Clifford-Hermite polynomials in fractional Clifford analysis*

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Abstract

In this paper we generalize radial and standard Clifford-Hermite polynomials to the new framework of fractional Clifford analysis with respect to the Riemann-Liouville derivative in a symbolic way. As main consequence of this approach, one does not require an a priori integration theory. Basic properties such as orthogonality relations, differential equations, and recursion formulas, are proven.

Keywords: Clifford analysis; Hermite polynomials; Fractional Calculus; Riemann-Liouville fractional derivatives; Fractional Dirac operator.

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1 Introduction

Clifford analysis is a generalization of classical complex analysis to the case of $\mathbb{R}^d, d \in \mathbb{N}$. At the heart of this theory lies the Dirac operator D, a conformally invariant first-order differential operator which plays the same role in classical Clifford analysis as the Cauchy-Riemann operator $\partial_{\overline{z}}$ does in Complex Analysis. Basic references for this mathematical field are [8, 11]. Recently (see [13, 18, 19]) the development of an extension of Clifford analysis to the fractional case started, i.e., to the case where the first order operator is defined via fractional derivatives - the so-called fractional Dirac operator.

The fractional Dirac operator allows to establish connections between fractional calculus and physics, due to its physical and geometrical interpretation. Physically, this fractional differential operator is related with some aspects of fractional quantum mechanics such as the derivation of the fractal Schrödinger type wave equation or the resolution of the gauge hierarchy problem. Geometrically, the fractional classical part of this operator may be identified with the scalar curvature in Riemannian geometry.

Orthogonal polynomials are an important tool in Clifford analysis. As chief examples, we name the Clifford-Hermite and Clifford-Gegenbauer polynomials. They were introduced in the PhD thesis of J. Cnops [5] (see also [8]) as a higher dimensional generalization of the classical Hermite and Gegenbauer polynomials. They have some interesting applications, e.g., in multi-dimensional wavelet analysis (see [3, 4]). This type of special functions has also been extended to the case of super Clifford analysis (see [9]). In [5] the author proved that under certain assumptions these are the only types of classical polynomials that can be generalized to the framework of Clifford analysis.

The aim of this paper is to extend Clifford-Hermite polynomials to the fractional setting. This is done taking into account their definition in the classical setting, which provide us with a canonical way to generalize them. Hereby, we have to ensure that the classical results still coincide with our results for particular choices of the fractional parameter. These fractional polynomials are polynomials in terms of the vector variable operator, or raising operator. Moreover, if we compare the fractional Clifford-Hermite polynomials with the classical

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Clifford-Hermite polynomials we observe that the main difference between these two classes of polynomials is in the ground state. Hence, in terms of algebra operators both classes coincide.

The authors would like to point out that there are some works exploring the connections between special functions and fractional calculus (see for example [6, 7]). However, as far as the authors are aware there are no attempts to study Hermite polynomials in higher dimensions via fractional derivatives.

The paper is organized as follows: first we present a brief introduction to Clifford analysis in a fractional setting (Section 2). In the following section the radial fractional Clifford-Hermite polynomials are defined and some of their basic properties are studied, namely, the orthogonality and recurrence relations, the differential equation, and a Rodrigues-type formula.

2 Preliminaries

2.1 Higher dimensional analysis

Complex analysis derives much of its advantages from the underlined algebraic structure of $\mathbb{C}$. However, its extension to higher dimensions always lose some structure. One appropriate multivariate analogue of classical complex analysis is Clifford analysis, a generalization based on the Cauchy-Riemann operator. This is a conformally invariant first order operator which factorize the Laplacian and is closely related to the Dirac operator introduced by Dirac in 1928. Although with this approach one loses commutativity, it still preserves enough geometric structure to make it interesting. For details about function theory on Clifford algebras we refer the interested reader to [8, 11].

Let $\{e_1, \cdots, e_d\}$ be the standard orthonormal basis of $\mathbb{R}^d$. The associated Clifford algebra $\mathbb{R}_{0,d}$ is defined as the free algebra generated by $\mathbb{R}^d$ modulo $x^2 = -\|x\|^2 e_\emptyset$, where $x \in \mathbb{R}^d$ and $e_\emptyset := 1$. The defining relation induces the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{i,j}, \quad i, j = 1, \ldots, d,$$

where $\delta_{i,j}$ denotes the Kronecker symbol. A basis for $\mathbb{R}_{0,d}$ is given by

$$\{e_\emptyset, e_A = e_{l_1} e_{l_2} \ldots e_{l_r} : \ A = \{l_1, \ldots, l_r\} \subseteq \{1, \ldots, d\} \ \& \ 1 \leq l_1 < \ldots < l_r \leq d\}.$$

Hence, each $a \in \mathbb{R}_{0,d}$ can be written in the form $a = \sum_A a_A \, e_A$, with $a_A \in \mathbb{R}$. In particular, we write $a = \sum_{i=1}^d e_i a_i \in \mathbb{R}^d$. We introduce the (Clifford) conjugation as $\overline{a} = \sum_A a_A \, \overline{e}_A$, where $\overline{ab} = \overline{b} \, \overline{a}$, and $\overline{e}_\emptyset = 1, \overline{e}_j = -e_j$ for $j = 1, \ldots, d$. We remark that each non-zero vector $a \in \mathbb{R}^d$ satisfies $a \, \overline{a} = \overline{a} a = \|a\|^2$ (the Euclidean norm in dimension d) so that it has an unique multiplicative inverse given by $a^{-1} = \frac{\overline{a}}{\|a\|^2}$.

The complex Clifford algebra is defined as the complexification of the real Clifford algebra $\mathbb{C}_d := \mathbb{C} \otimes_{\mathbb{C}} \mathbb{R}_{0,d}$. An element $w \in \mathbb{C}_d$ has the form $w = \sum_A w_A e_A$ with $w_A \in \mathbb{C}$. We remark that the imaginary unit i of $\mathbb{C}$ commutes with the basis elements, i.e., $ie_j = e_j i$ for all $j = 1, \ldots, d$. We define on $\mathbb{C}_d$ the Hermitian conjugation (an automorphism) as

$$a = \sum_A a_A e_A \mapsto a^\dagger := \sum_A \overline{a}_A \overline{e}_A,$$

where $\overline{e}_A$ represents the Clifford conjugation and $\overline{a}_A$ the complex conjugation.

Given an open domain $\Omega \subset \mathbb{R}^d$ we define a $\mathbb{C}_d-$valued function f over Ω as $f = \sum_A e_A f_A$, where $f_A : \Omega \to \mathbb{C}_d$. Properties such as continuity, L_2, etc. will be understood component-wisely. Next, we recall the Euclidean Dirac operator $Df = \sum_{j=1}^d e_j \, \partial_{x_j} f$, which factorizes the d-dimensional Euclidean Laplacian, i.e., $D^2 f = -\Delta f = -\sum_{j=1}^d \partial_{x_j}^2 f$. A $\mathbb{C}_d$-valued function f is called *left-monogenic* on $\Omega \subset \mathbb{R}^d$ if it satisfies $Df = 0$ on Ω (resp. *right-monogenic* if it satisfies $fD = 0$ on Ω).

2.2 Fractional Calculus

The most widely known fractional derivative is the so-called Riemann-Liouville derivative. Its definition appears as a result of the unification of the notions of integer-order integration and differentiation, and has the following expression (up to a normalizing constant):

$$_0D^\alpha f(x) = D^\alpha f(x) = \frac{d^m}{dx^m} \int_0^x (x - \tau)^{m-1-\alpha} \, f(\tau) \, d\tau, \tag{1}$$

with $m-1 \leq \alpha < m$ and $m \in \mathbb{N}$. The previous definition requires the function f to be m times continuously differentiable which corresponds, in some sense, to a narrow class of functions; however, this class is very important for applications since a large number of dynamical processes is smooth enough and does not allow for discontinuities. Understanding this fact is important for a proper use of methods of the fractional calculus in applications. The more so as the Riemann-Liouville definition allows to weaken the condition on f namely, (1) exists for all $x > 0$ such that the integral can be differentiated m times. The weak condition on f in (1) is necessary, for example, when calculating the solution of the Abel equation. For more details about fractional calculus and applications we refer to [14, 15, 16].

In [10] the Riemann-Liouville fractional derivative (1) was successfully applied in the definition of the fractional correspondent of the Dirac operator in the context of Clifford analysis. There, the author proved that under some specific conditions the fractional Dirac operator can be written as

$$D^\alpha f = \sum_{j=1}^{d} e_j \; D_j^\alpha f = \sum_{j=1}^{d} e_j \; (D_j + Y_j)\, f,$$

with $D_j := \frac{\partial}{\partial x_j}$ the classical derivative with respect to x_j. Here, $Y_j = \frac{\alpha-1}{x_j-\xi_j}$, $0 < \alpha < 1$, and $\xi = (\xi_1, \ldots, \xi_d)$ is the observer time-vector. In [19] the author showed that this simplification leads to the osp(1|2) case. Indeed, let us denote $\mathbf{x} = \sum_{j=1}^{d} e_j \mathbf{x}_j = \sum_{j=1}^{d} e_j (x_j - \xi_j)$. Hence,

$$D_i^\alpha(\mathbf{x}_j u) = \delta_{ij} u + \mathbf{x}_j D_i^\alpha u,$$

so that the fractional Weyl relation between the fractional derivatives and the vectors (viewed as operators) is given by

$$[D_i^\alpha, \mathbf{x}_j] = \delta_{ij}. \tag{2}$$

This leads to the commutator relation

$$\begin{aligned} [D^\alpha, \mathbf{x}]u &= D^\alpha \mathbf{x} u - \mathbf{x} D^\alpha u \\ &= \sum_{i,j=1}^{d} e_{ij} \left(\delta_{ij} u + \mathbf{x}_j D_i^\alpha u - \mathbf{x}_i D_j^\alpha u\right) \\ &= -du - 2\sum_{i<j} e_{ij} (\mathbf{x}_i D_j^\alpha - \mathbf{x}_j D_i^\alpha) u = (-d - 2\Gamma^\alpha) u, \end{aligned} \tag{3}$$

where $\Gamma^\alpha := \sum_{i<j} e_{ij} (\mathbf{x}_i D_j^\alpha - \mathbf{x}_j D_i^\alpha)$ denotes the fractional Gamma operator, while for the anti-commutator we obtain

$$\{D^\alpha, \mathbf{x}\} u = D^\alpha \mathbf{x} u + \mathbf{x} D^\alpha u = -du - 2\sum_{i=1}^{d} \mathbf{x}_i D_i^\alpha u = (-d - 2\mathbb{E}^\alpha) u,$$

that is to say

$$D^\alpha \mathbf{x} = -\mathbf{x} D^\alpha - d - 2\mathbb{E}^\alpha, \tag{4}$$

where $\mathbb{E}^\alpha := \sum_{i=1}^{d} \mathbf{x}_i D_i^\alpha$ represents the Euler operator. From the previous conclusions we have that the two natural operators D^α and $\mathbf{x}$, considered as elements of the odd part of the algebra, generate a finite-dimensional Lie superalgebra in the algebra of endomorphisms generated by the partial fractional Riemann-Liouville derivatives, the basic *vector variables* $\mathbf{x}_j$ (seen as multiplication operators), and the basis of the Clifford algebra e_j (for more details see [19]). Before we proceed we recall the definition of Lie superalgebra (for more details about Lie superalgebras and their connections with Clifford analysis see [2, 12]).

Definition 2.1 *A Lie superalgebra $\mathfrak{g}$ (over $\mathbb{R}$ or $\mathbb{C}$) is a $\mathbb{Z}_2-$graded vector space, that is, a direct sum of two vector spaces $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ equipped with a graded bracket $[[\cdot, \cdot]]$, satisfying*

- *the $\mathbb{Z}_2-$grading: $[[a_i,\ a_j]] \in \mathfrak{g}_{i+j \bmod 2}$;*
- *the graded antisymmetry: $[[a_i,\ a_j]] = (-1)^{ij}\ [[a_j,\ a_i]]$;*

- *the generalized Jacobi identity:*

$$(-1)^{ik}\ [[a_i,[[a_j,\ a_k]]]]+(-1)^{ji}\ [[a_j,[[a_k,\ a_i]]]]+(-1)^{kj}\ [[a_k,[[a_i,\ a_j]]]]=0,$$

with $a_i \in \mathfrak{g}_i$, $a_j \in \mathfrak{g}_j$, $a_k \in \mathfrak{g}_k$.

Now, a $\mathbb{C}_d$-valued function f is called *fractional left-monogenic on* $\Omega \subset \mathbb{R}^d$ if it satisfies there $D^\alpha f = 0$ (resp. *fractional right-monogenic* if it satisfies $fD^\alpha = 0$). We observe that due to the definition of the fractional Dirac operator we have

$$D^\alpha \left(\prod_{j=1}^{d} (x_j - \xi_j)^{1-\alpha} \right) = 0,$$

i.e., $\prod_{j=1}^{d}(x_j-\xi_j)^{1-\alpha}$ is a fractional monogenic function, the so-called *ground state*. We remark that the fractional power $(\xi_i - x_i)^{1-\alpha}$ should be understood in the following way

$$(\xi_j - x_j)^{1-\alpha} = \begin{cases} \exp((1-\alpha)\ \ln|\xi_j - x_j|); & \xi_j > x_j \\ 0; & \xi_j = x_j \\ \exp((1-\alpha)\ \ln|\xi_j - x_j| + i\alpha\pi); & \xi_j < x_j \end{cases}.$$

In order to simplify the notation we denote the ground state for the fractional Dirac operator by

$$[1](\mathbf{x}) := \prod_{j=1}^{d} \mathbf{x}_j^{1-\alpha} = \prod_{j=1}^{d} (x_j - \xi_j)^{1-\alpha}, \tag{5}$$

in analogy with the behavior of a constant function under the action of the Euclidean Dirac operator. Furthermore, the fractional Gaussian function is defined as

$$G^\alpha(\mathbf{x}) = \exp\left(-\frac{\sum_{j=1}^{d} \mathbf{x}_j^2}{2} \right). \tag{6}$$

Its action on the ground state is given by

$$G^\alpha(\mathbf{x})\ [1](\mathbf{x}) = \prod_{j=1}^{d} \mathbf{x}_j^{1-\alpha}\ \exp\left(-\frac{\mathbf{x}_j^2}{2} \right), \tag{7}$$

and, as expected, it verifies

$$D^\alpha\ G^\alpha\ [1] = -\mathbf{x}\ G^\alpha\ [1], \qquad\qquad \mathbb{E}^\alpha\ G^\alpha\ [1] = \mathbf{x}^2\ G^\alpha\ [1], \tag{8}$$

Remark 2.2 *During this paper we restrict ourselves to the case where the fractional parameter* α *belongs to the interval* $]0,1[$. *The general case can be easily reduced to this one by means of* $\alpha = [\alpha] + \tilde{\alpha}$, *with* $[\alpha] \in \mathbb{N}$ *and* $\tilde{\alpha} \in]0,1[$.

3 Radial fractional Clifford-Hermite polynomials

3.1 Definition

Let us search for formal series G_k^α in d variables such that

$$\sum_{k=0}^{\infty} \frac{(e_0\ \mathbf{x}_0)^k}{k!}\ G_k^\alpha(\mathbf{x})\ [1](\mathbf{x}_0 e_0 + \mathbf{x}) = \sum_{k=0}^{\infty} \frac{(e_0\ \mathbf{x}_0)^k\ [1](\mathbf{x}_0)}{k!}\ G_k^\alpha(\mathbf{x})\ [1](\mathbf{x})$$

is fractional monogenic in $d+1$ variables, where $[1](\mathbf{x}_0) := (x_0 - \xi_0)^{1-\alpha}$ and G_0^α is set as $G_0^\alpha := G^\alpha$ the fractional Gaussian in dimension d (see (7)). We have then,

$$\begin{aligned} 0 &= (e_0\ D_0^\alpha + D^\alpha) \sum_{k=0}^{\infty} \frac{(e_0\ \mathbf{x}_0)^k\ [1](\mathbf{x}_0)}{k!}\ G_k^\alpha(\mathbf{x})\ [1](\mathbf{x}) \\ &= -\sum_{k=1}^{\infty} \frac{(e_0\ \mathbf{x}_0)^{k-1}\ [1](\mathbf{x}_0)}{(k-1)!}\ G_k^\alpha(\mathbf{x})\ [1](\mathbf{x}) + \sum_{k=0}^{\infty} \frac{(-1)^k\ (e_0\ \mathbf{x}_0)^k\ [1](\mathbf{x}_0)}{k!}\ D^\alpha\ G_k^\alpha(\mathbf{x})\ [1](\mathbf{x}), \end{aligned}$$

with D^α being the fractional Dirac operator in d variables. From this, we immediately obtain the recursion relation

$$G^\alpha_{k+1}(\mathbf{x})\ [1](\mathbf{x}) \ = (-1)^{k+1} D^\alpha \left(G^\alpha_k(\mathbf{x})\ [1](\mathbf{x})\right) \ = \dots \ = (-1)^{\sum_{j=2}^{k+1} j} (D^\alpha)^{k+1} \left(G^\alpha_0(\mathbf{x})\ [1](\mathbf{x})\right). \tag{9}$$

Since $D^\alpha G^\alpha_0\ [1] = -\mathbf{x} G^\alpha_0\ [1]$ ($G^\alpha_0 = G^\alpha$!) the action of the fractional Dirac operator on $G^\alpha_0\ [1]$ results in a fractional homogeneous polynomial multiplying by $G^\alpha_0\ [1]$. Hence, we re-write $G^\alpha_k\ [1]$ in the form

$$G^\alpha_k(\mathbf{x}) = H^\alpha_k(\mathbf{x})\ G^\alpha(\mathbf{x}). \tag{10}$$

We call the subsequent polynomials $H^\alpha_k(\mathbf{x})$ (acting as operators) the radial fractional Clifford-Hermite polynomials in d variables. For the explicit determination of the G^α_k and hence, of the fractional Clifford-Hermite polynomials, we further need the following relations

$$[\mathbb{E}^\alpha, \mathbf{x}] = \sum_{i,j=1}^d e_i \mathbf{x}_j [D^\alpha_i, \mathbf{x}_j] = \sum_{i,j=1}^d e_i \mathbf{x}_j \delta_{i,j} = \mathbf{x} \quad \Rightarrow \quad \mathbb{E}^\alpha \mathbf{x} = \mathbf{x}\mathbb{E}^\alpha + \mathbf{x}. \tag{11}$$

By relation (4) we have, for all $k \geq 2$

$$\begin{aligned} D^\alpha \mathbf{x}^k &= (D^\alpha \mathbf{x})\mathbf{x}^{k-1} \ = (-\mathbf{x}D^\alpha - d - 2\mathbb{E}^\alpha)\mathbf{x}^{k-1} \\ &= -\mathbf{x}(-\mathbf{x}D^\alpha - d - 2\mathbb{E}^\alpha)\mathbf{x}^{k-2} + (-d - 2\mathbb{E}^\alpha)\mathbf{x}^{k-1} \\ &= \mathbf{x}^2 D^\alpha \mathbf{x}^{k-2} + 2\mathbf{x}\mathbb{E}^\alpha \mathbf{x}^{k-2} - 2\mathbb{E}^\alpha \mathbf{x}^{k-1} \\ &= \mathbf{x}^2 D^\alpha \mathbf{x}^{k-2} + 2\mathbf{x}\mathbb{E}^\alpha \mathbf{x}^{k-2} - 2(\mathbf{x}\mathbb{E}^\alpha + \mathbf{x})\mathbf{x}^{k-2} \\ &= \mathbf{x}^2 D^\alpha \mathbf{x}^{k-2} - 2\mathbf{x}^{k-1}. \end{aligned}$$

Applying recursively this formula one gets for $k = 2s$ even,

$$D^\alpha \mathbf{x}^{2s} = \mathbf{x}^{2s} D^\alpha - 2s\mathbf{x}^{2s-1}, \tag{12}$$

while for $k = 2s + 1$ odd we have

$$D^\alpha \mathbf{x}^{2s+1} = \mathbf{x}^{2s} D^\alpha \mathbf{x} - 2s\mathbf{x}^{2s} = -\mathbf{x}^{2s+1} D^\alpha - (d + 2s)\mathbf{x}^{2s} - 2\mathbf{x}^{2s}\mathbb{E}^\alpha, \tag{13}$$

Easy calculations show the first fractional Hermite polynomials in d variables to be

$$\begin{array}{ll} H^\alpha_0(\mathbf{x}) = 1, & H^\alpha_3(\mathbf{x}) = -\mathbf{x}^3 - (d+2)\mathbf{x}, \\ H^\alpha_1(\mathbf{x}) = \mathbf{x}, & H^\alpha_4(\mathbf{x}) = \mathbf{x}^4 + 2(d+2)\mathbf{x}^2 + d(d+2), \\ H^\alpha_2(\mathbf{x}) = -\mathbf{x}^2 - d, & H^\alpha_5(\mathbf{x}) = \mathbf{x}^5 + 2(d+4)\mathbf{x}^3 + (d+2)(d+4)\mathbf{x}. \end{array}$$

We observe that H^α_k are, up to sign, equal to $\mathbf{x}^k + p_{k-2}(\mathbf{x}), (k \geq 2)$, where p_{k-2} are polynomials of degree $k-2$. From (12) and (13) we conclude that H^α_{2l} only contains even powers of $\mathbf{x}$ while H^α_{2l+1} only contains odd ones, and therefore, the Hermite polynomials are polynomials of the vector variable operator. In conclusion, we have the following Rodrigues-type formula for the polynomials H^α_k acting as multiplicative operators arising from a recursive application of (9).

Lemma 3.1 *For $k \in \mathbb{N}$ we have that*

$$H^\alpha_k(\mathbf{x}) = (-1)^k\ (G^\alpha)^{-1}(\mathbf{x})\ (D^\alpha)^k\ G^\alpha(\mathbf{x}). \tag{14}$$

Indeed, from relations (9) and (10) we get the analogous of (14) for the case of Hermite functions:

$$\begin{aligned} \tilde{H}^\alpha_{k+1}(\mathbf{x}) \ &:= \ H^\alpha_{k+1}(\mathbf{x}) G^\alpha(\mathbf{x})\ [1](\mathbf{x}) \\ &= \ (-1)^{k+1}\ D^\alpha \left(H^\alpha_k(\mathbf{x})\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\right). \end{aligned} \tag{15}$$

From (15) we conclude that the fractional powers of $\mathbf{x}$ only appear in the Hermite functions, i.e., when the powers of the vector variable operator act on the ground state. This fact can be easily verified in the following first fractional Hermite functions in d variables

$$\begin{aligned}
\tilde{H}_0^\alpha(\mathbf{x}) &= \prod_{j=1}^d \mathbf{x}_j^{1-\alpha} \ \exp\left(-\frac{\mathbf{x}_j^2}{2}\right), \\
\tilde{H}_1^\alpha(\mathbf{x}) &= \sum_{j=1}^d e_j \left[\mathbf{x}_j^{2-\alpha} \ \exp\left(-\frac{\mathbf{x}_j^2}{2}\right) \prod_{i=1, i\neq j}^d \mathbf{x}_i^{1-\alpha} \ \exp\left(-\frac{\mathbf{x}_i^2}{2}\right)\right], \\
\tilde{H}_2^\alpha(\mathbf{x}) &= -\sum_{j=1}^d \left[\mathbf{x}_j^{3-\alpha} \ \exp\left(-\frac{\mathbf{x}_j^2}{2}\right) \prod_{i=1, i\neq j}^d \mathbf{x}_i^{1-\alpha} \ \exp\left(-\frac{\mathbf{x}_i^2}{2}\right)\right] - d\prod_{i=1}^d \mathbf{x}_i^{1-\alpha} \ \exp\left(-\frac{\mathbf{x}_i^2}{2}\right).
\end{aligned}$$

3.2 Recurrence relations

In order to establish a recurrence formula, we will represent the fractional Clifford-Hermite polynomials by

$$H_{2l}^\alpha(\mathbf{x}) = \sum_{j=0}^l a_{2j}^{2l} \ \mathbf{x}^{2j}, \qquad\qquad H_{2l+1}^\alpha(\mathbf{x}) = \sum_{j=0}^l a_{2j+1}^{2l+1} \ \mathbf{x}^{2j+1}. \tag{16}$$

From (15) we get

$$\begin{aligned}
H_{2l}^\alpha(\mathbf{x}) \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x}) &= D^\alpha \left(H_{2l-1}^\alpha(\mathbf{x}) \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x})\right) \\
&= \sum_{j=0}^{l-1} a_{2j+1}^{2l-1} \ D^\alpha \ (\mathbf{x}^{2j+1} \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x})) \\
&= \sum_{j=0}^{l-1} a_{2j+1}^{2l-1} \ \left[-(2j+d)\mathbf{x}^{2j} - 2\mathbf{x}^{2j}\mathbb{E}^\alpha - \mathbf{x}^{2j+1}D^\alpha\right] \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x}) \\
&= \sum_{j=0}^{l-1} a_{2j+1}^{2l-1} \ \left[-(2j+d)\mathbf{x}^{2j} - 2\mathbf{x}^{2j+2} + \mathbf{x}^{2j+2}\right] \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x}) \\
&= -\left[\sum_{j=0}^{l-1} a_{2j+1}^{2l-1} \ (2j+d)\mathbf{x}^{2j} + \mathbf{x}\sum_{j=0}^{l-1} a_{2j+1}^{2l-1}\mathbf{x}^{2j+1}\right] \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x}) \\
&= -\left[\sum_{j=0}^{l-1} a_{2j+1}^{2l-1}(2j+d)\mathbf{x}^{2j} + \mathbf{x} \ H_{2l-1}^\alpha(\mathbf{x})\right] \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x}).
\end{aligned} \tag{17}$$

Hence, we regard the first term as the formal derivative of H_{2l-1}^α, viewed as a polynomial in $\mathbf{x}$, which we will represent by $(D^\alpha H_{2l-1}^\alpha)(\mathbf{x})$. Thus, we get

$$H_{2l}^\alpha(\mathbf{x}) = (D^\alpha - \mathbf{x}) \ H_{2l-1}^\alpha(\mathbf{x}). \tag{18}$$

In a similar way we obtain for the fractional Clifford-Hermite polynomials of odd degree that

$$\begin{aligned}
H_{2l+1}^\alpha(\mathbf{x}) \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x}) &= -D^\alpha \ (H_{2l}^\alpha(\mathbf{x}) \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x})) \\
&= -\sum_{j=0}^l a_{2j}^{2l} \ \left[-2j \ \mathbf{x}^{2j-1} + \mathbf{x}^{2j}D^\alpha\right] \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x}) \\
&= -\sum_{j=0}^l a_{2j}^{2l} \ \left[-2j \ \mathbf{x}^{2j-1} - \mathbf{x}^{2j+1}\right] \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x}) \\
&= \left[\sum_{j=0}^l a_{2j}^{2l} \ 2(j+1)\mathbf{x}^{2j-1} + \mathbf{x} \ H_{2l}^\alpha\right] \ G^\alpha(\mathbf{x}) \ [1](\mathbf{x}),
\end{aligned} \tag{19}$$

where again the first term can be seen as $(D^\alpha H_{2l}^\alpha)(\mathbf{x})$, the formal derivative of H_{2l}^α, viewed as a polynomial in $\mathbf{x}$. Furthermore

$$H_{2l+1}^\alpha(\mathbf{x}) = -\left(D^\alpha - \mathbf{x}\right) \ H_{2l}^\alpha(\mathbf{x}). \tag{20}$$

Moreover, a comparison between the LHS and RHS of (17) and (19), respectively, leads to the following recurrence relations for the coefficients a_{2j}^{2l} and a_{2j+1}^{2l+1},

$$\sum_{j=0}^{l} a_{2j}^{2l}\ \mathbf{x}^{2j} = -\sum_{j=0}^{l-1} a_{2j+1}^{2l-1}\ \left[(2j+d)\ \mathbf{x}^{2j} + \mathbf{x}^{2j+2}\right],$$
$$\sum_{j=0}^{l} a_{2j+1}^{2l+1}\ \mathbf{x}^{2j+1} = \sum_{j=0}^{l} a_{2j}^{2l}\ \left[2(j+1)\ \mathbf{x}^{2j-1} + \mathbf{x}^{2j+1}\right],$$

there is to say,

$$a_{2j}^{2l} = -(2j+d)\ a_{2j+1}^{2l-1} - a_{2j-1}^{2l-1}, \qquad a_{2j+1}^{2l+1} = 2(j+1)\ a_{2j+2}^{2l} + a_{2j}^{2l}. \tag{21}$$

Example 3.2 *From the first fractional Hermite polynomials we obtain the coefficients*

$$\begin{aligned}
&a_1^1 = a_0^0,\\
&a_0^2 = -d\ a_1^1, \qquad a_2^2 = -a_1^1,\\
&a_1^3 = 2\ a_2^2 + a_0^2, \quad a_3^3 = a_2^2,\\
&a_0^4 = -d\ a_1^3, \qquad a_2^4 = -(2+d)\ a_3^3 - a_1^3, \qquad a_4^4 = -a_3^3,\\
&a_1^5 = 2\ a_2^4 + a_0^4, \quad a_3^5 = 4\ a_4^4 + a_2^4, \qquad a_5^5 = a_4^4.
\end{aligned} \tag{22}$$

Furthermore, we obtain the following recurrence lemma:

Lemma 3.3 *For the coefficients a_{2j}^{2l} and a_{2j+1}^{2l+1} we have the following recurrence relations*

$$2j\ a_{2j}^{2l} = -2l\ a_{2j-1}^{2l-1}, \qquad (2j+d)\ a_{2j+1}^{2l+1} = (2l+d)\ \ a_{2j}^{2l}. \tag{23}$$

Proof: The proof is done by induction on l. It is clear that (23) is true for $l = 0, 1$. Suppose now that these relations are valid for $l \in \mathbb{N}$. Hence, for a_{2j+1}^{2l+1} we have

$$\begin{aligned}
(2j+d)\ a_{2j+1}^{2l+1} &\underbrace{=}_{(21)} (2j+d)\ 2(j+1)\ a_{2j+2}^{2l} + (2j+d)\ a_{2j}^{2l}\\
&\underbrace{=}_{(23)} (2j+d)\left[-2l\ a_{2j+1}^{2l-1} + a_{2j}^{2l}\right] = -2l(2j+d)\ a_{2j+1}^{2l-1} + (2j+d)\ a_{2j}^{2l}\\
&\underbrace{=}_{(21)} 2l(a_{2j}^{2l} + a_{2j-1}^{2l-1}) + (2j+d)\ a_{2j}^{2l} = 2j\ a_{2j}^{2l} + 2l\ a_{2j-1}^{2l-1} + (2l+m)\ a_{2j}^{2l}\\
&\underbrace{=}_{(23)} -2l\ a_{2j-1}^{2l-1} + 2l\ a_{2j-1}^{2l-1} + (2l+m)\ a_{2j}^{2l} = (2l+d)\ a_{2j}^{2l}.
\end{aligned}$$

The case of a_{2j}^{2l} is similar.

■

Corollary 3.4 *The coefficients of the fractional Clifford-Hermite polynomials $H_k^\alpha(\mathbf{x})$ (see (16)) take the following form*

$$a_{2j}^{2l} = \frac{\left(-\frac{1}{2}(2l+d-2)\right)_j\ (-l)_j\ (-2)^{l-j}\ \left(\frac{d}{2}\right)_{l-j}}{j!\ \left(-\frac{1}{2}(2j+d-2)\right)_j}\ a_0^0, \qquad a_{2j+1}^{2l+1} = \frac{2l+d}{2j+d}\ a_{2j}^{2l}, \tag{24}$$

where $(\cdot)_j$ denotes the Pochhammer symbol.

Proof: We introduce the auxiliary notations

$$A(l,j) = -(2j+d), \qquad B(j,l) = -1, \qquad C(l,j) = 2(j+1),$$

$$D(l,j) = 1, \qquad E(l,j) - \tfrac{2l}{2j}, \qquad F(l,j) = \tfrac{2l+d}{2j+d}.$$

Then, relations (21) and (23) can be rewritten, respectively, as

$$a_{2j}^{2l} = A(l,j)\ a_{2j+1}^{2l-1} + B(l,j)\ a_{2j-1}^{2l-1}, \qquad a_{2j+1}^{2l+1} = C(l,j)\ a_{2j+2}^{2l} + D(l,j)\ a_{2j}^{2l},$$

$$a_{2j}^{2l} = E(l,j)\ a_{2j-1}^{2l-1}, \qquad a_{2j+1}^{2l+1} = F(l,j)\ a_{2j}^{2l}.$$

Replacing sequentially we obtain

$$\begin{aligned}
a_{2j}^{2l} &\underbrace{=}_{(23)} E(l,j)\ a_{2j-1}^{2l-1} \\
&\underbrace{=}_{(23)} E(l,j)\ F(l-1,j-1)\ a_{2j-2}^{2l-2} \\
&\underbrace{=}_{(23)} E(l,j)\ F(l-1,j-1)\ E(l-1,j-1)\ a_{2j-3}^{2l-3} \\
&= \cdots \\
&= \prod_{k=1}^{j} E(l-(k-1),j-(k-1))\ F(l-k,j-k)\ a_0^{2l-2j} \\
&= \frac{\left(-\frac{1}{2}(2l+d-2)\right)_j\ (-l)_j}{j!\ \left(-\frac{1}{2}(2j+d-2)\right)_j}\ a_0^{2l-2j}. \qquad (25)
\end{aligned}$$

In a similar way, we get for a_0^{2l-2j}

$$\begin{aligned}
a_0^{2l} &\underbrace{=}_{(21)} A(l,0)\ a_1^{2l-1} \\
&\underbrace{=}_{(23)} A(l,0)\ F(l-1,0)\ a_0^{2l-2} \\
&\underbrace{=}_{(21)} A(l,0)\ F(l-1,0)\ A(l-1,0)\ a_1^{2l-3} \\
&\underbrace{=}_{(23)} A(l,0)\ F(l-1,0)\ A(l-1,0)\ F(l-2,0)\ a_0^{2l-4} \\
&= \cdots \\
&= \prod_{k=1}^{l} A(k,0)\ F(k-1,0)\ a_0^0 \\
&= (-2)^{l-j}\ \left(\frac{d}{2}\right)_{l-j}\ a_0^0. \qquad (26)
\end{aligned}$$

Relation (24) arises from combining (25) with (26). For the case of a_{2j+1}^{2l+1} we additionally use the fact that $a_{2j+1}^{2l+1} = F(l,j)\ a_{2j}^{2l}$, together with (24).

■

Corollary 3.5 *The following differential relations for the fractional Clifford-Hermite polynomials are valid*

$$D^\alpha H_{2l}^\alpha(\mathbf{x}) = 2l\ H_{2l-1}^\alpha(\mathbf{x}), \qquad D^\alpha H_{2l+1}^\alpha(\mathbf{x}) = -(2l+d)\ H_{2l}^\alpha(\mathbf{x}). \qquad (27)$$

Proof: For the first relation we recall that

$$H_{2l}^\alpha(\mathbf{x}) = \sum_{j=0}^{l} a_{2j}^{2l}\ \mathbf{x}^{2j}, \qquad H_{2l+1}^\alpha(\mathbf{x}) = \sum_{j=0}^{l} a_{2j+1}^{2l+1}\ \mathbf{x}^{2j+1}.$$

Hence,

$$D^\alpha\ H_{2l}^\alpha(\mathbf{x}) \ = \sum_{j=0}^{l} a_{2j}^{2l}\ D^\alpha\ \mathbf{x}^{2j} \ = -\sum_{j=0}^{l} a_{2j}^{2l}\ 2j\ \mathbf{x}^{2j-1} \ \underbrace{=}_{(23)} 2l\ \sum_{j=0}^{l} a_{2j-1}^{2l-1}\ \mathbf{x}^{2j-1} \ = 2l\ H_{2l-1}^\alpha(\mathbf{x}).$$

In a similar way we prove the second relation.

■

3.3 Differential equation

In the classical setting the Hermite polynomials $H_k(x)$ are solutions of the differential equation

$$(-\Delta - xD - \mu_k)\ H_k(x) = 0,$$

where $\mu_{2l} = 2l$ and $\mu_{2l+1} = 2l + d$ (see [1]). Here, we aim to build the analogue of this equation in the fractional setting. This leads to the following result, which reflects the formal resemblance with the classical approach.

Theorem 3.6 *The radial fractional Clifford-Hermite polynomial $H_k^\alpha(\mathbf{x})$ is a solution of the fractional differential equation*

$$\left((D^\alpha)^2 - \mathbf{x}D^\alpha - \mu_k^\alpha\right)\ H_k^\alpha(\mathbf{x}) = 0,$$

with $\mu_{2l}^\alpha = 2l$ and $\mu_{2l+1}^\alpha = 2l + d$.

Proof: We start with the case $k = 2l$. We apply $D^\alpha - \mathbf{x}$ to both members of the first equality in (27). Taking into account the formal relations introduced in Subsection 3.2, it yields

$$(D^\alpha - \mathbf{x})\ D^\alpha\ H_{2l}^\alpha(\mathbf{x}) = (D^\alpha - \mathbf{x})\ (2l\ H_{2l-1}^\alpha)(\mathbf{x}) \quad \Leftrightarrow \quad \left((D^\alpha)^2 - \mathbf{x}\ D^\alpha\right)\ H_{2l}^\alpha(\mathbf{x}) = 2l\ (D^\alpha - \mathbf{x})\ H_{2l-1}^\alpha(\mathbf{x})$$

which implies, by (18), the desired result

$$\left((D^\alpha)^2 - \mathbf{x}D^\alpha - 2l\alpha\right)\ H_{2l}^\alpha(\mathbf{x}) = 0.$$

The proof for $k = 2l + 1$ is similar and therefore, it will be omitted.

■

3.4 Orthogonality relations

In this subsection we show the orthogonality between Hermite polynomials of different degrees, that is to say

$$\langle H_k^\alpha, H_n^\alpha \rangle := \int_{\mathbb{R}^d} H_k^\alpha(\mathbf{x}) H_n^\alpha(\mathbf{x})\ d\mu(\mathbf{x}) = 0, \quad k \neq n, \tag{28}$$

with $d\mu(\mathbf{x}) := G^\alpha(\mathbf{x})\ [1](\mathbf{x})\ d\mathbf{x}$ being the metric. Remark that despite the ground state being complex valued this bilinear form is real valued. Taking into account Lemma 3.1 we have

$$H_k^\alpha\ H_n^\alpha\ G^\alpha\ [1]\ = (-1)^k\ (-1)^n\ (G^\alpha)^{-1}\ (D^\alpha)^k\ G^\alpha\ (G^\alpha)^{-1}\ (D^\alpha)^n\ G^\alpha\ [1]\ = (-1)^{k+n}\ (G^\alpha)^{-1}\ (D^\alpha)^{k+n}\ G^\alpha\ [1] \tag{29}$$

so that $\langle H_k^\alpha, H_n^\alpha \rangle = \langle H_n^\alpha, H_k^\alpha \rangle$. Thus, it is enough to prove the cases in which $0 \leq k < n$. We will start by proving this relation for the case $0 = k < n$, and complete the proof of the orthogonality relations by induction over k. Then,

$$\langle H_0^\alpha, H_n^\alpha \rangle\ := \int_{\mathbb{R}^d} \{H_0^\alpha(\mathbf{x})\,[H_n^\alpha(\mathbf{x})\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})]\}\ d\mu(\mathbf{x})\ = \pm \int_{\mathbb{R}^d} [(D^\alpha)^n\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})]\ d\mu(\mathbf{x}) = 0.$$

Now, we proceed with the induction over k, under the assumption that the orthogonality relations hold for all $0 \leq k' < k$. Since $H_k^\alpha(\mathbf{x}) = \mathbf{x}^k +$ (terms of degree $\leq k - 2$), it is enough to prove that

$$\int_{\mathbb{R}^d} \{\mathbf{x}^k\,[H_n^\alpha(\mathbf{x})\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})]\}\ d\mu(\mathbf{x}) = 0$$

whenever $k < n$. We have the relation $D^\alpha \mathbb{E}^\alpha - \mathbb{E}^\alpha D^\alpha = -D^\alpha$. By Lemma 3.1 we get

$$\begin{aligned} \mathbf{x}^k\,[H^\alpha(\mathbf{x})\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})] &= \pm \mathbf{x}^k\ (D^\alpha)^n\ G^\alpha(\mathbf{x})\ [1](\mathbf{x}) \\ &= \pm \mathbf{x}^{k-1}\ (\mathbf{x}D^\alpha)\ (D^\alpha)^{n-1}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x}) \\ &= \pm \mathbf{x}^{k-1}\ (-D^\alpha \mathbf{x} - 2\mathbb{E}^\alpha - d)\ (D^\alpha)^{n-1}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x}). \end{aligned} \tag{30}$$

We observe that

$$\pm d \int_{\mathbb{R}^d} \{\mathbf{x}^{k-1}\ (D^\alpha)^{n-1}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\}\ d\mu(\mathbf{x})\ = 0 \tag{31}$$

under our hypothesis. For the remaining terms we get

$$\begin{aligned}
&\pm\, \mathbf{x}^{k-1}\{(-D^\alpha \mathbf{x} - 2\mathbb{E}^\alpha)\ (D^\alpha)^{n-1}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\} \\
&= \pm \mathbf{x}^{k-1}\{(-D^\alpha(\mathbf{x}D^\alpha) - 2\mathbb{E}^\alpha D^\alpha)\ (D^\alpha)^{n-2}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\} \\
&= \pm \mathbf{x}^{k-1}\{(-D^\alpha(-D^\alpha \mathbf{x} - 2\mathbb{E}^\alpha - d) - 2(D^\alpha \mathbb{E}^\alpha + D^\alpha)]\ (D^\alpha)^{n-2}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\} \\
&= \pm \mathbf{x}^{k-1}\{\big((D^\alpha)^2 \mathbf{x} + (d-2) D^\alpha\big)\ (D^\alpha)^{n-2}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\} \\
&= \pm \mathbf{x}^{k-1}\{(D^\alpha)^2 \mathbf{x}\ (D^\alpha)^{n-2}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x}) + (d-2)(D^\alpha)^{n-1}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\}.
\end{aligned}$$

Hence,

$$\int_{\mathbb{R}^d} \{\mathbf{x}^k\, [H^\alpha(\mathbf{x})\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})]\}\ d\mu(\mathbf{x}) = \pm \int_{\mathbb{R}^d} \{\mathbf{x}^{k-1}\ \big(\big((D^\alpha)^2 \mathbf{x}\big)\ (D^\alpha)^{n-2}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\big)\ d\mu(\mathbf{x}).$$

For n even, and repeating this procedure we end up with

$$\begin{aligned}
\int_{\mathbb{R}^d} \{\mathbf{x}^{k-1}\ \big(\big((D^\alpha)^2 \mathbf{x}\big)\ (D^\alpha)^{n-2}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\big)\}\ d\mu(\mathbf{x}) &= \int_{\mathbb{R}^d} \{\mathbf{x}^{k-1}\ ((D^\alpha)^n \mathbf{x}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x}))\}\ d\mu(\mathbf{x}) \\
&= -\int_{\mathbb{R}^d} \{\mathbf{x}^{k-1}\ \big((D^\alpha)^{n+1}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\big)\}\ d\mu(\mathbf{x}) = 0,
\end{aligned}$$

by hypothesis. For n odd, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} \{\mathbf{x}^{k-1}\ \big(\big((D^\alpha)^2 \mathbf{x}\big)\ (D^\alpha)^{n-2}\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\big)\}\ d\mu(\mathbf{x}) &= \int_{\mathbb{R}^d} \{\mathbf{x}^{k-1}\ \big((D^\alpha)^{n-1} \mathbf{x}(D^\alpha\ G^\alpha(\mathbf{x})\ [1](\mathbf{x}))\big)\}\ d\mu(\mathbf{x}) \\
&= -\int_{\mathbb{R}^d} \{\mathbf{x}^{k-1}\ \big((D^\alpha)^{n-1} \mathbf{x}^2\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\big)\}\ d\mu(\mathbf{x}) \\
&= -\int_{\mathbb{R}^d} \{\mathbf{x}^{k-1}\ \big((D^\alpha)^{n-1} \mathbb{E}^\alpha\ G^\alpha(\mathbf{x})\ [1](\mathbf{x})\big)\}\ d\mu(\mathbf{x}) = 0,
\end{aligned}$$

again by hypothesis since the Euler operator preserves the degree of homogeneity of $G^\alpha(\mathbf{x})\ [1](\mathbf{x})$.

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