Math. J. Okayama Univ. **59** (2017), 141–147

ON THE $(1-C_2)$ CONDITION

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Abstract. In this paper, we give some results on $(1-C_2)-$modules and $1-$continuous modules.

1. Introduction

All rings are associated with identity, and all modules are unital right modules. By M_R, $({}_RM)$ we indicate that M is a right (left) module over a ring R. The Jacobson radical, the uniform dimension and the endomorphism ring of M are denoted by $J(M), u-dim(M)$ and $End(M)$, respectively. For a module M (over a ring R), we consider the following conditions:

$(1-C_1)$ Every uniform submodule of M is essential in a direct summand of M.

(C_1) Every submodule of M is essential in a direct summand of M.

(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand of M.

(C_3) For any direct summands A, B of M with $A \cap B = 0$, $A \oplus B$ is also a direct summand of M.

A module M is defined to be a $(1-C_1)-$*module* if it satisfies the condition$(1-C_1)$. If M satisfies (C_1), then M is said to be a *CS−module* (or an *extending module*). M is defined to be a *continuous module* if it satisfies the conditions (C_1) and (C_2) . If M satisfies (C_1) and (C_3), then M is said to be a *quasi −continuous module*. We call a module M a $(C_2)-$*module* if it satisfies the condition (C_2). We have the following implications:

Injective ⇒ quasi −injective ⇒ continuous ⇒ quasi −continuous ⇒ CS ⇒ $(1-C_1)$,

and $(C_2) \Rightarrow (C_3)$.

For a set A and a module M, $M^{(A)}$ denotes the direct sum of $\mid A \mid$ copies of M. A module M is called a (*countably*) $\sum -$*quasi* − *injective* if $M^{(A)}$ (resp. $M^{(\mathbb{N})}$) is a quasi − injective −module for every set A (note that $\mathbb{N}$ denotes the set of all natural numbers). Similarly, a module M is called a (*countably*) $\sum -(1-C_1)$ if $M^{(A)}$ (resp. $M^{(\mathbb{N})}$) is a $(1-C_1)-$module for every set A.

Mathematics Subject Classification. Primary 16D50; Secondary 16P20.
Key words and phrases. injective module, continuous module, uniform module, UC module, distributive module.

In Section 2, we give several properties on the $(1-C_2)-$modules, (strongly) $1-$continuous modules, and discuss the question of when a $1-$continuous module is continuous ($(1-C_2)-$ module is $(C_2)-$module)?

2. $(1-C_2)$ CONDITION

In this section, we consider the following condition for a module M.

$(1-C_2)$ Every uniform submodule isomorphic to a direct summand of M is itself a direct summand of M.

A module M is defined to be a $(1-C_2)-$*module* if it satisfies the condition $(1-C_2)$. If M satisfies $(1-C_1)$ and $(1-C_2)$ conditions, then M is said to be a $1-$*continuouis module.* M is defined to be a *strongly* $1-$*continuous module* if it satisfies the conditions (C_1) and $(1-C_2)$. A ring R is called a *right* (*left*) $1-$*continuous ring* if R_R (resp. ${}_RR$) is a $1-$continuous module. We have the following implications:

$$\text{Continuous} \Rightarrow \text{strongly } 1-\text{continuous} \Rightarrow 1-\text{continuous},$$
$$\text{and } (C_2) \Rightarrow (1-C_2).$$

Remark 2.1. By [4, Corollary 7.8], let M be a right $R-$module with finite uniform dimension, M is a $(1-C_1)-$ module if and only if M is CS. Therefore, M has finite uniform dimension then M is a $1-$continuous module if and only if M is strongly $1-$continuous. In general, if M satisfies the condition $(1-C_2)$, M may not satisfy the condition (C_2). By the definitions $(1-C_2)-$module, $1-$continuous module and strongly $1-$continuous module, we have:

Lemma 2.2. *Let M be a right $R-$module and N is a direct summand of M. If M is a $(1-C_2)-$module ($1-$continuous, strongly $1-$continuous) then N is also $(1-C_2)-$module (resp. $1-$continuous, strongly $1-$continuous).*

Theorem 2.3. *Let $U=\oplus_{i=1}^{n}U_i$ where each U_i is a uniform module, then the following conditions are equivalent:*

(i) U is a $(C_2)-$module;

(ii) U is a $(1-C_2)-$module and U satisfies the condition (C_3).

Proof. (i) $\Longrightarrow$ (ii). It is obvious

(ii) $\Longrightarrow$ (i). We show that U is a $(C_2)-$module, i.e., for two submodules X, Y of U, with $X \cong Y$ and Y is a direct summand of U, X is also a direct summand of U. Note that Y is a closed submodule of M, there is a subset F of $\{1,..,n\}$ such that $Y \oplus (\oplus_{i\in F}U_i)$ is an essential submodule of U. But Y, $\oplus_{i\in F}U_i$ are direct summands of U and U satisfies the condition (C_3), we imply $Y \oplus (\oplus_{i\in F}U_i) = U$. If $F=\{1,..,n\}$ then $X=Y=0$, as desired.

If $F \neq \{1,..,n\}$ and set $J=\{1,..,n\}\backslash F$, then $U = Y \oplus (\oplus_{i\in F}U_i) = (\oplus_{i\in J}U_i) \oplus (\oplus_{i\in F}U_i)$. Hence, $X \cong Y \cong U/\oplus_{i\in F}U_i \cong \oplus_{i\in J}U_i = Z$. Suppose that $J=\{1,..,k\}$ with $1 \le k \le n$, i.e., $Z = U_1 \oplus .. \oplus U_k$. Let $\varphi: Z \longrightarrow X$, and set $X_i = \varphi(U_i)$ then $X_i \cong U_i$ for any $i=1,..,k$. We imply $X=\varphi(Z)=$

$\varphi(U_1 \oplus .. \oplus U_k) = \varphi(U_1) \oplus .. \oplus \varphi(U_k) = X_1 \oplus .. \oplus X_k$. By X_i is a uniform submodule of U, $X_i \cong U_i$ with U_i is a direct summand of U and U is a $(1-C_2)-$module, X_i is also a direct summand of U for any $i = 1, .., k$. But U satisfies the condition (C_3), $X = X_1 \oplus .. \oplus X_k$ is a direct summand of U. Hence U is a $(C_2)-$module, proving (i). □

Theorem 2.4. *Let $U = \oplus_{i=1}^{n} U_i$ where each U_i is a uniform module, then the following conditions are equivalent:*

(i) U is a continuous module;

(ii) U is a $1-$continuous module.

Proof. (i) $\Longrightarrow$ (ii). It is obvious.

(ii) $\Longrightarrow$ (i). We show that $S = End(U_i)$ is a local ring for any $i = 1, .., n$. We first prove a claim that U_i does not embed in a proper submodule of U_i. Let $f : U_i \longrightarrow U_i$ be a monomorphism with $f(U_i)$ is a proper submodule of U_i. Set $f(U_i) = V$, then $V \neq 0$, proper submodule of U_i and $V \cong U_i$. By hypothesis, U_i is a $(1-C_2)-$module, and hence V is a direct summand of U_i, i.e., U_i is not uniform module, a contradiction. Therefore, U_i does not embed in a proper submodule of U_i.

Let $g \in S$ and suppose that g is not an isomorphism. It suffices to show that $1-g$ is an isomorphism. Note that, g is not a monomorphism. Then, since $Keg(g)$ is a nonzero submodule, it is essential in the uniform module U_i. We always have $Keg(g) \cap Keg(1-g) = 0$, it follows that $Ker(1-g) = 0$, i.e. $1-g$ is a monomorphism. But U_i does not embed in a proper submodule of U_i, $1-g$ must be onto, and so $1-g$ is an isomorphism, as required.

Let $U_{ij} = U_i \oplus U_j$ with $i, j \in \{1, .., n\}$ and $i \neq j$. We show that U_{ij} satisfies the condition (C_3), i.e., for two direct summands S_1, S_2 of U_{ij} with $S_1 \cap S_2 = 0$, $S_1 \oplus S_2$ is also a direct summand of U_{ij}. Note that, since $u-dim(U_{ij}) = 2$, the following cases are trivial:

1) Either one of the S_i' has uniform dimension 2, consequently the other S_i is zero, or

2) One of the S_i' is zero

Hence we consider the case that both S_1, S_2 are uniform. We prove that U_i does not embed in a proper submodule of U_j. Let $h : U_i \longrightarrow U_j$ be a monomorphism with $h(U_i)$ is a proper submodule of U_j. Set $h(U_i) = L$, then $L \neq 0$, proper submodule of U_j and $L \cong U_i$. By hypothesis, U is a $(1-C_2)-$module and U_{ij} is a direct summand of U, U_{ij} is also $(1-C_2)-$module. Note that L is a uniform submodule of U_{ij} and $L \cong U_i$ with U_i is a direct summanmd of U_{ij}, L is also direct summand of U_{ij}. Set $U_{ij} = L \oplus L'$, then by modularity we get $U_j = L \oplus L''$ with $L'' = U_j \cap L'$. Note that L'' is also proper submodule of U_j and $L'' \neq 0$, hence U_j is not uniform module, a contradiction. Therefore U_i does not embed in a proper submodule of U_j.

Similary, U_j does not embed in a proper submodule of U_i. Note that, U_i (and U_j) does not embed in a proper submodule of U_i (resp. U_j).

Note that, $End(U_i)$ and $End(U_j)$ are local rings, by Azumaya's Lemma ([1, 12.6, 12.7]), we have $U_{ij} = S_2 \oplus K = S_2 \oplus U_i$ or $S_2 \oplus K = S_2 \oplus U_j$. Since i and j can interchange with each other, we need only consider one of the two possibilities. Let us consider the case $U_{ij} = S_2 \oplus K = S_2 \oplus U_i = U_i \oplus U_j$. Then it follows $S_2 \cong U_j$. Write $U_{ij} = S_1 \oplus H = S_1 \oplus U_i$ or $S_1 \oplus H = S_1 \oplus U_j$.

If $U_{ij} = S_1 \oplus H = S_1 \oplus U_i$, then by modularity we get $S_1 \oplus S_2 = S_1 \oplus W$ where $W = (S_1 \oplus S_2) \cap U_i$. From here we get $W \cong S_2$, this means U_i contains a copy of $S_2 \cong U_j$. By U_j does not embed in a proper submodule of U_i, we must have $W = U_i$, and hence $S_1 \oplus S_2 = U_i \oplus U_j = U_{ij}$.

If $U_{ij} = S_1 \oplus H = S_1 \oplus U_j$, then by modularity we get $S_1 \oplus S_2 = S_1 \oplus W'$ where $W' = (S_1 \oplus S_2) \cap U_j$. From here we get $W' \cong S_2$, this means U_j contains a copy of $S_2 \cong U_j$. By U_j does not embed in a proper submodule of U_j, we must have $W' = U_j$, and hence $S_1 \oplus S_2 = U_{ij}$.

Thus U_{ij} satisfies (C_3). Note that, U_{ij} is a direct summand of U and U is a CS $-$module (by U has finite dimension and U is a $(1 - C_1)-$module, thus U is CS $-$module), U_{ij} is also CS $-$module, and hence U_{ij} is a quasi $-$continuous module for any $i, j \in \{1, .., n\}$ and $i \neq j$.

Now, by [6, Corollary 11], thus U is a quasi $-$continuous module. By Theorem 2.3, U is a continuous module, proving (i). □

Corollary 2.5. *Let $U = \oplus_{i=1}^{n} U_i$ where each U_i is a uniform module, then the following conditions are equivalent:*

(i) U is a $\sum -$quasi $-$injective module;

(ii) U is a $1-$continuous module, countably $\sum -(1 - C_1)-$module.

Proof. (i) $\Longrightarrow$ (ii). It is obvious.

(ii) $\Longrightarrow$ (i). By Theorem 2.4, U is a continuous module. By [7, Proposition 2.5], U is a $\sum -$quasi $-$injective module, proving (i). □

A right $R-$module M is called *distributive* if for any submodule A, B, C of M then $A \cap (B + C) = A \cap B + A \cap C$. We say that, M is a *UC $-$module* if each of its submodule has a unique closure in M.

Theorem 2.6. *Let $U = \oplus_{i=1}^{n} U_i$ where each U_i is a uniform module. Assume that U is a distributive module, then the following conditions are equivalent:*

(i) U is a $(C_2)-$module;

(ii) U is a $(1 - C_2)-$module.

Proof. (i) $\Longrightarrow$ (ii). It is obvious.

(ii) $\Longrightarrow$ (i). Similar proof of Theorem 2.4, U_i does not embed in a proper submodule of U_j for any $i, j \in \{1, .., n\}$ and $S = End(U_i)$ is a uniform module for any $i \in \{1, .., n\}$. We first prove a claim that, if S_1 and S_2 are direct summands of U with $u - dim(S_1) = 1$, $u - dim(S_2) = n - 1$

and $S_1 \cap S_2 = 0$, then $S_1 \oplus S_2 = U$. By Azumaya's Lemma, we have $U = S_2 \oplus K = S_2 \oplus U_i$. Suppose that $i = 1$, i.e., $U = S_2 \oplus U_1 = (\oplus_{i=2}^n U_i) \oplus U_1$.

Write $U = S_1 \oplus H = S_1 \oplus (\oplus_{i \in I} U_i)$ with I being a subset of $\{1, .., n\}$ and $card(I) = n - 1$. There are cases:

Case 1. If $1 \notin I$, $U = S_1 \oplus (U_2 \oplus .. \oplus U_n) = U_1 \oplus (U_2 \oplus .. \oplus U_n)$. Then it follows from $S_1 \cong U_1$. By modularity we get $S_1 \oplus S_2 = S_2 \oplus V$ where $V = (S_1 \oplus S_2) \cap U_1$. From here we get $V \cong S_1$, this means U_1 contains a copy of $S_1 \cong U_1$. By U_1 does not embed in a proper submodule of U_1, we must have $V = U_1$, and hence $S_1 \oplus S_2 = S_2 \oplus U_1 = U$.

Case 2. If $1 \in I$, there exist $k \neq 1$ such that $k = \{1, .., n\} \backslash I$, $U = S_1 \oplus (\oplus_{i \in I} U_i) = U_k \oplus (\oplus_{i \in I} U_i)$. Then it follows $S_1 \cong U_k$. By modularity we get $S_1 \oplus S_2 = S_2 \oplus V'$ where $V' = (S_1 \oplus S_2) \cap U_1$. From here we get $V' \cong S_1$, this means U_1 contains a copy of $S_1 \cong U_k$. By U_k does not embed in a proper submodule of U_1, we must have $V' = U_1$, and hence $S_1 \oplus S_2 = U$, as required.

We aim show next that U satisfies the condition (C_3), i.e., for two direct summands of X_1, X_2 of U with $X_1 \cap X_2 = 0$, $X_1 \oplus X_2$ is also direct summand of U. By Azumaya's Lemma, we have $U = X_1 \oplus K = X_1 \oplus (\oplus_{i \in J} U_i) = (\oplus_{i \in F} U_i) \oplus (\oplus_{i \in J} U_i)$ (where $F = \{1, .., n\} \backslash J$) and $U = X_2 \oplus L = X_2 \oplus (\oplus_{j \in D} U_j) = (\oplus_{j \in E} U_j) \oplus (\oplus_{j \in D} U_j)$ (where $E = \{1, .., n\} \backslash D$). We imply $X_1 \cong \oplus_{i \in F} U_i$ and $X_2 \cong \oplus_{j \in E} U_j$. Suppose that $E = \{1, .., t\}$ and let $\varphi : \oplus_{j=1}^t U_j \longrightarrow X_2$ be an isomorphism and set $Y_j = \varphi(U_j)$, we have $Y_j \cong U_j$ and $X_2 = \oplus_{j=1}^t Y_j$. By hypothesis X_2 is a direct summand of U, thus Y_j is also direct summand of U for any $j \in \{1, .., t\}$. We show that $X_1 \oplus X_2 = X_1 \oplus (Y_1 \oplus .. \oplus Y_t)$ is a direct summand of U.

We prove that $X_1 \oplus Y_1$ is a direct summand of U. By Azumaya's Lemma, we have $U = Y_1 \oplus W = Y_1 \oplus (\oplus_{p \in P} U_p) = U_\alpha \oplus (\oplus_{p \in P} U_p)$, with P is a subset of $\{1, .., n\}$ such that $card(P) = n - 1$ and $\alpha = \{1, .., n\} \backslash P$. Note that, $card(P \cap J) \geq card(J) - 1 = m$. Suppose that $\{1, .., m\} \subseteq (P \cap J)$, i.e., $U = (X_1 \oplus (U_1 \oplus .. \oplus U_m)) \oplus U_\beta = Z \oplus U_\beta$ with $\beta = J \backslash \{1, .., m\}$ and $Z = X_1 \oplus (U_1 \oplus .. \oplus U_m)$. By U is a distributive module, we have $Z \cap Y_1 = (X_1 \oplus (U_1 \oplus .. \oplus U_m)) \cap Y_1 = (X_1 \cap Y_1) \oplus ((U_1 \oplus .. \oplus U_m) \cap Y_1) = 0$. Note that, Z, Y_1 are direct summands of U with $u - dim(Z) = n - 1$ and $u - dim(Y_1) = 1$, $U = Z \oplus Y_1 = (X_1 \oplus (U_1 \oplus .. \oplus U_m)) \oplus Y_1 = (X_1 \oplus Y_1) \oplus (U_1 \oplus .. \oplus U_m)$. Therefore, $X_1 \oplus Y_1$ is a direct summand of U. By induction, we have $X_1 \oplus X_2 = X_1 \oplus (Y_1 \oplus .. \oplus Y_t) = (X_1 \oplus Y_1 \oplus .. \oplus Y_{t-1}) \oplus Y_t$ is a direct summand of U. Thus U satisfies the condition (C_3).

Finally, we show that U satisfies the condition (C_2). By hypothesis (ii) and U satisfies (C_3), thus U is a $(1-C_2)-$module (see Theorem 2.3), proving (i). □

Theorem 2.7. *Let $U_1, .., U_n$ be uniform local modules such that U_i does not embed in $J(U_j)$ for any $i, j = 1, .., n$. If $U = \oplus_{i=1}^n U_i$ is a UC distributive module then it is a continuous module.*
Proof. We first prove a claim that U is a CS module. Let A be a uniform closed submodule of U. Let the $X_i = A \cap U_i$ for any $i \in \{1, .., n\}$. Suppose that $X_i = 0$ for every $i \in \{1, .., n\}$. By hypothesis, U is a distributive module, we have $A = A \cap (U_1 \oplus .. \oplus U_n) = X_1 \oplus .. \oplus X_n = 0$, a contradiction. Therefore, there exists a $X_t \neq 0$, i.e., $A \cap U_t \neq 0$. By property A and U_t are closed uniform submodules of U, thus X_t is an essential submodule of A and X_t is also essential submodule of U_t. Hence A and U_t are closure of X_t in U, U is an UC module we get $A = U_t$. This implies that A is a direct summand of U, i.e., U is a $(1 - C_1)-$module. By U has finite dimension, U is CS module (see [4, Corollary 7.8]), as required.

We aim to show next that $S = End(U_l)$ is a local ring for any $l \in \{1, .., n\}$. Let $f \in S$ and suppose that f is not an isomorphism. It suffices to show that $1 - f$ is an isomorphism.

Suppose that, f is a monomorphism. Then f is not onto, and $f : U_l \longrightarrow J(U_l)$ is an embedding, a contradiction. Thus f is not a monomorphism. Then, since $Ker(f)$ is a nonzero submodule, it is essential in the uniform local module U_l. Thus, since we always have $Ker(f) \cap Ker(1 - f) = 0$, it follows that $Ker(1 - f) = 0$, i.e., $1 - f$ is a monomorphism. But, since U_l does not embed in $J(U_l)$, $1 - f$ must be onto, and so $1 - f$ is an isomorphism. Thus, S is a local ring.

Now, we show that U is a $(1 - C_2)-$module, i.e., for two uniform submodules V, W of U, with $V \cong W$ and W is a direct summand of U, V is also a direct summand of U. By Azumaya's Lemma, we have $U = W \oplus W' = W \oplus (\oplus_{j \in J} U_j) = U_k \oplus (\oplus_{j \in J} U_j)$ where J is a subset of $\{1, .., n\}$ with $card(J) = n - 1$ and $k = \{1, .., n\} \backslash J$. Hence $V \cong W \cong U_k$. Let V^* be a closure of V in U. By U is a CS module, thus V^* is a direct summand of U. Similarly, there exists $s \in \{1, .., n\}$ such that $V^* = U_s$, this means U_s contains a copy of $W \cong U_k$. If V is a proper submodule of U_s, then U_k embed in $J(U_s)$, a contradiction. We must have $V = U_s$, and hence V is a direct summand of U. Thus, U is a $(1 - C_2)-$module, i.e., U is a $1-$continuous module (by U is a CS module).

Finally, by Theorem 2.4 thus U is a continuous module. □

Corollary 2.8. *Let $U_1, .., U_n$ be uniform local modules such that U_i does not embed in $J(U_j)$ for any $i, j = 1, .., n$. If $U = \oplus_{i=1}^n U_i$ is a UC, distributive module then the following conditions are equivalent:*

(i) U is a $\sum -$quasi $-$injective module;

(ii) U is a countably $\sum -(1 - C_1)-$module.

Proof. (i) $\Longrightarrow$ (ii). It is obvious.

(ii) $\Longrightarrow$ (i). By Theorem 2.7, U is a continuous module. By [7, Proposition 2.5], U is a $\sum -$quasi $-$injective module, proving (i). □

Acknowledgement

We would like thank the referee for carefully reading this note and for many useful comments

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(*Received June 20, 2013*)
(*Accepted June 18, 2015*)