \big[B_1\ B_2\ \cdots \left(A_1{}^i \sqcup^{j-1} A_2\right) B_2 + \\ + \left(A_1{}^{i-1} \sqcup^j A_2\right) B_1 \cdots \left(A_1{}^0 \sqcup^n A_2\right) B_2\big],$$

where

$$A_1{}^r \sqcup^0 A_2 = A_1^r, \quad A_1{}^0 \sqcup^s A_2 = A_2^s,$$

$$A_1{}^r \sqcup^s A_2 = A_1 \left(A_1^{r-1} \sqcup^s A_2\right) + A_2 \left(A_1^r \sqcup^{s-1} A_2\right),$$

for $r, s \geq 1$, is full row rank.

2) Σ is *modally reachable* if the matrix

$$\begin{bmatrix} I_\delta - A_1 z_1 - A_2 z_2 & B_1 z_1 + B_2 z_2 \end{bmatrix}$$

is ℓFP.

3) Σ is *modally observable* if the matrix

$$\begin{bmatrix} I_\delta - A_1 z_1 - A_2 z_2 \\ C \end{bmatrix}$$

is rFP.

There exists also the notion of locally observable 2D systems that we will not consider in this paper. For one-dimensional (1D) systems, the notions 1) and 2) (and the corresponding observability notions) presented in the above definitions are equivalent. Such equivalence is stated in the PBH test [6]. However, this does not happen in the 2D case. There are systems which are locally reachable (observable) but not modally reachable (observable) and vice-versa (see [4]).

III. Input-state-output representations of 2D finite support convolutional codes

Definition III.1 ([10], [11]). A $2D$ *(finite support) convolutional* code $\mathcal{C}$ of rate k/n is a free $\mathbb{F}[z_1,z_2]$-submodule of $\mathbb{F}[z_1,z_2]^n$, where k is the rank of $\mathcal{C}$. A full column rank matrix $G(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{n\times k}$ whose columns constitute a basis for $\mathcal{C}$, i.e., such that

$$\begin{aligned}\mathcal{C} &= \mathrm{Im}_{\mathbb{F}[z_1,z_2]} G(z_1,z_2)\\ &= \big\{\hat{v}(z_1,z_2)\in\mathbb{F}[z_1,z_2]^n \mid \hat{v}(z_1,z_2)=G(z_1,z_2)\hat{u}(z_1,z_2),\\ &\qquad \text{for some } \hat{u}(z_1,z_2)\in\mathbb{F}[z_1,z_2]^k\big\},\end{aligned}$$

is called *an encoder* of $\mathcal{C}$. The elements of $\mathcal{C}$ are called *codewords*.

Two full column rank matrices $G(z_1,z_2)$ and $\bar{G}(z_1,z_2)$ in $\mathbb{F}[z_1,z_2]^{n\times k}$ are *equivalent encoders* if they generate the same 2D convolutional code, i.e., if

$$\mathrm{Im}_{\mathbb{F}[z_1,z_2]} G(z_1,z_2) = \mathrm{Im}_{\mathbb{F}[z_1,z_2]} \bar{G}(z_1,z_2),$$

which happens if and only if there exists a unimodular matrix $U(z_1,z_2)\in\mathbb{F}[z_1,z_2]^{k\times k}$ such that (see [10], [11])

$$G(z_1,z_2)U(z_1,z_2) = \bar{G}(z_1,z_2).$$

Note that the fact that two equivalent encoders differ by unimodular matrices also implies that the primeness properties of the encoders of a code are preserved, i.e., if $\mathcal{C}$ admits a rFP (rZP) encoder then all its encoders are rFP (rZP). A 2D finite support convolutional code that admits rFP encoders is called *noncatastrophic* and is named *basic* if all its encoders are rZP.

Let us now consider a first quarter plane 2D system Σ defined in (1). For $(i,j)\in\mathbb{N}^2$, define

$$v(i,j) = \begin{bmatrix} y(i,j)\\ u(i,j)\end{bmatrix} \in \mathbb{F}^n$$

to be the *code vector*. We will only consider the finite support input-output trajectories, $(v(i,j))_{(i,j)\in\mathbb{N}^2}$ of (1). However, if the system (1) produces a finite support input-output trajectory with corresponding state trajectory $\hat{x}(z_1,z_2)$ having infinite support, this would make the system remain indefinitely excited, which is a situation that we want to avoid. Thus, we will restrict ourselves to finite support input-output trajectories $(\hat{u}(z_1,z_2),\hat{y}(z_1,z_2))$ with corresponding state $\hat{x}(z_1,z_2)$ also having finite support. We call such trajectories $(\hat{u}(z_1,z_2),\hat{y}(z_1,z_2))$ *finite-weight input-output* trajectories and the triple $(\hat{x}(z_1,z_2),\hat{u}(z_1,z_2),\hat{y}(z_1,z_2))$ *finite-weight* trajectories. Note that not all finite support input-output trajectories have finite weight. The following result asserts that the set of finite-weight trajectories of (1) forms a 2D finite support convolutional code.

Theorem III.2 ([8]). *The set of finite-weight input-output trajectories of (1) is a* $2D$ *finite support convolutional code of rate* k/n.

It is worth mentioning that this approach is different from the one adopted in [5] where the codewords are constituted by the output trajectories $\hat{y}(z_1,z_2)$ of a system.

We denote by $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$ the 2D finite support convolutional code whose codewords are the finite-weight input-output trajectories of the system $\Sigma = (A_1,A_2,B_1,B_2,C,D)$. Moreover, Σ is called an *input-state-output (ISO) representation* (see [8]) of $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$.

Next we will show how the properties of reachability and observability of ISO representations, stated in Definition II.5, reflect on the structure of the corresponding code.

Theorem III.3 ([8]). *Let* $\Sigma = (A_1,A_2,B_1,B_2,C,D)$ *be a* $2D$ *system. If* Σ *is modally observable then* $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$ *is noncatastrophic and its codewords are the finite support input-output trajectories of* Σ.

In case the ISO representation is modally reachable a necessary and sufficient condition can be stated for the noncatastrophicity of the corresponding code. To show that we need first to introduce the following technical lemma.

Lemma III.4. *Let* $\Sigma = (A_1,A_2,B_1,B_2,C,D)$ *be a* $2D$ *system and* $X(z_1,z_2)$ *the corresponding matrix defined in (2). Then* Σ *is modally reachable if and only if the matrix* $X(z_1,z_2)$ *is* ℓFP.

Proof. Let $\hat{u}_1(z_1,z_2)$ in $\mathbb{F}(z_1,z_2)^\delta$ and $\hat{u}_2(z_1,z_2)$ in $\mathbb{F}(z_1,z_2)^{n-k}$ such that

$$\begin{bmatrix}\hat{u}_1(z_1,z_2)^T & \hat{u}_2(z_1,z_2)^T\end{bmatrix} X(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{1\times(\delta+n)},$$

which is equivalent to

$$\begin{aligned}&\hat{u}_1(z_1,z_2)^T\begin{bmatrix} I_\delta - A_1z_1 - A_2z_2 & -B_1z_1-B_2z_2 & 0\end{bmatrix}\\ &+\hat{u}_2(z_1,z_2)^T\begin{bmatrix}-C & -D & I_{n-k}\end{bmatrix}\in\mathbb{F}[z_1,z_2]^{1\times(\delta+n)}.\end{aligned}$$

Therefore $\hat{u}_2(z_1,z_2)$ must be polynomial and the result follows from 2) of Lemma II.2. □

Theorem III.5. *Let* $\Sigma = (A_1,A_2,B_1,B_2,C,D)$ *be a modally reachable* $2D$ *system with* k *inputs,* $n-k$ *outputs and dimension* δ. Σ *is modally observable if and only if* $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$ *is noncatastrophic.*

Proof. From Theorem III.3, we just need to prove that if $\mathcal{C}(A_1,A_2,B_1,B_2,C,D)$ is noncatastrophic then Σ is modally observable. Let us assume that Σ is not modally observable. Then, from Lemma II.2, there exists a nonconstant $d(z_1,z_2)\in\mathbb{F}[z_1,z_2]$ which is a common factor of all $\delta\times\delta$ minors of $\begin{bmatrix} I_\delta - A_1z_1 - A_2z_2\\ -C\end{bmatrix}$.

Let $L(z_1, z_2)$ in $\mathbb{F}[z_1, z_2]^{\delta \times k}$ and $G(z_1, z_2)$ in $F[z_1, z_2]^{n \times k}$ such that

$$X(z_1, z_2) \begin{bmatrix} L(z_1, z_2) \\ G(z_1, z_2) \end{bmatrix} = 0,$$

with $X(z_1, z_2)$ defined in (2) and where $\begin{bmatrix} L(z_1, z_2) \\ G(z_1, z_2) \end{bmatrix}$ is rFP and $G(z_1, z_2)$ is an encoder of $\mathcal{C}(A_1, A_2, B_1, B_2, C, D)$.

Note that from Lemma III.4, $X(z_1, z_2)$ is ℓFP and all $(\delta + n - k) \times (\delta + n - k)$ minors of $X(z_1, z_2)$ whose corresponding submatrices include $\begin{bmatrix} I_\delta - A_1 z_1 - A_2 z_2 \\ -C \end{bmatrix}$ have also $d(z_1, z_2)$ as common factor. Therefore, by Lemma II.4, all $k \times k$ minors of $G(z_1, z_2)$ have $d(z_1, z_2)$ as common factor which implies that $G(z_1, z_2)$ is not rFP and consequently $\mathcal{C}(A_1, A_2, B_1, B_2, C, D)$ is not catastrophic. □

If S is an invertible constant matrix, it is said that the systems $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ and

$$\tilde{\Sigma} = \left(SA_1S^{-1}, SA_2S^{-1}, SB_1, SB_2, CS^{-1}, D\right)$$

are algebraically equivalent [4], and it is easy to see that they represent the same code, as stated in the following lemma.

Lemma III.6 ([8]). *Let $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ be a 2D system with dimension δ and S a $\delta \times \delta$ invertible constant matrix. Then*

$$\mathcal{C}(A_1, A_2, B_1, B_2, C, D) = \\ = \mathcal{C}\left(SA_1S^{-1}, SA_2S^{-1}, SB_1, SB_2, CS^{-1}, D\right).$$

An ISO representation of a 2D convolutional code is said to be minimal if it has minimal dimension among all the ISO representations of the code. Locally reachability is a necessary condition for minimality, as we will see next.

Definition III.7 ([4]). A system

$$\Sigma = \left(\tilde{A}_1, \tilde{A}_2, \tilde{B}_1, \tilde{B}_2, \tilde{C}, \tilde{D}\right)$$

of dimension δ is in the *Kalman reachability form* if

$$\tilde{A}_1 = \begin{bmatrix} \tilde{A}_{11}^{(1)} & \tilde{A}_{12}^{(1)} \\ 0 & \tilde{A}_{22}^{(1)} \end{bmatrix}, \quad \tilde{A}_2 = \begin{bmatrix} \tilde{A}_{11}^{(2)} & \tilde{A}_{12}^{(2)} \\ 0 & \tilde{A}_{22}^{(2)} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \tilde{B}_1^{(1)} \\ 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} \tilde{B}_1^{(2)} \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix},$$

where $\tilde{A}_{11}^{(1)}$ and $\tilde{A}_{11}^{(2)}$ are $\delta \times \delta$ matrices, $\tilde{B}_1^{(1)}$ and $\tilde{B}_1^{(2)}$ are $\delta \times k$ matrices, $\tilde{C}_1$ is a $(n-k) \times \delta$ matrix, with the remaining matrices of suitable dimensions, and

$$\Sigma_1 = \left(\tilde{A}_{11}^{(1)}, \tilde{A}_{11}^{(2)}, \tilde{B}_1^{(1)}, \tilde{B}_1^{(2)}, \tilde{C}_1, \tilde{D}\right)$$

is a locally reachable system, which is the largest reachable subsystem of Σ.

Lemma III.8 ([8]). *Let $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ be a system with the corresponding 2D finite support convolutional code $\mathcal{C}(A_1, A_2, B_1, B_2, C, D)$. Let S be an invertible constant matrix such that $\tilde{\Sigma} = \left(SA_1S^{-1}, SA_2S^{-1}, SB_1, SB_2, CS^{-1}, D\right)$ is in the Kalman reachability form and let*

$$\tilde{\Sigma}_1 = \left(\tilde{A}_{11}^{(1)}, \tilde{A}_{11}^{(2)}, \tilde{B}_1^{(1)}, \tilde{B}_1^{(2)}, \tilde{C}_1, D\right)$$

be the largest reachable subsystem of $\tilde{\Sigma}$. Then

$$\mathcal{C}(A_1, A_2, B_1, B_2, C, D) = \mathcal{C}\left(\tilde{A}_{11}^{(1)}, \tilde{A}_{11}^{(2)}, \tilde{B}_1^{(1)}, \tilde{B}_1^{(2)}, \tilde{C}_1, D\right).$$

IV. Series concatenation of 2D convolutional codes

In this section we will study 2D convolutional codes that result from series interconnection of two systems representations of other 2D convolutional codes. We will consider the interconnection model defined in [2] for 1D systems.

Let

$$\Sigma_1 = \left(A_1^{(1)}, A_2^{(1)}, B_1^{(1)}, B_2^{(1)}, C^{(1)}, D^{(1)}\right)$$

and

$$\Sigma_2 = \left(A_1^{(2)}, A_2^{(2)}, B_1^{(2)}, B_2^{(2)}, C^{(2)}, D^{(2)}\right)$$

be two ISO representations of the 2D convolutional codes $\mathcal{C}_1$ and $\mathcal{C}_2$, of rate k/m and m/n, respectively. Represent by $u^{(1)}$ and $u^{(2)}$ the input vectors of Σ_1 and Σ_2, by $x^{(1)}$ and $x^{(2)}$ the state vectors of Σ_1 and Σ_2 and by $y^{(1)}$ and $y^{(2)}$ the output vectors of Σ_1 and Σ_2, respectively. Let us consider the series interconnection of Σ_1 and Σ_2 by feeding the code vectors $v^{(1)} = \begin{bmatrix} y^{(1)} \\ u^{(1)} \end{bmatrix}$ of Σ_1 as inputs of Σ_2 as represented in Figure 1.

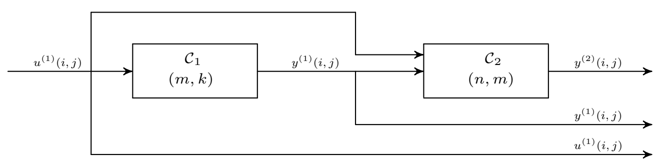

Fig. 1. Series concatenation of $\mathcal{C}_1$ and $\mathcal{C}_2$

Theorem IV.1. *Let*

$$\mathcal{C}_1 = \mathcal{C}\left(A_1^{(1)}, A_2^{(1)}, B_1^{(1)}, B_2^{(1)}, C^{(1)}, D^{(1)}\right)$$

and

$$\mathcal{C}_2 = \mathcal{C}\left(A_1^{(2)}, A_2^{(2)}, B_1^{(2)}, B_2^{(2)}, C^{(2)}, D^{(2)}\right)$$

be two 2D convolutional codes of rate k/m and m/n, respectively. Then the series interconnection of $\Sigma_1 = \left(A_1^{(1)}, A_2^{(1)}, B_1^{(1)}, B_2^{(1)}, C^{(1)}, D^{(1)}\right)$ and $\Sigma_2 = \left(A_1^{(2)}, A_2^{(2)}, B_1^{(2)}, B_2^{(2)}, C^{(2)}, D^{(2)}\right)$ by considering the inputs of Σ_2 to be the code vectors of Σ_1 and with code vector $v = v^{(2)}$, where $v^{(2)}$ is the code vector of Σ_2, is a 2D convolutional code with ISO representation

$$\Sigma = (A_1, A_2, B_1, B_2, C, D),$$

given by

$$A_1 = \begin{bmatrix} A_1^{(2)} & B_{11}^{(2)} C^{(1)} \\ 0 & A_1^{(1)} \end{bmatrix}, \ A_2 = \begin{bmatrix} A_2^{(2)} & B_{21}^{(2)} C^{(1)} \\ 0 & A_2^{(1)} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} B_{12}^{(2)} + B_{11}^{(2)} D^{(1)} \\ B_1^{(1)} \end{bmatrix}, \ B_2 = \begin{bmatrix} B_{22}^{(2)} + B_{21}^{(2)} D^{(1)} \\ B_2^{(1)} \end{bmatrix},$$

$$C = \begin{bmatrix} C^{(2)} & D_1^{(2)} C^{(1)} \\ 0 & C^{(1)} \end{bmatrix}, \ D = \begin{bmatrix} D_1^{(2)} D^{(1)} + D_2^{(2)} \\ D^{(1)} \end{bmatrix},$$

where $B_1^{(2)} = \begin{bmatrix} B_{11}^{(2)} & B_{12}^{(2)} \end{bmatrix}$, $B_2^{(2)} = \begin{bmatrix} B_{21}^{(2)} & B_{22}^{(2)} \end{bmatrix}$ *and* $D_2^{(2)} = \begin{bmatrix} D_1^{(2)} & D_2^{(2)} \end{bmatrix}$, *with* $B_{11}^{(2)} \in \mathbb{F}^{\delta_2 \times (m-k)}$, $B_{12}^{(2)} \in \mathbb{F}^{\delta_2 \times k}$, $D_1^{(2)} \in \mathbb{F}^{(n-m)\times(m-k)}$ *and* $D_2^{(2)} \in \mathbb{F}^{(n-m)\times k}$, *with* δ_2 *being the dimension of* Σ_2.

The next theorem gives conditions to ensure the (modal) observability of the code obtained by concatenation.

Theorem IV.2. *Let* $\Sigma_1 = \left(A_1^{(1)}, A_2^{(1)}, B_1^{(1)}, B_2^{(1)}, C^{(1)}, D^{(1)}\right)$ *and* $\Sigma_2 = \left(A_1^{(2)}, A_2^{(2)}, B_1^{(2)}, B_2^{(2)}, C^{(2)}, D^{(2)}\right)$ *be two 2D systems of dimension* δ_1 *and* δ_2, *respectively. If* Σ_1 *and* Σ_2 *are modally observable, then the series interconnection,* Σ, *of* Σ_1 *and* Σ_2 *(defined as in Theorem IV.1) is modally observable.*

Proof. Assume that Σ_1 and Σ_2 are modally observable. Attending to Theorem IV.1 and Lemma II.2, we have to prove that the matrix

$$Y(z_1, z_2) = \begin{bmatrix} I_{\delta_2} - A_1^{(2)} z_1 - A_2^{(2)} z_2 & B_{11}^{(2)} C^{(1)} z_1 - B_{21}^{(2)} C^{(1)} z_2 \\ 0 & I_{\delta_1} - A_1^{(1)} z_1 - A_2^{(1)} z_2 \\ -C^{(2)} & -D_1^{(2)} C^{(1)} \\ 0 & -C^{(1)} \end{bmatrix} \tag{3}$$

is rFP. Let $\hat{u}(z_1, z_2) \in \mathbb{F}(z_1, z_2)^{\delta_1 + \delta_2}$ be such that

$$Y(z_1, z_2)\hat{u}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{\delta_1 + \delta_2 + n - k}.$$

Suppose that $\hat{u}(z_1, z_2) = \begin{bmatrix} \hat{u}_2(z_1, z_2)^T & \hat{u}_1(z_1, z_2)^T \end{bmatrix}^T$ with $\hat{u}_2(z_1, z_2) \in \mathbb{F}(z_1, z_2)^{\delta_2}$. Then

$$\begin{bmatrix} I_{\delta_1} - A_1^{(1)} z_1 - A_2^{(1)} z_2 \\ -C^{(1)} \end{bmatrix} \hat{u}_1(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{\delta_1 + m - k}$$

and, since Σ_1 is modally observable, $\hat{u}_1(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{\delta_1}$.

On the other hand,

$$\begin{bmatrix} I_{\delta_2} - A_1^{(2)} z_1 - A_2^{(2)} z_2 \\ -C^{(2)} \end{bmatrix} \hat{u}_2(z_1, z_2) + \begin{bmatrix} B_{11}^{(2)} C^{(1)} z_1 - B_{21}^{(2)} C^{(1)} z_2 \\ -D_1^{(2)} C^{(1)} \end{bmatrix} \hat{u}_1(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{\delta_2 + n - m}$$

which implies that

$$\begin{bmatrix} I_{\delta_2} - A_1^{(2)} z_1 - A_2^{(2)} z_2 \\ -C^{(2)} \end{bmatrix} \hat{u}_2(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{\delta_2 + n - m}$$

and, since Σ_2 is modally observable, $\hat{u}_2(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{\delta_2}$, and therefore $\hat{u}(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{\delta_1 + \delta_2}$. Thus Σ is modally observable. ☐

The next example shows that it is not sufficient that the systems Σ_1 and Σ_2 are modally / locally reachable to get the system obtained by series interconnection modally / locally reachable.

Example IV.3. Let α be a primitive element of the Galois field $\mathbb{F} = GF(8)$ with $\alpha^3 + \alpha + 1 = 0$, and consider the 2D systems $\Sigma_1 = \left(A_1^{(1)}, A_2^{(1)}, B_1^{(1)}, B_2^{(1)}, C^{(1)}, D^{(1)}\right)$ and $\Sigma_2 = \left(A_1^{(2)}, A_2^{(2)}, B_1^{(2)}, B_2^{(2)}, C^{(2)}, D^{(2)}\right)$, where

$$A_1^{(1)} = A_2^{(1)} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^2 \end{bmatrix}, \qquad B_1^{(1)} = B_2^{(1)} = \begin{bmatrix} 1 & 0 \\ 0 & \alpha^6 \end{bmatrix},$$

$$C^{(1)} = \begin{bmatrix} \alpha^4 & \alpha^3 \end{bmatrix}, \qquad D^{(1)} = \begin{bmatrix} 1 & \alpha^4 \end{bmatrix},$$

$$A_1^{(2)} = A_2^{(2)} = \begin{bmatrix} \alpha^4 & 1 \\ \alpha^3 & 0 \end{bmatrix}, \qquad B_1^{(2)} = B_2^{(2)} = \begin{bmatrix} 1 & 0 & 1 \\ \alpha & 1 & \alpha \end{bmatrix},$$

and $C^{(2)}$ and $D^{(2)}$ are arbitrary matrices, and $\Sigma = (A_1, A_2, B_1, B_2, C, D)$ be the 2D system obtained by series interconnection of Σ_1 and Σ_2 as defined in Theorem IV.1, with

$$A_1 = A_2 = \begin{bmatrix} \alpha^4 & 1 & \alpha^4 & \alpha^3 \\ \alpha^3 & 0 & \alpha^5 & \alpha^4 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha^2 \end{bmatrix} \text{ and } B_1 = B_2 = \begin{bmatrix} 1 & \alpha^5 \\ \alpha^3 & \alpha^6 \\ 1 & 0 \\ 0 & \alpha^6 \end{bmatrix}.$$

It is easy to see that the matrices

$$M(z_1, z_2) = \begin{bmatrix} I_2 - A_1^{(1)} z_1 - A_2^{(1)} z_2 & B_1^{(1)} z_1 + B_2^{(1)} z_2 \end{bmatrix}$$

and

$$N(z_1, z_2) = \begin{bmatrix} I_2 - A_1^{(2)} z_1 - A_2^{(2)} z_2 & B_1^{(2)} z_1 + B_2^{(2)} z_2 \end{bmatrix}$$

are ℓFP. In fact,

$$M(z_1, z_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & 0 \\ 0 & \alpha^3 \end{bmatrix} = I_2, \qquad N(z_1, z_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha^4 & 1 \\ \alpha^2 & \alpha \\ 0 & 0 \end{bmatrix} = I_2,$$

which means that $M(z_1, z_2)$ and $N(z_1, z_2)$ are rZP, and therefore, they are also ℓFP.

But the matrix

$$\begin{bmatrix} I_4 - A_1 z_1 - A_2 z_2 & B_1 z_1 + B_2 z_2 \end{bmatrix}$$

is not ℓFP. In fact, there exists

$$\hat{u}(z_1, z_2) \in \mathbb{F}(z_1, z_2)^{1\times 4} \setminus \mathbb{F}[z_1, z_2]^{1\times 4}$$

such that

$$\hat{u}(z_1,z_2)\begin{bmatrix} I_4 - A_1z_1 - A_2z_2 & B_1z_1 + B_2z_1 \end{bmatrix} \in \mathbb{F}[z_1,z_2]^{1\times 6}.$$

Just consider

$$\hat{u}(z_1,z_2) = \frac{1}{1+\alpha(z_1+z_2)}\begin{bmatrix} 1 & z_1+z_2 & \alpha(z_1+z_2)^2 & 0 \end{bmatrix},$$

which is not polynomial, and

$$\begin{aligned} &\hat{u}(z_1,z_2)\begin{bmatrix} I_4 - A_1z_1 - A_2z_2 & B_1z_1 + B_2z_2 \end{bmatrix} \\ &= \big[1+\alpha^2(z_1+z_2) \quad 0 \quad \alpha^4(z_1+z_2)(1+\alpha^4(z_1+z_2)) \\ &\quad \alpha^3(z_1+z_2) \quad (z_1+z_2)(1+z_1+z_2) \quad \alpha^5(z_1+z_2)\big], \end{aligned}$$

which is polynomial. Then

$$\begin{bmatrix} I_4 - A_1z_1 - A_2z_2 & B_1z_1 + B_2z_2 \end{bmatrix}$$

is not ℓFP, which means that Σ is not modally reachable.

Moreover, it is easy to see that the systems Σ_1 and Σ_2 are locally reachable; their reachability matrices are, respectively, (taking out the repeated columns):

$$\mathcal{R}_1 = \begin{bmatrix} 1 & 0 & \alpha & 0 & \alpha^2 & 0 & \alpha^3 & 0 \\ 0 & \alpha^6 & 0 & \alpha & 0 & \alpha^3 & 0 & \alpha^5 \end{bmatrix}$$

$$\mathcal{R}_2 = \begin{bmatrix} 1 & 0 & 1 & \alpha^2 & 1 & \alpha^2 & \alpha^4 & \alpha^4 & \alpha^4 & \alpha^6 & 1 & \alpha^6 \\ \alpha & 1 & \alpha & \alpha^3 & 0 & \alpha^3 & \alpha^5 & \alpha^3 & \alpha^5 & 1 & 1 & 1 \end{bmatrix}$$

which have full row rank. But the reachability matrix of the system obtained by series interconnection is (taking out the repeated columns)

$$\mathcal{R} = \begin{bmatrix} 1 & \alpha^5 & \alpha^3 & \alpha^6 & \alpha & \alpha^2 & \alpha & \alpha^3 \\ \alpha^3 & \alpha^6 & \alpha^2 & 1 & 0 & \alpha^3 & \alpha^5 & \alpha^4 \\ 1 & 0 & \alpha & 0 & \alpha^2 & 0 & \alpha^3 & 0 \\ 0 & \alpha^6 & 0 & \alpha & 0 & \alpha^3 & 0 & \alpha^5 \end{bmatrix}$$

which has rank 3. Therefore is not locally reachable.

V. Conclusion

In this paper we have considered 2D convolutional codes that result from series interconnection of two 2D systems. We have showed that the interconnection of two modally observable 2D systems is also modally observable. However, the same does not happen when we consider the reachability properties. These structural properties reflect on the structure of the corresponding code and on the minimality properties of the representations. Thus, more investigation must be done to determine conditions for ISO representations that interconnected in series will determine a locally and modally reachable systems.

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