Solutions of Tikhonov Functional Equations and Applications to Multiplication Operators on Szegö Spaces†

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Abstract. We consider a natural representation of solutions for Tikhonov functional equations. This will be done by applying the theory of reproducing kernels to the approximate solutions of general bounded linear operator equations (when defined from reproducing kernel Hilbert spaces into general Hilbert spaces), by using the Hilbert-Schmidt property and tensor product of Hilbert spaces. As a concrete case, we shall consider generalized fractional functions formed by the quotient of Bergman functions by Szegö functions considered from the multiplication operators on the Szegö spaces.

Mathematics Subject Classification (2000). 47A52, 46E22, 30C40, 32A36, 47A15.

Keywords. Reproducing kernel, Moore-Penrose generalized inverse, Tikhonov regularization, Hilbert-Schmidt operator, tensor product of Hilbert spaces, generalized fractional function, Bergman space, Szegö space, multiplication operator.

1. Introduction

In this paper, we shall consider a natural representation of solutions for Tikhonov functional equations by applying the theory of reproducing kernels. This will give us approximate solutions for general bounded linear operator equations, when considered from reproducing kernel Hilbert spaces into general Hilbert spaces.

This work was supported in part by the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e a Tecnologia"), through CIDMA - Center for Research and Development in Mathematics and Applications (http://cidma.mat.ua.pt/), within project UID/MAT/04106/2013.

† Accepted author's manuscript (AAM) published in [Complex Analysis and Operator Theory 10(8) (2016), 1705–1723] [DOI: 10.1007/s11785-016-0545-4] The final publication is available at Springer via http://dx.doi.org/10.1007/s11785-016-0545-4

In view of that, besides presenting a general introduction on the area in this section, we include in the next section some known background on the subject (which is mainly based on the original ideas from [18] and the updated work [19]). In Section 3 we will analyse, in detail, the solution of a Tikhonov functional equation by using complete orthonormal systems. In Section 4, as a typical and concrete application of the proposed theory, we will discuss and deduce generalized fractional functions, built by Bergman and Szegö functions through the consideration of the multiplication operators on the Szegö spaces. This will be achieved on the basis a an application of the main result of Section 3, about Tikhonov regularization on concrete reproducing kernel Hilbert spaces. Finally, in Section 5 we will analyse a completeness condition in the Szegö space and auxiliary characterizations of certain properties which are needed to consider when applying the general method proposed in this paper.

As it is well-known, a large class of problems can be formulated (or translated) into a single equation

$$Lf = \mathbf{d}, \tag{1.1}$$

where $L : H \to \mathcal{H}$ is a linear operator acting between Hilbert spaces H and $\mathcal{H}$, $\mathbf{d}$ is some given data and f is the unknown element. In case L is invertible by the inverse $L^{-1} : \mathcal{H} \to H$, the solution of (1.1) is given by $f = L^{-1}\mathbf{d}$. However, the equation may be "inconsistent" or L^{-1} may not exist. In such case, sometimes we can take profit of generalized inverses to still "directly" obtain solutions of (1.1). Namely, by constructing the so-called best approximate solution (or least square solution of minimum norm) given by

$$f = L^{\dagger}\mathbf{d}, \tag{1.2}$$

where $L^{\dagger}$ is the Moore-Penrose generalized inverse of L. We recall that for a bounded linear operator $L : H \to \mathcal{H}$, an operator $L^{\dagger}$ is called the Moore-Penrose generalized inverse of L if $LL^{\dagger}L = L$, $L^{\dagger}LL^{\dagger} = L^{\dagger}$, $(LL^{\dagger})^* = LL^{\dagger}$ and $(L^{\dagger}L)^* = L^{\dagger}L$, where T^* denotes the adjoint operator of T. It is clear that if L^{-1} exists, then $L^{\dagger} = L^{-1}$. However, it is very often that $L^{\dagger}\mathbf{d}$ does not depend continuously on the right-hand side $\mathbf{d}$, and in such case we have to regularize (1.1) – as explained below.

When we are refereing to the possibility of (1.1) to be inconsistent, we are considering that it may be arising from an ill-posed problem. We recall that a problem is called well-posed (in Hadamard sense) if: (i) it is solvable, (ii) its solution is unique, (iii) its solution depends continuously on the system parameters (i.e., arbitrary small perturbation of the data cannot cause arbitrary large perturbation of the solution). Otherwise, the problem is called ill-posed.

To analyse (1.1) when it is ill-posed, we can look for a minimizer of the least squares functional

$$\mathcal{F}_L(f) := \|Lf - \mathbf{d}\|^2_{\mathcal{H}}, \tag{1.3}$$

where $\|\cdot\|_{\mathcal{H}}$ denotes the norm of $\mathcal{H}$. However, if L has a nontrivial kernel, then the minimization problem will not have a unique solution. We may choose the minimum-norm solution in this case. If L is invertible but L^{-1} is not continuous, then any perturbation included in $\mathbf{d}$ can generate an arbitrarily large difference in f. In this case, in view to obtain a stable solution, we must regularize the problem.

One of the most famous methods of regularization is the so-called *Tikhonov regularization* (see [16, 20, 21]). It consists in replacing the least squares minimization problem, associated with the previous functional $\mathcal{F}_L$ in (1.3), by a new minimization problem for a new suitably chosen Tikhonov functional. Here, for such choice, the key idea is to balance $\mathcal{F}_L(f)$ against additional information, e.g. on the norm of the solution f, by minimizing

$$\mathcal{F}_L^\lambda(f) := \|Lf - \mathbf{d}\|_{\mathcal{H}}^2 + \lambda\|f\|_H^2, \tag{1.4}$$

where λ is a positive regularization parameter (which has to be chosen in an appropriate way).

The minimization of $\mathcal{F}_L^\lambda(f)$, in H, leads to the so-called *regularized solution* f_λ (which depends on the parameter λ). From (1.1) and (1.4), we realize the behaviour of such solutions: at small λ's, a solution of the last minimization problem is near to a solution of the initial (ill-posed) problem; at large values of λ, the new minimization problem is well-posed, but its solution is far from the solutions of the initial ill-posed problem. Thus, the main objective of regularization is to incorporate additional information about the desired solution in order to stabilize the initial problem and find a useful and stable solution. In general it is not straightforward to prove the existence or the uniqueness of a minimizer for a Tikhonov functional. Anyway, there are restricted conditions under which extra information can be ensured (e.g., in the case of strictly convex Tikhonov functionals, the corresponding minimization problem admits a unique solution). Anyway, in concrete cases, a major problem is still the identification of such a minimizer.

2. Known background

In the present work, we are devoted to analyse the type of problems presented in the last section, but when considering H to be a reproducing kernel Hilbert space H_K (with an associated reproducing kernel K, as a two variable complex valued function, defined on $E \times E$, for some set E). In such a richer structure, some additional techniques can be considered and implemented. In what remains from this section, we will revise what is already known and relevant for this issue, in our reproducing kernel Hilbert space setting context (cf. [18, 19]).

Thus, from now on we will be considering L to be any bounded linear operator from a reproducing kernel Hilbert space H_K into a Hilbert space $\mathcal{H}$. Having in mind the above framework, for any element $\mathbf{d}$ of $\mathcal{H}$, at a first instance, we are now interested in looking for

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}}. \tag{2.1}$$

It is clear that now we would like to consider consequent operator equations, generalized solutions and corresponding generalized inverses within the framework of $f \in H_K$ and $\mathbf{d} \in \mathcal{H}$, having in mind the equation (1.1).

The problem of determining (2.1) turns out to be characterized, within the reproducing kernels theory framework, in the sense of the next (very important) proposition. In view of this, let us built the usual auxiliary reproducing kernel, with the help of the adjoint L^*:

$$k(p,q) = (L^*LK(\cdot,q), L^*LK(\cdot,p))_{H_K} \qquad \text{on} \qquad E \times E. \tag{2.2}$$

Notice that the reproducing property of K allows us to identify the range of L^*L with the new reproducing kernel Hilbert space H_k, generated by k and given in (2.2); for further details, we refer the reader to [18, Chapter 4].

Proposition 2.1. *For any element* $\mathbf{d}$ *of* $\mathcal{H}$*, there exists a function* $\tilde{f}$ *in* H_K *satisfying*

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} = \|L\tilde{f} - \mathbf{d}\|_{\mathcal{H}} \tag{2.3}$$

if and only if

$$L^*\mathbf{d} \in H_k. \tag{2.4}$$

Furthermore, when there exists a function $\tilde{f}$ *satisfying (2.3), there exists a uniquely determined function that minimizes the norms in* H_K *among the functions satisfying the equality, and its function* $f_{\mathbf{d}}$ *is represented as follows:*

$$f_{\mathbf{d}}(p) = (L^*\mathbf{d}, L^*LK(\cdot,p))_{H_k} \quad \textit{on} \quad E. \tag{2.5}$$

Here, the adjoint operator L^* of L, as we see from

$$(L^*\mathbf{d})(p) = (L^*\mathbf{d}, K(\cdot,p))_{H_K} = (\mathbf{d}, LK(\cdot,p))_{\mathcal{H}},$$

is represented by the known elements $\mathbf{d}, L, K(p,q)$, and $\mathcal{H}$. From this proposition, we realize that the initial problem is considered in a very natural way by the theory of reproducing kernels. In particular, note that the adjoint operator is represented in a useful way (which will be very important in our reasoning later on). The extremal function $f_{\mathbf{d}}$ is the image of Moore-Penrose generalized inverse, $L^\dagger\mathbf{d}$, associated with the equation $Lf = \mathbf{d}$.

Note also the following simple facts that may be referred in our delicate problems.

Proposition 2.2. *In Proposition 2.1, if* L *is injective, then* $f_{\mathbf{d}}(p)$ *is the image of the Moore-Penrose generalized inverse for* $\mathbf{d}$ *if and only if it satisfies the equation*

$$L^*Lf_{\mathbf{d}} = L^*\mathbf{d} \tag{2.6}$$

and it is determined uniquely. Furthermore, if L^* *is injective, then the Moore-Penrose generalized inverse coincides with the usual inverse; that is,*

$$Lf_{\mathbf{d}} = \mathbf{d}. \tag{2.7}$$

Meanwhile, if L^* is injective, then $\{Lf : f \in H_K\}$ is dense in $\mathcal{H}$ and so, for any $\mathbf{d} \in \mathcal{H}$,

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} = 0. \tag{2.8}$$

In the opposite case, we observe the following simple fact for the existence of the proper Moore-Penrose generalized inverse.

Corollary 2.3. *If L^* is not injective and $\mathbf{d} \in \mathcal{H}$ has the Moore-Penrose generalized inverse (that is $L^*\mathbf{d} \in H_k$ in Proposition 2.1 and if $\mathbf{d}$ is not an image of L), then*

$$\min_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} = \|\mathbf{d}_2\|_{\mathcal{H}} > 0, \tag{2.9}$$

for the $\mathbf{d}_2$ part of $\mathbf{d}$ belonging to the kernel of the operator L^*.

Note that the criteria (2.4) is involved, specially when the data contain error or noises in some practical cases. Anyway, in this paper, we will be able to realize the complicated structure of $f_{\mathbf{d}}$. Thus, we shall need to consider error estimates, when $\mathbf{d}$ contains error or noises. For this fundamental issue, we have already at our disposal the following auxiliary conclusions.

Proposition 2.4. *Let $L : H_K \to \mathcal{H}$ be a bounded linear operator defined from any reproducing kernel Hilbert space H_K into any Hilbert space $\mathcal{H}$, let $\lambda > 0$, and define the inner product*

$$\langle f_1, f_2 \rangle_{H_{K_\lambda}} = \lambda \langle f_1, f_2 \rangle_{H_K} + \langle Lf_1, Lf_2 \rangle_{\mathcal{H}}$$

for $f_1, f_2 \in H_K$. Then $(H_K, \langle \cdot, \cdot \rangle_{H_{K_\lambda}})$ is a reproducing kernel Hilbert space whose reproducing kernel is given by

$$K_\lambda(p, q) = [(\lambda + L^*L)^{-1} K_q](p).$$

Here, $K_\lambda(p, q)$ is the uniquely determined solution $\tilde{K}_\lambda(p, q)$ of the functional equation

$$\tilde{K}_\lambda(p, q) + \frac{1}{\lambda}(L\tilde{K}_q, LK_p)_{\mathcal{H}} = \frac{1}{\lambda}K(p, q), \tag{2.10}$$

(that is corresponding to the Fredholm integral equation of the second kind for many concrete cases). Moreover, we are using

$$\tilde{K}_q = \tilde{K}_\lambda(\cdot, q) \in H_K \quad \textit{for} \quad q \in E, \qquad K_p = K(\cdot, p) \quad \textit{for} \quad p \in E.$$

Proposition 2.5. *The Tikhonov functional*

$$H_K \ni f \mapsto \left\{\lambda \|f\|_{H_K}^2 + \|Lf - \mathbf{d}\|_{\mathcal{H}}^2\right\} \in \mathbb{R}$$

attains the minimum and the minimum is attained only by the function $f_{\mathbf{d},\lambda} \in H_K$ such that

$$(f_{\mathbf{d},\lambda})(p) = \langle \mathbf{d}, LK_\lambda(\cdot, p) \rangle_{\mathcal{H}}. \tag{2.11}$$

Furthermore, $(f_{\mathbf{d},\lambda})(p)$ satisfies

$$|(f_{\mathbf{d},\lambda})(p)| \le \sqrt{\frac{K(p, p)}{2\lambda}} \|\mathbf{d}\|_{\mathcal{H}}. \tag{2.12}$$

For up-to-date versions of the Tikhonov regularization by using the theory of reproducing kernels, and some concrete classes of operators, see [1, 2, 3, 5, 6, 7, 8]. Cf. also [9, 16, 20, 21] for the classical references.

Notice that computations on the kernels $K_\lambda(p,q)$, in Proposition 2.4, which are determined by the equation (2.10), were done already by H. Fujiwara and T. Matuura in specific explicit analytical and numerical cases. In particular, H. Fujiwara (cf. [10, 11, 12]) succeeded to handle the corresponding outstanding case of the real and numerical inversion of the Laplace transform. In fact, H. Fujiwara gave the solutions in 600 digits precision for 6000 simultaneous linear equations which were representing a discretization of (2.10). Therefore, the computation of the kernel $K_\lambda(p,q)$ may be considered as a possible task by using computers.

In this paper, we will give a definite representation of the Tikhonov extremal function in Proposition 2.5 by using complete orthogonal systems in the Hilbert spaces H_K and $\mathcal{H}$.

3. The solution of the Tikhonov functional equation

As previously mentioned, we shall now give a natural representation of the Tikhonov extremal function $f_{\mathbf{d},\lambda}$, considered in Proposition 2.5, by using complete orthogonal systems in the Hilbert spaces H_K and $\mathcal{H}$.

Let $\{\Phi_\mu\}_{\mu=0}^{\infty}$ and $\{\Psi_\nu\}_{\nu=0}^{\infty}$ be any fixed complete orthonormal systems of the Hilbert spaces $\mathcal{H}$ and H_K, respectively. Therefore, from the form (2.10) and the representation (2.11), in Proposition 2.5, of the Tikhonov extremal function $f_{\mathbf{d},\lambda}$, we assume the representation

$$L\tilde{K}_\lambda(\cdot,q) = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} d_{\mu,\nu}\Phi_\mu \otimes \overline{\Psi_\nu(q)}, \tag{3.1}$$

in the sense of the tensor product $\mathcal{H}\otimes\overline{H_K}$. Here, we are using the usual notation $\overline{V} := (V, +, \overline{\cdot})$ for the well-known complex conjugate vector space of a vector space $V := (V, +, \cdot)$, with "+" and "$\cdot$" being the addition and multiplication maps on V, and where the complex conjugate multiplication $\overline{\cdot} : \mathbb{C}\times V \to V$ is defined in the obvious way by $c \mathbin{\overline{\cdot}} v := \overline{c}\cdot v$ (with $c\in\mathbb{C}, v\in V$).

In the expansion of the reproducing kernel $K(p,q)$,

$$K(p,q) = \sum_{n=0}^{\infty}\Psi_n(p)\overline{\Psi_n(q)}, \tag{3.2}$$

since $LK(\cdot,q)$ belongs to $\mathcal{H}$, we can set

$$L\Psi_n(\cdot) = \sum_{m=0}^{\infty} D_{n,m}\Phi_m \tag{3.3}$$

with the uniquely determined constants $\{D_{n,m}\}$ satisfying, for any fixed n,

$$\sum_{m=0}^{\infty} |D_{n,m}|^2 < \infty.$$

Then, we obtain

$$(L\tilde{K}_q, LK_p)_{\mathcal{H}} = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{n=0}^{\infty} d_{\mu,\nu}\overline{D_{n,\mu}}\Psi_n(p)\overline{\Psi_\nu(q)}. \tag{3.4}$$

Indeed, by repeating the application of the Schwarz inequality, we derive the estimate:

$$\begin{aligned} |(L\tilde{K}_q, LK_p)_{\mathcal{H}}| \quad &\leq \quad \left(\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} |d_{\mu,\nu}|^2\right)^{1/2} \left(\sum_{n=0}^{\infty}\sum_{\mu=0}^{\infty} |D_{n,\mu}|^2\right)^{1/2} \\ &\quad \cdot K(p,p)^{1/2}K(q,q)^{1/2}. \end{aligned} \tag{3.5}$$

Note that the natural condition

$$\sum_{n=0}^{\infty}\sum_{\mu=0}^{\infty} |D_{n,\mu}|^2 < \infty \tag{3.6}$$

is equivalent to the circumstance of the operator L be a Hilbert-Schmidt operator from H_K into $\mathcal{H}$.

By applying L, from (3.4) we obtain

$$L(L\tilde{K}_q, LK_p)_{\mathcal{H}} = \sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{n=0}^{\infty} d_{\mu,\nu}\overline{D_{n,\mu}} \sum_{m=0}^{\infty} D_{n,m}\Phi_m \otimes \overline{\Psi_\nu(q)}. \tag{3.7}$$

Here, note that, by repeating the Schwarz inequality, it follows

$$\begin{aligned} &\sum_{m=0}^{\infty}\left(|\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{n=0}^{\infty} d_{\mu,\nu}\overline{D_{n,\mu}}D_{n,m}\overline{\Psi_\nu(q)}|\right)^2 \\ &\qquad \leq \left(\sum_{\mu=0}^{\infty}\sum_{\nu=0}^{\infty} |d_{\mu,\nu}|^2\right) \left(\sum_{n=0}^{\infty}\sum_{\mu=0}^{\infty} |D_{n,\mu}|^2\right)^2 K(q,q). \end{aligned} \tag{3.8}$$

Now, by comparing the coefficients of Φ_μ, we obtain, for any μ:

$$\lambda\sum_{\nu=0}^{\infty} d_{\mu,\nu}\overline{\Psi_\nu(q)} + \sum_{m=0}^{\infty}\sum_{\nu=0}^{\infty}\sum_{n=0}^{\infty} d_{m,\nu}\overline{D_{n,m}}D_{n,\mu}\overline{\Psi_\nu(q)} = \sum_{n=0}^{\infty} D_{n,\mu}\overline{\Psi_n(q)}. \tag{3.9}$$

On the other hand, by comparing the coefficients of $\overline{\Psi_\nu(q)}$, we obtain, for any μ, ν:

$$\lambda d_{\mu,\nu} + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_{m,\nu}\overline{D_{n,m}}D_{n,\mu} = D_{\nu,\mu}. \tag{3.10}$$

Therefore, by setting

$$\sum_{n=0}^{\infty} \overline{D_{n,m}} D_{n,\mu} = A_{m,\mu},$$

we have just obtained the infinite equations

$$\lambda d_{\mu,\nu} + \sum_{m=0}^{\infty} d_{m,\nu} A_{m,\mu} = D_{\nu,\mu}. \tag{3.11}$$

We are assuming the existence and uniqueness of the coefficients $\{d_{\mu,\nu}\}$, satisfying

$$\sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} |d_{\mu,\nu}|^2 < \infty. \tag{3.12}$$

Here, note that in any finite case of $\{d_{\mu,\nu}\}_{\mu,\nu=0}^{N}$, we can determine the coefficients $\{d_{\mu,\nu}\}_{\mu,\nu=0}^{N}$ by using those equations. This is because, for any $N > 0$, the coefficient matrix $[\lambda I_N + A_{m,\mu}]$ is positive definite Hermitian (for the unit matrix $[I_N]$). However, here, we have to assume that all $\{d_{\mu,\nu}\}_{\mu,\nu=N+1}^{\infty}$ are zero; that is, we approximate the solutions by $\{d_{\mu,\nu}\}_{\mu,\nu=0}^{N}$.

Thus, we obtain the representation of the Tikhonov extremal function in the following way.

Theorem 3.1. *Assume that the operator L is a Hilbert-Schmidt operator and the equations (3.11) have the solutions $\{d_{\mu,\nu}\}$ satisfying (3.12). Then, for any*

$$\mathbf{d} = \sum_{\mu=0}^{\infty} b_\mu \Phi_\mu \in \mathcal{H},$$

the Tikhonov extremal function $(f_{\mathbf{d},\lambda})(p)$ (in the sense of Proposition 2.5) is given by

$$(f_{\mathbf{d},\lambda})(p) = \sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} b_\mu \overline{d_{\nu,\mu}} \Psi_\nu(p) \tag{3.13}$$

and the following fundamental estimates hold:

$$\begin{aligned} |(f_{\mathbf{d},\lambda})(p)| &\leq \left(\sum_{\mu=0}^{\infty} |b_\mu|^2 \right)^{1/2} \left(\sum_{\mu=0}^{\infty} | \sum_{\nu=0}^{\infty} \overline{d_{\nu,\mu}} \Psi_\nu(p)|^2 \right)^{1/2} \\ &\leq \left(\sum_{\mu=0}^{\infty} |b_\mu|^2 \right)^{1/2} \left(\sum_{\mu=0}^{\infty} \sum_{\nu=0}^{\infty} |d_{\nu,\mu}|^2 \right)^{1/2} K(p,p)^{1/2} \end{aligned}$$

and

$$||f_{\mathbf{d},\lambda}||_{H_K}^2 = \sum_{\nu=0}^{\infty} | \sum_{\mu=0}^{\infty} b_\mu \overline{d_{\nu,\mu}}|^2. \tag{3.14}$$

In the previous result, it is important to stress that $d_{\nu,\mu}$ are depending on λ. Moreover, if H_K is finite dimensional then the infinite equations (3.11) are always explicitly solved and so, for such case, it is enough to check the convergence (3.12). Obviously, if both spaces H_K and $\mathcal{H}$ are finite dimensional, then we obtain the complete representation for the Tikhonov extremal function $(f_{\mathbf{d},\lambda})(p)$.

Notice also that many concrete Hilbert-Schmidt operators may be considered from self-adjoint compact operators related to ordinary and partial differential equations. Moreover, in these cases, the transform constants $D_{m,n}$ are diagonal and consequently the equations (3.11) may be solved easily (see [15] for possible corresponding concrete examples). On this last line, we also recall that in many cases, arising from the applications, we need to consider concrete integral operators on the usual L_2 Lebesgue spaces, and not on reproducing kernel Hilbert spaces. Nevertheless, this is not an obstacle since we can take into account that the L_2 Lebesgue spaces may be related to the norm of reproducing kernel Hilbert spaces.

4. On the Tikhonov regularization for the approximations of quotients of Bergman functions by Szegö functions

This section is devoted to give a very surprising and concrete application of Theorem 3.1, which enables us to consider approximations of quotients of Bergman functions by Szegö functions. As the reader will easily identify later on, a key part in the deduction of such results is taken by a simple trick of using multiplication operators on the Szegö space. Thus, basically, we will consider an equation with the structure of (1.1), but where in the right-hand side we incorporate already the unknown element, multiplied by the known data (cf. (4.5)). This simple consideration, will allow us to obtain approximations of the announced quotients based on associated multiplication operations and a detailed concretization of Theorem 3.1.

4.1. Associated developments

Prior to enter directly with the concrete deduction for the approximations of quotients of Bergman functions by Szegö functions, let us use this subsection to make some general remarks, about previous works, somehow near to this issue.

Although later on we will be just focused on the unit disk case, we would like to mention that more general regions can be considered as well. Relevant properties are also already known e.g. when considering an N-ply connected regular domain D with disjoint analytic Jordan curves. Namely, we have the basic and deep relation, for the Bergman kernel K and the Szegö kernel $\hat{K}$ on D,

$$K(z,\overline{u}) \gg 4\pi\hat{K}(z,\overline{u})^2$$

–i.e., the left-hand side minus the right-hand side is a positive definite quadratic form function– which was given by D.A. Hejhal (cf. [14]). This profound result has a great historical background and we may also find its roots in the works of G.F.B. Riemann, F. Klein, S. Bergman, G. Szegö, Z. Nehari, M.M. Schiffer, P.R. Garabedian and D.A. Hejhal. It seems that any elementary proof is impossible. However, the result will, in particular, mean the following fairly simple inequality: For two functions φ and ψ of $H_2(D)$, the Szegö space or the Hardy 2-space on D, we obtain the generalized isoperimetric inequality

$$\frac{1}{\pi}\int\int_D |\varphi(z)\psi(z)|^2 dxdy \leq \frac{1}{2\pi}\int_{\partial D} |\varphi(z)|^2 |dz| \frac{1}{2\pi}\int_{\partial D} |\psi(z)|^2 |dz|,$$

and we can determine completely the cases which attain the equality in here. This result was given in the thesis [17] of the second author, published already in 1979. At that time, the author realized the importance of the abstract and general theory of reproducing kernels by N. Aronszajn. In that paper, the core part was to determine the equality statement in the above inequality. Surprisingly enough, some deep and general independently proof appeared 26 years later by A. Yamada [22]. Moreover, notice that for the special case $\varphi \equiv \psi \equiv 1$, for the plane measure $m(D)$ of D and the length ℓ of the boundary, we have the isoperimetric inequality

$$4\pi m(D) \leq \ell^2.$$

Now, we find an important meaning or application of the inequality. That is, when we fix any element ψ of $H_2(D)$, the multiplication operator from $H_2(D)$ into the Bergman space,

$$\varphi \longmapsto \varphi(z)\psi(z), \tag{4.1}$$

is bounded. Therefore, by Proposition 2.5 and Theorem 3.1, we can give the approximate representation of the generalized fractional functions

$$\frac{\mathbf{F}(z)}{\psi(z)}, \tag{4.2}$$

for any Bergman function $\mathbf{F}(z)$ on the domain D, in the sense of Tikhonov regularization; that is, we can consider the best approximation problem for the functions $\mathbf{F}(z)/\psi(z)$ by the functions in $H_2(D)$. In particular, note that the elements (4.2) are, in general, meromorphic functions. For a global theory about fractional functions we would like to refer the reader to [4].

4.2. Tikhonov fractional functions

In our general situation in Section 3, we are now considering the Tikhonov generalized fractional functions for the Bergman space and the Szegö space on regular domains (in the sense just identified in the previous subsection). As a previous note about one of the inherent difficulties in such application, we notice that the multiplication operator (4.1) is not Hilbert-Schmidt.

The Szegö space $\mathcal{H}_S$ with the reproducing kernel

$$K_S(z,\bar{u}) = \sum_{n=0}^{\infty} z^n \bar{u}^n = \frac{1}{1 - z\bar{u}} \tag{4.3}$$

is given by the Hardy H_2 space. Thus, we will be in fact focused on the classical Hardy space of the unit disk. Then, the norm is given by

$$\|f(z)\|^2_{\mathcal{H}_S} = \frac{1}{2\pi}\int_{|z|=1} |f(z)|^2 |dz| = \sum_{n=0}^{\infty} |c_n|^2, \tag{4.4}$$

with $f(z) = \sum_{n=0}^{\infty} c_n z^n$. For any fixed $f_0(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathcal{H}_S$, we shall consider the bounded linear operator $L : \mathcal{H}_S \to \mathcal{H}_B$, defined by

$$L(f) = f_0 f. \tag{4.5}$$

The Bergman kernel is given by

$$K_B(z, \bar{u}) = \frac{1}{(1 - z\bar{u})^2} \tag{4.6}$$

and the norm is given by

$$\|f(z)\|^2_{\mathcal{H}_B} = \frac{1}{\pi}\int\int_{|z|<1} |f(z)|^2 dxdy = \sum_{n=0}^{\infty} \frac{|b_n|^2}{n+1}, \tag{4.7}$$

for $f(z) = \sum_{n=0}^{\infty} b_n z^n$. Therefore, when comparing with the general situation of Section 3, we are here using

$$\mathcal{H} = \mathcal{H}_B, \qquad H_K = \mathcal{H}_S,$$
$$\Phi_\mu(z) = \sqrt{\mu+1}\, z^\mu, \qquad \Psi_\nu(z) = z^\nu,$$

respectively. Then, by comparing the Taylor coefficients, we know that

$$D_{n,m} = \frac{a_{m-n}}{\sqrt{m+1}}. \tag{4.8}$$

Here, and in the sequel, we shall understand the coefficients a_j as follows:

$$a_{m-n} = 0, \quad m < n.$$

By setting

$$(L\tilde{K}_\lambda(\cdot, \bar{u}))(z) = f_0(z)\tilde{K}_\lambda(z, \bar{u}) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d^*_{m,n} z^m \bar{u}^n, \tag{4.9}$$

we consider the equations

$$\begin{aligned} \lambda d^*_{n,m} \quad &+ \quad d^*_{0,m} \cdot (a_n \overline{a_0}) \\ &+ \quad d^*_{1,m} \cdot \frac{1}{2}\left(a_n\overline{a_1} + a_{n-1}\overline{a_0}\right) \\ &+ \quad d^*_{2,m} \cdot \frac{1}{3}\left(a_n\overline{a_2} + a_{n-1}\overline{a_1} + a_{n-2}\overline{a_0}\right) \\ &+ \quad \cdots \qquad\qquad\qquad\qquad\qquad = a_{n-m}. \end{aligned} \tag{4.10}$$

Then, we obtain

Theorem 4.1. *For any $\mathbf{F}(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{H}_B$, in the Bergman space, and for any fixed $f_0(z) = \sum_{n=0}^{\infty} a_n z^n$ (in the Szegö space $\mathcal{H}_S$), the Tikhonov fractional function $(f_{\mathbf{F},\lambda})(z)$ in the sense of Proposition 2.5 for $H_K = \mathcal{H}_S$, $\mathcal{H} = \mathcal{H}_B$, and $Lf = f_0 \cdot f$ is given by*

$$(f_{\mathbf{F},\lambda})(z) = \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} b_j \frac{1}{\sqrt{j+1}} \overline{d_{j,n}^*} \right) z^n. \tag{4.11}$$

Here, the constants $\{d_{j,n}^*\}$ satisfying (4.9) and (4.10) exist and are unique. Moreover, we obtain the fundamental estimates:

$$\begin{aligned} |(f_{\mathbf{F},\lambda})(z)| &\leq ||\mathbf{F}||_{\mathcal{H}_B} \, ||f_0(\cdot)\tilde{K}_\lambda(\cdot, \bar{z})||_{\mathcal{H}_B} \\ &= \left(\sum_{n=0}^{\infty} \frac{|b_n|^2}{n+1} \right)^{1/2} \left(\sum_{n=0}^{\infty} \frac{1}{n+1} \left| \sum_{m=0}^{\infty} \overline{d_{n,m}^*} z^m \right|^2 \right)^{1/2}. \end{aligned} \tag{4.12}$$

In particular, $\{d_{n,m}^*\}$ are depending on λ and

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \left| \sum_{m=0}^{\infty} \overline{d_{n,m}^*} z^m \right|^2 < \infty, \quad \textit{for } z \textit{ such that } |z| < 1.$$

Proof. At first, we shall consider the representation (4.9). Note here that this function $f_0(z)\tilde{K}_\lambda(z, \bar{u})$ exists and it belongs to the Bergman space, for any fixed u in the unit disc, and its complex conjugate belongs to the Szegö space, for any fixed z of the unit disc. Therefore, the expansion converges uniformly on any compact set of the unit disc and we can use the expansion formally in the following formal computations.

For the function $f_0(z)K_S(z, \bar{u})$, we can also expand it in the Taylor series with respect to the two variable analytic function of z and $\bar{u}$.

The crucial point is that for the analyticity of the functions, they are expanded in Taylor series and the inner product in the functional equation (2.10) may be computed explicitly. Here, we do not need the expansion (3.3) and, of course, we do not need the assumption on the Hilbert-Schmidt operator property.

Then, at first, by comparing the coefficients of any z^n $(n \geq 0)$, we obtain:

$$\begin{aligned} \lambda \left(\sum_{m=0}^{\infty} d_{n,m}^* \overline{u}^m \right) &+ \left(\sum_{m=0}^{\infty} d_{0,m}^* \overline{u}^m \right) \cdot (a_n \overline{a_0}) \\ &+ \left(\sum_{m=0}^{\infty} d_{1,m}^* \overline{u}^m \right) \cdot \frac{1}{2} (a_n \overline{a_1} + a_{n-1} \overline{a_0}) \\ &+ \left(\sum_{m=0}^{\infty} d_{2,m}^* \overline{u}^m \right) \cdot \frac{1}{3} (a_n \overline{a_2} + a_{n-1} \overline{a_1} + a_{n-2} \overline{a_0}) \\ &\vdots \end{aligned}$$

$$
\begin{aligned}
&+ \left(\sum_{m=0}^{\infty} d^*_{n,m} \overline{u}^m \right) \cdot \frac{1}{n+1} \left(a_n \overline{a_n} \; + a_{n-1} \overline{a_{n-1}} + ... + a_0 \overline{a_0} \right) \\
&+ \left(\sum_{m=0}^{\infty} d^*_{n+1,m} \overline{u}^m \right) \cdot \frac{1}{n+2} \left(a_n \overline{a_{n+1}} \; + a_{n-1} \overline{a_n} + ... + a_0 \overline{a_1} \right) \\
&+ \left(\sum_{m=0}^{\infty} d^*_{n+2,m} \overline{u}^m \right) \cdot \frac{1}{n+3} \left(a_n \overline{a_{n+2}} \; + a_{n-1} \overline{a_{n+1}} + ... + a_0 \overline{a_2} \right) \\
&+ \cdots \qquad\qquad\qquad\qquad\qquad\qquad\qquad = \sum_{j=0}^{n} a_j \overline{u}^{n-j}.
\end{aligned}
$$

Next, for any fixed n, by comparing the coefficients of any $\overline{u}^m$ $(m \geq 0)$, we obtain the desired equations (4.10).

□

In the particular case of $f_0(z) = a_m z^m$ $(m \geq 0)$, we see that all theses equations for $\{d^*_{n,m}\}$ may be solved explicitly as

$$
d^*_{n+m,m} = \frac{a_m}{\lambda + \frac{|a_m|^2}{n+m+1}}
$$

and therefore we obtain the following result.

Theorem 4.2. *For any* $\mathbf{F}(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{H}_B$, *and for any fixed* $f_0(z) = a_m z^m \in \mathcal{H}_S$ *(with* $m \geq 0$ *and* $a_m \neq 0$*), the Tikhonov fractional function* $(f_{\mathbf{F},\lambda})(z)$ *in the sense of Theorem 3.1 for* $H_K = \mathcal{H}_S$, $\mathcal{H} = \mathcal{H}_B$, *and* $Lf = f_0 \cdot f$ *is given by*

$$
(f_{\mathbf{F},\lambda})(z) = \sum_{n=0}^{\infty} \frac{\bar{a}_m b_{n+m}}{\lambda(n+m+1) + |a_m|^2} z^n. \tag{4.13}
$$

It seems that even when $f_0 \equiv 1$, the result is interesting. Furthermore, for $m > 1$ the fractional function

$$
\frac{\mathbf{F}(z)}{f_0(z)}
$$

is, in general, a meromorphic function.

We notice that when we use the solutions $\{d^*_{n,m}\}$ for any fixed m, by letting run n, and we consider a finite size matrix with $n \leq N$, the corresponding matrix is not Hermitian and to prove the non-singularity of the matrix seems to be not obvious. For example, when

$$
f_0(z) = a_0 + a_1 z, \tag{4.14}
$$

we have the following matrix, for $N = 3$,

$$\begin{bmatrix} \lambda + |a_0|^2 & \frac{1}{2}a_0\overline{a_1} & 0 & 0 \\ a_0\overline{a_1} & \lambda + \frac{1}{2}(|a_0|^2 + |a_1|^2) & \frac{1}{3}a_0\overline{a_1} & 0 \\ 0 & \frac{1}{2}a_1\overline{a_0} & \lambda + \frac{1}{3}(|a_0|^2 + |a_1|^2) & \frac{1}{4}a_0\overline{a_1} \\ 0 & 0 & \frac{1}{3}a_1\overline{a_0} & \lambda + \frac{1}{4}(|a_0|^2 + |a_1|^2) \end{bmatrix}. \tag{4.15}$$

Even when $f_0(z) = a_0 + a_1 z$, it seems impossible to solve the equations completely for $\{d_{n,m}\}$ or $\{d^*_{n,m}\}$.

Meanwhile, by comparing the two matrices corresponding to the coefficients of $\{d^*_{n,m}\}$ and $\{d_{n,m}\}$, we obtain the following result, as a generalization of the matrix (4.15).

Theorem 4.3. *The coefficient matrices of $\{d^*_{n,m}\}$ of any size $N > 1$ are regular (i.e., have a non-zero determinant).*

Proof. Indeed, in Section 3, the coefficients $A_{m,n}$ are calculated as follows:

$$\begin{aligned} A_{0,n} &= (a_n\overline{a_0})\frac{1}{\sqrt{n+1}} \\ A_{1,n} &= (a_n\overline{a_1} + a_{n-1}\overline{a_0})\frac{1}{2}\cdot\frac{1}{\sqrt{n+1}} \\ A_{2,n} &= (a_n\overline{a_2} + a_{n-1}\overline{a_1} + a_{n-2}\overline{a_0})\frac{1}{3}\cdot\frac{1}{\sqrt{n+1}} \\ &\vdots \\ A_{n,n} &= (a_n\overline{a_n} + a_{n-1}\overline{a_{n-1}} + \ldots + a_0\overline{a_0})\frac{1}{n+1}\cdot\frac{1}{\sqrt{n+1}} \\ A_{n+1,n} &= (a_n\overline{a_{n+1}} + a_{n-1}\overline{a_n} + \ldots + a_0\overline{a_1})\frac{1}{n+2}\cdot\frac{1}{\sqrt{n+1}} \\ A_{n+2,n} &= (a_n\overline{a_{n+2}} + a_{n-1}\overline{a_{n+1}} + \ldots + a_0\overline{a_2})\frac{1}{n+3}\cdot\frac{1}{\sqrt{n+1}} \\ &\vdots \end{aligned}$$

by the transform

$$d_{m,n} = d^*_{m,n}\frac{1}{\sqrt{m+1}}, \tag{4.16}$$

which corresponds to the representation (4.9). Therefore, from the results of $\{d_{m,n}\}$, we have the desired result. □

4.3. Some remarks on the technical difficulties of the method

From the concrete cases in Theorem 4.2, we see the delicate point in the general theorem of Section 3, that is, in the Bergman and Szegö case, the kernel $f_0(\cdot)\tilde{K}_\lambda(\cdot, \bar{z})$ does not belong to the tensor product $\mathcal{H} \otimes \overline{H_K}$, because we can see that in general (4.16) does not belong to ℓ^2. Therefore, as in Theorem 3.1, we cannot, in general, derive the following estimate in Theorem 4.1:

$$\left(\sum_{n=0}^{\infty}\frac{|b_n|^2}{n+1}\right)^{1/2}\left(\sum_{n=0}^{\infty}\frac{1}{n+1}\left|\sum_{m=0}^{\infty}\overline{d_{n,m}^{*}}z^m\right|^2\right)^{1/2}$$
$$\leq \left(\sum_{n=0}^{\infty}\frac{|b_n|^2}{n+1}\right)^{1/2}\left(\sum_{n=0}^{\infty}\frac{1}{n+1}\sum_{m=0}^{\infty}|d_{n,m}^{*}|^2\right)^{1/2}\left(\frac{1}{1-|z|^2}\right)^{1/2}.$$

Thus, the basic natural assumption (3.1) is containing a delicate problem, in some concrete cases, concerning the tensor product $\mathcal{H}\otimes\overline{H_K}$, and this realizes the intrinsic difficulty within general criteria admitting the expansion (3.1).

Moreover, for the representation in Section 2, for $f = f_0 \in \mathcal{H}_S$,

$$(L^*\mathbf{F})(z) = \sum_{i=0}^{\infty}\sum_{n=i}^{\infty}\frac{1}{n+1}b_n\bar{a}_{n-i}z^i, \tag{4.17}$$

we have, for $g(z) = \sum_{i=0}^{\infty} A_i\xi^i \in \mathcal{H}_S$,

$$\begin{aligned}
(L^*Lg)(z) &= (g, L^*LK_z)_{\mathcal{H}_S} \\
&= \left(\sum_{i=0}^{\infty}A_i\xi^i, \sum_{i=0}^{\infty}\sum_{m=i}^{\infty}\sum_{k=0}^{m}\left(\frac{a_{m-k}\bar{a}_{m-i}(\bar{z})^k}{m+1}\right)\xi^i\right)_{\mathcal{H}_S} \\
&= \sum_{i=0}^{\infty}A_i\overline{\left\{\sum_{m=i}^{\infty}\sum_{k=0}^{m}\left(\frac{a_{m-k}\bar{a}_{m-i}(\bar{z})^k}{m+1}\right)\right\}} \\
&= \sum_{i=0}^{\infty}A_i\left\{\sum_{m=i}^{\infty}\sum_{k=0}^{m}\left(\frac{\bar{a}_{m-k}a_{m-i}z^k}{m+1}\right)\right\} \\
&= \sum_{i=0}^{\infty}\sum_{m=i}^{\infty}\sum_{n=0}^{m}\frac{A_n\bar{a}_{m-i}a_{m-n}}{m+1}z^i.
\end{aligned} \tag{4.18}$$

We thus obtain the completeness condition

$$\sum_{n=i}^{\infty}\frac{1}{n+1}b_n\bar{a}_{n-i} = \sum_{m=i}^{\infty}\sum_{n=0}^{m}\frac{A_n\bar{a}_{m-i}a_{m-n}}{m+1} \tag{4.19}$$

for the existence of the generalized fractional function (and we must must clarify that to look for the solutions $\{A_n\}$ of (4.19) is not an easy task).

5. Szegö space completeness condition and the injectivity issue

In this final section we will be considering the problems of obtaining conditions to ensure the above important issues of having the completeness condition within the Szegö space and the injectivity of the operator L^*.

Let P denote the set of polynomials in z. We need to use the following well-known corollary to Beurling's theorem on invariant subspaces (see, for example, [13, p. 84, Corollary 7.3]):

Proposition 5.1. *Let $f(z) \in \mathcal{H}_S$. Then $f(z)$ is an outer function if and only if $Pf = \{p(z)f(z) \colon p \in P\}$ is dense.*

Having in mind the above presented theory, it is now relevant to put the question when is the range of the multiplication operator dense in $\mathcal{H}_S$; i.e., is it the operator L^* injective? Our answer is assembled in the following result.

Theorem 5.2. *Let $\psi \in \mathcal{H}_S$. The adjoint L^* of the multiplication operator $L \colon \mathcal{H}_S \to \mathcal{H}_B$, $LF = \psi F$, is injective if and only if ψ is an outer function.*

Proof. From (4.17), $L^*F = 0$ for $F(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{H}_B$ if and only if

$$\sum_{n=i}^{\infty} \frac{1}{n+1} b_n \overline{a}_{n-i} = 0, \quad i = 0, 1, \ldots. \tag{5.1}$$

Putting

$$\hat{F}(z) = \sum_{n=0}^{\infty} \frac{1}{n+1} b_n z^n,$$

we see that $\hat{F} \in \mathcal{H}_S$ and it is obvious that $F = 0$ if and only if $\hat{F} = 0$. Then, the condition (5.1) is rewritten as $(\hat{F}, z^i\psi)_{\mathcal{H}_S} = 0$, $i = 0, 1, \ldots$. Since this is equivalent to the condition that $\hat{F} \perp P\psi$, we conclude that L^* is injective if and only if $P\psi$ is dense in $\mathcal{H}_S$. Thus, from Proposition 5.1, we see that L^* is injective if and only if ψ is an outer function. □

In the particular case of $f_0(z) = a_m z^m$ $(m \geq 0)$, we can obtain, from Theorem 4.2, the following result, because the Moore-Penrose generalized inverses are obtained by taking the limit

$$\lim_{\lambda \to 0} (f_{\mathbf{F},\lambda})(z) := (f_{\mathbf{F},0})(z),$$

and the limit functions have to belong to the Szegö space. Thus, to conclude, we obtain:

Corollary 5.3. *For any $\mathbf{F}(z) = \sum_{n=0}^{\infty} b_n z^n \in \mathcal{H}_B$ and for any fixed $f_0(z) = a_m z^m \in \mathcal{H}_S$ $(m \geq 0,\ a_m \neq 0)$, the generalized fractional function $(f_{\mathbf{F},0})(z)$ in the sense of the Moore-Penrose (cf. Section 1) for $H_K = \mathcal{H}_S$, $\mathcal{H} = \mathcal{H}_B$, and $Lf = f_0 \cdot f$ exists if and only if*

$$f_{\mathbf{F},0} \in \mathcal{H}_S;$$

that is,

$$\sum_{n=0}^{\infty} |b_{n+m}|^2 < \infty,$$

and then it is given by

$$(f_{\mathbf{F},0})(z) = \frac{1}{a_m} \sum_{n=0}^{\infty} b_{n+m} z^n = \frac{1}{a_m z^m} \left(\mathbf{F}(z) - \sum_{n=0}^{m-1} b_n z^n \right). \tag{5.2}$$

In particular, notice that in the viewpoint of Corollary 2.3 and Theorem 5.2, we have,

$$\mathbf{F}(z) - f_0(z)(f_{\mathbf{F},0})(z) = \sum_{n=0}^{m-1} b_n z^n \not\equiv 0, \quad m > 0,$$

that is, as expected, the generalized fractional functions in the sense of the Moore-Penrose are, in general, not coinciding with the usual fractional function, i.e.

$$(f_{\mathbf{F},0})(z) \neq \frac{\mathbf{F}(z)}{f_0(z)}.$$

As a concluding remark, we would like to stress that the last result corroborates what was explained in Section 1 about consequent approximate and exact solutions. Moreover, it exhibits, in explicit form, what is the difference between those solutions for the present case. Finally, as a major result in this section, we have obtained the identification of $f_{\mathbf{F},0}$, as well as a characterization of the conditions for its existence.

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