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NOETHER'S THEOREM FOR HIGHER-ORDER VARIATIONAL PROBLEMS OF HERGLOTZ TYPE

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Abstract. We approach higher-order variational problems of Herglotz type from an optimal control point of view. Using optimal control theory, we derive a generalized Euler–Lagrange equation, transversality conditions, DuBois–Reymond necessary optimality condition and Noether's theorem for Herglotz's type higher-order variational problems, valid for piecewise smooth functions.

1. **Introduction.** The generalized variational problem proposed by Herglotz in 1930 [2] can be formulated as follows:

$$\begin{gathered} z(b) \longrightarrow \text{extr} \\ \text{with } \dot{z}(t) = L(t, x(t), \dot{x}(t), z(t)), \quad t \in [a,b], \\ \text{subject to } z(a) = \gamma, \quad \gamma \in \mathbb{R}. \end{gathered} \tag{H_1}$$

The Herglotz variational problem consists in the determination of trajectories $x(\cdot)$ subject to some initial condition $x(a) = \alpha$, $\alpha \in \mathbb{R}^m$, that extremize (maximize or minimize) the value $z(b)$, where $L \in C^1([a,b] \times \mathbb{R}^{2m+1}; \mathbb{R})$. While in [2] the admissible functions are $x(\cdot) \in C^2([a,b]; \mathbb{R}^m)$ and $z(\cdot) \in C^1([a,b]; \mathbb{R})$, here we consider ($H_1$) in the wider class of functions $x(\cdot) \in PC^1([a,b]; \mathbb{R}^m)$ and $z(\cdot) \in PC^1([a,b]; \mathbb{R})$, where the notation PC stands for "piecewise continuous" (for the precise meaning of piecewise continuity and piecewise differentiability see, e.g., [3, Sec. 1.1]).

It is clear to see that Herglotz's problem (H_1) reduces to the classical fundamental problem of the calculus of variations (see, e.g., [12]) if the Lagrangian L does not depend on the z variable: if $\dot{z}(t) = L(t, x(t), \dot{x}(t))$, $t \in [a,b]$, then (H_1) is equivalent to the classical variational problem

$$\int_a^b L(t, x(t), \dot{x}(t)) dt \longrightarrow \text{extr}. \tag{1}$$

Herglotz proved that a necessary optimality condition for a pair $(x(\cdot), z(\cdot))$ to be an extremizer of the generalized variational problem (H_1) is given by

$$\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t), z(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t), z(t)) + \frac{\partial L}{\partial z}(t, x(t), \dot{x}(t), z(t)) \frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t), z(t)) = 0, \tag{2}$$

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$t \in [a, b]$. The equation (2) is known as the generalized Euler–Lagrange equation. Observe that for the classical problem of the calculus of variations (1) one has $\frac{\partial L}{\partial z} = 0$ and equation (2) reduces to the classical Euler–Lagrange equation

$$\frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t)) = 0.$$

In [6] we have introduced higher-order variational problems of Herglotz type and obtained a generalized Euler–Lagrange equation and transversality conditions for these problems. In particular, we considered the problem of determining the trajectories $x(\cdot)$ such that

$$\begin{gathered} z(b) \longrightarrow \text{extr} \\ \text{with } \dot{z}(t) = L(t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t)), \quad t \in [a, b], \\ \text{subject to } z(a) = \gamma, \quad \gamma \in \mathbb{R}. \end{gathered} \tag{H_n}$$

We proved that if a pair $(x(\cdot), z(\cdot))$ is an extremizer of the higher-order problem (H_n), then it satisfies the higher-order generalized Euler–Lagrange equation

$$\sum_{j=0}^{n}(-1)^j \frac{d^j}{dt^j}\left(\psi_z(t)\frac{\partial L}{\partial x^{(j)}}\langle x, z\rangle_n(t)\right) = 0, \quad t \in [a, b],$$

and the transversality conditions $\psi_j(b) = \psi_j(a) = 0$, for $j = 1, \ldots, n$, where

$$\begin{cases} \psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z}\langle x, z\rangle_n(\theta)d\theta} \\ \psi_j(t) = \sum_{i=0}^{n-j}(-1)^{i+1}\frac{d^i}{dt^i}\left(\psi_z(t)\frac{\partial L}{\partial x^{(i+j)}}\langle x, z\rangle_n(t)\right), \quad j = 1, \ldots, n. \end{cases}$$

While in [6] the admissible functions are $x(\cdot) \in C^{2n}([a, b]; \mathbb{R}^m)$ and $z(\cdot) \in C^1([a, b]; \mathbb{R})$, here we consider ($H_n$) in the wider class of functions $x(\cdot) \in PC^n([a, b]; \mathbb{R}^m)$ and $z(\cdot) \in PC^1([a, b]; \mathbb{R})$.

One of the most important results in optimal control theory is Pontryagin's maximum principle proved by Pontryagin et al. in [5]. This principle provides conditions for optimization problems with differential equations as constraints. The maximum principle is still widely used for solving problems of control and other problems of dynamic optimization. Moreover, basic necessary optimality conditions from classical calculus of variations follow from Pontryagin's maximum principle.

One of the problems of optimal control, in Bolza form, is the following one:

$$\begin{gathered} \mathcal{J}(x(\cdot), u(\cdot)) = \int_a^b f(t, x(t), u(t))dt + \phi(x(b)) \longrightarrow \text{extr} \\ \text{subject to } \dot{x}(t) = g(t, x(t), u(t)), \end{gathered} \tag{P}$$

with some initial condition on x, where $f \in C^1([a, b] \times \mathbb{R}^m \times \Omega; \mathbb{R})$, $\phi \in C^1(\mathbb{R}^m; \mathbb{R})$, $g \in C^1([a, b] \times \mathbb{R}^m \times \Omega; \mathbb{R}^m)$, $x \in PC^1([a, b]; \mathbb{R}^m)$ and $u \in PC([a, b]; \Omega)$, with $\Omega \subseteq \mathbb{R}^r$ an open set. In the literature of optimal control, x and u are frequently called the state and control variables, respectively, while ϕ is known as the payoff or salvage term. Note that the classical problem of the calculus of variations (1) is a particular case of problem (P) with $\phi(x) \equiv 0$, $g(t, x, u) = u$ and $\Omega = \mathbb{R}^m$. In this work we show how the results on the higher-order variational problem of Herglotz (H_n) obtained in [6] can be generalized by using the theory of optimal control. The technique used consists in rewriting the generalized higher-order variational problem of Herglotz (H_n) as a standard optimal control problem (P), and then to apply available results of optimal control theory. For the first-order case we refer the reader to [7].

The paper is organized as follows. In Section 2 we present the necessary concepts and results from optimal control theory: Pontryagin's maximum principle (Theorem 2.1); the DuBois–Reymond condition of optimal control (Theorem 2.3); and the Noether theorem of optimal control (Theorem 2.5). Our main results are given in Section 3: we extend the higher-order Euler–Lagrange equation and the transversality conditions for problem (H_n) found in [6] to admissible functions $x(\cdot) \in PC^n([a,b];\mathbb{R}^m)$ and $z(\cdot) \in PC^1([a,b];\mathbb{R})$ (Theorem 3.3); we obtain a DuBois–Reymond necessary optimality condition for problem (H_n) (Theorem 3.5); and we generalize the Noether theorem to higher-order variational problems of Herglotz type (Theorem 3.7). We end with Section 4 of conclusions.

2. Preliminaries.

We begin this section by stating the well known Pontryagin's maximum principle, which is a first-order necessary optimality condition.

Theorem 2.1 (Pontryagin's maximum principle for problem (P) [5]). *If a pair $(x(\cdot), u(\cdot))$ with $x \in PC^1([a,b];\mathbb{R}^m)$ and $u \in PC([a,b];\Omega)$ is a solution to problem (P) with the initial condition $x(a) = \alpha$, $\alpha \in \mathbb{R}^m$, then there exists $\psi \in PC^1([a,b];\mathbb{R}^m)$ such that the following conditions hold:*

- *the optimality condition*

$$\frac{\partial H}{\partial u}(t, x(t), u(t), \psi(t)) = 0; \tag{3}$$

- *the adjoint system*

$$\begin{cases} \dot{x}(t) = \frac{\partial H}{\partial \psi}(t, x(t), u(t), \psi(t)) \\ \dot{\psi}(t) = -\frac{\partial H}{\partial x}(t, x(t), u(t), \psi(t)); \end{cases} \tag{4}$$

- *and the transversality condition*

$$\psi(b) = grad(\phi(x))(b); \tag{5}$$

where the Hamiltonian H is defined by

$$H(t, x, u, \psi) = f(t, x, u) + \psi \cdot g(t, x, u). \tag{6}$$

Definition 2.2 (Pontryagin extremal to (P)). A triplet $(x(\cdot), u(\cdot), \psi(\cdot))$ with $x \in PC^1([a,b];\mathbb{R}^m)$, $u \in PC([a,b];\Omega)$ and $\psi \in PC^1([a,b];\mathbb{R}^m)$ is called a Pontryagin extremal to problem (P) if it satisfies the optimality condition (3), the adjoint system (4) and the transversality condition (5).

Theorem 2.3 (DuBois–Reymond condition of optimal control [5]). *If $(x(\cdot), u(\cdot), \psi(\cdot))$ is a Pontryagin extremal to problem (P), then the Hamiltonian (6) satisfies the equality*

$$\frac{dH}{dt}(t, x(t), u(t), \psi(t)) = \frac{\partial H}{\partial t}(t, x(t), u(t), \psi(t)), \quad t \in [a,b].$$

The famous Noether theorem [4] is another fundamental tool of the calculus of variations [11], optimal control [8, 9, 10] and modern theoretical physics [1]. It states that when an optimal control problem is invariant under a one parameter family of transformations, then there exists a corresponding conservation law: an expression that is conserved along all the Pontryagin extremals of the problem (see [8, 9, 10] and references therein). Here we use Noether's theorem as found in [8], which is formulated for problems of optimal control in Lagrange form, that is, for problem (P) with $\phi \equiv 0$. In order to apply the results of [8] to the Bolza problem (P), we rewrite it in the following equivalent Lagrange form:

$$\begin{gathered} \mathcal{I}(x(\cdot), y(\cdot), u(\cdot)) = \int_a^b [f(t, x(t), u(t)) + y(t)]\, dt \longrightarrow \text{extr}, \\ \begin{cases} \dot{x}(t) = g\left(t, x(t), u(t)\right), \\ \dot{y}(t) = 0, \end{cases} \\ x(a) = \alpha, \; y(a) = \frac{\phi(x(b))}{b-a}. \end{gathered} \tag{7}$$

Before presenting the Noether theorem for the optimal control problem (P), we need to define the concept of invariance. Here we apply the notion of invariance found in [8] to the equivalent optimal control problem (7). In Definition 2.4 we use the little-o notation.

Definition 2.4 (Invariance of problem (P) cf. [8])**.** Let h^s be a one-parameter family of invertible C^1 maps

$$\begin{gathered} h^s : [a,b] \times \mathbb{R}^m \times \Omega \longrightarrow \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^r, \\ h^s(t,x,u) = \left(\mathcal{T}^s(t,x,u), \mathcal{X}^s(t,x,u), \mathcal{U}^s(t,x,u)\right), \\ h^0(t,x,u) = (t,x,u) \text{ for all } (t,x,u) \in [a,b] \times \mathbb{R}^m \times \Omega. \end{gathered}$$

Problem (P) is said to be invariant under transformations h^s if for all $(x(\cdot), u(\cdot))$ the following two conditions hold:

(i)

$$\begin{aligned} &\left[f \circ h^s(t,x(t),u(t)) + \frac{\phi(x(b))}{b-a} + \xi s + o(s)\right] \frac{d\mathcal{T}^s}{dt}(t,x(t),u(t)) \\ &\qquad\qquad = f(t,x(t),u(t)) + \frac{\phi(x(b))}{b-a} \end{aligned} \tag{8}$$

for some constant ξ;

(ii)

$$\frac{d\mathcal{X}^s}{dt}(t,x(t),u(t)) = g \circ h^s(t,x(t),u(t)) \frac{d\mathcal{T}^s}{dt}(t,x(t),u(t)). \tag{9}$$

The next result can be easily obtained from the Noether theorem proved by Torres in [8] and Pontryagin's maximum principle (Theorem 2.1).

Theorem 2.5 (Noether's theorem for the optimal control problem (P))**.** *If problem* (P) *is invariant in the sense of Definition 2.4, then the quantity*

$$(b-t)\xi + \psi(t) \cdot X(t,x(t),u(t)) - \left[H(t,x(t),u(t),\psi(t)) + \frac{\phi(x(b))}{b-a}\right] \cdot T(t,x(t),u(t))$$

is constant in t along every Pontryagin extremal $(x(\cdot), u(\cdot), \psi(\cdot))$ of problem (P)*, where H is defined by* (6) *and*

$$\begin{aligned} T(t,x(t),u(t)) &= \frac{\partial \mathcal{T}^s}{\partial s}(t,x(t),u(t)) \bigg|_{s=0}, \\ X(t,x(t),u(t)) &= \frac{\partial \mathcal{X}^s}{\partial s}(t,x(t),u(t)) \bigg|_{s=0}. \end{aligned}$$

3. Main results.

We begin by introducing some definitions for the higher-order variational problem of Herglotz (H_n).

Definition 3.1 (Admissible pair to problem (H_n))**.** We say that $(x(\cdot), z(\cdot))$ with $x(\cdot) \in PC^n([a,b];\mathbb{R}^m)$ and $z(\cdot) \in PC^1([a,b];\mathbb{R})$ is an admissible pair to problem (H_n) if it satisfies the equation

$$\begin{gathered} \dot{z}(t) = L(t, x(t), \dot{x}(t), \cdots, x^{(n)}(t), z(t)), \quad t \in [a,b], \\ \text{with } z(a) = \gamma \in \mathbb{R}. \end{gathered}$$

Definition 3.2 (Extremizer to problem (H_n))**.** We say that an admissible pair $(x^*(\cdot), z^*(\cdot))$ is an extremizer to problem (H_n) if $z(b) - z^*(b)$ has the same signal for all admissible pairs $(x(\cdot), z(\cdot))$ that satisfy $\|z - z^*\|_0 < \epsilon$ and $\|x - x^*\|_0 < \epsilon$ for some positive real ϵ, where $\|y\|_0 = \max\limits_{a \le t \le b} |y(t)|$.

We now present a necessary condition for a pair $(x(\cdot), z(\cdot))$ to be a solution (extremizer) to problem (H_n). The following result generalizes [6] by considering a more general class of functions. To simplify notation, we use the operator $\langle \cdot, \cdot \rangle_n$, $n \in \mathbb{N}$, defined by $\langle x, z\rangle_n(t) := (t, x(t), \dot{x}(t), \ldots, x^{(n)}(t), z(t))$. When there is no possibility of ambiguity, we sometimes suppress arguments.

Theorem 3.3 (Higher-order Euler–Lagrange equation and transversality conditions for problem (H_n))**.** *If $(x(\cdot), z(\cdot))$ is an extremizer to problem (H_n) that satisfies the inicial conditions*

$$x(a) = \alpha_0, \ldots, x^{(n-1)}(a) = \alpha_{n-1}, \quad \alpha_0, \ldots, \alpha_{n-1} \in \mathbb{R}^m, \tag{10}$$

then the Euler–Lagrange equation

$$\sum_{j=0}^{n} (-1)^j \frac{d^j}{dt^j} \left(\psi_z(t) \frac{\partial L}{\partial x^{(j)}} \langle x, z\rangle_n(t) \right) = 0 \tag{11}$$

holds, for $t \in [a, b]$, where

$$\begin{cases} \psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z} \langle x,z\rangle_n(\theta) d\theta} \\ \psi_j(t) = \sum_{i=0}^{n-j} (-1)^{i+1} \frac{d^i}{dt^i} \left(\psi_z(t) \frac{\partial L}{\partial x^{(i+j)}} \langle x, z\rangle_n(t) \right), \quad j = 1, \ldots, n. \end{cases} \tag{12}$$

Moreover, the following transversality conditions hold:

$$\psi_j(b) = 0, \quad j = 1, \ldots, n. \tag{13}$$

Proof. Observe that the higher-order problem of Herglotz (H_n) is a particular case of problem (P) when we consider a $n+1$ coordinates state variable $(x_0, x_1, \ldots x_{n-1}, z)$ with $x_0 = x$, $x_1 = \dot{x}$, …, $x_{n-1} = x^{(n-1)}$, a control $u = x^{(n)}$ and choose $f \equiv 0$ and $\phi(x_0, \ldots, x_{n-1}, z) = z$. The higher-order problem of Herglotz can now be described as an optimal control problem as follows:

$$z(b) \longrightarrow \text{extr}$$

$$\begin{cases} \dot{x}_0(t) = x_1(t), \\ \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = x_3(t), \\ \qquad \vdots \\ \dot{x}_{n-2}(t) = x_{n-1}(t), \\ \dot{x}_{n-1}(t) = u(t), \\ \dot{z}(t) = L(t, x_0(t), \ldots, x_{n-1}(t), u(t), z(t)), \end{cases} \tag{14}$$

$$z(a) = \gamma, \quad \gamma \in \mathbb{R}.$$

Observe that since we consider problem (H_n) subject to the initial conditions (10), then $x_0(a) = \alpha_0, \ldots, x_{n-1}(a) = \alpha_{n-1}$ with $\alpha_0, \ldots, \alpha_{n-1}$ given $\mathbb{R}^m$ vectors. From Pontryagin's Maximum Principle for problem (P) (Theorem 2.1) there are $(\psi_1, \ldots, \psi_n, \psi_z) \in PC^1([a, b]; \mathbb{R}^{n\times m+1})$ such that the following conditions hold:

- the optimality condition

$$\frac{\partial H}{\partial u}(t, x_0(t), \ldots, x_{n-1}(t), u(t), z(t), \psi_1(t), \ldots, \psi_n(t), \psi_z(t)) = 0, \tag{15}$$

- the adjoint system

$$\begin{cases} \dot{x}_{j-1}(t) = \frac{\partial H}{\partial \psi_j}(t, x_0(t), \ldots, x_{n-1}(t), u(t), z(t), \psi_1(t), \ldots, \psi_n(t), \psi_z(t)), \ j = 1, \ldots, n \\ \dot{\psi}_1(t) = -\frac{\partial H}{\partial x_0}(t, x_0(t), \ldots, x_{n-1}(t), u(t), z(t), \psi_1(t), \ldots, \psi_n(t), \psi_z(t)) \\ \dot{\psi}_j(t) = -\frac{\partial H}{\partial x_{j-1}}(t, x_0(t), \ldots, x_{n-1}(t), u(t), z(t), \psi_1(t), \ldots, \psi_n(t), \psi_z(t)), \ j = 2, \ldots, n \\ \dot{\psi}_z(t) = -\frac{\partial H}{\partial z}(t, x_0(t), \ldots, x_{n-1}(t), u(t), z(t), \psi_1(t), \ldots, \psi_n(t), \psi_z(t)) \end{cases} \tag{16}$$

- the transversality conditions

$$\begin{cases} \psi_j(b) = 0, \quad j = 1, \ldots, n, \\ \psi_z(b) = 1, \end{cases} \tag{17}$$

where the Hamiltonian H is defined by

$$H(t, x_0, \ldots, x_{n-1}, u, z, \psi_1, \ldots, \psi_n, \psi_z) \\ = \psi_1 \cdot x_1 + \ldots + \psi_{n-1} \cdot x_{n-1} + \psi_n \cdot u + \psi_z \cdot L(t, x_0, \ldots, x_{n-1}, u, z).$$

Observe that the optimality condition (15) implies that $\psi_n = -\psi_z \frac{\partial L}{\partial u}$ and that the adjoint system (16) implies that

$$\begin{cases} \dot{\psi}_1 = -\psi_z \frac{\partial L}{\partial x_0}, \\ \dot{\psi}_j = -\psi_{j-1} - \psi_z \frac{\partial L}{\partial x_{j-1}}, \quad \text{for } j = 2, \ldots, n, \\ \dot{\psi}_z = -\psi_z \frac{\partial L}{\partial z}. \end{cases}$$

Hence, ψ_z is solution of a first-order linear differential equation, which is solved using an integrand factor to find that $\psi_z(t) = ke^{-\int_a^t \frac{\partial L}{\partial z} d\theta}$ with k a constant. From the last transversality condition in (17), we obtain that $k = e^{\int_a^b \frac{\partial L}{\partial z} d\theta}$ and, consequently,

$$\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z} d\theta}.$$

Note also that for $j = n$ we obtain $\dot{\psi}_n = -\psi_{n-1} - \psi_z \frac{\partial L}{\partial x_{n-1}}$, which is equivalent to

$$\psi_{n-1} = \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_n}\right) - \psi_z \frac{\partial L}{\partial x_{n-1}}.$$

By differentiation of the previous expression, we obtain that

$$\dot{\psi}_{n-1} = \frac{d^2}{dt^2}\left(\psi_z \frac{\partial L}{\partial x_n}\right) - \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_{n-1}}\right)$$

and noting that $\dot{\psi}_{n-1} = -\psi_{n-2} - \psi_z \frac{\partial L}{\partial x_{n-2}}$, we find an expression for ψ_{n-2}:

$$\psi_{n-2} = -\frac{d^2}{dt^2}\left(\psi_z \frac{\partial L}{\partial x_n}\right) + \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_{n-1}}\right) - \psi_z \frac{\partial L}{\partial x_{n-2}}.$$

Similarly, we obtain that

$$\psi_{n-3} = \frac{d^3}{dt^3}\left(\psi_z \frac{\partial L}{\partial x_n}\right) - \frac{d^2}{dt^2}\left(\psi_z \frac{\partial L}{\partial x_{n-1}}\right) + \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_{n-2}}\right) - \psi_z \frac{\partial L}{\partial x_{n-3}}.$$

Applying the same argument to the next multipliers and noting that $\psi_1 = -\dot{\psi}_2 - \psi_z \frac{\partial L}{\partial x_1}$, we have

$$\dot{\psi}_1 = -\psi_z \frac{\partial L}{\partial x_0} \\ = (-1)^n \frac{d^n}{dt^n}\left(\psi_z \frac{\partial L}{\partial x_n}\right) + (-1)^{n-1} \frac{d^{n-1}}{dt^{n-1}}\left(\psi_z \frac{\partial L}{\partial x_{n-1}}\right) + \cdots - \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_1}\right)$$

or, equivalently,

$$(-1)^n \frac{d^n}{dt^n}\left(\psi_z \frac{\partial L}{\partial x_n}\right) + (-1)^{n-1} \frac{d^{n-1}}{dt^{n-1}}\left(\psi_z \frac{\partial L}{\partial x_{n-1}}\right) + \cdots - \frac{d}{dt}\left(\psi_z \frac{\partial L}{\partial x_1}\right) + \psi_z \frac{\partial L}{\partial x_0} = 0.$$

Rewriting previous equation in terms of problem (H_n) and in the form of a summation, one gets

$$\sum_{j=0}^{n} (-1)^j \frac{d^j}{dt^j}\left(\psi_z \frac{\partial L}{\partial x^{(j)}}\right) = 0$$

as intended. Observe also that from the previous argumentation we were also able to derive expressions for the multipliers:

$$\psi_j = \sum_{i=0}^{n-j} (-1)^i \frac{d^i}{dt^i} \left(-\psi_z \frac{\partial L}{\partial x^{(i+j)}} \right), \quad j = 1, \ldots, n,$$

which together with (17) leads to the transversality conditions

$$\sum_{i=0}^{n-j} (-1)^i \frac{d^i}{dt^i} \left(\psi_z \frac{\partial L}{\partial x^{(i+j)}} \right) \bigg|_{t=b} = 0, \quad j = 1, \ldots, n.$$

This concludes the proof. □

Definition 3.4 (Extremal to problem (H_n))**.** We say that an admissible pair $(x(\cdot), z(\cdot))$ is an extremal to problem (H_n) if it satisfies the Euler–Lagrange equation (11) and the transversality conditions (13).

Next we present two new important results: the DuBois–Reymond condition and the Noether theorem for the higher-order variational problem of Herglotz (H_n).

Theorem 3.5 (DuBois–Reymond condition for problem (H_n))**.** *If $(x(\cdot), z(\cdot))$ is an extremal to problem (H_n), then*

$$\frac{d}{dt} \left(\sum_{i=1}^{n} \psi_i(t) x^{(i)}(t) + \psi_z(t) L\langle x, z \rangle_n(t) \right) = \psi_z(t) \frac{\partial L}{\partial t} \langle x, z \rangle_n(t),$$

where $\psi_z(t)$ and $\psi_i(t)$ are defined in (12).

Proof. Rewrite (H_n) as optimal control problem (14) and apply Theorem 2.3. □

Definition 3.6 (Invariance for problem (H_n))**.** Let h^s be a one-parameter family of invertible C^1 maps $h^s : [a, b] \times \mathbb{R}^m \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$,

$$h^s(t, x(t), z(t)) = (\mathcal{T}^s\langle x, z \rangle_n(t), \mathcal{X}^s\langle x, z \rangle_n(t), \mathcal{Z}^s\langle x, z \rangle_n(t)),$$
$$h^0(t, x, z) = (t, x, z), \quad \forall (t, x, z) \in [a, b] \times \mathbb{R}^m \times \mathbb{R}.$$

Problem (H_n) is said to be invariant under the transformations h^s if for all admissible pairs $(x(\cdot), z(\cdot))$ the following two conditions hold:

(i)

$$\left(\frac{z(b)}{b-a} + \xi s + o(s) \right) \frac{d\mathcal{T}^s}{dt} \langle x, z \rangle_n(t) = \frac{z(b)}{b-a}, \text{ for some constant } \xi; \tag{18}$$

(ii)

$$\frac{d\mathcal{Z}^s}{dt}\langle x, z \rangle_n(t) = L \Bigg(\mathcal{T}^s\langle x, z \rangle_n(t), \mathcal{X}^s\langle x, z \rangle_n(t), \frac{d\mathcal{X}^s}{d\mathcal{T}^s}\langle x, z \rangle_n(t), \ldots$$
$$\ldots, \frac{d^n\mathcal{X}^s}{d(\mathcal{T}^s)^n}\langle x, z \rangle_n(t), \mathcal{Z}^s\langle x, z \rangle_n(t) \Bigg) \frac{d\mathcal{T}^s}{dt}\langle x, z \rangle_n(t), \tag{19}$$

where

$$\frac{d\mathcal{X}^s}{d\mathcal{T}^s}\langle x, z \rangle_n(t) = \frac{\frac{d\mathcal{X}^s}{dt}\langle x, z \rangle_n(t)}{\frac{d\mathcal{T}^s}{dt}\langle x, z \rangle_n(t)} \text{ and } \frac{d^i\mathcal{X}^s}{d(\mathcal{T}^s)^i}\langle x, z \rangle_n(t) = \frac{\frac{d}{dt}\left(\frac{d^{i-1}\mathcal{X}^s}{d(\mathcal{T}^s)^{i-1}}\langle x, z \rangle_n(t) \right)}{\frac{d\mathcal{T}^s}{dt}\langle x, z \rangle_n(t)} \tag{20}$$

for $i = 2, \ldots, n$.

Next we present the main result of this paper.

Theorem 3.7 (Noether's Theorem for problem (H_n)). *If problem* (H_n) *is invariant in the sense of Definition 3.6, then the quantity*

$$\sum_{i=1}^{n} \psi_i(t) X_{i-1}\langle x, z\rangle_n(t) + \psi_z(t) Z\langle x, z\rangle_n(t) - \left(\sum_{i=1}^{n} \psi_i(t) x^{(i)}(t) + \psi_z(t) L\langle x, z\rangle_n(t) \right) T\langle x, z\rangle_n(t)$$

is constant in t along every extremal to problem (H_n)*, where*

$$T = \left.\frac{\partial \mathcal{T}^s}{\partial s}\right|_{s=0}, \quad X_0 = \left.\frac{\partial \mathcal{X}^s}{\partial s}\right|_{s=0}, \quad Z = \left.\frac{\partial \mathcal{Z}^s}{\partial s}\right|_{s=0},$$
$$X_i = \frac{d}{dt} X_{i-1} - x^{(i)} \frac{d}{dt} \left(\left.\frac{\partial \mathcal{T}^s}{\partial s}\right|_{s=0} \right) \quad \text{for } i = 1, \ldots, n-1,$$

ψ_i is defined by (12) *and $\psi_z(t) = e^{\int_t^b \frac{\partial L}{\partial z} d\theta}$.*

Proof. As before, we deal with problem (H_n) in its equivalent optimal control form (14). We now prove that if problem (H_n) is invariant in the sense of Definition 3.6, then (14) is invariant in the sense of Definition 2.4. First, observe that if (18) holds, then (8) holds for (14) with $f \equiv 0$ and $\phi(x_0, \ldots, x_{n-1}, z) = z$. Second, note that the control system of (14) defines $\mathcal{U}^s := \frac{d\mathcal{X}^s_{n-1}}{d\mathcal{T}^s}$ and $\mathcal{X}^s_i := \frac{d\mathcal{X}^s_{i-1}}{d\mathcal{T}^s}$, that is,

$$\begin{cases} \frac{d\mathcal{X}^s_{i-1}}{dt}\langle x, z\rangle_n(t) = \mathcal{X}^s_i\langle x, z\rangle_n(t) \frac{d\mathcal{T}^s}{dt}\langle x, z\rangle_n(t), \quad i = 1, \ldots, n-1, \\ \frac{d\mathcal{X}^s_{n-1}}{dt}\langle x, z\rangle_n(t) = \mathcal{U}^s\langle x, z\rangle_n(t) \frac{d\mathcal{T}^s}{dt}\langle x, z\rangle_n(t). \end{cases} \tag{21}$$

This means that if (19) and (21) hold, then there is also invariance in the sense of (9) and problem (14) is invariant in the sense of Definition 2.4. This invariance gives conditions to apply Theorem 2.5 to problem (14), which assures that the quantity

$$(b-t)\xi + \sum_{i=1}^{n} \psi_i(t) X_{i-1}\langle x, z\rangle_n(t) + \psi_z(t) Z\langle x, z\rangle_n(t) - \left[\sum_{i=1}^{n} \psi_i(t) x_i(t) + \psi_z(t) L\langle x, z\rangle_n(t) + \frac{\phi(x(b))}{b-a} \right] T\langle x, z\rangle_n(t),$$

where $X_i = \frac{\partial}{\partial s} \frac{d^i \mathcal{X}^s}{d(\mathcal{T}^s)^i}\Big|_{s=0}$ is constant in t along every Pontryagin extremal of problem (14). This means that the quantity

$$(b-t)\xi - \frac{\phi(x(b))}{b-a} T\langle x, z\rangle_n(t) + \sum_{i=1}^{n} \psi_i(t) X_{i-1}\langle x, z\rangle_n(t) + \psi_z(t) Z\langle x, z\rangle_n(t) - \left[\sum_{i=1}^{n} \psi_i(t) x^{(i)}(t) + \psi_z(t) L\langle x, z\rangle_n(t) \right] T\langle x, z\rangle_n(t)$$

is constant in t along every extremal of problem (H_n). Observe that $X_0 = \frac{\partial \mathcal{X}^s}{\partial s}\Big|_{s=0}$, which together with (20) leads to

$$X_i = \frac{\partial}{\partial s}\frac{d^i\mathcal{X}^s}{d(\mathcal{T}^s)^i}\bigg|_{s=0} = \frac{\partial}{\partial s}\left(\frac{\frac{d}{dt}\left(\frac{d^{i-1}\mathcal{X}^s}{d(\mathcal{T}^s)^{i-1}}\right)}{\frac{d\mathcal{T}^s}{dt}}\right)\Bigg|_{s=0}$$
$$= \frac{d}{dt}\left(\frac{\partial}{\partial s}\frac{d^{i-1}\mathcal{X}^s}{d(\mathcal{T}^s)^{i-1}}\bigg|_{s=0}\right) - x^{(i)}\frac{d}{dt}\left(\frac{\partial \mathcal{T}^s}{\partial s}\bigg|_{s=0}\right)$$
$$= \frac{d}{dt}X_{i-1} - x^{(i)}\frac{d}{dt}\left(\frac{\partial \mathcal{T}^s}{\partial s}\bigg|_{s=0}\right).$$

To end the proof we only need to prove that the quantity

$$(b-t)\xi - \frac{z(b)}{b-a}T\langle x,z\rangle_n(t) \tag{22}$$

is a constant. From the invariance condition (18), we know that

$$(z(b)+\xi(b-a)s+o(s))\,\frac{d\mathcal{T}^s}{dt}\langle x,z\rangle_n(t) = z(b).$$

Integrating from a to t we conclude that

$$\begin{aligned}(z(b)+\xi(b-a)s+o(s))\,&\mathcal{T}^s\langle x,z\rangle_n(t)\\ &= z(b)(t-a) + (z(b)+\xi(b-a)s+o(s))\,\mathcal{T}^s\langle x,z\rangle_n(a).\end{aligned} \tag{23}$$

Differentiating (23) with respect to s, and then putting $s=0$, we obtain:

$$\xi(b-a)t + z(b)T\langle x,z\rangle_n(t) = \xi(b-a)a + z(b)T\langle x,z\rangle_n(a). \tag{24}$$

We conclude from (24) that expression (22) is the constant

$$(b-a)\xi - \frac{z(b)T\langle x,z\rangle_n(a)}{b-a}.$$

The proof is complete. □

4. **Conclusion.** We investigated the higher-order variational problem of Herglotz from an optimal control point of view. The higher-order generalized Euler–Lagrange equation and the transversality conditions proved in [6] were obtained in the wider class of piecewise admissible functions. Moreover, we proved two important new results: a DuBois–Reymond necessary condition and Noether's theorem for higher-order variational problems of Herglotz type.

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