DIFFRACTION BY A HALF-PLANE WITH DIFFERENT FACE IMPEDANCES ON AN OBSTACLE PERPENDICULAR TO THE BOUNDARY*

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Abstract

The paper is devoted to study classes of plane wave diffraction problems by a region which involves a crack with impedance boundary conditions. Conditions on the wave number and impedance parameters are found to ensure the well-posedness of the problems in a scale of Bessel potential spaces. Under such conditions, representations of the solutions are also obtained upon the consideration of some associated operators which, in a sense, combine operators of Wiener-Hopf and Hankel type.

AMS Subject Classification: 35J05; 35B30; 35C15; 35J25; 35P25; 47A20; 47A53; 47A68; 47B35; 47G30; 78A45.

Keywords: Helmholtz equation, wave diffraction, impedance boundary value problems, potential method, oscillating symbols, Wiener-Hopf operator, Hankel operator.

1 Introduction

The diffraction by regions containing cracks is a well recognized important problem in applied mathematics. Recently, several works increased significantly the knowledge about

*Accepted authors manuscript (AAM) published in [Communications in Mathematical Analysis 17(2) (2014), 45–65]. The final publication is available at Mathematical Research Publishers (MRP) via http://www.math-res-pub.org/cma

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the solutions of such problems and the most appropriate spaces framework where to interpret these solutions. As a matter of fact, a great part of the recent interest goes to the mathematical analysis of the weak formulation of such kind of problems, and to an eventual possibility to increase regularity of the solution in a Sobolev spaces context. E.g., the works [7, 8, 9, 10, 11, 12, 13, 14, 19, 35, 36, 38, 39, 40] follow this line of research, and worked out already a corresponding detailed mathematical analysis about several classes of diffraction problems (but mostly without cracks).

In the present work, we go further on our own research (cf. [12, 14]) as it concerns the knowledge about the regularity of solutions of wave diffraction problems in the half-plane with a crack geometry, by considering in here (possible different) boundary impedance conditions on the crack. More specifically, we will derive seven different possible conditions on the wave number and on the impedance paramentes such that the unique solutions for the considered problems are obtained in Bessel potential spaces $H^{1+\varepsilon}$, for a smoothness parameter $\varepsilon \in [0, 1/2)$.

It is worth mentioning that the present methods are different from other somehow more classical works which are based only on the Wiener-Hopf method and its generalizations. Important tools in our technique are the Green formula and a convenient combination of pseudo-differential operators with other types of operators. More precisely, we use here a potential theory approach combined with the use of concrete extension operators, and a corresponding integral description of the problems. Thus, this is also different from the most classical approach due to the Malyuzhinets method [33], and corresponding representations of solutions by using the so-called Sommerfeld integral transform (cf., e.g., [32]). It is also clear that the present crack diffraction problems present an increase of the difficulties when compared with the corresponding problems of wave diffraction by a half-plane. The main difficulty is related to the different geometries of these two classes of problems, which for the crack case is originating much more difficult operators and equations for deriving the solutions of the problems.

2 Formulation of the problems

We will now introduce the general notation which will allow the mathematical formulation of the problem. Let $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space of all rapidly vanishing functions and $\mathcal{S}'(\mathbb{R}^n)$ the dual space of tempered distributions on $\mathbb{R}^n$. The Bessel potential space $H^s(\mathbb{R}^n)$, with $s \in \mathbb{R}$, is formed by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|\varphi\|_{H^s(\mathbb{R}^n)} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathcal{F}\varphi\|_{L_2(\mathbb{R}^n)}$ is finite. As the notation indicates, $\|\cdot\|_{H^s(\mathbb{R}^n)}$ is a norm for the space $H^s(\mathbb{R}^n)$ which makes it a Banach space. Here, $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^n$.

For a given Lipshitz domain $\mathcal{D}$, on $\mathbb{R}^n$, we denote by $\widetilde{H}^s(\mathcal{D})$ the closed subspace of $H^s(\mathbb{R}^n)$ whose elements have supports in $\overline{\mathcal{D}}$, and $H^s(\mathcal{D})$ denotes the space of generalized functions on $\mathcal{D}$ which have extensions into $\mathbb{R}^n$ that belong to $H^s(\mathbb{R}^n)$. The space $\widetilde{H}^s(\mathcal{D})$ is endowed with the subspace topology, and on $H^s(\mathcal{D})$ we introduce the norm of the quotient space $H^s(\mathbb{R}^n)/\widetilde{H}^s(\mathbb{R}^n\backslash\overline{\mathcal{D}})$. Throughout the paper we will use the notation $\mathbb{R}^n_\pm := \{x = (x_1, \ldots, x_{n-1}, x_n) \in \mathbb{R}^n : \pm x_n > 0\}$. Note that the spaces $H^0(\mathbb{R}^n_+)$ and $\widetilde{H}^0(\mathbb{R}^n_+)$ can be identified, and we will denote them by $L_2(\mathbb{R}^n_+)$.

Let $\Omega := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 \in \mathbb{R}\}$, $\Gamma_1 := \{(x_1, 0) : x_1 \in \mathbb{R}\}$, and $\Gamma_2 := \{(0, x_2) : x_2 \in \mathbb{R}\}$. Let further $C := \{(x_1, 0) : 0 < x_1 < a\} \subset \Gamma_1$ for a certain positive number a and $\Omega_C := \Omega \backslash \overline{C}$. Clearly, $\partial\Omega = \Gamma_2$ and $\partial\Omega_C = \Gamma_2 \cup C$.

For our purposes below we introduce further notations: $\Omega_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ and $\Omega_2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 < 0\}$, then $\partial\Omega_{\mathrm{j}} = \mathcal{S}_{\mathrm{j}} \cup \mathcal{S}$, for j $= 1, 2$, where $\mathcal{S} := \{(x_1, 0) : x_1 \geq 0\} \subset \Gamma_1$, $\mathcal{S}_1 := \{(0, x_2) : x_2 \geq 0\} \subset \Gamma_2$, and $\mathcal{S}_2 := \{(0, x_2) : x_2 \leq 0\} \subset \Gamma_2$. Finally, we introduce the following unit normal vectors $n_1 = \overrightarrow{(0, -1)}$ on Γ_1 and $n_2 = \overrightarrow{(-1, 0)}$ on Γ_2.

Let $\varepsilon \in [0, \frac{1}{2})$. We are interested in studying the problem of existence and uniqueness of an element $u \in H^{1+\varepsilon}(\Omega_C)$, such that

$$\left(\Delta + k^2\right) u = 0 \quad \text{in} \quad \Omega_C, \tag{2.1}$$

and u satisfies one of the following two representative boundary conditions:

$$\begin{cases} [\partial_{n_1} u]_C^+ - p_1 [u]_C^+ = g_1 & \text{on } C, \\ [\partial_{n_1} u]_C^- + p_2 [u]_C^- = g_2 & \text{on } C, \end{cases} \quad \text{and} \quad \begin{cases} [u]_{\mathcal{S}_1}^+ = h_1 & \text{on } \mathcal{S}_1, \\ [u]_{\mathcal{S}_2}^+ = h_2 & \text{on } \mathcal{S}_2, \end{cases} \tag{2.2}$$

$$\begin{cases} [\partial_{n_1} u]_C^+ - p_1 [u]_C^+ = g_1 & \text{on } C, \\ [\partial_{n_1} u]_C^- + p_2 [u]_C^- = g_2 & \text{on } C, \end{cases} \quad \text{and} \quad \begin{cases} [\partial_{n_2} u]_{\mathcal{S}_1}^+ = f_1 & \text{on } \mathcal{S}_1, \\ [\partial_{n_2} u]_{\mathcal{S}_2}^+ = f_2 & \text{on } \mathcal{S}_2; \end{cases} \tag{2.3}$$

here the wave number $k \in \mathbb{C} \setminus \mathbb{R}$ and the numbers $p_{\mathrm{j}} \in \mathbb{C}$ are given. The elements $[u]_{\mathcal{S}_{\mathrm{j}}}^+$ and $[\partial_{n_2} u]_{\mathcal{S}_{\mathrm{j}}}^+$ denote the Dirichlet and the Neumann traces on $\mathcal{S}_{\mathrm{j}}$, j $= 1, 2$, respectively, while by $[u]_C^{\pm}$ we denote the Dirichlet traces on C from both sides of the screen and by $[\partial_{n_1} u]_C^{\pm}$ we denote the Neumann traces on C from both sides of the crack.

Throughout the paper on the given data we assume that $h_{\mathrm{j}} \in H^{1/2+\varepsilon}(\mathcal{S}_{\mathrm{j}})$, $f_{\mathrm{j}} \in H^{-1/2+\varepsilon}(\mathcal{S}_{\mathrm{j}})$, and $g_{\mathrm{j}} \in H^{-1/2+\varepsilon}(C)$, for j $= 1, 2$. Furthermore, we suppose that they satisfy the following compatibility conditions:

$$\chi_a(g_1^+ - g_1^-) \in r_C \widetilde{H}^{-1/2+\varepsilon}(C), \tag{2.4}$$

and

$$\chi_0\left(g_1 + r_C f_1 \circ e^{i\frac{\pi}{2}}\right), \; \chi_0\left(g_2 - r_C f_2 \circ e^{-i\frac{\pi}{2}}\right) \in r_C \widetilde{H}^{-1/2+\varepsilon}(C). \tag{2.5}$$

Here, r_C denotes the restriction operator to C and $\chi_a(x) := \chi_0(a - x)$, where $\chi_0 \in C^\infty([0, a])$, such that $\chi_0(x) \equiv 1$ for $x \in [0, a/3]$ and $\chi_0(x) \equiv 0$ for $x \in [2a/3, a]$.

From now on we will refer to:

- Problem $\mathcal{P}_{I-D}$ as the one given by the conditions (2.1), (2.2), (2.4);
- Problem $\mathcal{P}_{I-N}$ as the one given by the conditions (2.1), (2.3), (2.4), and (2.5).

Note that the compatibility conditions (2.4) and (2.5) are necessary for the well-posedness of the problems $\mathcal{P}_{I-D}$ and $\mathcal{P}_{I-N}$ but only in the case $\varepsilon = 0$. For $\varepsilon \in (0, \frac{1}{2})$ the compatibility conditions are superfluous.

3 The fundamental solution and potentials

We start this section by proving the uniqueness result for the problems in consideration.

Theorem 3.1. *If* $\varepsilon \in [0, \frac{1}{2})$ *and one of the following situations holds:*

(a) $(\mathfrak{Re}\,k)(\mathfrak{Im}\,k) > 0, \quad \mathfrak{Im}\,p_1 \geq 0, \quad \mathfrak{Im}\,p_2 \geq 0,$

(b) $(\mathfrak{Re}\,k)(\mathfrak{Im}\,k) < 0, \quad \mathfrak{Im}\,p_1 \leq 0, \quad \mathfrak{Im}\,p_2 \leq 0,$

(c) $|\mathfrak{Im}\,k| > |\mathfrak{Re}\,k|, \quad \mathfrak{Re}\,p_1 \leq 0, \quad \mathfrak{Re}\,p_2 \leq 0,$

(c′) $|\mathfrak{Im}\,k| = |\mathfrak{Re}\,k|, \quad \mathfrak{Re}\,p_1 < 0, \quad \mathfrak{Re}\,p_2 < 0,$

(d) $\mathfrak{Re}\,k = 0, \quad (\mathfrak{Im}\,p_1)(\mathfrak{Im}\,p_2) > 0,$

(e) $\mathfrak{Im}\,p_1 \neq 0,\ (\mathfrak{Im}\,k)^2 - (\mathfrak{Re}\,k)^2 + 2(\mathfrak{Re}\,k)(\mathfrak{Im}\,k)\frac{\mathfrak{Re}\,p_1}{\mathfrak{Im}\,p_1} > 0,$
$\mathfrak{Re}\,p_2 \leq \mathfrak{Re}\,p_1 \frac{\mathfrak{Im}\,p_2}{\mathfrak{Im}\,p_1},$

(f) $\mathfrak{Im}\,p_2 \neq 0,\ (\mathfrak{Im}\,k)^2 - (\mathfrak{Re}\,k)^2 + 2(\mathfrak{Re}\,k)(\mathfrak{Im}\,k)\frac{\mathfrak{Re}\,p_2}{\mathfrak{Im}\,p_2} > 0,$
$\mathfrak{Re}\,p_1 \leq \mathfrak{Re}\,p_2 \frac{\mathfrak{Im}\,p_1}{\mathfrak{Im}\,p_2},$

then problems $\mathcal{P}_{I-D}$ *and* $\mathcal{P}_{I-N}$ *have at most one solution.*

Proof. The proof is standard and uses the Green's formula (being sufficient to consider the case $\varepsilon = 0$). Let R be a sufficiently large positive number and $B(R)$ be the disk centered at the origin with the radius R. Set $\Omega_R := \Omega_C \cap B(R)$. Note that the domain Ω_R has a piecewise smooth boundary S_R including both sides of C and denote by $n(x)$ the outward unit normal vector at the non-singular points $x \in S_R$.

Let u be a solution of the homogeneous problem. Then the first Green's identity for u and its complex conjugate $\overline{u}$ in the domain Ω_R, together with zero boundary conditions on S_R yields

$$\int_{\Omega_R} \Big[|\nabla u|^2 - k^2 |u|^2\Big] dx = p_1 \int_C |[u]^+|^2 dx \; + \; p_2 \int_C |[u]^-|^2 dx + \int_{\partial B(R) \cap \Omega_C} (\partial_n u)\,\overline{u}\, dS_R\,. \quad (3.1)$$

From the real and imaginary parts of the last identity, we obtain

$$\int_{\Omega_R} \Big[|\nabla u|^2 + \Big((\mathfrak{Im}\,k)^2 - (\mathfrak{Re}\,k)^2\Big)|u|^2\Big] dx \; - \; \mathfrak{Re}\,p_1 \int_C |[u]^+|^2 dx \; - \; \mathfrak{Re}\,p_2 \int_C |[u]^-|^2 dx$$
$$= \mathfrak{Re} \int_{\partial B(R) \cap \Omega_C} (\partial_n u)\,\overline{u}\, dS_R\,, \qquad (3.2)$$

$$-2(\mathfrak{Re}\,k)(\mathfrak{Im}\,k) \int_{\Omega_R} |u|^2\, dx \; - \; \mathfrak{Im}\,p_1 \int_C |[u]^+|^2 dx \; - \; \mathfrak{Im}\,p_2 \int_C |[u]^-|^2 dx$$
$$= \mathfrak{Im} \int_{\partial B(R) \cap \Omega_C} (\partial_n u)\,\overline{u}\, dS_R\,. \qquad (3.3)$$

Now, note that since $u \in H^1(\Omega_C)$ there exists a monotonic sequence of positive numbers $\{R_{\mathrm{j}}\}$, such that $R_{\mathrm{j}} \to \infty$ as $\mathrm{j} \to \infty$ and

$$\lim_{\mathrm{j}\to\infty} \int_{\partial B(R_{\mathrm{j}})\cap\Omega_C} (\partial_n u)\bar{u}\, dS_{R_{\mathrm{j}}} = 0. \tag{3.4}$$

For each of the conditions (a), (b), and (d) the expression in the left hand side of (3.3) and for the condition (c) and (c′) the expression in the left hand side of (3.2) are monotonic with respect to R, and therefore (3.4) implies that the limits at infinity of the expressions in the left and right hand sides of (3.2) and (3.3) exist and are equal to 0. Thus $u = 0$ in Ω_C.

For the condition (e), equalities (3.2) and (3.3) give us

$$\int_{\Omega_R} \Big[|\nabla u|^2 + \Big((\Im\mathrm{m}\, k)^2 - (\Re\mathrm{e}\, k)^2 + 2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k)\frac{\Re\mathrm{e}\, p_1}{\Im\mathrm{m} p_1}\Big)|u|^2\Big] dx$$
$$+\Big(\Re\mathrm{e}\, p_1 \frac{\Im\mathrm{m}\, p_2}{\Im\mathrm{m}\, p_1} - \Re\mathrm{e}\, p_2\Big) \int_C |[u]^-|^2 dx$$
$$= \Re\mathrm{e} \int_{\partial B(R)\cap\Omega_C} (\partial_n u)\bar{u}\, dS_R \;-\; \frac{\Re\mathrm{e}\, p_1}{\Im\mathrm{m} p_1} \Im\mathrm{m} \int_{\partial B(R)\cap\Omega_C} (\partial_n u)\bar{u}\, dS_R\,,$$

with the expression in the left hand side which is also monotonic with respect to R, and therefore (3.4) implies that limit at infinity exists and equals to 0. Similar arguments work for the condition (f). Thus $u = 0$ in Ω_C also for the conditions (e) and (f). □

Now, without loss of generality, we assume that $\Im\mathrm{m}\, k > 0$; the complementary case $\Im\mathrm{m}\, k < 0$ is treated with obvious modifications. Let us denote the standard fundamental solution of the Helmholtz equation (in two dimensions) by

$$\mathcal{K}(x) \quad := \quad -\frac{i}{4} H_0^{(1)}(k|x|)\,,$$

where $H_0^{(1)}(k|x|)$ is the Hankel function of the first kind of order zero (cf. [20, §3.4]). Furthermore, we introduce the single and double layer potentials on Γ_{j}:

$$\begin{aligned} V_{\mathrm{j}}(\psi)(x) &= \int_{\Gamma_{\mathrm{j}}} \mathcal{K}(x-y)\psi(y) d_y\Gamma_{\mathrm{j}}\,, \quad x \notin \Gamma_{\mathrm{j}}\,, \\ W_{\mathrm{j}}(\varphi)(x) &= \int_{\Gamma_{\mathrm{j}}} [\partial_{n_j(y)}\mathcal{K}(x-y)]\varphi(y) d_y\Gamma_{\mathrm{j}}\,, \quad x \notin \Gamma_{\mathrm{j}}\,, \end{aligned}$$

where $\mathrm{j} = 1, 2$ and ψ, φ are density functions. Note that for $\mathrm{j} = 1$ sometimes we will write $\mathbb{R}$ instead of Γ_1. In this case, for example, the single layer potential defined above has the form

$$V_1(\psi)(x_1, x_2) = \int_{\mathbb{R}} \mathcal{K}(x_1 - y, x_2)\psi(y) dy, \quad x_2 \neq 0.$$

Set $\mathbb{R}^2_\pm := \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_2 \gtrless 0\}$ and let us first consider the operators $V := V_1$ and $W := W_1$.

Theorem 3.2 (Cf., e.g., [10]). *The single and double layer potentials V and W are continuous operators*

$$V : H^s(\mathbb{R}) \to H^{s+1+\frac{1}{2}}(\mathbb{R}^2_\pm), \qquad W : H^{s+1}(\mathbb{R}) \to H^{s+1+\frac{1}{2}}(\mathbb{R}^2_\pm) \tag{3.5}$$

for all $s \in \mathbb{R}$ and satisfy following well-known limit relations:

$$\begin{aligned} &[V(\psi)]^+_{\mathbb{R}} = [V(\psi)]^-_{\mathbb{R}} =: \mathcal{H}(\psi), \qquad [\partial_n V(\psi)]^\pm_{\mathbb{R}} =: [\mp\frac{1}{2}I](\psi), \\ &[W(\varphi)]^\pm_{\mathbb{R}} =: [\pm\frac{1}{2}I](\varphi), \qquad [\partial_n W(\varphi)]^+_{\mathbb{R}} = [\partial_n W(\varphi)]^-_{\mathbb{R}} =: \mathcal{L}(\varphi), \end{aligned} \tag{3.6}$$

where

$$\mathcal{H}(\psi)(z) := \int_{\mathbb{R}} \mathcal{K}(z-y)\psi(y)dy, \quad z \in \mathbb{R}, \tag{3.7}$$

$$\mathcal{L}(\varphi)(z) := \lim_{\mathbb{R}^2_+ \ni x \to z \in \mathbb{R}} \partial_{n(x)} \int_{\mathbb{R}} [\partial_{n(y)}\mathcal{K}(y-x)]\varphi(y)dy, \quad z \in \mathbb{R}, \tag{3.8}$$

and I denotes the identity operator.

In our further considerations we will use the even and odd extension operators defined by

$$\ell^e\varphi(y) = \begin{cases} \varphi(y), & y \in \mathbb{R}_\pm \\ \varphi(-y), & y \in \mathbb{R}_\mp \end{cases} \qquad \text{and} \qquad \ell^o\varphi(y) = \begin{cases} \varphi(y), & y \in \mathbb{R}_\pm \\ -\varphi(-y), & y \in \mathbb{R}_\mp \end{cases},$$

respectively.

Remark 3.3 (Cf. [19]). The following operators

$$\begin{aligned} &\ell^e : H^{\varepsilon+\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon+\frac{1}{2}}(\mathbb{R}), \qquad \ell^o : r_{\mathbb{R}_\pm}\widetilde{H}^{\varepsilon+\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon+\frac{1}{2}}(\mathbb{R}), \\ &\ell^o : H^{\varepsilon-\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon-\frac{1}{2}}(\mathbb{R}), \qquad \ell^e : r_{\mathbb{R}_\pm}\widetilde{H}^{\varepsilon-\frac{1}{2}}(\mathbb{R}_\pm) \longrightarrow H^{\varepsilon-\frac{1}{2}}(\mathbb{R}), \end{aligned}$$

are continuous for all $\varepsilon \in [0, 1/2)$.

Lemma 3.4 (Cf. [10]). *If $\varepsilon \in [0, \frac{1}{2})$, then*

$$r_{\Gamma_2} \circ V \circ \ell^o\psi = 0, \qquad r_{\Gamma_2} \circ W \circ \ell^o\tilde{\varphi} = 0,$$

$$r_{\Gamma_2} \circ \partial_{n_2} V \circ \ell^e\tilde{\psi} = 0, \qquad r_{\Gamma_2} \circ \partial_{n_2} W \circ \ell^e\varphi = 0$$

for all $\psi \in H^{\varepsilon-\frac{1}{2}}(\mathcal{S})$, $\tilde{\psi} \in r_{\mathcal{S}}\widetilde{H}^{\varepsilon-\frac{1}{2}}(\mathcal{S})$, $\varphi \in H^{\varepsilon+\frac{1}{2}}(\mathcal{S})$, and $\tilde{\varphi} \in r_{\mathcal{S}}\widetilde{H}^{\varepsilon+\frac{1}{2}}(\mathcal{S})$.

Note that analogous results are valid for the operators V_2 and W_2.

4 The problems in the form of Wiener-Hopf plus Hankel equations

In the present section, we will equivalently write our problems in the form of single equations characterized by particular operators which will depend on algebraic sums of Wiener-Hopf and Hankel operators. These operators will be originated by an appropriate use of certain pseudodifferential operators (introduced in the last section), and also a convenient use of odd and even extension operators. In view to formalize these operators later on, we will now introduce the reflection operator J given by the rule

$$J\psi(y) = \psi(-y) \quad \text{for all} \quad y \in \mathbb{R}.$$

Lemma 4.1. *Let one of the conditions (a)–(f) of Theorem 3.1 be satisfied, then operators*

$$\Phi_{\mathrm{j}} := \frac{1}{2}I + p_{\mathrm{j}}\mathcal{H} : H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}) \longrightarrow H^{-\frac{1}{2}+\varepsilon}(\mathbb{R}), \quad \mathrm{j} = 1, 2, \tag{4.1}$$

are invertible.

Proof. Note that the invertibility of the operators (4.1) can be easily derived from the analogous results on operators $\mathcal{L} - \frac{p_{\mathrm{j}}}{2}I$ from [10, Section 5], using the mapping properties of the operator $\mathcal{H}$ and the identities $\mathcal{L}\mathcal{H} = \mathcal{H}\mathcal{L} = -\frac{1}{4}I$; cf. [7, 8, 10]. □

From now on we assume that one of the conditions (a)–(f) of Theorem 3.1 is satisfied.

4.1 The $\mathcal{P}_{I-D}$ problem

We start by considering the boundary value problem $\mathcal{P}_{I-D}$ in the following equivalent form: Find $u_{\mathrm{j}} \in H^{1+\varepsilon}(\Omega_{\mathrm{j}})$, $\mathrm{j} = 1, 2$, such that

$$\left(\Delta + k^2\right)u_{\mathrm{j}} = 0 \qquad \text{in} \qquad \Omega_j, \tag{4.2}$$

$$\left[u_{\mathrm{j}}\right]^{+}_{\mathcal{S}_{\mathrm{j}}} = h_{\mathrm{j}} \qquad \text{on} \qquad \mathcal{S}_{\mathrm{j}}, \tag{4.3}$$

$$[\partial_{n_1}u_1]^{+}_{C} - p_1\,[u_1]^{+}_{C} = g_1, \quad [\partial_{n_1}u_2]^{-}_{C} + p_2\,[u_2]^{-}_{C} = g_2 \qquad \text{on} \qquad C, \tag{4.4}$$

and

$$[u_1]^{+}_{C^c} - [u_2]^{-}_{C^c} = 0, \quad [\partial_{n_1}u_1]^{+}_{C^c} - [\partial_{n_1}u_2]^{-}_{C^c} = 0 \quad \text{on} \quad C^c, \tag{4.5}$$

where $C^c = \mathcal{S}\backslash\overline{C}$.

Let us consider the following functions

$$u_1 := -V_1\Phi_1^{-1}\ell^o r_{\mathcal{S}}\psi + H_1 \quad \text{in } \Omega_1, \tag{4.6}$$

and

$$u_2 := V_1\Phi_2^{-1}\ell^o r_{\mathcal{S}}(\psi - \varphi) + H_2 \quad \text{in } \Omega_2, \tag{4.7}$$

where

$$H_1 := -V_1\Phi_1^{-1}\left(\ell^o(\ell_+ g_1 + 2p_1[W_2(\ell^e h_1)]^{+}_{\mathcal{S}})\right) + 2W_2(\ell^e h_1) \quad \text{in } \Omega_1,$$

and

$$H_2 := V_1\Phi_2^{-1}\left(\ell^o(\ell_+g_2 - 2p_2[W_2(\ell^e h_2)]_{\mathcal{S}}^+)\right) + 2W_2(\ell^e h_2) \quad \text{in } \Omega_2;$$

here ψ and φ are arbitrary elements of the spaces $\widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$ and $\widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$, respectively; $\ell_+g_1 \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{S})$ is any fixed extension of $g_1 \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{C})$, while $\ell_+g_2 \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{S})$ denotes the extension of $g_2 \in H^{-\frac{1}{2}+\varepsilon}(\mathcal{C})$ which satisfies the condition $r_{\mathcal{C}^c}(\ell_+g_1 - \ell_+g_2) = 0$. Note that such extension exists due to the compatibility condition (2.4). Furthermore, the operators Φ_1^{-1} and Φ_2^{-1} denote the inverse of the operators Φ_1 and Φ_2, respectively, cf. (4.1). Note also that, the functions H_1 and H_2 are well defined and are known.

Using the results from Section 3 and noting that $\Phi_{\mathrm{j}}^{-1}\ell^o = \ell^o r_{\mathcal{S}}\Phi_{\mathrm{j}}^{-1}\ell^o$ (cf. [10]) it is easy to verify that u_{j}, j $= 1,2$, belong to the spaces $H^{1+\varepsilon}(\Omega_{\mathrm{j}})$ and satisfy equations (4.2)-(4.4). Thus it remains to satisfy the conditions (4.5). The first condition of (4.5) gives us

$$-r_{\mathcal{C}^c}\mathcal{H}(\Phi_1^{-1} + \Phi_2^{-1})\ell^o r_S\psi + r_{\mathcal{C}^c}\mathcal{H}\Phi_2^{-1}\ell^o r_S\varphi = H_{ID}^1,$$

where

$$H_{ID}^1 := [H_2]_{\mathcal{C}^c}^- - [H_1]_{\mathcal{C}^c}^+ \in H^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c).$$

From the second condition of (4.5), we have

$$0 = [\partial_{n_1}u_1]_{\mathcal{C}^c}^+ - [\partial_{n_1}u_2]_{\mathcal{C}^c}^- = [\partial_{n_1}u_1]_{\mathcal{C}^c}^+ - p_1[u_1]_{\mathcal{C}^c}^+ - [\partial_{n_1}u_2]_{\mathcal{C}^c}^- - p_2[u_2]_{\mathcal{C}^c}^- + (p_1+p_2)[u_1]_{\mathcal{C}^c}^+,$$

which leads us to the following equation

$$-(p_1+p_2)r_{\mathcal{C}^c}\mathcal{H}\Phi_1^{-1}\ell^o r_S\psi + r_{\mathcal{C}^c}\ell^o r_S\varphi = H_{ID}^2,$$

where

$$H_{ID}^2 := -(p_1+p_2)[H_1]_{\mathcal{C}^c}^+ \in H^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c).$$

Thus we arrived to the following equation

$$r_{\mathcal{C}^c}\mathcal{K}\ell^o r_{\mathcal{S}}\Upsilon = H_{ID}, \tag{4.8}$$

where

$$\mathcal{K} := \begin{pmatrix} -\mathcal{H}(\Phi_1^{-1}+\Phi_2^{-1}) & \mathcal{H}\Phi_2^{-1} \\ -(p_1+p_2)\mathcal{H}\Phi_1^{-1} & I \end{pmatrix}, \quad \Upsilon := \begin{pmatrix} \psi \\ \varphi \end{pmatrix},$$

and $H_{ID} := (H_{ID}^1, H_{ID}^2)^\top$ is a known vector function. Therefore we need to investigate the invertibility of the operator

$$r_{\mathcal{C}^c}\mathcal{K}\ell^o r_{\mathcal{S}} : \begin{array}{c} \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \\ \oplus \\ \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \end{array} \longrightarrow \begin{array}{c} H^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \\ \oplus \\ H^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c) \end{array}. \tag{4.9}$$

With the help of the operator J and the shift convolution operators

$$\mathrm{Op}(\tau_{\pm a}) := \mathcal{F}^{-1}\tau_{\pm a}\cdot\mathcal{F},$$

where $\tau_b(\xi) := e^{ib\xi}$, $\xi \in \mathbb{R}$, we equivalently reduce the problem to the invertibility of the operator

$$r_{\mathbb{R}_+}\mathcal{K}_{--} : \begin{array}{c} \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \end{array} \longrightarrow \begin{array}{c} H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \end{array}, \tag{4.10}$$

where

$$\mathcal{K}_{--} := \mathcal{K}\mathrm{diag}\{I - \mathrm{Op}(\tau_{-2a})J, I - \mathrm{Op}(\tau_{-2a})J\}.$$

Let us note here that because of Theorem 3.1 and having in mind the exhibited limit relations of the potentials, we already know that $\mathrm{Ker}\, r_{\mathbb{R}_+}\mathcal{K}_{--} = \{0\}$.

4.2 The $\mathcal{P}_{I-N}$ problem

As in the previous subsection, the boundary value problem $\mathcal{P}_{I-N}$ can be equivalently rewritten in the following form: Find $u_j \in H^{1+\varepsilon}(\Omega_j)$, j = 1,2, such that

$$\left(\Delta + k^2\right)u_j = 0 \qquad \text{in} \qquad \Omega_j, \tag{4.11}$$

$$\left[\partial_{n_2}u_j\right]^+_{\mathcal{S}_j} = f_j \qquad \text{on} \qquad \mathcal{S}_j, \tag{4.12}$$

$$[\partial_{n_1}u_1]^+_{\mathcal{C}} - p_1\,[u_1]^+_{\mathcal{C}} = g_1, \quad [\partial_{n_1}u_2]^-_{\mathcal{C}} + p_2\,[u_2]^-_{\mathcal{C}} = g_2 \qquad \text{on} \qquad \mathcal{C}, \tag{4.13}$$

and

$$[u_1]^+_{\mathcal{C}^c} - [u_2]^-_{\mathcal{C}^c} = 0, \quad [\partial_{n_1}u_1]^+_{\mathcal{C}^c} - [\partial_{n_1}u_2]^-_{\mathcal{C}^c} = 0 \quad \text{on} \quad \mathcal{C}^c, \tag{4.14}$$

where $\mathcal{C}^c = \mathcal{S}\backslash\overline{\mathcal{C}}$.

Let us consider the following functions

$$u_1 := -V_1\Phi_1^{-1}\ell^e r_{\mathcal{S}}\psi + F_1 \quad \text{in } \Omega_1, \tag{4.15}$$

and

$$u_2 := V_1\Phi_2^{-1}\ell^e r_{\mathcal{S}}(\psi - \varphi) + F_2 \quad \text{in } \Omega_2, \tag{4.16}$$

where

$$F_1 := -V_1\mathcal{B}_1^{-1}\left(\ell^e(\ell_+ g_1 + 2[\partial_{n_1}V_2(\ell^o f_1)]^+_{\mathcal{S}})\right) - 2V_2(\ell^o f_1) \quad \text{in } \Omega_1,$$

and

$$F_2 := V_1\mathcal{B}_2^{-1}\left(\ell^e(\ell_+ g_2 + 2[\partial_{n_1}V_2(\ell^o f_2)]^+_{\mathcal{S}})\right) - 2V_2(\ell^o f_2) \quad \text{in } \Omega_2;$$

here ψ and φ are arbitrary elements of the spaces $\widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$ and $\widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathcal{C}^c)$, respectively, as above. Note also that, the functions F_1 and F_2 are well defined (cf. Remark 3.3 and compatibility conditions (2.5)) and are known.

Using the results from Section 3 and noting that $\Phi_j^{-1}\ell^e = \ell^e r_{\mathcal{S}}\Phi_j^{-1}\ell^o$ (cf. [10]) it we have that u_j, j = 1,2 belong to the spaces $H^{1+\varepsilon}(\Omega_j)$ and satisfy the equations (4.2)-(4.4). The first condition of (4.5) gives us

$$-r_{\mathcal{C}^c}\mathcal{H}(\Phi_1^{-1} + \Phi_2^{-1})\ell^e r_s\psi + r_{\mathcal{C}^c}\mathcal{H}\Phi_2^{-1}\ell^e r_s\varphi = F^1_{IN},$$

where

$$F^1_{IN} := [F_2]^-_{C^c} - [F_1]^+_{C^c} \in H^{\frac{1}{2}+\varepsilon}(C^c).$$

On the other hand, the second condition of (4.14) leads us to

$$0 = [\partial_{n_1} u_1]^+_{C^c} - [\partial_{n_1} u_2]^-_{C^c} = [\partial_{n_1} u_1]^+_{C^c} - p_1[u_1]^+_{C^c} - [\partial_{n_1} u_2]^-_{C^c} - p_2[u_2]^-_{C^c} + (p_1+p_2)[u_1]^+_{C^c},$$

which gives rise to the equation

$$-(p_1+p_2) r_{C^c}\mathcal{H}\Phi_1^{-1}\ell^e r_s\psi + r_{C^c}\ell^e r_s\varphi = F^2_{IN},$$

where

$$F^2_{IN} := -(p_1+p_2)[F_1]^+_{C^c} \in H^{\frac{1}{2}+\varepsilon}(C^c).$$

Thus, altogether, we arrived to the following equation

$$r_{C^c}\mathcal{K}\ell^e r_{\mathcal{S}}\Upsilon = F_{IN}, \tag{4.17}$$

where

$$\mathcal{K} := \begin{pmatrix} -\mathcal{H}(\Phi_1^{-1}+\Phi_2^{-1}) & \mathcal{H}\Phi_2^{-1} \\ -(p_1+p_2)\mathcal{H}\Phi_1^{-1} & I \end{pmatrix}, \quad \Upsilon := \begin{pmatrix} \psi \\ \varphi \end{pmatrix},$$

and $F_{IN} := (F^1_{IN}, F^2_{IN})^\top$ is a known vector function. Therefore, by similar arguments as above, we need to investigate the invertibility of the operator

$$r_{\mathbb{R}_+}\mathcal{K}_{++} : \begin{array}{c} \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \end{array} \longrightarrow \begin{array}{c} H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+). \end{array} \tag{4.18}$$

where

$$\mathcal{K}_{++} := \mathcal{K}\mathrm{diag}\{I+\mathrm{Op}(\tau_{-2a})J, I+\mathrm{Op}(\tau_{-2a})J\}.$$

and having the property $\mathrm{Ker}\, r_{\mathbb{R}_+}\mathcal{K}_{++} = \{0\}$.

5 Analysis of Wiener-Hopf plus and minus Hankel operators

In this section we will consider general operators with the global structure of Wiener-Hopf plus and minus Hankel operators, and we will recall – in an appropriate framework for our purposes – some known operator relations between these operators and Wiener-Hopf operators.

In view of this, let us also recall that two bounded linear operators T and S (acting between Banach spaces) are said to be *equivalent* if $T = ESF$ for some boundedly invertible operators E and F. In such a case we will write $T \sim S$. In addition, when the use of identity extension operators is needed in combination with the related operators T and S, such corresponding relations are referred to as a *(toplinear) equivalence after extension* (see [2, 18] for a detailed description about such operator relations).

Let us define

$$\Lambda^s_\pm(\xi) := (\xi \pm i)^s = (1+\xi^2)^{\frac{s}{2}} \exp\{s\, i \arg(\xi \pm i)\},$$

with a branch chosen in such a way that $\arg(\xi \pm i) \to 0$ as $\xi \to +\infty$, i.e., with a cut along the negative real axis (see Example 1.7 in [22] for additional information about the properties of these functions). In addition, we will also use the notation

$$\zeta(\xi) := \frac{\Lambda_-(\xi)}{\Lambda_+(\xi)} = \frac{\xi - i}{\xi + i}, \quad \xi \in \mathbb{R}.$$

Lemma 5.1. [22, §4] *Let $s, r \in \mathbb{R}$, and consider the operators*

$$\begin{array}{rcl} \Lambda_+^s(D) & = & (D+i)^s \\ \Lambda_-^s(D) & = & r_{\mathbb{R}_+}(D-i)^s \ell^{(r)}, \end{array}$$

where $(D \pm i)^{\pm s} = \mathcal{F}^{-1}(\xi \pm i)^{\pm s} \cdot \mathcal{F}$, and $\ell^{(r)} : H^r(\mathbb{R}_+) \to H^r(\mathbb{R})$ is any bounded extension operator in these spaces (which particular choice does not change the definition of $\Lambda_-^s(D)$).

These operators arrange isomorphisms in the following space settings

$$\begin{array}{rcl} \Lambda_+^s(D) & : & \widetilde{H}^r(\mathbb{R}_+) \to \widetilde{H}^{r-s}(\mathbb{R}_+), \\ \Lambda_-^s(D) & : & H^r(\mathbb{R}_+) \to H^{r-s}(\mathbb{R}_+) \end{array}$$

(for any $s, r \in \mathbb{R}$).

Bearing in mind the purpose of this section, let $A_{\mathrm{ij}} = \mathrm{Op}(a_{\mathrm{ij}}) = \mathcal{F}^{-1} a_{\mathrm{ij}} \cdot \mathcal{F}$ and $B_{\mathrm{ij}} = \mathrm{Op}(b_{\mathrm{ij}})$ be pseudodifferential operators of order $\mu_{\mathrm{ij}} \in \mathbb{R}$; thus, $\langle\cdot\rangle^{-\mu_{\mathrm{ij}}} a_{\mathrm{ij}}, \langle\cdot\rangle^{-\mu_{\mathrm{ij}}} b_{\mathrm{ij}} \in L^\infty(\mathbb{R})$, where $\langle\xi\rangle := (1+\xi^2)^{\frac{1}{2}}$ and $\mathrm{i}, \mathrm{j} = 1, 2$. Since the operators $r_{\mathbb{R}_+}(A_{\mathrm{ij}} + B_{\mathrm{ij}} J)$ arrange continuous maps

$$r_{\mathbb{R}_+}(A_{\mathrm{ij}} + B_{\mathrm{ij}} J) : \widetilde{H}^s(\mathbb{R}_+) \to H^{s-\mu_{\mathrm{ij}}}(\mathbb{R}_+)$$

for all $s \in \mathbb{R}$, then 2×2 matrix operator

$$A + BJ = \begin{pmatrix} A_{11} + B_{11}J & A_{12} + B_{12}J \\ A_{21} + B_{21}J & A_{22} + B_{22}J \end{pmatrix}, \quad A = (A_{\mathrm{ij}})_{\mathrm{i,j}=1,2}, \ B = (B_{\mathrm{ij}})_{\mathrm{i,j}=1,2}$$

arrange continuous maps

$$r_{\mathbb{R}_+}(A+BJ) : \begin{array}{c} \widetilde{H}^{-\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \end{array} \to \begin{array}{c} H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \\ \oplus \\ H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \end{array}$$

where $A_{\mathrm{ij}} = \mathrm{Op}(a_{\mathrm{ij}})$, $B_{\mathrm{ij}} = \mathrm{Op}(\tau_{-2a} a_{\mathrm{ij}})$, for $\mathrm{i}, \mathrm{j} = 1, 2$, and

$$\begin{array}{rcl} a_{11}(\xi) & := & -\sigma(\mathcal{H})(\xi)([\sigma(\Phi_1)(\xi)]^{-1} + [\sigma(\Phi_2)(\xi)]^{-1}), \\ a_{12}(\xi) & := & \sigma(\mathcal{H})(\xi)[\sigma(\Phi_2)(\xi)]^{-1}, \\ a_{21}(\xi) & := & -(p_1 + p_2)\sigma(\mathcal{H})(\xi)[\sigma(\Phi_1)(\xi)]^{-1}, \\ a_{22}(\xi) & := & 1. \end{array}$$

Recall that the complete symbols of the pseudodifferential operators $\mathcal{H}$ and Φ_{j} are (cf. [10, 11]):

$$\sigma(\mathcal{H})(\xi) = -\frac{i}{2w(\xi)} \quad \text{and} \quad \sigma(\Phi_{\mathrm{j}})(\xi) = \frac{1}{2} - p_{\mathrm{j}} \frac{i}{2w(\xi)}, \tag{5.1}$$

where $w = w(\xi) := (\varrho^2 + \rho^2)^{\frac{1}{4}}(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2})$, with

$$\varrho = \varrho(\xi) := (\Re e\, k)^2 - (\Im m\, k)^2 - \xi^2,$$
$$\rho := 2(\Re e\, k)(\Im m\, k)$$

and

$$\alpha := \begin{cases} \arctan\frac{\rho}{|\varrho|} & \text{if} \quad \varrho > 0,\ \rho > 0 \\ \frac{\pi}{2} & \text{if} \quad \varrho = 0,\ \rho > 0 \\ \pi - \arctan\frac{\rho}{|\varrho|} & \text{if} \quad \varrho < 0,\ \rho > 0 \\ \pi & \text{if} \quad \rho = 0 \\ 2\pi - \arctan\frac{|\rho|}{|\varrho|} & \text{if} \quad \varrho > 0,\ \rho < 0 \\ \frac{3\pi}{2} & \text{if} \quad \varrho = 0,\ \rho < 0 \\ \pi + \arctan\frac{|\rho|}{|\varrho|} & \text{if} \quad \varrho < 0,\ \rho < 0 \end{cases}. \tag{5.2}$$

Lemma 5.1 allows us to construct an equivalence relation between $r_{\mathbb{R}_+}(A + BJ)$ and

$$r_{\mathbb{R}_+}(\mathcal{A} + \mathcal{B}J) : [L_2(\mathbb{R}_+)]^2 \to [L_2(\mathbb{R}_+)]^2, \tag{5.3}$$

which is explicitly given by the following identity

$$r_{\mathbb{R}_+}(\mathcal{A} + \mathcal{B}J) := \operatorname{diag}\{\Lambda_-^{\frac{1}{2}+\varepsilon}, \Lambda_-^{\frac{1}{2}+\varepsilon}\}\, r_{\mathbb{R}_+}(A + BJ)\, \operatorname{diag}\{\Lambda_+^{\frac{1}{2}-\varepsilon}, \Lambda_+^{-\frac{1}{2}-\varepsilon}\}, \tag{5.4}$$

where $\mathcal{A} := (\mathcal{A}_{\rm ij})_{\rm i,j=1,2}$, $\mathcal{B} := (\mathcal{B}_{\rm ij})_{\rm i,j=1,2}$, with

$$\mathcal{A}_{\rm ij} := (D - i)^{\frac{1}{2}+\varepsilon} A_{\rm ij}(D + i)^{-r_{\rm j}}, \quad \mathcal{B}_{\rm ij} := (D - i)^{\frac{1}{2}+\varepsilon} B_{\rm ij} J (D + i)^{-r_{\rm j}} J, \tag{5.5}$$

for $r_1 := -\frac{1}{2} + \varepsilon$, $r_2 := \frac{1}{2} + \varepsilon$. Due to the fact that $\Lambda_-^{s-\mu} : H^{s-\mu}(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$ and $\Lambda_+^{-s} : L_2(\mathbb{R}_+) \to \widetilde{H}^s(\mathbb{R}_+)$ are invertible operators (cf. Lemma 5.1), the identity (5.4) shows that

$$r_{\mathbb{R}_+}(A + BJ) \sim r_{\mathbb{R}_+}(\mathcal{A} + \mathcal{B}J).$$

Note that

$$\Lambda_+^s(-\xi) = \Lambda_-^s(\xi)\, e^{s\pi i}, \qquad \Lambda_-^s(-\xi) = \Lambda_+^s(\xi)\, e^{-s\pi i}$$

which in particular allow us to describe the operators $\mathcal{A}_{\rm ij}$ and $\mathcal{B}_{\rm ij}$ and their symbols in the following way

$$\begin{aligned} \mathcal{A}_{\rm ij} &= \operatorname{Op}(\tilde{a}_{\rm ij}), \qquad & \tilde{a}_{\rm ij}(\xi) &= \Lambda_-^{\frac{1}{2}+\varepsilon}(\xi)\, a_{\rm ij}(\xi)\, \Lambda_+^{-r_j}(\xi), \\ \mathcal{B}_{\rm ij} &= \operatorname{Op}(\tilde{b}_{\rm ij}), \qquad & \tilde{b}_{\rm ij}(\xi) &= \Lambda_-^{\frac{1}{2}+\varepsilon}(\xi)\, b_{\rm ij}(\xi)\, \Lambda_+^{-r_j}(-\xi) = \Lambda_-^{\frac{1}{2}+\varepsilon-r_j}(\xi)\, b_{\rm ij}(\xi)\, e^{-r_j\pi i}. \end{aligned}$$

In particular, we have $\sigma(\mathcal{A})(\xi) = (\tilde{a}_{\rm ij}(\xi))_{\rm i,j=1,2}$ with

$$\tilde{a}_{11}(\xi) = \langle\xi\rangle \zeta^{\varepsilon}(\xi)\, a_{11}(\xi), \qquad \tilde{a}_{12}(\xi) = \zeta^{\frac{1}{2}+\varepsilon}(\xi)\, a_{12}(\xi),$$
$$\tilde{a}_{21}(\xi) = \langle\xi\rangle \zeta^{\varepsilon}(\xi)\, a_{21}(\xi), \qquad \tilde{a}_{22}(\xi) = \zeta^{\frac{1}{2}+\varepsilon}(\xi)\, a_{22}(\xi),$$

and $\sigma(\mathcal{B})(\xi) = (\tilde{b}_{\rm ij}(\xi))_{\rm i,j=1,2}$, where

$$\begin{array}{rcl}
\tilde{b}_{11}(\xi) & = & i(\xi - i)\tau_{-2a}(\xi) e^{-\varepsilon\pi i} a_{11}(\xi), \\
\tilde{b}_{12}(\xi) & = & -i\,\tau_{-2a}(\xi)\, e^{-\varepsilon\pi i}\, a_{12}(\xi), \\
\tilde{b}_{21}(\xi) & = & i\,(\xi - i)^{-1}\tau_{-2a}(\xi)\, e^{-\varepsilon\pi i}\, a_{21}(\xi), \\
\tilde{b}_{22}(\xi) & = & -i\tau_{-2a}(\xi)\, e^{-\varepsilon\pi i}\, a_{22}(\xi).
\end{array}$$

Thus

$$r_{\mathbb{R}_+}\mathcal{K}_{++} \sim r_{\mathbb{R}_+}(\mathcal{A} + \mathcal{B}J) \quad \text{and} \quad r_{\mathbb{R}_+}\mathcal{K}_{--} \sim r_{\mathbb{R}_+}(\mathcal{A} - \mathcal{B}J). \tag{5.6}$$

Further, let us consider a pseudodifferential operator $\mathrm{Op}(\Xi)$ with 4×4 matrix symbol $\Xi(\xi)$ partitioned into four 2×2 blocks $\alpha_{\rm ij}$, $\rm i, j = 1, 2$:

$$\Xi(\xi) := \begin{pmatrix} \alpha_{11}(\xi) & \alpha_{12}(\xi) \\ \alpha_{21}(\xi) & \alpha_{22}(\xi) \end{pmatrix} \tag{5.7}$$

with

$$\begin{array}{rcl}
\alpha_{11}(\xi) & := & \sigma(\mathcal{A})(\xi) - \sigma(\mathcal{B})(\xi)[\sigma(\mathcal{A})(-\xi)]^{-1}\sigma(\mathcal{B})(-\xi), \\
\alpha_{12}(\xi) & := & -\sigma(\mathcal{B})(\xi)\,[\sigma(\mathcal{A})(-\xi)]^{-1}, \\
\alpha_{21}(\xi) & := & [\sigma(\mathcal{A})(-\xi)]^{-1}\,\sigma(\mathcal{B})(-\xi), \\
\alpha_{22}(\xi) & := & [\sigma(\mathcal{A})(-\xi)]^{-1}.
\end{array}$$

The direct calculation shows that α_{11} is the null matrix, i.e., $\alpha_{11}(\xi) \equiv 0$, while

$$\begin{array}{rcl}
\alpha_{12}(\xi) & = & -i\,\tau_{-2a}(\xi)\, e^{\varepsilon\pi i} \zeta^{\frac{1}{2}+\varepsilon}(\xi) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\alpha_{21}(\xi) & = & i\,\tau_{2a}(\xi)\, e^{\varepsilon\pi i} \zeta^{\varepsilon}(\xi) \begin{pmatrix} -\zeta^{-\frac{1}{2}}(\xi) & 0 \\ 0 & \zeta^{\frac{1}{2}}(\xi) \end{pmatrix}, \\
\alpha_{22}(\xi) & = & \zeta^{\varepsilon}(\xi) e^{2\varepsilon\pi i} \dfrac{1}{\rho(\xi)} \begin{pmatrix} -\zeta^{-\frac{1}{2}}(\xi) a_{22}(\xi) & \zeta^{-\frac{1}{2}}(\xi) a_{12}(\xi) \\ -\langle\xi\rangle a_{21}(\xi) & \langle\xi\rangle a_{11}(\xi) \end{pmatrix},
\end{array}$$

where

$$\rho(\xi) := \langle\xi\rangle\zeta^{-\frac{1}{2}}(\xi)(-a_{11}(\xi)a_{22}(\xi) + a_{21}(\xi)a_{12}(\xi)) = \frac{\langle\xi\rangle\zeta^{-\frac{1}{2}}(\xi)\sigma(\mathcal{H})(\xi)}{\sigma(\Phi_1)(\xi)\sigma(\Phi_2)(\xi)}.$$

Under the above conditions it is straightforward to conclude that

$$r_{\mathbb{R}_+}\mathrm{Op}(\Xi) : [L_2(\mathbb{R}_+)]^4 \to [L_2(\mathbb{R}_+)]^4 \tag{5.8}$$

is a continuous operator. Moreover, it is easy to see that the determinant of the symbol of this operator, as well as $\rho(\xi)$, is always nonzero, for all $\xi \in \mathbb{R}$.

The importance of the operator $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ is clarified by the next result.

Theorem 5.2. *(i) The operators*

$$r_{\mathbb{R}_+}\mathcal{A} \pm r_{\mathbb{R}_+}\mathcal{B}J : [L_2(\mathbb{R}_+)]^2 \to [L_2(\mathbb{R}_+)]^2$$

(defined in (5.3)–(5.5)) are both invertible if and only if the operator $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ *(given in (5.8)) is invertible.*

(ii) The operators $r_{\mathbb{R}_+}\mathcal{A} + r_{\mathbb{R}_+}\mathcal{B}J$ *and* $r_{\mathbb{R}_+}\mathcal{A} - r_{\mathbb{R}_+}\mathcal{B}J$ *have both the Fredholm property if and only if* $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ *has the Fredholm property. In addition, if all three operators are Fredholm, their Fredholm indices satisfy the identity*

$$Ind\,(r_{\mathbb{R}_+}\mathcal{A} + r_{\mathbb{R}_+}\mathcal{B}J) + Ind\,(r_{\mathbb{R}_+}\mathcal{A} - r_{\mathbb{R}_+}\mathcal{B}J) = Ind\, r_{\mathbb{R}_+}\mathrm{Op}(\Xi)\,. \tag{5.9}$$

We would like to notice here that this theorem is a consequence of a even stronger result which basically states that $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ is (toplinear) equivalent after extension to a diagonal block matrix operator whose diagonal entries are the operators $r_{\mathbb{R}_+}\mathcal{A} + r_{\mathbb{R}_+}\mathcal{B}J$ and $r_{\mathbb{R}_+}\mathcal{A} - r_{\mathbb{R}_+}\mathcal{B}J$. Moreover, it is interesting to clarify that all the necessary operators to identify such (toplinear) equivalence after extension relation can be written explicitly (see [15, 16, 17, 18]).

Having in mind the Theorem 5.2, now we would like to investigate the Wiener-Hopf operator

$$r_{\mathbb{R}_+}\mathrm{Op}(\Xi) : [L_2(\mathbb{R}_+)]^4 \to [L_2(\mathbb{R}_+)]^4. \tag{5.10}$$

We then realize that Ξ belongs to the C^*-algebra of the semi-almost periodic two by two matrix functions on the real line ($[SAP(\mathbb{R})]^{4\times4}$); see [41]. We recall that $[SAP(\mathbb{R})]^{4\times4}$ is the smallest closed subalgebra of $[L^\infty(\mathbb{R})]^{4\times4}$ that contains the (classical) algebra of (four by four) *almost periodic elements* ($[AP]^{4\times4}$) and the (four by four) continuous matrices with possible jumps at infinity.

Due to a known characterization of the structure of $[SAP(\mathbb{R})]^{4\times4}$ (see [4, 5, 41]), we can choose a continuous function on the real line, say γ, such that $\gamma(-\infty) = 0$, $\gamma(+\infty) = 1$, and

$$\Xi = (1-\gamma)\Xi_l + \gamma\Xi_r + \Xi_0,$$

where Ξ_0 is a continuous four by four matrix function with zero limit at infinity, and Ξ_l and Ξ_r are matrices with almost periodic elements, uniquely determined by Ξ, and that in our case have the following form (due to the behavior of Ξ at $\pm\infty$):

$$\Xi_l = \begin{pmatrix} 0 & 0 & i\,\tau_{-2a}\,e^{-\varepsilon\pi i} & 0 \\ 0 & 0 & 0 & i\,\tau_{-2a}\,e^{-\varepsilon\pi i} \\ i\,\tau_{2a}\,e^{-\varepsilon\pi i} & 0 & \frac{1}{2} & 0 \\ 0 & -i\,\tau_{2a}\,e^{-\varepsilon\pi i} & -\frac{(p_1+p_2)}{2} & -1 \end{pmatrix},$$

$$\Xi_r = \begin{pmatrix} 0 & 0 & -i\,\tau_{-2a}\,e^{\varepsilon\pi i} & 0 \\ 0 & 0 & 0 & -i\,\tau_{-2a}\,e^{\varepsilon\pi i} \\ -i\,\tau_{2a}\,e^{\varepsilon\pi i} & 0 & \frac{1}{2}\,e^{2\varepsilon\pi i} & 0 \\ 0 & i\,\tau_{2a}\,e^{\varepsilon\pi i} & \frac{(p_1+p_2)}{2}\,e^{2\varepsilon\pi i} & -\,e^{2\varepsilon\pi i} \end{pmatrix}.$$

It is worth noting here that so far we have assumed

$$\omega(\xi) \to i|\xi| \text{ as } \xi \to \pm\infty, \qquad \zeta^{\nu}(\xi) \to 1 \text{ as } \xi \to \infty \qquad \text{and} \qquad \zeta^{\nu}(\xi) \to e^{-2\pi\nu i} \text{ as } \xi \to -\infty.$$

For a given Banach algebra (with unit element) $\mathcal{M}$, by $\mathcal{GM}$ we will denote the collection of all invertible elements of $\mathcal{M}$.

Definition 5.3 (See, e.g., [23] or §6.3 in [6])**.** An invertible almost periodic matrix function $\Phi \in \mathcal{G}[AP]^{4\times 4}$ admits a *canonical right AP-factorization* if

$$\Phi = \Phi^{-}\Phi^{+}, \tag{5.11}$$

where $\Phi^{\pm} \in \mathcal{G}[AP^{\pm}]^{4\times 4}$, with $AP^{\pm}$ denoting the intersection of AP with the non-tangential limits of functions in $H^{\infty}(\mathbb{C}_{\pm})$ (the set of all bounded and analytic functions in $\mathbb{C}_{\pm}$).

Proposition 5.4. *(Cf., e.g., [6, Proposition 2.22]) Let $A \subset (0,\infty)$ be an unbounded set and let*

$$\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$$

be a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$ as $\alpha \to \infty$. If $\varphi \in AP$, then the limit

$$M(\varphi) := \lim_{\alpha\to\infty} \frac{1}{|I_\alpha|}\int_{I_\alpha} \varphi(x)\,dx$$

exists, is finite, and is independent of the particular choice of the family $\{I_\alpha\}$.

Definition 5.5. (i) For any $\varphi \in AP$, the number that has just been introduced in Proposition 5.4, $M(\varphi)$, is called the *Bohr mean value* (or simply the *mean value*) of φ. In the matrix case the Bohr mean value is defined entry-wise.

(ii) For a matrix function $\Phi \in \mathcal{G}[AP]^{4\times 4}$ admitting a canonical right AP factorization (5.11), we may define the new matrix

$$\mathbf{d}(\Phi) := M(\Phi^{-})M(\Phi^{+}), \tag{5.12}$$

which is known as the *geometric mean* of Φ.

It is worth mentioning that (5.12) is independent of the particular choice of the (canonical) right AP factorization of Φ, and that this definition is consistent with the corresponding one for the scalar case (which can be defined in a somehow more global way).

Theorem 5.6. *For $\varepsilon \in [0, \frac{1}{2})$, the operator $r_{\mathbb{R}_+}\mathrm{Op}(\Xi) : [L_2(\mathbb{R}_+)]^2 \to [L_2(\mathbb{R}_+)]^2$, with Ξ given by (5.7), is a Fredholm operator with zero Fredholm index.*

Proof. The matrices Ξ_l and Ξ_r admit the following right canonical AP-factorizations:

$$\Xi_l = (\Xi_l)_l(\Xi_l)_r, \quad \text{and} \quad \Xi_r = (\Xi_r)_l(\Xi_r)_r, \tag{5.13}$$

where

$$(\Xi_l)_l = \begin{pmatrix} e^{-2\varepsilon\pi i} & 0 & 2i\,\tau_{-2a}\,e^{-\varepsilon\pi i} & 0 \\ 0 & e^{-2\varepsilon\pi i} & -(p_1+p_2)i\tau_{-2a}\,e^{-\varepsilon\pi i} & -i\,\tau_{-2a}\,e^{-\varepsilon\pi i} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(\Xi_l)_r = \begin{pmatrix} 2 & 0 & 0 & 0 \\ p_1+p_2 & -1 & 0 & 0 \\ i\,\tau_{2a}\,e^{-\varepsilon\pi i} & 0 & \frac{1}{2} & 0 \\ 0 & -i\,\tau_{2a}\,e^{-\varepsilon\pi i} & -\frac{p_1+p_2}{2} & -1 \end{pmatrix},$$

$$(\Xi_r)_l = \begin{pmatrix} 1 & 0 & -2\,i\,\tau_{-2a}\,e^{-\varepsilon\pi i} & 0 \\ 0 & 1 & -(p_1+p_2)\,i\,\tau_{-2a}\,e^{-\varepsilon\pi i} & i\,\tau_{-2a}\,e^{-\varepsilon\pi i} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$(\Xi_r)_r = \begin{pmatrix} 2 & 0 & 0 & 0 \\ p_1+p_2 & 1 & 0 & 0 \\ -i\,\tau_{2a}\,e^{\varepsilon\pi i} & 0 & \frac{1}{2}\,e^{2\varepsilon\pi i} & 0 \\ 0 & i\,\tau_{2a}\,e^{\varepsilon\pi i} & \frac{p_1+p_2}{2}\,e^{2\varepsilon\pi i} & -e^{2\varepsilon\pi i} \end{pmatrix}$$

(in which the necessary factor properties are evident; cf. Definition 5.3).

Having built the factorizations (5.13), we can now apply Theorem 3.2 in [23] or Theorem 10.11 in [6] in view of proving the Fredholm property for $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$. Indeed, within our case of $\Xi \in \mathcal{G}[SAP(\mathbb{R})]^{4\times 4}$ and whose local representatives at infinity admit canonical right AP-factorizations (5.13), applying that theorems we have that $r_{\mathbb{R}_+}\mathrm{Op}(\Xi)$ is a Fredholm operator if and only if

$$\mathrm{sp}\left[\mathbf{d}^{-1}(\Xi_r)\,\mathbf{d}\,(\Xi_l)\right] \cap (-\infty, 0] = \emptyset\,,$$

where $\mathrm{sp}\left[\mathbf{d}^{-1}(\Xi_r)\,\mathbf{d}\,(\Xi_l)\right]$ stands for the set of eigenvalues of the matrix

$$\mathbf{d}^{-1}(\Xi_r)\,\mathbf{d}\,(\Xi_l) := [\mathbf{d}\,(\Xi_r)]^{-1}\,\mathbf{d}\,(\Xi_l)\,.$$

Noticing that directly from the definition of the Bohr mean value we have $M(c) = c$ for any complex constant c and $M(\tau_{\pm 2a}) = 0$, it follows

$$\begin{aligned}
\mathbf{d}(\Xi_l) &= M[(\Xi_l)_l]\, M[(\Xi_l)_r] \\
&= \begin{pmatrix} e^{-2\varepsilon\pi i} & 0 & 0 & 0 \\ 0 & e^{-2\varepsilon\pi i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ p_1+p_2 & -1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{p_1+p_2}{2} & -1 \end{pmatrix} \\
&= \begin{pmatrix} 2e^{-2\varepsilon\pi i} & 0 & 0 & 0 \\ (p_1+p_2)e^{-2\varepsilon\pi i} & -e^{-2\varepsilon\pi i} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{p_1+p_2}{2} & -1 \end{pmatrix},
\end{aligned}$$

$$\mathbf{d}(\Xi_r) = M[(\Xi_r)_l]\, M[(\Xi_r)_r] = \begin{pmatrix} 2 & 0 & 0 & 0 \\ p_1+p_2 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}e^{2\varepsilon\pi i} & 0 \\ 0 & 0 & \frac{p_1+p_2}{2}e^{2\varepsilon\pi i} & e^{2\varepsilon\pi i} \end{pmatrix}.$$

As a consequence,

$$\mathbf{d}^{-1}(\Xi_r)\,\mathbf{d}(\Xi_l) = \begin{pmatrix} e^{-2\varepsilon\pi i} & 0 & 0 & 0 \\ 0 & -e^{-2\varepsilon\pi i} & 0 & 0 \\ 0 & 0 & e^{-2\varepsilon\pi i} & 0 \\ 0 & 0 & -(p_1+p_2)e^{-2\varepsilon\pi i} & -e^{2\varepsilon\pi i} \end{pmatrix}$$

and

$$\mathrm{sp}\left[\mathbf{d}^{-1}(\Xi_r)\,\mathbf{d}(\Xi_l)\right] \cap (-\infty, 0] = \{\pm e^{-2\varepsilon\pi i}\} \cap (-\infty, 0] = \emptyset, \tag{5.14}$$

which allows us to conclude that $r_{\mathbb{R}_+}\mathrm{Op}(\Xi) : [L_2(\mathbb{R}_+)]^4 \to [L_2(\mathbb{R}_+)]^4$ is a Fredholm operator (for the case under consideration of $0 \le \varepsilon < 1/2$; cf. Section 2).

The zero Fredholm index is obtained from the formula (cf. Theorem 10.21 in [6])

$$Ind\, r_{\mathbb{R}_+}\mathrm{Op}(\Xi) = -ind\,[\det \Xi] - \sum_{\mathrm{j}=1}^{4} \Big(\frac{1}{2} - \Big\{\frac{1}{2} - \frac{1}{2\pi} arg\, \xi_{\mathrm{j}}\Big\}\Big),$$

where $ind\,[\det \Xi]$ denotes the Cauchy index of the determinant of Ξ, the numbers $\xi_{\mathrm{j}} \in \mathbb{C}\backslash(-\infty, 0]$, $\mathrm{j} = 1, \dots, 4$ are the eigenvalues of the matrix $\mathbf{d}^{-1}(\Xi_r)\,\mathbf{d}(\Xi_l)$ and $\{\cdot\}$ stands for the fractional part of a real number. □

From the derived condition (5.14), it follows that if we would allow the case $\varepsilon = 1/2$ then our operators would not have the Fredholm property (and therefore would not be invertible operators).

Corollary 5.7. *Let $\varepsilon \in [0, \frac{1}{2})$. The Wiener-Hopf plus and minus Hankel operators* (4.10) *and* (4.18) *(related to the problems $\mathcal{P}_{I-D}$ and $\mathcal{P}_{I-N}$) are invertible operators.*

Proof. Bearing in mind the equivalence relations (5.6), we have:

$$\begin{aligned} \dim \operatorname{CoKer} r_{\mathbb{R}_+} \mathcal{K}_{++} &= \dim \operatorname{CoKer} r_{\mathbb{R}_+} (\mathcal{A} + \mathcal{B}J), && (5.15) \\ \dim \operatorname{Ker} r_{\mathbb{R}_+} \mathcal{K}_{++} &= \dim \operatorname{Ker} r_{\mathbb{R}_+} (\mathcal{A} + \mathcal{B}J). && (5.16) \end{aligned}$$

and

$$\begin{aligned} \dim \operatorname{CoKer} r_{\mathbb{R}_+} \mathcal{K}_{--} &= \dim \operatorname{CoKer} r_{\mathbb{R}_+} (\mathcal{A} - \mathcal{B}J), && (5.17) \\ \dim \operatorname{Ker} r_{\mathbb{R}_+} \mathcal{K}_{--} &= \dim \operatorname{Ker} r_{\mathbb{R}_+} (\mathcal{A} - \mathcal{B}J). && (5.18) \end{aligned}$$

From Theorem 5.2 and Theorem 5.6, we obtain that $r_{\mathbb{R}_+}(\mathcal{A} + \mathcal{B}J)$ and $r_{\mathbb{R}_+}(\mathcal{A} - \mathcal{B}J)$ are Fredholm operators. Moreover, recalling that $\operatorname{Ker} r_{\mathbb{R}_+} \mathcal{K}_{++} = \{0\}$ and $\operatorname{Ker} r_{\mathbb{R}_+} \mathcal{K}_{--} = \{0\}$, from identities (5.9), (5.15)–(5.18) and Theorem 5.6 it follows

$$\begin{aligned} 0 &= \mathit{Ind}\, r_{\mathbb{R}_+}(\mathcal{A} + \mathcal{B}J) + \mathit{Ind}\, r_{\mathbb{R}_+}(\mathcal{A} - \mathcal{B}J) \\ &= \mathit{Ind}\, r_{\mathbb{R}_+} \mathcal{K}_{++} + \mathit{Ind}\, r_{\mathbb{R}_+} \mathcal{K}_{--} \\ &= (0 - \dim \operatorname{CoKer} r_{\mathbb{R}_+} \mathcal{K}_{++}) + (0 - \dim \operatorname{CoKer} r_{\mathbb{R}_+} \mathcal{K}_{--}) \,. \end{aligned}$$

Thus, we have

$$\dim \operatorname{CoKer} r_{\mathbb{R}_+} \mathcal{K}_{++} = \dim \operatorname{CoKer} r_{\mathbb{R}_+} \mathcal{K}_{--} = 0$$

and so we reach to the conclusion that both operators in (4.10) and (4.18) are invertible. □

Due to a direct combination of the results of sections 3 and 4, and Corollary 5.7, we now obtain the main conclusion of the present work for the problems in consideration.

Theorem 5.8. *Let $\varepsilon \in [0, \frac{1}{2})$ and assume that one of the conditions (a)–(f) of Theorem 3.1 is satisfied.*

(i) *The Problem $\mathcal{P}_{I-D}$ has a unique solution which is representable as a pair (u_1, u_2) defined by the formulas* (4.6) *and* (4.7)*, where the components φ and ψ of the unique solution Υ of the equation* (4.8) *are used.*

(ii) *The Problem $\mathcal{P}_{I-N}$ has a unique solution which is representable as a pair (u_1, u_2) defined by the formulas* (4.15) *and* (4.16)*, where the components φ and ψ of the unique solution Υ of the equation* (4.17) *are used.*

Moreover, all these problems are well-posed (since the resolvent operators are continuous).

Acknowledgments

L.P. Castro was supported in part by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e a Tecnologia"), within project PEst-OE/MAT/UI4106/2014. D. Kapanadze was supported by Shota Rustaveli National Science Foundation within grant YS/4/5-102/13 with the number 24/04.

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