SIGN CHANGING SOLUTIONS OF A SEMILINEAR EQUATION ON HEISENBERG GROUP

JIANQING CHEN AND EUGÉNIO M. ROCHA

ABSTRACT. This paper is concerned with the existence of multiple solutions to the semilinear equation $\Delta_H u(\xi)+|u(\xi)|^{\frac{4}{q-2}}u(\xi)+g(\xi,u(\xi))=0$ in a bounded domain Ω of the Heisenberg group $\mathbb{H}^N$ with Dirichlet boundary condition. By using the variational method, we prove that this problem possesses at least one positive solution and at least one sign changing solution for a class of $g(\xi,u)$.

1. Introduction

In this paper, we study the existence and multiplicity of nontrivial solutions of the following problem

$$\begin{cases} -\Delta_H u(\xi) & = |u(\xi)|^{\frac{4}{q-2}}u(\xi)+g(\xi,u(\xi)), \ \xi\in\Omega, \\ u(\xi) & = 0, \quad \xi\in\partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{H}^N$ is a bounded domain with smooth boundary and $\mathbb{H}^N$ is the Heisenberg group. The Δ_H is the Kohn Laplacian on the Heisenberg group and $q=2N+2$ is the homogeneous dimension of $\mathbb{H}^N$ and g is a suitable lower order perturbation. The exponent $4/(q-2)$ in (1.1) is a critical exponent for the semilinear Dirichlet problem of the Kohn Laplacian, as well as the exponent $4/(N-2)$ is critical for the semilinear equation $-\Delta u=|u|^{4/(N-2)}u+h(u)$ in a domain of $\mathbb{R}^N$ with Dirichlet boundary condition. The purpose of the present paper is to prove that for suitable $g(\xi,u)$, problem (1.1) has at least one positive solution and at least one sign changing solution.

Equations like (1.1) arise naturally in the study of the Yamabe problem for a Cauchy-Riemann manifold $(\mathcal{N},\Lambda)$, which is the problem of finding a contact form $\tilde{\Lambda}$ on $\mathcal{N}$ with constant scalar curvature and which is equivalent to find a C^∞ positive function u on $\mathcal{N}$ such that $\tilde{\Lambda}=u^{4/(q-2)}\Lambda$ and

$$-\Delta_{\mathcal{N}}u+au=|u|^{\frac{4}{q-2}}u,$$

where a is a suitable constant and $\Delta_{\mathcal{N}}$ is the sub-Laplacian on $\mathcal{N}$, see also [19] for more details.

Before stating the main results of the present paper, we give some notions about the Heisenberg group. If we denote $\xi=(x,y,t)$ with $x,\ y\in\mathbb{R}^N$ and

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$t \in \mathbb{R}$, then the Heisenberg group $\mathbb{H}^N$ is identified with $\mathbb{R}^{2N+1}$ under the group composition: for all $\xi = (x, y, t)$ and $\xi' = (x', y', t')$,

$$\xi \circ \xi' = (x + x', y + y', t + t' + 2(x \cdot y' - x' \cdot y)),$$

where "·" denotes the inner product in $\mathbb{R}^N$. And for $\xi \in \mathbb{H}^N$, the left translations on $\mathbb{H}^N$ is defined by

$$\tau_\xi : \mathbb{H}^N \to \mathbb{H}^N, \qquad \tau_\xi(\xi') = \xi \circ \xi'.$$

For $\lambda > 0$, a family of dilation on $\mathbb{H}^N$ is defined by

$$\delta_\lambda : \mathbb{H}^N \to \mathbb{H}^N, \qquad \delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t).$$

The Kohn Laplacian Δ_H on $\mathbb{H}^N$ is defined as

$$\Delta_H = \sum_{j=1}^{N} (X_j^2 + Y_j^2),$$

where

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}.$$

For every $u \in C_0^\infty(\Omega)$, the subelliptic gradient is defined as

$$\nabla_H u = (X_1 u, \cdots, X_N u, Y_1 u, \cdots, Y_N u).$$

The closure of $C_0^\infty(\Omega)$ under the norm $\int_\Omega |\nabla_H \cdot|^2 d\xi$ is denoted by $S_0^{1,2}(\Omega)$. In $S_0^{1,2}(\mathbb{H}^N)$, the following Sobolev type inequality holds(see e.g. [12, 13]): there exists $C_q > 0$ such that

$$\|u\|_{2q/(q-2)} \le C_q \|u\|_{S_0^{1,2}(\mathbb{H}^N)}, \quad \text{for all} \quad u \in S_0^{1,2}(\mathbb{H}^N), \tag{1.2}$$

where $\|\cdot\|_{2q/(q-2)}$ is the norm in $L^{2q/(q-2)}$. The number $2q/(q-2)$ is the critical Sobolev exponent, since for bounded domain Ω and $2 < p < 2q/(q-2)$, $S_0^{1,2}(\Omega)$ is compactly embedded into $L^p(\Omega)$, while this inclusion is only continuous if $p = 2q/(q-2)$.

Although Δ_H is not elliptic at any point of $\mathbb{H}^N$, the above-mentioned properties underline some similarities between Δ_H and the classical Laplacian Δ on Euclidean space $\mathbb{R}^N$. We point out that the Poisson equation with critical exponent of the form

$$-\Delta u = |u|^{\frac{4}{N-2}} u + \mu u, \quad u \in W_0^{1,2}(D), \quad D \subset \mathbb{R}^N \tag{1.3}$$

has been studied extensively after the pioneer work of Brezis-Nirenberg [1], see [6] for the existence and multiplicity results of (1.3). The existence of sign changing solutions of (1.3) has been obtained in [7]. While for (1.1), only few result is known about the existence and non-existence of solutions. We mention that when $g(\xi, u) \equiv 0$, Citti et al.[10] has proven that (1.1) has a nontrivial solution if Ω has at least a nontrivial homology group, see also Lanconelli [20]. When $g(\xi, u) = au + f(\xi, u)$ with suitable assumptions of f, Citti [9] has proven a Brezis-Nirenberg type result of (1.1). We also refer the interested readers to [2, 3, 4] for other related existence and nonexistence results of semilinear Kohn Lapalace equations on Heisenberg group. However, to our best knowledge, we have not seen any multiplicity results for Kohn Lapalace equation with critical exponent, nor sign changing solutions to the Kohn Lapalace equation with critical exponent.

The purpose of the present paper is to study the existence of positive and sign changing solutions of (1.1) under suitable assumptions of g. Throughout this paper, we study the model case $g(\xi,u)=\mu|\xi|_H^{\alpha}u$ with $\alpha>0$ and $|\xi|_H$ the distance in $\mathbb{H}^N$. We always assume that $0\in\Omega$ and simply write (1.1) as

$$-\Delta_H u=|u|^{\frac{4}{q-2}}u+\mu|\xi|_H^{\alpha}u,\quad u\in S_0^{1,2}(\Omega). \tag{1.4}$$

A solution of (1.4) is equivalent to a critical point of the functional

$$L(u)=\frac{1}{2}\int_\Omega\left(|\nabla_H u|^2-\mu|\xi|_H^{\alpha}|u|^2\right)d\xi-\frac{1}{2^*}\int_\Omega|u|^{2^*}d\xi,\quad u\in S_0^{1,2}(\Omega),$$

where $2^*=2q/(q-2)$. We say that $u\in S_0^{1,2}(\Omega)$ is a positive solution of (1.4) if u is a critical point of L and $u\geq 0$ but $u\not\equiv 0$; $u\in S_0^{1,2}(\Omega)$ is said to be a sign changing solution of (1.4) if u is a critical point of L and $u=u^+-u^-$ with $u^+\not\equiv 0$ and $u^-\not\equiv 0$, where $u^+=\max\{u(\xi),0\}$ and $u^-=\max\{-u(\xi),0\}$. For $\alpha>0$, we denote

$$\mu_1=\inf\left\{\int_\Omega|\nabla_H u|^2d\xi\ :\ \int_\Omega|\xi|_H^{\alpha}|u|^2d\xi=1\right\}.$$

Then Lemma 2.1 implies that $\mu_1>0$. The main result of the present paper is the following theorem.

Theorem 1.1. *If $0<\mu<\mu_1$, then*

- **(i):** *equation (1.4) has at least one positive solution provided $0<\alpha<q-4$;*
- **(ii):** *equation (1.4) has at least one positive solution and one sign changing solution provided $0<\alpha<\frac{q}{2}-3$.*

The method of proving Theorem 1.1 is variational. Since the embedding relation in (1.2) is not compact, we can not use the standard variational argument. However, as observed in Brezis-Nirenberg [1], the extremal function of inequality (1.2) should play an important role. Jerison et al. [18] has proven that the following minimum

$$S=\inf_{u\in S_0^{1,2}(\mathbb{H}^N)}\frac{\int_{\mathbb{H}^N}|\nabla_H u|^2d\xi}{\left(\int_{\mathbb{H}^N}|u|^{2^*}d\xi\right)^{\frac{2}{2^*}}} \tag{1.5}$$

is achieved by

$$\omega(x,y,t)=\frac{C_0}{(t^2+(1+|x|^2+|y|^2)^2)^{(q-2)/4}},\quad (x,y,t)\in\mathbb{H}^N,$$

where C_0 is a suitable positive constant. Moreover positive solutions of

$$-\Delta_H u=|u|^{\frac{4}{q-2}}u,\quad u\in S_0^{1,2}(\mathbb{H}^N)$$

can be obtained from $\omega(x, y, t)$ with a rescaling and are of the form

$$\omega_{\lambda,\xi'} = \lambda^{\frac{q-2}{2}} \omega(\delta_\lambda(\tau_{\xi'}^{-1})), \quad \lambda > 0, \quad \xi' \in \mathbb{H}^N. \tag{1.6}$$

With suitable modifications of $\omega_{\lambda,\xi'}$, Citti [9] has proven the existence of one positive solution of (1.1). We will prove the existence of one positive solution and one sign changing solution of (1.4).

This paper is organized as follows. In section 2, we give some preliminaries. Particular attention is focused on several integral estimates for solutions of (1.4), which will play an important role in the study of multiple solutions of (1.4). In section 3, we will prove Theorem 1.1 by studying suitable minimization problems. We emphasize that unlike the method used in Citti [9], we use neither Palais-Smale sequence, nor Ekeland variational principle.

Notations: Throughout this paper, the norm in $S_0^{1,2}(\Omega)$ is denoted by $\|\cdot\|$. All integrals are taken over Ω unless stated otherwise. A ball in $\mathbb{H}^N$ with center at ξ and radius R is denoted by $B(\xi, R)$. $O(\varepsilon^\beta)$ means that $|O(\varepsilon^\beta)\varepsilon^{-\beta}| \le C$; $o(\varepsilon^\beta)$ means $|o(\varepsilon^\beta)\varepsilon^{-\beta}| \to 0$ as $\varepsilon \to 0$, and $o(1)$ is an infinitesimal value. The $\to$ denotes strong convergence and the $\rightharpoonup$ denotes weak convergence.

2. Preliminaries

We start with a compact embedding relation from $S_0^{1,2}(\Omega)$ into $L^2(\Omega, |\xi|_H^\alpha d\xi)$, where $L^2(\Omega, |\xi|_H^\alpha d\xi)$ is a weighted Sobolev space.

Lemma 2.1. *Let $\Omega \subset \mathbb{H}^N$ be a bounded open domain. Then $S_0^{1,2}(\Omega)$ is compactly embedded into $L^2(\Omega, |\xi|_H^\alpha d\xi)$.*

Proof. Note that $\alpha > 0$. We can get the conclusion by a combination of [21, Lemma 3.2] and [8, Lemma 2.6]. The detailed proof is omitted. □

Next, let $\omega_{\lambda,\xi'}$ be defined as in (1.6). $\omega_{\lambda,0}$ is simply denoted by ω_λ. Then from the characterization of S, we may deduce that

$$\int_{\mathbb{H}^N} |\nabla \omega_\lambda|^2 d\xi = \int_{\mathbb{H}^N} |\omega_\lambda|^{2^*} d\xi = S^{\frac{q}{2}}. \tag{2.1}$$

Choose a $R > 0$ such that $B(0, 2R) \subset \Omega$ and a function $\phi \in C_0^1(B(0, 2R))$ such that $\phi(\xi) \equiv 1$ for $\xi \in B(0, R)$ and $\phi(\xi) \equiv 0$ for $\xi \in \Omega \backslash B(0, 2R)$. Denote

$$v_\lambda(\xi) := \phi(\xi)\omega_\lambda(\xi).$$

Lemma 2.2. [11] *(Heisenberg polar coordinates) Let $f : \mathbb{R}^+ \to \mathbb{R}$ and $R > 0$. Then*

$$\int_{B(0,R)} f(|\xi|_H) d\xi = \int_0^R f(\rho)\rho^{q-1} d\rho, \qquad q = 2N + 2,$$

whenever one of the previous integral exists.

Lemma 2.3. [9] *Let v_λ be defined as above. Then v_λ satisfies the following estimates: as $\lambda \to +\infty$,*

$$\int_\Omega |\nabla_H v_\lambda|^2 d\xi = S^{\frac{q}{2}} + O(\lambda^{-(q-2)}) \tag{2.2}$$

and

$$\int_\Omega |v_\lambda|^{2^*} d\xi = S^{\frac{q}{2}} + O(\lambda^{-q}). \tag{2.3}$$

Moreover, we can prove the following lemma, which will paly an important role in the proof of Theorem 1.1.

Lemma 2.4. *Let $0 < \alpha < q - 4$. Then as $\lambda \to +\infty$,*

$$\int_\Omega |\xi|_H^\alpha |v_\lambda|^2 d\xi = O(\lambda^{-(2+\alpha)}).$$

Proof. Note that

$$\begin{aligned}
\int_\Omega |\xi|_H^\alpha |v_\lambda|^2 d\xi &= \int_{|\xi|_H < 2R} |\xi|_H^\alpha \omega_\lambda^2(\delta_\lambda(\xi)) d\xi \\
&= \lambda^{-2-\alpha} \int_{|\eta|_H < 2\lambda R} |\eta|_H^\alpha \omega^2(\eta) d\eta \\
&= \lambda^{-2-\alpha} \left(\int_{|\eta|_H < 1} |\eta|_H^\alpha \omega^2(\eta) d\eta + \int_{1 < |\eta|_H < 2\lambda R} |\eta|_H^\alpha \omega^2(\eta) d\eta \right) \\
&\leq C\lambda^{-2-\alpha} \left(1 + \int_1^{2\lambda R} \rho^{4-q+\alpha-1} d\rho \right) \\
&= O((\lambda^{-1})^{2+\alpha}) + O((\lambda^{-1})^{q-2}) \quad \text{for } \lambda \quad \text{large enough.}
\end{aligned}$$

Therefore $0 < \alpha < q - 4$ implies that for λ large enough,

$$\int_\Omega |\xi|_H^\alpha |v_\lambda|^2 d\xi = O(\lambda^{-(2+\alpha)}).$$

The proof is complete. □

Next, we prove a regularity result for the solutions of (1.4). The idea is originated from Brezis-Kato[5], see also Struwe [22]. The following lemma will play a key role in the process of getting sign changing solutions.

Lemma 2.5. *If $u \in S_0^{1,2}(\Omega)$ is a solution of (1.4), then $u \in L^r(\Omega)$ for each $r \in (1, +\infty)$.*

Proof. Since u is a weak solution of (1.4), we test the equation with a test function $\varphi = u \min\{|u|^{2s}, m^2\}$, where $s \geq 0$ and $m > 1$. Then we have that

$$\begin{aligned}
\int \nabla_H u \nabla_H (u \min\{|u|^{2s}, m^2\}) d\xi &= \int |u|^{2^*} \min\{|u|^{2s}, m^2\} d\xi \\
&\quad + \mu \int |\xi|_H^\alpha u^2 \min\{|u|^{2s}, m^2\} d\xi.
\end{aligned}$$

For each sufficiently large $K > 0$, we deduce that

$$\begin{aligned}
&\int |\nabla_H (u \min\{|u|^s, m\}) d\xi \\
&\leq (2s+2) \int |u|^{2^*} \min\{|u|^{2s}, m^2\} d\xi + C \int u^2 \min\{|u|^{2s}, m^2\} d\xi \\
&= (2s+2) \int_{|u| \leq K} |u|^{2^*} \min\{|u|^{2s}, m^2\} d\xi + C \int u^2 \min\{|u|^{2s}, m^2\} d\xi \\
&\qquad (2s+2) \int_{|u|>K} |u|^{2^*} \min\{|u|^{2s}, m^2\} d\xi \\
&\leq (2s+2) \text{meas}(\Omega) K^{2^*+2s} + C \int u^2 \min\{|u|^{2s}, m^2\} d\xi \\
&\qquad (2s+2) \left(\int_{|u|>K} |u|^{2^*} d\xi \right)^{\frac{2^*-2}{2^*}} \left(\int (u \min\{|u|^{2s}, m^2\})^{2^*} d\xi \right)^{\frac{2}{2^*}} \\
&\leq C + \frac{1}{2} \int |\nabla_H (u \min\{|u|^s, m\})|^2 d\xi + C \int u^2 \min\{|u|^{2s}, m^2\} d\xi,
\end{aligned}$$

which implies that

$$\int |\nabla_H (u \min\{|u|^s, m\}) d\xi \leq 4(s+1) \text{meas}(\Omega) K^{2^*+2s} + C_1 \int u^2 \min\{|u|^{2s}, m^2\} d\xi.$$

Letting $m \to +\infty$, we obtain that

$$\int |\nabla_H (u|u|^s)|^2 d\xi \leq 4(s+1) \text{meas}(\Omega) K^{2^*+2s} + C_1 \int |u|^{2(s+1)} d\xi.$$

Now iterate, letting $s_0 = 0$, $s_j + 1 = (s_{j-1} + 1)\frac{q}{q-2}$, if $j \geq 1$, to obtain the conclusion. $\square$

3. Existence of a positive solution

In this section, we will prove the existence of at least one positive solution of (1.4). The $0 < \mu < \mu_1$ will be assumed throughout this section. Define another functional

$$I(u) = \int |\nabla_H u|^2 d\xi - \mu \int |\xi|_H^\alpha |u|^2 d\xi - \int |u|^{2^*} d\xi, \qquad u \in S_0^{1,2}(\Omega)$$

and denote the Nehari set

$$\mathcal{M} = \left\{ u \in S_0^{1,2}(\Omega) \backslash \{0\} \ : \ I(u) = 0 \right\}.$$

Define the following minimization problem

$$c_1 = \inf_{u \in \mathcal{M}} L(u). \tag{3.1}$$

Next, we will prove that c_1 is achieved and a minimizer of (3.1) is a positive solution of (1.4). The following several lemmas will be useful in what follows.

Lemma 3.1. *For any $u \in S_0^{1,2}(\Omega)\backslash\{0\}$, there is a unique $\theta(u) > 0$ such that $\theta(u)u \in \mathcal{M}$. Moreover, if $I(u) < 0$, then $0 < \theta(u) < 1$.*

Proof. For any $u \in S_0^{1,2}(\Omega)\backslash\{0\}$ and $\theta > 0$,

$$L(\theta u) = \frac{\theta^2}{2}\int\left(|\nabla_H u|^2 - \mu|\xi|_H^{\alpha}|u|^2\right)d\xi - \frac{\theta^{2^*}}{2^*}\int|u|^{2^*}d\xi.$$

Hence

$$\frac{\partial}{\partial\theta}L(\theta u) = \theta\left(\int\left(|\nabla_H u|^2 - \mu|\xi|_H^{\alpha}|u|^2\right)d\xi - \theta^{2^*-2}\int|u|^{2^*}d\xi\right),$$

which implies that there is a unique

$$\theta(u) = \left(\int\left(|\nabla_H u|^2 - \mu|\xi|_H^{\alpha}|u|^2\right)d\xi\right)^{\frac{1}{2^*-2}}\left(\int|u|^{2^*}d\xi\right)^{\frac{1}{2-2^*}}$$

such that $\theta(u)u \in \mathcal{M}$. Clearly $\theta(u) > 0$ since $0 < \mu < \mu_1$. Moreover if $I(u) < 0$, i.e., $\int\left(|\nabla_H u|^2 - \mu|\xi|_H^{\alpha}|u|^2\right)d\xi < \int|u|^{2^*}d\xi$, then we know that $0 < \theta(u) < 1$. □

Lemma 3.2. *Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ be such that $u_n \rightharpoonup u$ weakly in $S_0^{1,2}(\Omega)$ and $I(u_n) \to d$, but u_n does not converge strongly to u in $S_0^{1,2}(\Omega)$. Then one of the following conclusions holds:*

- *(1) $d \geq \frac{1}{q}S^{\frac{q}{2}}$ provided $u = 0$;*
- *(2) $d > L(\theta(u)u)$ provided $u \neq 0$ and $I(u) < 0$;*
- *(3) $d \geq L(\theta(u)u) + \frac{1}{q}S^{\frac{q}{2}}$ in the case that $u \neq 0$ and $I(u) \geq 0$.*

Proof. The idea is originated from Hirano et al.[17]. Note that $u_n \rightharpoonup u$ in $S_0^{1,2}(\Omega)$, by Lemma 2.1, $\int|\xi|_H^{\alpha}|u_n - u|^2 d\xi \to 0$ as $n \to \infty$. We may assume that as $n \to \infty$,

$$\int|\nabla_H(u_n - u)|^2 d\xi \to a^2 \quad \text{and} \quad \int|u_n - u|^{2^*}d\xi \to b^{2^*}.$$

Since u_n does not converge strongly to u in $S_0^{1,2}(\Omega)$, we have that $a \neq 0$.

(1) In the case of $u = 0$, we deduce from Brezis-Lieb lemma that

$$\begin{aligned} 0 = I(u_n) &= I(u) + \int|\nabla_H(u_n - u)|^2 d\xi - \int|u_n - u|^{2^*}d\xi + o(1) \\ &= \int|\nabla_H(u_n - u)|^2 d\xi - \int|u_n - u|^{2^*}d\xi + o(1). \end{aligned}$$

Combining this with $a \neq 0$, one deduces that $b \neq 0$. It is deduced from the Sobolev inequality that $a^2 = b^{2^*} \geq S^{\frac{q}{2}}$. Therefore

$$d \geq \frac{1}{2}a^2 - \frac{1}{2^*}b^{2^*} \geq \frac{1}{q}S^{\frac{q}{2}}.$$

(2) When $u \neq 0$ and $I(u) < 0$, we denote $J(\theta) = L(\theta u)$,

$$\beta(\theta) = \frac{a^2}{2}\theta^2 - \frac{b^{2^*}}{2^*}\theta^{2^*} \quad \text{and} \quad \gamma(\theta) = J(\theta) + \beta(\theta).$$

From Lemma 3.1, one has a $\theta(u)$ with $0<\theta(u)<1$ such that $\theta(u)u\in\mathcal{M}$. Since $J'(1)<0$ and $\gamma'(1)=0$, we have $\beta'(1)\geq 0$ and then $a^2-b^{2^*}\geq 0$. Hence from

$$\beta'(\theta)=a^2\theta-b^{2^*}\theta^{2^*-1}>\theta(1-\theta^{2^*-2})b^{2^*},$$

one deduces that $\beta'(\theta)>0$ for $0<\theta<1$. So we have that $\beta(\theta(u))>0$. Therefore

$$d=\gamma(1)\geq\gamma(\theta(u)u)=L(\theta(u)u)+\beta(\theta(u))>L(\theta(u)u).$$

This proves the second statement of the Lemma.

(3) Now we prove the third statement. Firstly we consider the case of $u\neq 0$ and $I(u)=0$. Similar to those proofs in case (1), we have that $b\neq 0$ and $a^2=b^{2^*}\geq S^{\frac{q}{2}}$. Brezis-Lieb lemma implies that

$$\begin{aligned} d+o(1)&=L(u_n)\\ &=L(u)+\frac{1}{2}\int|\nabla_H(u_n-u)|^2d\xi-\frac{1}{2^*}\int|u_n-u|^{2^*}d\xi+o(1)\\ &\geq L(\theta(u)u)+\frac{1}{q}S^{\frac{q}{2}}, \end{aligned}$$

where we have used the fact that $u\neq 0$ and $I(u)=0$ imply $\theta(u)=1$.

Secondly we prove the case of $u\neq 0$ and $I(u)>0$. Noticing that $\theta(u)>1$, we claim that $b\neq 0$. Indeed if $b=0$, then from $\gamma'(1)=0$, we have that $J'(1)=-a^2<0$, which contradicts $\theta(u)>0$ and $J'(\theta)>0$ for all $0<\theta<\theta(u)$. So we have that $b\neq 0$. Denoting $\theta_*=(a^2/b^{2^*})^{1/(2^*-2)}$, we know that β attains its maximum at θ_* and $\beta'(\theta)>0$ for $0<\theta<\theta_*$ and $\beta'(\theta)<0$ for $\theta>\theta_*$. Moreover $\beta(\theta_*)=\frac{1}{q}S^{\frac{q}{2}}$. Next we claim that $\theta_*\leq\theta(u)$. Suppose that this is not the case, i.e., $1<\theta(u)<\theta_*$. From $0>\gamma'(\theta)=J'(\theta)+\beta'(\theta)$ for all $\theta>1$, we have that $J'(\theta)<-\beta'(\theta)<0$ for $\theta\in(1,\theta_*)$, which contradicts $1<\theta(u)<\theta_*$ and $J'(\theta(u))=0$. So we have proven that $\theta_*\leq\theta(u)$. Hence we obtain that

$$d=\gamma(1)\geq\gamma(\theta_*)=L(\theta_*u)+\beta(\theta_*)\geq L(\theta(u)u)+\frac{1}{q}S^{\frac{q}{2}}.$$

These complete the proof of the third statement and hence we have proven Lemma 3.2. □

Lemma 3.3. *If* $0<\alpha<q-4$, *then* $0<c_1<\frac{1}{q}S^{\frac{q}{2}}$.

Proof. It is easy to see that $c_1>0$. To prove that $c_1<\frac{1}{q}S^{\frac{q}{2}}$, the idea is to find an element in $\mathcal{M}$ such that the value of L at this element is strictly less than $\frac{1}{q}S^{\frac{q}{2}}$. As observed by Brezis-Nirenberg [1], the extremal function $\omega(x,y,t)$ of inequality (1.3) will play an important role. From Lemma 3.1, we know that for v_λ, there is a

$$\theta(v_\lambda)=\left(\int\left(|\nabla_H v_\lambda|^2-\mu|\xi|_H^\alpha|v_\lambda|^2\right)d\xi\right)^{\frac{1}{2^*-2}}\left(\int|v_\lambda|^{2^*}d\xi\right)^{\frac{1}{2-2^*}}$$

such that $\theta(v_\lambda)v_\lambda \in \mathcal{M}$. We claim that there is $\lambda_0 > 0$ such that

$$L(\theta(v_{\lambda_0})v_{\lambda_0}) < \frac{1}{q} S^{\frac{q}{2}}.$$

Indeed, note that from the definition of v_λ,

$$\begin{aligned} L(\theta(v_\lambda)v_\lambda) &= \left(\frac{1}{2} - \frac{1}{2^*}\right) \int |\theta(v_\lambda)v_\lambda|^{2^*} d\xi \\ &= \left(\frac{1}{2} - \frac{1}{2^*}\right) (\theta(v_\lambda))^{2^*} \int |v_\lambda|^{2^*} d\xi \\ &= \frac{1}{q} \left(\int \left(|\nabla_H v_\lambda|^2 - \mu |\xi|_H^\alpha |v_\lambda|^2 \right) d\xi \right)^{\frac{2^*}{2^*-2}} \left(\int |v_\lambda|^{2^*} d\xi \right)^{\frac{2}{2-2^*}} d\xi \\ &= \frac{1}{q} \left(\int \left(|\nabla_H v_\lambda|^2 - \mu |\xi|_H^\alpha |v_\lambda|^2 \right) d\xi \right)^{\frac{q}{2}} \left(\int |v_\lambda|^{2^*} d\xi \right)^{\frac{2-q}{2}} \\ &= \frac{1}{q} \left(S^{\frac{q}{2}} + O((\lambda^{-1})^q) - O((\lambda^{-1})^{2+\alpha}) \right)^{\frac{q}{2}} \left(S^{\frac{q}{2}} + O((\lambda^{-1})^{q-2}) \right)^{\frac{2-q}{2}} \\ &= \frac{1}{q} S^{\frac{q}{2}} + O((\lambda^{-1})^{q-2}) - O((\lambda^{-1})^{2+\alpha}) \end{aligned}$$

as λ large enough. Therefore we know that there is a $\lambda_0 > 0$ sufficiently large such that

$$L(\theta(v_{\lambda_0})v_{\lambda_0}) < \frac{1}{q} S^{\frac{q}{2}}.$$

This completes the proof of the lemma. □

Theorem 3.4. *If $0 < \mu < \mu_1$ and $0 < \alpha < q - 4$, then there is a positive function $\psi_1 \in S_0^{1,2}(\Omega)$ such that $c_1 = L(\psi_1)$ and ψ_1 is a positive solution of (1.4).*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ be such that $L(u_n) \to c_1$ as $n \to \infty$. From Lemma 3.3, one knows that $0 < c_1 < \frac{1}{q} S^{\frac{q}{2}}$. Next, from $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}$ we have that

$$\int \left(|\nabla_H u_n|^2 - \mu |\xi|_H^\alpha |u_n|^2 \right) d\xi = \int |u_n|^{2^*} d\xi.$$

Note also that for n large enough,

$$\begin{aligned} & c_1 + o(1) = L(u_n) \\ & = \frac{1}{2} \int \left(|\nabla_H u_n|^2 - \mu |\xi|_H^\alpha |u_n|^2 \right) d\xi - \frac{1}{2^*} \int |u_n|^{2^*} d\xi \\ & = \frac{1}{q} \int \left(|\nabla_H u_n|^2 - \mu |\xi|_H^\alpha |u_n|^2 \right) d\xi \\ & \geq \frac{1}{q} \left(1 - \frac{\mu}{\mu_1} \right) \int |\nabla_H u_n|^2 d\xi. \end{aligned} \tag{3.2}$$

On the other hand, using Sobolev inequality, one has that

$$\begin{aligned}\left(1-\frac{\mu}{\mu_1}\right)\int|\nabla_H u_n|^2 d\xi &\le \int\left(|\nabla_H u_n|^2-\mu|\xi|_H^{\alpha}|u_n|^2\right)d\xi\\ &=\int|u_n|^{2^*}d\xi\le\left(S^{-1}\int|\nabla_H u_n|^2 d\xi\right)^{\frac{2^*}{2}}.\end{aligned}$$

Therefore

$$\left(\int|\nabla_H u_n|^2 d\xi\right)^{\frac{2^*}{2}-1}\ge S^{\frac{2^*}{2}}\left(1-\frac{\mu}{\mu_1}\right). \tag{3.3}$$

Then (3.2) and (3.3) imply that there are positive constants C_1, C_2 such that

$$C_1\le\int|\nabla_H u_n|^2 d\xi\le C_2. \tag{3.4}$$

Going if necessary to a subsequence, we may assume that $(u_n)_{n\in\mathbb{N}}$ converges weakly to v in $S_0^{1,2}(\Omega)$. Assume that $(u_n)_{n\in\mathbb{N}}$ does not converge strongly to v in $S_0^{1,2}(\Omega)$, we obtain from Lemma 3.2 that one of the following three cases occurs:

(i): if $v=0$, then $c_1\ge\frac{1}{q}S^{\frac{q}{2}}$;
(ii): if $v\ne 0$ and $I(v)<0$, then $c_1>L(\theta(v)v)$;
(iii): if $v\ne 0$ and $I(v)\ge 0$, then $c_1\ge L(\theta(v)v)+\frac{1}{q}S^{\frac{q}{2}}$.

Since in the case of $v\ne 0$, $\theta(v)v\in\mathcal{M}$ implies that $L(\theta(v)v)\ge c_1$. We obtain from the fact of $c_1<\frac{1}{q}S^{\frac{q}{2}}$ that any one of the above-mentioned three cases will not occur. This means that $(u_n)_{n\in\mathbb{N}}$ converges strongly to v in $S_0^{1,2}(\Omega)$. It is deduced from (3.4) that $v\ne 0$ and $v\in\mathcal{M}$. Hence $L(v)=c_1$. Using Lagrange multiplier rule, one has a $\tilde{\theta}\in\mathbb{R}$ such that

$$L'(v)=\tilde{\theta}I'(v).$$

From $\langle L'(v),v\rangle=I(v)=0$ and

$$\langle I'(v),v\rangle=2\int_\Omega\left(|\nabla_H v|^2-\mu|\xi|_H^{\alpha}|v|^2\right)d\xi-2^*\int|v|^{2^*}d\xi=(2-2^*)\int|v|^{2^*}d\xi<0,$$

one obtains that $\tilde{\theta}=0$ and v is a nontrivial solution of (1.4). Note that if $(u_n)_{n\in\mathbb{N}}\subset\mathcal{M}$ and $L(u_n)\to c_1$, then $(|u_n|)_{n\in\mathbb{N}}\subset\mathcal{M}$ and $L(|u_n|)\to c_1$, we may assume that $v\ge 0$. Hence the proof is complete by choosing $\psi_1\equiv v$. □

4. Existence of sign changing solution

In this section, we prove that (1.4) has also at least one sign changing solution. Note that $\mathbb{H}^N$ is identified with $\mathbb{R}^{2N+1}$, we know that if $u\in S_0^{1,2}(\Omega)$, then $u^+,\ u^-\in S_0^{1,2}(\Omega)$, where $u^+=\max\{u(\xi),0\}$ and $u^-=\max\{-u(\xi),0\}$. Denote

$$\mathcal{M}_*=\{u=u^+-u^-\in S_0^{1,2}(\Omega)\ :\ u^+\in\mathcal{M}\text{ and }u^-\in\mathcal{M}\}$$

and set

$$c_2 = \inf_{u \in \mathcal{M}_*} L(u). \tag{4.1}$$

Next, we will prove that c_2 is achieved and a minimizer is a sign changing solution of (1.4). To attain this goal, we need several additional lemmas.

Lemma 4.1. *If $0 < \mu < \mu_1$ and $0 < \alpha < \frac{q}{2} - 3$, then $c_2 < c_1 + \frac{1}{q} S^{\frac{q}{2}}$.*

Proof. The strategy is to find a suitable element in $\mathcal{M}_*$ such that the value of L at this element is strictly less than $c_1 + \frac{1}{q} S^{\frac{q}{2}}$. Let ψ_1 be the positive solution obtained in the previous section, we claim that there are $a_0 > 0$ and $b_0 \in \mathbb{R}$ such that $a_0\psi_1 + b_0 v_\lambda \in \mathcal{M}_*$. Indeed, denote $\zeta(s) = \psi_1 + s v_\lambda$ with $s \in \mathbb{R}$, and we define $s_1 \in [-\infty, \infty)$ and $s_2 \in (-\infty, \infty]$ by $s_1 = \inf\{s \in \mathbb{R} \ : \ (\zeta(s))^+ \neq 0\}$ and $s_2 = \inf\{s \in \mathbb{R} \ : \ (\zeta(s))^- \neq 0\}$. Since $\theta((\zeta(s))^+) - \theta((\zeta(s))^-) \to \infty$ as $s \to s_1 + 0$ and $\theta((\zeta(s))^+) - \theta((\zeta(s))^-) \to -\infty$ as $s \to s_2 + 0$, there exists $s_0 \in (s_1, s_2)$ such that $\theta((\zeta(s_0))^+) = \theta((\zeta(s_0))^-)$, which implies that there are $a_0 > 0$ and $b_0 \in \mathbb{R}$ such that $a_0\psi_1 + b_0 v_\lambda \in \mathcal{M}_*$.

Next, we claim that there is $\lambda_* > 0$ such that

$$\sup_{a>0,\ b\in\mathbb{R}} L(a\psi_1 + b v_{\lambda_*}) < c_1 + \frac{1}{q} S^{\frac{q}{2}}. \tag{4.2}$$

Note that

$$\begin{aligned} L(a\psi_1 + b v_\lambda) \ &= \frac{1}{2}\int |\nabla_H (a\psi_1 + b v_\lambda)|^2 d\xi - \frac{\mu}{2}\int |\xi|_H^\alpha |a\psi_1 + b v_\lambda|^2 d\xi \\ &\quad - \frac{1}{2^*}\int |a\psi_1 + b v_\lambda|^{2^*} d\xi. \end{aligned}$$

We can infer that there is $R_0 > 0$ such that $L(a\psi_1 + b v_\lambda) \le 0$ for all $a \ge 0$ and $b \in \mathbb{R}$ with $a^2 + b^2 \ge R_0$ and for all $\lambda > 1$. Hence it is sufficient to estimate $\sup_{a>0,\ b\in\mathbb{R}} L(a\psi_1 + b v_\lambda)$ in the case that $a^2 + b^2 \le R_0$. Since ψ_1 is a solution of (1.4), we have that as λ large enough,

$$\begin{aligned} & L(a\psi_1 + b v_\lambda) \\ &= L(a\psi_1) + L(b v_\lambda) + ab \int \left(\nabla_H \psi_1 \nabla_H v_\lambda - \mu |\xi|_H^\alpha \psi_1 v_\lambda\right) d\xi \\ &\quad + \frac{1}{2^*}\int \left(|a\psi_1|^{2^*} + |b v_\lambda|^{2^*} - |a\psi_1 + b v_\lambda|^{2^*}\right) d\xi \\ &\le L(a\psi_1) + L(b v_\lambda) + ab \int |\psi_1|^{\frac{4}{q-2}} \psi_1 v_\lambda d\xi \\ &\quad + C\int \left(|\psi_1|^{2^*-1} v_\lambda + |v_\lambda|^{2^*-1}\psi_1\right) d\xi \\ &\le L(\psi_1) + \frac{1}{q} S^{\frac{q}{2}} - O((\lambda^{-1})^{2+\alpha}) \\ &\quad + C\int |\psi_1|^{2^*-1} v_\lambda d\xi + C\int \psi_1 v_\lambda^{2^*-1} d\xi, \end{aligned} \tag{4.3}$$

where we have used the fact that

$$\int \left(\nabla_H \psi_1 \nabla_H v_\lambda - \mu |\xi|_H^\alpha \psi_1 v_\lambda\right) d\xi = \int |\psi_1|^{\frac{4}{q-2}} \psi_1 v_\lambda d\xi.$$

Next, choosing $\gamma_1 \in (1, \frac{q}{q-2})$, we know from Lemma 2.5 that

$$\int |\psi_1|^{2^*-1} v_\lambda d\xi \le \left(\int v_\lambda^{\gamma_1} d\xi\right)^{\frac{1}{\gamma_1}} \left(\int |\psi_1|^{\frac{(2^*-1)\gamma_1}{\gamma_1-1}}\right)^{\frac{\gamma_1-1}{\gamma_1}}$$

holds. Since

$$\begin{aligned}\int_\Omega v_\lambda^{\gamma_1} d\xi &= \int_{|\xi|_H<2R} (\omega_\lambda(\delta_\lambda(\xi)))^{\gamma_1} d\xi = \lambda^{\frac{q-2}{2}\gamma_1 - q} \int_{|\eta|_H<2\lambda R} \omega^{\gamma_1}(\eta) d\eta \\ &= \lambda^{\frac{q-2}{2}\gamma_1 - q} \left(\int_{|\eta|_H<1} \omega^{\gamma_1}(\eta) d\eta + \int_{1<|\eta|_H<2\lambda R} \omega^{\gamma_1}(\eta) d\eta\right) \\ &= \lambda^{\frac{q-2}{2}\gamma_1 - q} \left(C + \int_1^{2\lambda R} \frac{\rho^{q-1} d\rho}{\rho^{(q-2)\gamma_1}}\right) \\ &= O((\lambda^{-1})^{\frac{q-2}{2}\gamma_1}) \quad \text{as } \lambda \text{ sufficiently large,}\end{aligned}$$

we obtain that

$$\int |\psi_1|^{2^*-1} v_\lambda d\xi = o((\lambda^{-1})^{2+\alpha}). \tag{4.4}$$

Similarly from $0 < \alpha < \frac{q}{2} - 3$, we can choose a $\gamma_2 > 1$ such that as λ sufficiently large,

$$\int \psi_1 v_\lambda^{2^*-1} d\xi = O((\lambda^{-1})^{\frac{q}{\gamma_2} - \frac{q+2}{2}}) = o((\lambda^{-1})^{2+\alpha}). \tag{4.5}$$

It is now deduced from (4.3), (4.4) and (4.5) that there is a $\lambda_* > 0$ such that (4.2) holds. The proof is complete. □

Lemma 4.2. *There exists a $\psi_2 \in \mathcal{M}_*$ such that $L(\psi_2) = c_2$.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_*$ be such that $L(u_n) \to c_2$. Then $(u_n)_{n\in\mathbb{N}} \subset \mathcal{M}_*$ implies that we have that

$$\int \left(|\nabla_H u_n^+|^2 - \mu|\xi|_H^\alpha |u_n^+|^2\right) d\xi = \int |u_n^+|^{2^*} d\xi$$

and

$$\int \left(|\nabla_H u_n^-|^2 - \mu|\xi|_H^\alpha |u_n^-|^2\right) d\xi = \int |u_n^-|^{2^*} d\xi.$$

Using a proof similar to those in the proof of (3.2), (3.3) and (3.4), we can obtain that

$$0 < \inf_n \|u_n^+\| \le \sup_n \|u_n^+\| < \infty, \qquad 0 < \inf_n \|u_n^-\| \le \sup_n \|u_n^-\| < \infty.$$

Therefore we may assume that as $n \to \infty$, $L(u_n^+) \to d_1$ and $L(u_n^-) \to d_2$, $c_2 = d_1 + d_2$ and that $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ converge weakly to u^+ and u^-, respectively. If $u^+ = 0$ or $u^- = 0$, by Lemma 3.2, one has $c_2 \ge c_1 + \frac{1}{q} S^{\frac{q}{2}}$, which contradicts Lemma 4.1. Thus we have that both $u^+ \ne 0$ and $u^- \ne 0$. Using Lemma 3.1 again, we have one of the following:

(I1): $(u_n^+)_{n\in\mathbb{N}}$ converges strongly to u^+ in $S_0^{1,2}(\Omega)$;
(I2): if $I(u^+) < 0$, then $d_1 > L(\theta(u^+)u^+)$;

(I3): if $I(u^+) \geq 0$, then $d_1 \geq L(\theta(u^+)u^+) + \frac{1}{q}S^{\frac{q}{2}}$;

and we also have one of the following:

(II1): $(u_n^-)_{n\in\mathbb{N}}$ converges strongly to u^- in $S_0^{1,2}(\Omega)$;
(II2): if $I(u^-) < 0$, then $d_2 > L(\theta(u^-)u^-)$;
(II3): if $I(u^-) \geq 0$, then $d_2 \geq L(\theta(u^-)u^-) + \frac{1}{q}S^{\frac{q}{2}}$.

Next, we will prove that only the case (I1) and the case (II1) hold. Indeed, if (I1) and (II2) hold, then from $u^+ - \theta(u^-)u^- \in \mathcal{M}_*$, we have that

$$c_2 \leq L(u^+ - \theta(u^-)u^-) < d_1 + d_2 = c_2,$$

which is a contradiction. If (I1) and (II3) hold, then

$$c_1 + \frac{1}{q}S^{\frac{q}{2}} < L(u^+ - \theta(u^-)u^-) + \frac{1}{q}S^{\frac{q}{2}} < d_1 + d_2 = c_2,$$

which contradicts Lemma 4.1. Similarly, one can prove that the cases (II1) and (I2) do not hold, and the cases (II1) and (I3) do not hold either. If (I2) and (II2) hold, then from $\theta(u^+)u^+ - \theta(u^-)u^- \in \mathcal{M}_*$, we have that

$$c_2 \leq L(\theta(u^+)u^+ - \theta(u^-)u^-) < d_1 + d_2 = c_2,$$

which is again a contradiction. If (I2) and (II3) hold, then we have that $\theta(u^+)u^+ - \theta(u^-)u^- \in \mathcal{M}_*$ and hence

$$c_1 + \frac{1}{q}S^{\frac{q}{2}} < L(\theta(u^+)u^+ - \theta(u^-)u^-) + \frac{1}{q}S^{\frac{q}{2}} < d_1 + d_2 = c_2,$$

which again contradicts Lemma 4.1. Similarly, one can prove that the cases (II2) and (I3) do not hold. Finally, if the cases (I3) and (II3) hold, then

$$c_1 + \frac{2}{q}S^{\frac{q}{2}} < L(\theta(u^+)u^+ - \theta(u^-)u^-) + \frac{2}{q}S^{\frac{q}{2}} < d_1 + d_2 = c_2,$$

which also contradicts Lemma 4.1. Therefore we have proven that only the cases (I1) and (II1) hold. Hence $u^+,\ u^- \in \mathcal{M}$. Choosing $\psi_2 \equiv u^+ - u^-$, we have that $\psi_2 \in \mathcal{M}_*$ and $L(\psi_2) = c_2$. □

In the following, we are going to prove the ψ_2 is a sign changing solution of (1.4). Note that $\mathcal{M}_*$ is usually not a manifold, the usual Lagrange multiplier rule may not be applied.

Theorem 4.3. *If* $0 < \mu < \mu_1$ *and* $0 < \alpha < \frac{q}{2} - 3$, *then (1.4) has at least one sign changing solution.*

Proof. By Lemma 4.2, we have that $\psi_2 \in \mathcal{M}_*$ is sign changing and $L(\psi_2) = c_2$. Arguing by a contradiction, we assume that ψ_2 is not a critical point of L, i.e., $L'(\psi_2) \neq 0$. Note that for any $v \in \mathcal{M}$, $I'(v) \neq 0$ because

$$\begin{aligned} \langle I'(v), v\rangle &= 2\int_\Omega \left(|\nabla_H v|^2 - \mu|\xi|_H^\alpha |v|^2\right) d\xi - 2^* \int |v|^{2^*} d\xi \\ &= (2 - 2^*) \int |v|^{2^*} d\xi < 0. \end{aligned}$$

Define

$$\Phi(u) = L'(u) - \langle L'(u), \frac{I'(u)}{\|I'(u)\|}\rangle \frac{I'(u)}{\|I'(u)\|}, \qquad u \in \mathcal{M}.$$

Then we infer that $\Phi(\psi_2) \neq 0$. Let $\delta \in (0, \min\{\|\psi_2^+\|, \|\psi_2^-\|\}/3)$ such that $\|\Phi(v) - \Phi(\psi_2)\| \leq \frac{1}{2}\|\Phi(\psi_2)\|$ for each $v \in \mathcal{M}$ with $\|v - \psi_2\| \leq 2\delta$. Let $\chi : \mathcal{M} \to [0,1]$ be a Lipschitz mapping such that

$$\chi(v) = \begin{cases} 1, & \text{for } v \in \mathcal{M} \text{ with } \|v - \psi_2\| \leq \delta, \\ 0, & \text{for } v \in \mathcal{M} \text{ with } \|v - \psi_2\| \geq 2\delta. \end{cases}$$

Let $\eta :\ [0, s_0] \times \mathcal{M} \to \mathcal{M}$ be such that the solution of the differential equation $\eta(0, v) = v$,

$$\frac{d\eta(s,v)}{ds} = -\chi(\eta(s,v))\Phi(\eta(s,v))$$

for $(s, v) \in [0, s_0] \times \mathcal{M}$, where s_0 is some positive number. We set

$$r(\tau) = \theta((1-\tau)\psi_2^+ - \tau\psi_2^-)((1-\tau)\psi_2^+ - \tau\psi_2^-)$$

and $\varsigma(\tau) = \eta(s_0, r(\tau))$ for $0 \leq \tau \leq 1$. Here we recall that $\psi_2^+ = \max\{\psi_2, 0\}$ and $\psi_2^- = \max\{-\psi_2, 0\}$. If $\tau \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$, we have that

$$L(\varsigma(\tau)) \leq L(r(\tau)) = L(r(\tau)^+) + L(r(\tau)^-) < L(\psi_2^+) + L(\psi_2^-) = L(\psi_2)$$

and $L(\varsigma(\frac{1}{2})) = L(r(\frac{1}{2})) = L(\psi_2)$, i.e., $L(\varsigma(\tau)) < L(\psi_2)$ for all $\tau \in (0,1)$. Since $\theta(\varsigma(\tau)^+) - \theta(\varsigma(\tau)^-) \to -\infty$ as $\tau \to 0+$ and $\theta(\varsigma(\tau)^+) - \theta(\varsigma(\tau)^-) \to +\infty$ as $\tau \to 1-0$, there exists $\tau_1 \in (0,1)$ satisfying $\theta(\varsigma(\tau_1)^+) = \theta(\varsigma(\tau_1)^-)$. So we have $\varsigma(\tau_1) \in \mathcal{M}_*$ and hence $L(\varsigma(\tau_1)) < L(\psi_2)$, which is a contradiction. Thus we have proven that $L'(\psi_2) = 0$. Hence ψ_2 is a sign changing solution of (1.4). □

Proof of Theorem 1.1: The first statement of Theorem 1.1 follows directly from Theorem 3.4. The second statement of Theorem 1.1 follows from Theorem 4.3 and Theorem 3.4. The proof is complete.

Remark. We have proven that for $0 < \mu < \mu_1$ and $0 < \alpha < \frac{q}{2} - 3$, (1.4) possesses at least one positive solution and at least one sign changing solution. Since (1.4) is odd with respect to u, we have obtained that (1.4) has at least one pair of fixed sign solutions and at least one pair of sign changing solutions.

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(J. Chen) School of Mathematics and Computer Science, Fujian Normal University, Qishan Campus, Fuzhou 350108, P. R. China

E-mail address: `jqchen@fjnu.edu.cn`

(E. M. Rocha) Center for Research & Development in Mathematics and Applications, University of Aveiro, 3810-193, Portugal

E-mail address: `eugenio@ua.pt`