A REMARK ON THE LAVALLEE-SPEARMAN-WILLIAMS-YANG FAMILY OF QUADRATIC FIELDS

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Abstract. In [4], M. J. Lavallee, B. K. Spearman, K. S. Williams and Q. Yang introduced a certain parametric D_5-quintic polynomial and studied its splitting field. The present paper gives an infinite family of quadratic fields with class number divisible by 5 by using properties of its polynomial.

1. Introduction

One of the main topics in algebraic number theory is to study the structure of ideal class groups of number fields, particularly the divisibility of class numbers. In the case of quadratic fields, numerous related results are known by many authors. As was first shown by T. Nagell [7] (resp. by Y. Yamamoto [11]), there exist infinitely many imaginary (resp. real) quadratic fields with class numbers divisible by an arbitrary given integer n. After that, several authors gave such results (for example, [1], [10], [6], [9], [12], [5], and so on).

Every unramified cyclic extension of prime degree p of a quadratic field is normal over $\mathbb{Q}$ and is a D_p-extension of $\mathbb{Q}$, that is, a Galois extension whose Galois group is isomorphic to the dihedral group D_p of order $2p$. Therefore, it is given by roots of a polynomial of degree p with rational coefficients. Y. Kishi and K. Miyake [2] used some D_3-cubic polynomials to classify all quadratic fields whose class number is divisible by 3. Here, a D_3-cubic polynomial means a polynomial of degree 3 such that its minimal splitting field over $\mathbb{Q}$ is a D_3-extension of $\mathbb{Q}$. In the case $p = 5$, Kishi [3] found a family of D_5-quintic polynomials such that its minimal splitting fields over $\mathbb{Q}$ are unramified cyclic quintic extensions of quadratic subfields. But his methods is slightly complicated. In this paper, we will give such polynomials that are simpler and more efficient by using M. J. Lavallee, B. K. Spearman, K. S. Williams and Q. Yang's results.

For a number field K, let $D(K)$ denote the discriminant of K. For a polynomial f, let $d(f)$ (resp. $\mathrm{Spl}_{\mathbb{Q}}(f)$) denote the discriminant (resp. the minimal splitting field over $\mathbb{Q}$) of f.

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2. Main Theorem

We need the following well-known lemma before we state our main theorem.

Lemma 1 (Dedekind)**.** *Let F be an algebraic number field and $\mathfrak{D}$ be the different of F over $\mathbb{Q}$. Let $\mathfrak{p}$ be a prime divisor in F over a prime number p, and $\mathfrak{p}^d \parallel \mathfrak{D}$ and $\mathfrak{p}^e \parallel p$. Then*

(a) *if $p \nmid e$, then $d = e - 1$,*

(b) *if $p^v \parallel e$ ($v > 0$), then $e \le d \le ev + e - 1$.*

Proof. See, for example, [8, Chap. III, §2, (2.6) Theorem]. □

Theorem. *Let f be a D_5-quintic polynomial and put $L := \mathrm{Spl}_{\mathbb{Q}}(f)$. Moreover, let K be the unique quadratic subfield of L. Suppose that*

$$5^6 \nmid d(f) \quad and \quad q^4 \nmid d(f)$$

for every prime number $q \ne 5$. Then L is an unramified cyclic quintic extension of K. Especially, the class number of K is divisible by 5.

Proof. Let $F = \mathbb{Q}(\theta)$ where $f(\theta) = 0$. Then we have

$$d(f) = a^2 D(F)$$

for some integer a.

If L/K is unramified, then the class number of K is divisible by 5 by class field theory. So it suffices to show that if L/K is not unramified, then we have

$$5^6 \mid D(F) \quad \text{or} \quad q^4 \mid D(F)$$

for some prime number $q \ne 5$. Suppose that $\mathfrak{p}$ is a prime ideal of K which is ramified in L/K.

Case 1, where $5 \mid \mathfrak{p}$. Since $\mathfrak{p}$ is ramified in L/K and $[L : K] = 5$, 5 is wildly ramified in $F/\mathbb{Q}$. By Lemma 1 (b), it must hold that $5 \le d \le 9$. This implies that $5^5 \mid D(F)$. Now we may view $\mathrm{Gal}(L/K)$ as a subgroup of the symmetric group S_5 of order 5 because f is a quintic polynomial. Since $\mathrm{Gal}(L/K) \simeq D_5$ is contained in the alternative group A_5 of degree 5, $D(F)$ must be a square in $\mathbb{Z}$. Thus we have $5^6 \mid D(F)$.

Case 2, where $q \mid \mathfrak{p}$ with $q \equiv 1 \bmod 5$. Since $\mathfrak{p}$ is ramified in L/K and $[L : K] = 5$, q is tamely ramified in $F/\mathbb{Q}$. By Lemma 1 (a), we have $q^4 \parallel \mathfrak{D}$. In this case, we can easily verify that $q^4 \mid D(F)$. The proof is completed. □

3. The Lavallee-Spearman-Williams-Yang family of quadratic fields

By using our theorem, we will give an infinite family of quadratic fields with class number divisible by 5. This is based on Lavallee, Spearman, Williams and Yang's works [4]. Let us refer to them.

For an integer b, we define
$$f_b(x) := x^5 - 2x^4 + (b+2)x^3 - (2b+1)x^2 + bx + 1.$$
First they proved that f_b is irreducible for any $b \in \mathbb{Z}$ and
$$d(f_b) = (4b^3 + 28b^2 + 24b + 47)^2$$
(see [4, Lemmas 2.1 and 2.2]). They also proved the following.

Proposition 1 ([4, Lemmas 2.3 and 2.4]). *If* $-(4b^3 + 28b^2 + 24b + 47)$ *is not a square in* $\mathbb{Z}$*, then* $\mathrm{Spl}_{\mathbb{Q}}(f_b)$ *is a* D_5*-extension of* $\mathbb{Q}$ *and the unique quadratic subfield of* $\mathrm{Spl}_{\mathbb{Q}}(f_b)$ *is*
$$\mathbb{Q}(\sqrt{-(4b^3 + 28b^2 + 24b + 47)}).$$

Furthermore, there are infinitely many integers b such that $-(4b^3 + 28b^2 + 24b + 47)$ is square-free, is not a square in $\mathbb{Z}$, and the quadratic fields $\mathbb{Q}(\sqrt{-(4b^3 + 28b^2 + 24b + 47)})$ are distinct, as we can see in the proof of [4, Theorem]. From this, together with Proposition 1 and our main theorem, the following proposition is immediately obtained.

Proposition 2. *There exist infinitely many quadratic fields of the form*
$$\mathbb{Q}(\sqrt{-(4b^3 + 28b^2 + 24b + 47)}), \ b \in \mathbb{Z}$$
whose class number is divisible by 5.

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