Math. J. Okayama Univ. **59** (2017), 41–70

ON A NON-ABELIAN GENERALIZATION OF THE BLOCH–KATO EXPONENTIAL MAP

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Abstract. The present paper establishes a non-abelian generalization of the Bloch–Kato exponential map. Then, we relate p-adic polylogarithms introduced by Coleman to ℓ-adic polylogarithms introduced by Wojtkowiak. This formula is another analog of the Coleman–Ihara formula obtained by Nakamura, Wojtkowiak, and the author.

1. Introduction

Let p be a rational prime and let F be a finite extension of $\mathbb{Q}_p$ with the absolute Galois group $\mathscr{G}_F := \mathrm{Gal}(\overline{F}/F)$. Let $B_{\mathrm{dR}}^+, B_{\mathrm{dR}}, B_{\mathrm{crys}}$ be the period rings attached to F (cf. [8]).

For each de Rham representation V of $\mathscr{G}_F$, Spencer Bloch and Kazuya Kato constructed a natural isomorphism called *the Bloch–Kato exponential map*

$$\exp_V \colon D_{\mathrm{dR}}^0(V) \backslash D_{\mathrm{dR}}(V) \xrightarrow{\sim} H_e^1(F, V) \subset H^1(F, V),$$

where $H_e^1(F, V)$ is the exponential part of the continuous Galois cohomology $H^1(F, V)$ and $D_{\mathrm{dR}}^0(V)$, $D_{\mathrm{dR}}(V)$ are defined to be $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\mathscr{G}_F}$ and $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\mathscr{G}_F}$, respectively (cf. [2, Definition 3.10]). The map $\exp_V$, which generalizes the p-adic exponential map for abelian varieties over F, is one of powerful tools for studying cohomologies of algebraic varieties over F. Bloch and Kato related de Rham cohomology to étale cohomology of "motives" by using $\exp_V$ and formulated their Tamagawa number conjecture. *The Bloch–Kato logarithmic map*, denoted by $\log_V$, is defined to be the inverse of $\exp_V$. The aim of the present paper is to give non-abelian generalizations of $\exp_V$ and $\log_V$.

Let $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$ be the category of crystalline representations of $\mathscr{G}_F$ and let G be an affine group scheme in $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$ in the sense of Deligne (cf. [6, Paragraphe 5]). The ring of regular functions $\mathcal{O}(G)$ of G is a Hopf algebra object in the ind-category of $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$ by definition and the set of $\mathbb{Q}_p$-valued points $G(\mathbb{Q}_p)$ of G has a natural continuous action of $\mathscr{G}_F$. In Definition 12, we will define *the finite part* $H_f^1(F, G(\mathbb{Q}_p))$ of the continuous Galois cohomology $H^1(F, G(\mathbb{Q}_p))$ which generalizes the finite part $H_f^1(F, V)$

Mathematics Subject Classification. Primary 11R34; Secondary 14C30.
Key words and phrases. Bloch–Kato exponential map, Non-abelian p-adic Hodge theory, Coleman–Ihara formula.

of $H^1(F,V)$. We say that G is an algebraic group in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$ if $\mathcal{O}(G)$ is of finite type as a $\mathbb{Q}_p$-algebra. Let $\mathrm{MF}^{\mathrm{ad}}_F(\varphi)$ be the category of weakly admissible filtered φ-modules over F. Then, the Fontaine functor

$$V_{\mathrm{crys}}\colon \mathrm{MF}^{\mathrm{ad}}_F(\varphi) \longrightarrow \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F); D \mapsto V_{\mathrm{crys}}(D) := (D \otimes_{F_0} B_{\mathrm{crys}})^{\varphi=1} \cap \mathrm{Fil}^0(D \otimes_{F_0} B_{\mathrm{dR}})$$

defines an equivalence of Tannakian categories over $\mathbb{Q}_p$ (cf. [4, Thèoréme A]). Here, F_0 is the maximal subfield of F unramified over $\mathbb{Q}_p$. Therefore, V_{crys} induces an equivalence of the category of group schemes in $\mathrm{MF}^{\mathrm{ad}}_F(\varphi)$ and the category of group schemes in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$. We also denote by V_{crys} this equivalence by abuse of notation. The following theorem is our main result.

Theorem 1.1 (Theorem 3.9)**.** *Let H be an algebraic group in $\mathrm{MF}^{\mathrm{ad}}_F(\varphi)$ and let G be the algebraic group $V_{\mathrm{crys}}(H)$ in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$. Suppose that H satisfies the conditions* (Non-neg) *and* (Bij) *(see Definition* 8 *for the definitions of* (Non-neg) *and* (Bij)*). Then, there exists a canonical closed algebraic subgroup F^0H_F of $H_F := H \times_{\mathrm{Spec}(F_0)} \mathrm{Spec}(F)$ where $F_0 = F \cap \mathbb{Q}_p^{\mathrm{ur}}$. Furthermore, we suppose the following two conditions*:

(a) *For any right G-torsor X in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$, the set of $\mathbb{Q}_p$-rational points $X(\mathbb{Q}_p)$ is non-empty.*
(b) *The algebraic group H is either a unipotent algebraic group over F_0 in the usual sense or an algebraic group in the category of ordinary filtered φ-modules in the sense of Perrin-Riou (cf.* [18]*).*

Then, there exists a canonical injection

$$\exp_G\colon F^0H_F(F)\backslash H(F) \hookrightarrow H^1_f(F, G(\mathbb{Q}_p))$$

satisfying the following properties:

(i) *If H is ω_F-trivial (cf. Definition* 5 *and Subsection* 3.1*), then $\exp_G$ is a bijection.*
(ii) *If G is a vector group scheme attached to $V \in \mathrm{Obj}(\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F))$, then $\exp_G$ coincides with $\exp_V$ under the canonical identifications*

$$H^1_f(F, G(\mathbb{Q}_p)) = H^1_f(F,V) = H^1_e(F,V),\ F^0H_F(F)\backslash H(F) = D^0_{\mathrm{dR}}(V)\backslash D_{\mathrm{dR}}(V).$$

We call $\exp_G$ the generalized Bloch–Kato exponential map and define the generalized Bloch–Kato logarithmic map $\log_G$ to be its inverse when it exists.

As an application of Theorem 1.1, we give an analog of the Coleman–Ihara formula which is another version of [17, Theorem 1.1]. For each $z \in \mathbb{P}^1 \setminus \{0,1,\infty\}(\mathcal{O}_F)$ and for each positive integer N greater than 1, we will define an extension $\widetilde{F}(z,N)$ of F to be the kernel field of a certain 1-cocycle

associated with z in Definition 18 (2). Our analog of the Coleman–Ihara formula is then:

Theorem 1.2 (Corollary 2)**.** *Let N be a positive integer greater than 1. Suppose that p is odd and that F is unramified over $\mathbb{Q}_p$. Let σ_F be the arithmetic Frobenius automorphism of F. Then, we have the following formula for all $\sigma \in \mathscr{G}_{\widetilde{F}(z,N)}$:*

$$\ell i_N(z)(\sigma) = \frac{-1}{(N-1)!}\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\left\{\left(1-\frac{\sigma_F}{p^N}\right)\mathrm{Li}_N^{p\text{-adic}}(z)\right\}\phi_{F,N}^{CW}(\sigma)\right). \tag{1.1}$$

Here, $\ell i_N(z)\colon \mathscr{G}_F \to \mathbb{Q}_p$ is the ℓ-adic polylogarithm (cf. [16]*), $\phi_{F,N}^{CW}$ is the Coates–Wiles homomorphism (cf.* [2, Section 2]*), and $\mathrm{Li}_N^{p\text{-adic}}(z)$ is Coleman's p-adic polylogarithm (cf.* [3]*).*

Our study is motivated by the construction of *the fundamental commutative diagram* of Minhyong Kim (the diagram (1.3) below). We recall his work briefly. Let K be a number field, v a finite place of K dividing p, and K_v the v-adic completion of K with the ring $\mathcal{O}_{K_v}$ of integers. Let C be a smooth curve over $\mathcal{O}_K$ with a good compactification over $\mathcal{O}_{K_v}$. Let x be a K-valued point of C with good reduction at v and let $\bar{x}$ be a geometric point of C over x. We denote by $\pi_1^{\text{un-ét}}(C_{\overline{K}}, \bar{x})$ the unipotent completion over $\mathbb{Q}_p$ of the étale fundamental group of $C_{\overline{K}} := C \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(\overline{K})$ with the base point $\bar{x}$[1]. Kim studied a $\mathbb{Q}_p$-scheme $H_f^1(K, \pi_1^{\text{un-ét}}(C_{\overline{K}}, \bar{x}))$ whose set of R-valued points is defined to be the finite part $H_f^1(K, \pi_1^{\text{un-ét}}(C_{\overline{K}}, \bar{x})(R))$ of $H^1(K, \pi_1^{\text{un-ét}}(C_{\overline{K}}, \bar{x})(R))$ for each $\mathbb{Q}_p$-algebra R (cf. [13], [14]). The $\mathbb{Q}_p$-scheme $H_f^1(K, \pi_1^{\text{un-ét}}(C_{\overline{K}}, \bar{x}))$ is called *the Selmer variety* attached to C. A key ingredient in [14] is the morphism of $\mathbb{Q}_p$-schemes

$$\log_{C,v}\colon H_f^1(K_v, \pi_1^{\text{un-ét}}(C_{\overline{K}}, \bar{x})) \to R_{K_v/\mathbb{Q}_p}\left(F^0\pi_1^{\mathrm{dR}}(C_{K_v}, x)\backslash\pi_1^{\mathrm{dR}}(C_{K_v}, x)\right), \tag{1.2}$$

where $R_{K_v/\mathbb{Q}_p}$ is the Weil restriction functor, $\pi_1^{\mathrm{dR}}(C_{K_v}, x)$ is the de Rham fundamental group of $C_{K_v} := C \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spec}(K_v)$, and $F^0 = F^0\pi_1^{\mathrm{dR}}(C_{K_v}, x)$ is the "Hodge subgroup" of $\pi_1^{\mathrm{dR}}(C_{K_v}, x)$. He constructed $\log_{C,v}$ by rewriting both sides of (1.2) as (subspaces of) classifying spaces of torsors of fundamental groups. The morphism $\log_{C,v}$ fits into a commutative diagram

(1.3)

$$\begin{array}{ccccc} C(\mathcal{O}_K) & \longrightarrow & C(\mathcal{O}_{K_v}) & \longrightarrow & F^0\backslash\pi_1^{\mathrm{dR}}(C_{K_v}, x)(K_v) \\ \downarrow & & \downarrow & \overset{\log_{C,v}}{\nearrow} & \\ H_f^1(K, \pi_1^{\text{un-ét}}(C_{\overline{K}}, \bar{x}))(\mathbb{Q}_p) & \longrightarrow & H_f^1(K_v, \pi_1^{\text{un-ét}}(C_{\overline{K}}, \bar{x}))(\mathbb{Q}_p) & & \end{array}$$

[1]This group is referred as the unipotent fundamental group of $C_{\overline{K}}$ in [14].

which plays a central role in his study of the finiteness of $C(\mathcal{O}_K)$ which is a non-abelian generalization of the Coleman–Chabauty method (cf. [14, Introduction]). The proof of Theorem 1.1 is executed in a similar way as Kim's construction of $\log_{C,v}$, namely, we show that the classifying spaces of torsors of G and H in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$ and $\mathrm{MF}^{\mathrm{ad}}_F(\varphi)$ satisfying certain conditions are isomorphic to $H^1_f(F, G(\mathbb{Q}_p))$ and $F^0H_F(F)\backslash H(F)$, respectively.

The outline of this article is as follows. In Section 2, we introduce concepts of algebraic group and its torsor in a neutral rigid exact $\otimes$-category. In Subsection 3.1 and Subsection 3.2, we consider five rigid exact $\otimes$-categories and study classifying spaces of torsors in those categories. We define the generalized Bloch–Kato exponential map assuming Proposition 3.3 and define the generalized Bloch–Kato logarithmic map in Subsection 3.3. In Section 4, we recall the exponential map introduced in [15], [20] and study a relation between this exponential map and the logarithmic map introduced in Subsection 3.3. Then, we give a proof of Proposition 3.3. In Section 5, we prove Theorem 1.2.

Notation. In this article, p denotes a rational prime. For each field k, we fix its separable closure $\overline{k}$ and denote by $\mathscr{G}_k$ the absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ of k. Let A be a topological group with a continuous action of $\mathscr{G}_k$. We denote by ${}^{\sigma}a$ the result of the action of $\sigma \in \mathscr{G}_k$ on $a \in A$. For $i = 0, 1$, $H^i(k, A)$ denotes the i-th continuous Galois cohomology $H^i_{\mathrm{cont}}(\mathscr{G}_k, A)$.

Let X be a scheme over a field k and let R be a k-algebra. We denote by X_R or by $X \otimes_k R$ the base change $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$ of X to $\mathrm{Spec}(R)$ and $\mathcal{O}(X)$ denotes the ring of regular functions on X. Denote by Vec_k the category of finite dimensional k-vector spaces.

For an object X of a category $\mathcal{C}$, $[X]$ denotes the isomorphism class of X in $\mathcal{C}$.

2. Algebraic groups in a rigid exact $\otimes$-category

Let us fix a field k in this section. Recall that *a rigid exact $\otimes$-category over k* is a k-linear exact category $\mathcal{T}$ equipped with an associative, unitary, and commutative k-bilinear $\otimes$-structure $\otimes\colon \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ satisfying the following three conditions (cf. [7, Definition 1.1, Definition 1.7]):

(a) For each $M, N \in \mathrm{Obj}(\mathcal{T})$, the internal hom $\underline{\mathrm{Hom}}(M, N)$ exists.
(b) For all four objects $M_1, M_2, N_1, N_2 \in \mathrm{Obj}(\mathcal{T})$, the canonical morphism
$$\underline{\mathrm{Hom}}(M_1, N_1) \otimes \underline{\mathrm{Hom}}(M_2, N_2) \to \underline{\mathrm{Hom}}(M_1 \otimes M_2, N_1 \otimes N_2)$$
is an isomorphism.
(c) Any object in $\mathcal{T}$ is reflexive.

The unit object in $\mathcal{T}$ is denoted by $1_{\mathcal{T}}$. We say that $\mathcal{T}$ is *neutral* if the natural map $k \to \mathrm{End}_{\mathcal{T}}(1_{\mathcal{T}})$ is an isomorphism and if there exists a faithful exact k-linear $\otimes$-functor $\omega\colon \mathcal{T} \to \mathrm{Vec}_k$. Such an ω is called *a fiber functor* on $\mathcal{T}$. We denote by $\mathrm{Ind}(\mathcal{T})$ the ind-category of $\mathcal{T}$. Now on, we fix $\mathcal{T}$ a rigid exact $\otimes$-category over k.

Definition 1 (*cf.* [6, Paragraphe 5.3])**.**

(1) *A commutative algebra in* $\mathcal{T}$ is an object A of $\mathrm{Ind}(\mathcal{T})$ equipped with the unit $1_{\mathcal{T}} \to A$ and the multiplication $A \otimes A \to A$ which is associative, unitary, and commutative. Denote by $\mathrm{Alg}(\mathcal{T})$ the category of commutative algebras in $\mathcal{T}$.

(2) *An affine scheme in* $\mathcal{T}$ is an object in the opposite category $\mathrm{Alg}(\mathcal{T})^{\mathrm{op}}$ of $\mathrm{Alg}(\mathcal{T})$. The category of affine schemes in $\mathcal{T}$ is denoted by $\mathrm{Aff}(\mathcal{T})$. We denote by $\mathrm{Sp}(A)$ the image of $A \in \mathrm{Obj}(\mathrm{Alg}(\mathcal{T}))$ in $\mathrm{Aff}(\mathcal{T})$ under the natural equivalence $\mathrm{Alg}(\mathcal{T}) \cong \mathrm{Aff}(\mathcal{T})$.

(3) Let $\mathcal{T}'$ be a rigid exact $\otimes$-category over a field k' and let $D\colon \mathcal{T} \to \mathcal{T}'$ be a $\otimes$-functor in the sense of [7, Definition 1.8]. Then, we also denote by

$$D\colon \mathrm{Aff}(\mathcal{T}) \to \mathrm{Aff}(\mathcal{T}')$$

the functor defined by $\mathrm{Sp}(A) \mapsto \mathrm{Sp}(D(A))$.

Example 1. A commutative algebra in Vec_k is nothing but a commutative k-algebra in the usual sense. Hence, the category of affine schemes in Vec_k can be identified with the category of affine k-schemes.

Definition 2. Suppose that $\mathcal{T}$ is neutral and let ω be a fiber functor on $\mathcal{T}$. Let $f\colon X \to Y$ be a morphism of affine schemes in $\mathcal{T}$. For each property P of morphisms of k-schemes (resp. property P of k-schemes), we say that f is P (resp. X is P) if $\omega(f)\colon \omega(X) \to \omega(Y)$ (resp. $\omega(X)$) is P. Here, we identify the category of affine schemes in Vec_k with the category of affine k-schemes.

Remark 1. Recall that if $\mathcal{T}$ is a Tannakian category over k, then any two fiber functors are locally isomorphic for the fpqc topology (cf. [19, Theorem 3.2.3]). Therefore, in Definition 2, if the property P is local on the base field k in the fpqc topology (e. g. of finite type, closed immersion, open immersion, unipotent etc.), then the concepts defined in Definition 2 does not depend on the choice of the fiber functor ω.

For any two affine schemes $X = \mathrm{Sp}(A)$ and $Y = \mathrm{Sp}(B)$ in $\mathcal{T}$, we put $X \times Y := \mathrm{Sp}(A \otimes B)$. We define the empty scheme in $\mathcal{T}$ to be $\mathrm{Sp}(0)$. There exists the following standard functor from $\mathcal{T}$ to $\mathrm{Aff}(\mathcal{T})$.

Definition 3. We define the functor $\mathbb{A}\colon \mathcal{T} \to \mathrm{Aff}(\mathcal{T})$ by the equality $\mathbb{A}(M) := \mathrm{Sp}(\mathrm{Sym}(M^\vee))$. Here, $\mathrm{Sym}(M^\vee)$ is the symmetric algebra associated with $M^\vee$. We call such $\mathbb{A}(M)$ *an affine space in* $\mathcal{T}$.

Example 2. For any object V of Vec_k, $\mathbb{A}(V)$ is the affine scheme representing the functor $(k\text{-algebras}) \to (\mathrm{Sets})$; $R \mapsto V \otimes_k R$.

Lemma 2.1. *Suppose that $\mathcal{T}$ is neutral. Let ω be a fiber functor on $\mathcal{T}$ and let $X = \mathrm{Sp}(A)$ be an affine scheme in $\mathcal{T}$. If X is of finite type, then there exists a closed immersion $X \hookrightarrow \mathbb{A}(V)$ for some $V \in \mathrm{Obj}(\mathcal{T})$.*

Proof. We write A as a direct limit $\varinjlim_i A_i$ with $A_i \in \mathrm{Obj}(\mathcal{T})$. Since X is of finite type, for sufficiently large i, the image of $\omega(A_i)$ in $\omega(A)$ contains a set of generators of the k-algebra $\omega(A)$. Because the algebra homomorphism $\mathrm{Sym}(\omega(A_i)) \to \omega(A)$ is surjective, the induced morphism $X \hookrightarrow \mathbb{A}(A_i^\vee)$ is a closed immersion by definition. □

Definition 4. Let $\mathcal{T}$ be a neutral rigid exact $\otimes$-category over k.

(1) *An affine group scheme in* $\mathcal{T}$ is a group object in $\mathrm{Aff}(\mathcal{T})$. Further, we say that an affine group scheme G in $\mathcal{T}$ is *an algebraic group in* $\mathcal{T}$ if G is of finite type.
(2) Let X be an affine scheme in $\mathcal{T}$ and let G be an affine group scheme in $\mathcal{T}$. *A left action* of G on X is a functorial left action of G on X. In other words, it is a collection of actions of $G(Y) := \mathrm{Hom}_{\mathrm{Aff}(\mathcal{T})}(Y, G)$ on $X(Y) := \mathrm{Hom}_{\mathrm{Aff}(\mathcal{T})}(Y, X)$ from the left for each object Y of $\mathrm{Aff}(\mathcal{T})$ which are functorial in Y. We define *a right action* of G on X in a similar way.
(3) Let G be an affine group scheme in $\mathcal{T}$. *A left G-torsor in* $\mathcal{T}$ is a non-empty object X in $\mathrm{Aff}(\mathcal{T})$ equipped with a left action $a\colon G \times X \to X$ such that $a \times \mathrm{pr}_2\colon G \times X \to X \times X$ is an isomorphism of affine schemes in $\mathcal{T}$. We define *a right G-torsor in* $\mathcal{T}$ by replacing a left action by a right action. We denote by $H^1(\mathcal{T}, G)$ the set of isomorphism classes of right G-torsors in $\mathcal{T}$.
(4) Let G and G' be affine group schemes in $\mathcal{T}$. *A (G, G')-bitorsor in* $\mathcal{T}$ is an affine scheme X in $\mathcal{T}$ equipped with a left (resp. right) action of G (resp. G') such that X is a left G-torsor and a right G'-torsor.

We regard $H^1(\mathcal{T}, G)$ as a pointed set with the point $[G]$ consisting of trivial torsors. The following well-known fact is useful for later discussions.

Lemma 2.2 (cf. [22, Proposition 2.3.6 (iii)])**.** *Let k be a field and let G be an algebraic group over k. Let V be a k-vector space equipped with an algebraic action of G. Then, for each finite dimensional subspace W of V, there exists a finite dimensional subspace W' of V which contains W and is stable under the action of G.*

Proposition 2.3. *Let G be an algebraic group in $\mathcal{T}$ and let $X = \mathrm{Sp}(A)$ be an affine scheme in $\mathcal{T}$ equipped with a left action of G. If $\mathcal{T}$ is a neutral Tannakian category over k, then for each object M in $\mathcal{T}$ contained in A, there exists a G-stable object $N \subset A$ in $\mathcal{T}$ containing M. In particular, A is a direct limit of G-stable sub-objects of A in $\mathcal{T}$.*

Proof. By Tannakian duality, we may assume that $\mathcal{T}$ is the category $\mathrm{Rep}_k(\pi)$ of algebraic representations of a pro-algebraic group π over k on finite dimensional k-vector spaces. Then, A (resp. M) is an object in $\mathrm{Ind}(\mathrm{Rep}_k(\pi))$ (resp. $\mathrm{Rep}_k(\pi)$) and G is an algebraic group over k equipped with an algebraic action of π. We define a subspace N of A to be the intersection of subspaces of A which contain M and are stable under the action of G. Then, according to Lemma 2.2, N is a finite dimensional k-vector space. By definition, N contains M and is stable under the action of G. Moreover, this space is also stable under the action of π because M is stable under the action of π and N is characterized as the minimal k-vector subspace of A containing M and stable under the action of G. Hence, N is an object of $\mathrm{Rep}_k(\pi)$ and this completes the proof of the proposition. □

Let $\mathcal{T}'$ be a rigid exact $\otimes$-category over a field k' and let $D\colon \mathcal{T} \to \mathcal{T}'$ be a $\otimes$-functor. Then, for any affine group scheme G in $\mathcal{T}$, $D(G)$ is an affine group scheme in $\mathcal{T}'$. If D is faithful, then D sends each G-torsor to a $D(G)$-torsor. We define a notion named D-trivial as follows.

Definition 5. Let $D_i\colon \mathcal{T} \to \mathcal{T}_i,\ 1 \leq i \leq n$, be faithful $\otimes$-functors between rigid exact $\otimes$-categories and let G be an affine group scheme in $\mathcal{T}$. Let X be a right G-torsor in $\mathcal{T}$. We say that X is $(D_1, \ldots, D_n)$-*trivial* if $D_i(X)$ is a trivial $D_i(G)$-torsor for each i. When $n = 1$, we mean D_1-*trivial* for (D_1)-trivial. Define a subset $H^1_{D_1,\ldots,D_n}(\mathcal{T}, G)$ of $H^1(\mathcal{T}, G)$ by

$$H^1_{D_1,\ldots,D_n}(\mathcal{T}, G) := \mathrm{Ker}\left(H^1(\mathcal{T}, G) \to \prod_{i=1}^{n} H^1(\mathcal{T}_i, D_i(G))\right).$$

We say that G is $(D_1, \ldots, D_n)$-*trivial* if $H^1(\mathcal{T}, G) = H^1_{D_1,\ldots,D_n}(\mathcal{T}, G)$.

In other words, the pointed set $H^1_{D_1,\ldots,D_n}(\mathcal{T}, G)$ is canonically identified with the set of isomorphism classes of $(D_1, \ldots, D_n)$-trivial G-torsors.

Remark 2. Suppose that $\mathcal{T}$ is neutral and let $\omega\colon \mathcal{T} \to \mathrm{Vec}_k$ be a fiber functor on $\mathcal{T}$. Then, a G-torsor X is ω-trivial if and only if $\omega(X)$ has a k-rational point.

We recall the contraction and the pushforward of torsors.

Definition 6. ([11, Definition 1.3.1]). Let X (resp. X') be an affine scheme with a right (resp. left) action of an affine group scheme G in $\mathcal{T}$. Then,

we define *the contraction* $X \wedge^G X'$ to be the quotient of $X \times X'$ by the natural G-action defined by $(x, x') \mapsto (xg, g^{-1}x')$. If X' is an affine group scheme in $\mathcal{T}$ and the action of G on X' is induced by a group homomorphism $u\colon G \to X'$, we denote $X \wedge^G X'$ by $u_*(X)$ and call it *the pushforward* of X by u.

Remark 3. Let X (resp X') be a right G-torsor (resp. (G, G')-bitorsor).

(1) The contraction $X \wedge^G X'$ is a right G'-torsor. We call such a contraction the composition of torsors and call $X \wedge^G X'$ the composite X and X'.
(2) The composition is functorial in the following sense. Let $D\colon \mathcal{T} \to \mathcal{T}'$ be a faithful $\otimes$-functor between rigid exact $\otimes$-categories. Then, the quotient morphism $X \times X' \to X \wedge^G X'$ induces $D(G')$-equivariant morphism

$$c\colon D(X) \wedge^{D(G)} D(X') \to D(X \wedge^G X').$$

Since both hand sides are right $D(G')$-torsors, c is an isomorphism.

Finally, we introduce the condition (Unip).

Definition 7. Suppose that $\mathcal{T}$ is neutral and let ω be a fiber functor on $\mathcal{T}$. For each algebraic group G in $\mathcal{T}$, we say that G satisfies (Unip) if G satisfies the following condition: (**Unip**) $\omega(G)$ is a unipotent algebraic group over k.

3. Generalized Bloch–Kato exponential map and logarithmic map

In this section, we define the generalized Bloch–Kato exponential map and logarithmic map as maps between two classifying spaces of torsors in certain rigid exact $\otimes$-categories. We fix the following system of notation for the rest of this paper. Let F be a finite extension of $\mathbb{Q}_p$ and let F_0 be the maximal subfield of F unramified over $\mathbb{Q}_p$. Let $f := [F_0, \mathbb{Q}_p]$ be the residual degree of F over $\mathbb{Q}_p$.

3.1. **Classifying spaces of torsors in** $\mathrm{MF}_F(\varphi)$ **and in** $\mathrm{MF}_F^{\mathrm{ad}}(\varphi)$**.** In this and the next subsections, we give examples of classifying spaces of torsors in rigid exact $\otimes$-categories. Let $\mathrm{MF}_F(\varphi)$ be the category of finite dimensional filtered φ-modules over F. Recall that *a filtered φ-module* is an F_0-vector space M equipped with a Frobenius semi-linear endomorphism φ_M and a separated and exhaustive decreasing filtration $F^i(M \otimes_{F_0} F)$ of the F-vector space $M \otimes_{F_0} F$. *A Hodge-Tate weight of M* is an integer n satisfying $F^n(M \otimes_{F_0} F) \neq F^{n+1}(M \otimes_{F_0} F)$. This category has a natural $\otimes$-structure

and internal hom (cf. [9, 4.3.4]) and one can check that $\mathrm{MF}_F(\varphi)$ is a neutral rigid exact $\otimes$-category over $\mathbb{Q}_p$. We let

$$\omega_{F_0}\colon \mathrm{MF}_F(\varphi) \longrightarrow \mathrm{Vec}_{F_0}; M \mapsto M$$

be the forgetful functor. Note that ω_{F_0} is a $\otimes$-functor. We denote by MF_F the category of finite dimensional filtered F-vector spaces. Then, this category also has a natural $\otimes$-structure and internal hom (cf. loc. cit.) and is a neutral rigid exact $\otimes$-category over F. We also define *a Hodge-Tate weight of* M for each object of MF_F as we did. The correspondence $M \mapsto M \otimes_{F_0} F$ defines a faithful exact $\otimes$-functor of rigid exact $\otimes$-categories

$$\omega_F\colon \mathrm{MF}_F(\varphi) \longrightarrow \mathrm{MF}_F.$$

We fix an algebraic group H in $\mathrm{MF}_F(\varphi)$. We denote by $\varphi_H\colon H \to H$ the $\mathbb{Q}_p$-scheme endomorphism induced by the Frobenius endomorphism of the filtered φ-module $\mathcal{O}(H)$. Note that φ_H^f is an endomorphism of an F_0-scheme H because f is the residual degree of F_0 over $\mathbb{Q}_p$. Let $H_F := H \times_{\mathrm{Spec}(F_0)} \mathrm{Spec}(F)$. By definition, the following two inclusion relations are satisfied for all integers i and j:

$$F^i\mathcal{O}(H_F)F^j\mathcal{O}(H_F) \subset F^{i+j}\mathcal{O}(H_F), \tag{3.1}$$

$$\mathrm{cm}_H(F^i\mathcal{O}(H_F)) \subset \sum_{j+k=i} F^j\mathcal{O}(H_F) \otimes_F F^k\mathcal{O}(H_F). \tag{3.2}$$

Here, cm_H is the comultiplication of $\mathcal{O}(H)$. We sometimes identify $\omega_F(H)$ with H_F.

Definition 8. Let Y be an affine scheme in $\mathrm{MF}_F(\varphi)$.

(1) We say that Y satisfies the condition (Non-neg) if the following holds:
(**Non-neg**) All Hodge-Tate weights of $\mathcal{O}(Y)$ are non-negative, that is, we have $F^0(\mathcal{O}(Y_F)) = \mathcal{O}(Y_F)$.

(2) Suppose that Y is an affine group scheme in $\mathrm{MF}_F(\varphi)$. Then, we define the condition (Bij) as follows:
(**Bij**) The morphism of F_0-schemes $\varphi^f - 1\colon Y \to Y; x \mapsto \varphi^f(y)y^{-1}$ is an isomorphism.

Remark 4. Let Y be an affine scheme in $\mathrm{MF}_F(\varphi)$ and let $Y_F := Y \times_{\mathrm{Spec}(F_0)} \mathrm{Spec}(F)$. If Y satisfies the condition (Non-neg), then $F^i\mathcal{O}(Y_F)$ is an ideal of $\mathcal{O}(Y_F)$.

Suppose that H satisfies the condition (Non-neg). Then, $F^1\mathcal{O}(H_F)$ is a Hopf-ideal of $\mathcal{O}(H_F)$ by the inclusion relation (3.1) in the beginning of this subsection. We define a closed subgroup scheme F^0H_F of H_F to be

$\mathrm{Spec}(\mathcal{O}(H_F)/F^1\mathcal{O}(H_F))$. We say that an H-torsor Y in $\mathrm{MF}_F(\varphi)$ is non-negative if Y satisfies the condition (Non-neg). For each non-negative H-torsor Y, we put $F^0Y_F := \mathrm{Spec}(\mathcal{O}(Y_F)/F^1\mathcal{O}(Y_F))$. It can be checked that F^0Y_F is a torsor under F^0H_F in the usual sense. By definition, any ω_F-trivial torsor satisfies the condition (Non-neg).

Lemma 3.1. *Let Y be a right H-torsor in $\mathrm{MF}_F(\varphi)$ and let $Y_F := Y \times_{\mathrm{Spec}(F_0)} \mathrm{Spec}(F)$. If the underlying algebraic group of H is unipotent, then Y satisfies the condition* (Non-neg).

Proof. Let U_N be the closed subgroup scheme of GL_{N,F_0} consisting of upper triangle unipotent matrices and suppose that H is a closed subgroup scheme of U_N. Then, there exists $x_{i,j} \in \mathcal{O}(H), 1 \leq i < j \leq N$ such that $F_0[x_{i,j} | 1 \leq i < j \leq N] = \mathcal{O}(H)$ and

$$\mathrm{cm}_H(x_{i,j}) = x_{i,j} \otimes 1 + 1 \otimes x_{i,j} + \sum_{i<l<j} x_{i,l} \otimes x_{l,j}$$

for all $1 \leq i < j \leq N$. Since the algebraic group H over F_0 is unipotent, there exists a trivialization $H \cong Y$ of an H-torsor in Vec_{F_0}. We fix a trivialization $t\colon Y \xrightarrow{\sim} H$ and denote by $y_{i,j} \in \mathcal{O}(Y)$ the image of $x_{i,j}$ under $t^\sharp\colon \mathcal{O}(H) \xrightarrow{\sim} \mathcal{O}(Y)$. Then, we have the equality

$$\mathrm{ca}_{H,Y}(y_{i,j}) = y_{i,j} \otimes 1 + 1 \otimes x_{i,j} + \sum_{i<l<j} y_{i,l} \otimes x_{l,j}, \tag{3.3}$$

where $\mathrm{ca}_{H,Y}$ is the coaction of $\mathcal{O}(H)$ on $\mathcal{O}(Y)$. Let us denote by $\mathrm{ca}'_{H,Y}$ the isomorphism of filtered φ-modules $\mathcal{O}(Y) \otimes_{F_0} \mathcal{O}(Y) \xrightarrow{\sim} \mathcal{O}(Y) \otimes_{F_0} \mathcal{O}(H)$ defined by $y \otimes y' \mapsto (y \otimes 1)\mathrm{ca}_{H,Y}(y')$. We show that $y_{i,j} \in F^0\mathcal{O}(Y_F)$ by induction on $d := j - i$.

First, we suppose that $d = 1$. Then, by the equality (3.3), we have $\mathrm{ca}'_{H,Y}(1 \otimes y_{i,j} - y_{i,j} \otimes 1) = 1 \otimes x_{i,j}$. Since $1 \otimes x_{i,j} \in F^0(\mathcal{O}(Y_F) \otimes_F \mathcal{O}(H_F))$ by the condition (Non-neg), $1 \otimes y_{i,j} - y_{i,j} \otimes 1$ is also contained in $F^0(\mathcal{O}(Y_F) \otimes_F \mathcal{O}(Y_F))$. This implies that $y_{i,j} \in F^0\mathcal{O}(Y_F)$.

Next, we suppose that $d > 1$ and that $y_{i',j'} \in F^0\mathcal{O}(Y_F)$ if $i' - j' < d$. Then, by the induction hypothesis, $\mathrm{ca}'_{H,Y}(1 \otimes y_{i,j} - y_{i,j} \otimes 1) = 1 \otimes x_{i,j} + \sum_{i<l<j} y_{i,l} \otimes x_{l,j}$ is contained in $F^0(\mathcal{O}(Y_F) \otimes_F \mathcal{O}(H_F))$. Hence, we have $y_{i,j} \in F^0\mathcal{O}(Y_F)$ and this completes the proof of the lemma. □

Remark 5. In general, an H-torsor Y is not always to be non-negative. Let $F = \mathbb{Q}_p$ and let $H = \mathbb{G}_{m,\mathbb{Q}_p} = \mathrm{Spec}(\mathbb{Q}_p[t, t^{-1}])$ equipped with the trivial Frobenius action and the filtration defined by $F^i\mathcal{O}(H) = \mathcal{O}(H)$ if $i \leq 0$ and by $F^i\mathcal{O}(H) = 0$ if $i > 0$. Let $Y = \mathbb{G}_{m,\mathbb{Q}_p}$ equipped with the trivial Frobenius action, the filtration defined by $F^i\mathcal{O}(Y) = t^i\mathbb{Q}_p[t]$, and the right action of

H defined by the group structure of H. Then, Y is an H-torsor in $\mathrm{MF}_{\mathbb{Q}_p}(\varphi)$ which is not non-negative.

Definition 9. Let M be a filtered φ-module over F and let $f\colon M \otimes_{F_0} F \to M \otimes_{F_0} F$ be an F-linear automorphism. Then, we define the filtered φ-module M_f to be M equipped with the filtration $F^i(M_f \otimes_{F_0} F) := f(F^i(M \otimes_{F_0} F))$ and with the same Frobenius endomorphism.

Proposition 3.2. *Let H be an algebraic group in $\mathrm{MF}_F(\varphi)$ satisfying the conditions* (Non-neg) *and* (Bij). *Then, there exists a canonical isomorphism of pointed sets*

$$\Psi\colon H^1_{\omega_F,\omega_{F_0}}(\mathrm{MF}_F(\varphi), H) \xrightarrow{\sim} F^0H_F(F)\backslash H(F).$$

Proof. We repeat the same argument as in [14, Section 1]. First, we construct a map

$$\Psi\colon \{(\omega_{F_0},\omega_F)\text{-trivial } H\text{-torsors in } \mathrm{MF}^{\mathrm{ad}}_F(\varphi)\} \to F^0H_F(F)\backslash H(F).$$

Let Y be an (ω_{F_0},ω_F)-trivial right H-torsor in $\mathrm{MF}^{\mathrm{ad}}_F(\varphi)$. Then, we can take an F-rational point $p_{Y,d}$ of F^0Y_F because Y is ω_F-trivial. Further, by the same argument of [1, Corollary 3.2], we see that there exists a unique F_0-valued point $p_{Y,c}$ of Y invariant under φ^f. Here, for applying Besser's argument, we need the conditions that Y is ω_{F_0}-trivial and that H satisfies (Bij). We define $\Psi(Y)$ to be the class of $h \in H(F)$ in $F^0H_F(F)\backslash H(F)$ satisfying $p_{Y,c} = p_{Y,d}h$. One can check that the class of h in $F^0H_F(F)\backslash H(F)$ depends only on the isomorphism class of Y and that Ψ induces a well-defined injective map $H^1_{\omega_F,\omega_{F_0}}(\mathrm{MF}_F(\varphi), H) \hookrightarrow F^0H_F(F)\backslash H(F)$. By abuse of notation, we also denote this injection by Ψ.

Finally, we show the surjectivity of Ψ. Let h be an element of $H(F)$ and denote by $h^\sharp$ the ring automorphism of $\mathcal{O}(H_F)$ induced by the left multiplication of h. Then, the filtered φ-module $\mathcal{O}(H)_{h^\sharp}$ is a commutative algebra in $\mathrm{MF}_F(\varphi)$. We define H_h to be $\mathrm{Sp}(\mathcal{O}(H)_{h^\sharp})$. The affine scheme H_h is ω_{F_0}-trivial and the unit of H is the unique Frobenius invariant point of H_h. On the other hand, the closed subscheme $F^0H_{h,F}$ of $H_{h,F}$ coincides with $h^{-1}F^0H_F$ by construction. Therefore, we may take $p_{H_h,d}$ as h^{-1}. Hence, $\Psi(H_h) = h$ modulo $F^0H_F(F)$ and this implies the surjectivity of Ψ. □

Now, we assume that H is an algebraic group in the category $\mathrm{MF}^{\mathrm{ad}}_F(\varphi)$ of weakly admissible filtered φ-modules (cf. [9]). According to [4, Théorème A], the category $\mathrm{MF}^{\mathrm{ad}}_F(\varphi)$ is a neutral Tannakian category over $\mathbb{Q}_p$. Let $\mathrm{MF}^{\mathrm{ord}}_F(\varphi)$ be the category of ordinary filtered φ-modules in the sense of Perrin-Riou (cf. [18]). Recall that $M \in \mathrm{Obj}(\mathrm{MF}_F(\varphi))$ is said to be *ordinary* if its Hodge numbers coincide with Newton numbers counted with

multiplicity and that each ordinary filtered φ-module is weakly admissible. Perrin-Riou proved that $\mathrm{MF}_F^{\mathrm{ord}}(\varphi)$ is a neutral Tannakian category over $\mathbb{Q}_p$ (cf. loc. cit. Théorème 1.5). The following proposition is a key of this paper.

Proposition 3.3. *Let H be an algebraic group in $\mathrm{MF}_F^{\mathrm{ad}}(\varphi)$ satisfying the conditions* (Non-neg) *and* (Bij)*. Furthermore, we suppose one of the following conditions*:

(a) *The algebraic group H is an algebraic group in $\mathrm{MF}_F^{\mathrm{ord}}(\varphi)$.*
(b) *The algebraic group H satisfies the condition* (Unip)*.*

Then, the canonical injection

$$H^1_{\omega_{F_0},\omega_F}(\mathrm{MF}_F^{\mathrm{ad}}(\varphi),H)\hookrightarrow H^1_{\omega_{F_0},\omega_F}(\mathrm{MF}_F(\varphi),H)$$

is a bijection.

Remark 6. If an algebraic group H in $\mathrm{MF}_F^{\mathrm{ad}}(\varphi)$ satisfies the condition (Unip), then H is ω_F-trivial by Lemma 3.1. Furthermore, it is clear that H is ω_{F_0}-trivial in this case (cf. Remark 2). Hence, Proposition 3.3 states that we have

$$H^1(\mathrm{MF}_F^{\mathrm{ad}}(\varphi),H)\xrightarrow{\sim} H^1(\mathrm{MF}_F(\varphi),H)\cong F^0H_F(F)\backslash H(F) \tag{3.4}$$

if H satisfies three conditions (Non-neg), (Bij), and (Unip).

The proofs of Proposition 3.3 for the ordinary case (a) and the unipotent case (b) are given separately. The proof of the former case is simper than the latter case and we give a proof of that case in the last of this subsection. The proof of the latter case will be given in Subsection 4.3 depending on a totally different idea.

Before starting the proof of the ordinary case of Proposition 3.3, we recall the slope filtration of φ-modules. Let σ be the arithmetic Frobenius automorphism of F_0. Let M be a dualizable finite dimensional φ-module over F_0, that is, the F_0-linear homomorphism $M\to\sigma_*M$ induced by φ_M is an isomorphism. Then, by the Dieudonné–Manin classification, there exists a unique decomposition $M=\oplus_\alpha M^{[\alpha]}$ such that $M^{[\alpha]}$ is pure of slope α (cf. [12, Corollary 14.6.4], [18, Subsection 1.2]). We define *the slope filtration* $\{M_\alpha\}_{\alpha\in\mathbb{R}}$ of M by $M_\alpha:=\oplus_{\beta\leq\alpha}M^{[\beta]}$. By definition, this filtration is an increasing, separated, and saturated filtration of M. If M is ordinary, then M is dualizable and each slope of M is an integer. Thus, we regard the slope filtration of M as a filtration indexed by integers when M is ordinary.

Lemma 3.4. *Let $\mathcal{C}$ be a sub-Tannakian category of $\mathrm{MF}_F^{\mathrm{ord}}(\varphi)$ closed under subquotients and extensions. Let M be an object in $\mathcal{C}$ and let $f\colon M\otimes_{F_0}F\to M\otimes_{F_0}F$ be an F-linear automorphism. If f stabilizes the filtration $\{M_\alpha\otimes_{F_0}F\}_{\alpha\in\mathbb{Z}}$, then the filtered φ-module M_f is also an object in $\mathcal{C}$. In particular, M_f is weakly admissible.*

Proof. We prove this lemma by induction on $n := \sharp\{\text{Hodge numbers of } M\}$. If $n = 1$, then M is pure of slope α for some integer α. The Hodge filtration of M is give by the following equalities:

$$F^i(M \otimes_{F_0} F) = \begin{cases} M \otimes_{F_0} F & \text{if } i \leq \alpha, \\ 0 & \text{if } i > \alpha. \end{cases}$$

Hence, we have $M_f = M$ and the assertion holds for M.

Next, we show the general case. We assume that the assertion of the lemma holds for each $N \in \mathrm{Obj}(\mathcal{C})$ such that $\sharp\{\text{Hodge numbers of } N\} < n$. Let α be the maximum integer such that

$$F^\alpha(M \otimes_{F_0} F) = M \otimes_{F_0} F.$$

Then, according to [18, Lemma 2.5], M_α is pure of slope α and M_α is a sub-object of M in $\mathcal{C}$. By the assumption of the induction, the assertion of the lemma for $M' := (M/M_\alpha) \in \mathrm{Obj}(\mathcal{C})$ holds. Consider the following exact sequence of filtered φ-modules:

$$0 \to M_{\alpha,f} = M_\alpha \to M_f \to M'_{\bar{f}} \to 0$$

where $\bar{f} : M' \otimes_{F_0} F \to M' \otimes_{F_0} F$ is the F-linear automorphism induced by f. Since M_α and $M'_{\bar{f}}$ are ordinary, M_f is also ordinary (cf. [18, Proposition 2.4]). Furthermore, since M_α and $M'_{\bar{f}}$ are objects in $\mathcal{C}$, we see that M_f is also an object in $\mathcal{C}$. □

Proof of Proposition 3.3 *when H satisfies the condition* (b). By the proof of Proposition 3.2, it is sufficient to show that $\mathcal{O}(H)_{h^\sharp}$ is admissible for any $h \in H(F)$. According to Lemma 3.4, it is sufficient to show that $h^\sharp$ preserves the slope filtration of $\mathcal{O}(H_F)$. Let α be a slope of $\mathcal{O}(H)$. Since the comultiplication cm_H of $\mathcal{O}(H)$ is compatible with its slope filtration, $h^\sharp$ is factored as follows:

$$\mathcal{O}(H_F)_\alpha := \mathcal{O}(H)_\alpha \otimes_{F_0} F \xrightarrow{\mathrm{cm}_H} \sum_{\beta+\gamma=\alpha} \mathcal{O}(H_F)_\beta \otimes_F \mathcal{O}(H_F)_\gamma \xrightarrow{1\otimes h} \sum_{\beta+\gamma=\alpha} \mathcal{O}(H_F)_\gamma.$$

Note that each slope of $\mathcal{O}(H)$ is non-negative by the condition (Non-neg). Hence, we have $\sum_{\beta+\gamma=\alpha} \mathcal{O}(H_F)_\gamma = \sum_{\gamma=0}^{\alpha} \mathcal{O}(H_F)_\gamma = \mathcal{O}(H_F)_\alpha$. Thus, $h^\sharp$ preserves the filtration $\{\mathcal{O}(H_F)_\alpha\}_{\alpha\in\mathbb{Z}_{\geq 0}}$ and this completes the proof of Proposition 3.3 when the condition (a) holds. □

By Lemma 3.4 and the above proof, we obtain the following corollary.

Corollary 1. Let $\mathcal{C}$ be a sub-Tannakian category of $\mathrm{MF}_F^{\mathrm{ord}}(\varphi)$ stable under subquotients and extensions. Let H be an algebraic group in $\mathcal{C}$ satisfying the conditions (Non-neg) and (Bij). Then, the canonical injections

$$H^1_{\omega_{F_0},\omega_F}(\mathcal{C}, H) \hookrightarrow H^1_{\omega_{F_0},\omega_F}(\mathrm{MF}_F^{\mathrm{ad}}(\varphi), H) \hookrightarrow H^1_{\omega_{F_0},\omega_F}(\mathrm{MF}_F(\varphi), H)$$

are bijections.

3.2. **Classifying spaces of torsors in** $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$ **and in** $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$**.** The next example of a rigid exact $\otimes$-category is the category $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$ of continuous representations of $\mathscr{G}_F$ on finite dimensional $\mathbb{Q}_p$-vector spaces. We study fundamental properties of affine schemes and classifying spaces of torsors in this category.

Definition 10. Let X be an affine scheme in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$ of finite type. Choose $V \in \mathrm{Obj}(\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F))$ and a closed immersion $X \hookrightarrow \mathbb{A}(V)$ (cf. Lemma 2.1). Then, for any topological $\mathbb{Q}_p$-algebra R, we define the topology on $X(R)$ to be the relative topology defined by the inclusion $X(R) \hookrightarrow \mathbb{A}(V)(R) = V \otimes_{\mathbb{Q}_p} R$. Here, we equip $V \otimes_{\mathbb{Q}_p} R$ with the product topology on R.

Lemma 3.5. *The topology on $X(R)$ does not depend on the choice of a closed immersion $i : X \hookrightarrow \mathbb{A}(V)$.*

Proof. First, we consider the case when X is an affine space $\mathbb{A}(W)$ with $W \in \mathrm{Obj}(\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F))$ and i is induced by an injective $\mathbb{Q}_p[\mathscr{G}_F]$-homomorphism $W \hookrightarrow V$. In this case, the product topology on $W \otimes_{\mathbb{Q}_p} R$ coincides with the topology induced by $W \otimes_{\mathbb{Q}_p} R \hookrightarrow V \otimes_{\mathbb{Q}_p} R$ because $W \otimes_{\mathbb{Q}_p} R$ is a direct factor of $V \otimes_{\mathbb{Q}_p} R$ as an R-module.

Next, we consider the general case. Take another object V' of $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$ and a closed immersion $i : X \hookrightarrow \mathbb{A}(V')$. By replacing V' by $V \oplus V'$, we may assume that V is contained in V' and that there exists a commutative diagram of closed immersions

$$\begin{array}{ccc} X & \xrightarrow{\quad i \quad} & \mathbb{A}(V) \\ & {\scriptstyle i'}\searrow \quad \swarrow{\scriptstyle f} & \\ & \mathbb{A}(V') & \end{array}$$

such that f is induced by the inclusion $V \hookrightarrow V'$. Since the topology on $\mathbb{A}(V)$ coincides with the relative topology defined by f, the relative topology on $X(R)$ induced by i coincides with its relative topology defined by i'. □

Definition 11. Let X be an affine scheme in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$ of finite type. Then, for each topological $\mathbb{Q}_p$-algebra R equipped with a continuous action of $\mathscr{G}_F$, we define the action of $\sigma \in \mathscr{G}_F$ on $x \in X(R) = \mathrm{Hom}(\mathcal{O}(X), R)$ by the equality ${}^{\sigma}x(a) := {}^{\sigma}\left(x({}^{\sigma^{-1}}a)\right)$ for each $a \in \mathcal{O}(X)$.

Remark 7. We use the same notation as in Definition 11.

(1) The action of $\mathscr{G}_F$ on $X(R)$ is functorial in X. In other words, for any morphism $X \to Y$ between affine schemes in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$, the map $X(R) \to Y(R)$ is $\mathscr{G}_F$-equivariant.
(2) If $X = \mathbb{A}(V)$ with $V \in \mathrm{Obj}(\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F))$, then the action of $\mathscr{G}_F$ on $X(R) = V \otimes_{\mathbb{Q}_p} R$ coincides with the diagonal action. In particular, this action is continuous.

Lemma 3.6. *Let X and R be the same as in Definition* 11. *Then, the action of $\mathscr{G}_F$ on $X(R)$ is continuous.*

Proof. Choose $V \in \mathrm{Obj}(\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F))$ and a closed immersion $X \hookrightarrow \mathbb{A}(V)$. Recall that the inclusion $X(R) \hookrightarrow \mathbb{A}(V)(R)$ is $\mathscr{G}_F$-equivariant (cf. Remark 7 (1)). Furthermore, the action of $\mathscr{G}_F$ on $\mathbb{A}(V)(R)$ is continuous by Remark 7 (2). Since the topology on $X(R)$ is defined by this inclusion, we obtain the conclusion of the lemma. □

Let $\omega_{\mathbb{Q}_p} \colon \mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F) \to \mathrm{Vec}_{\mathbb{Q}_p}$ be the forgetful functor. Then, we can describe the classifying space of $\omega_{\mathbb{Q}_p}$-trivial torsors as follows:

Proposition 3.7. *Let G be an algebraic group in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$. Then, there exists a natural isomorphism of pointed sets*

$$\Phi \colon H^1_{\omega_{\mathbb{Q}_p}}(\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F), G) \xrightarrow{\sim} H^1(F, G(\mathbb{Q}_p)).$$

Proof. Let X be an $\omega_{\mathbb{Q}_p}$-trivial right G-torsor in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$. Since X is $\omega_{\mathbb{Q}_p}$-trivial, $X(\mathbb{Q}_p)$ is non-empty (cf. Remark 2) and is a right torsor under $G(\mathbb{Q}_p)$ in the usual sense. We fix an element x of the non-empty set $X(\mathbb{Q}_p)$. Then, for each $\sigma \in \mathscr{G}_F$, there exists a unique element $c_x(\sigma) \in G(\mathbb{Q}_p)$ satisfying ${}^{\sigma}x = xc_x(\sigma)$. By construction, c_x is a continuous 1-cocycle of $\mathscr{G}_F$ with coefficients in $G(\mathbb{Q}_p)$. By the standard argument, one can prove that the cohomology class $[c_x] \in H^1(F, G(\mathbb{Q}_p))$ of c_x depends only on the isomorphism class of X and $[c_x]$ determines the isomorphism class of X. We put $\Phi([X]) := [c_x]$. Then, Φ is injective because the isomorphism class of X is uniquely determined by the cohomology class $\Phi([X])$.

We shall check the surjectivity of Φ. Let $c \colon \mathscr{G}_F \to G(\mathbb{Q}_p)$ be a continuous 1-cocycle. For each $\sigma \in \mathscr{G}_F$, we define $a_c(\sigma) \colon G \to G$ by $a_c(\sigma)(g) := {}^{\sigma}g\, c(\sigma)$. Since c is a 1-cocycle, a_c defines a left action of $\mathscr{G}_F$ on G. We define G_c to be G equipped with the action of $\mathscr{G}_F$ defined by a_c. Further, we equip G_c with the natural right action of G defined by translations. We shall show that G_c is a right torsor under G in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$. It is sufficient to show that G_c is an affine scheme in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$. According to Proposition 2.3, there exists a collection of finite dimensional sub-$\mathbb{Q}_p$-vector spaces $\{V_\lambda\}_\lambda$ of $\mathcal{O}(G)$

stable under the actions of $\mathscr{G}_F$ and G such that $\mathcal{O}(G) = \cup_\lambda V_\lambda$. Then, V_λ is also stable under the new action of $\mathscr{G}_F$ defined by a_c. Therefore, G_c is an affine scheme in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$. By the construction of G_c and the definition of Φ, we have $\Phi([G_c]) = [c]$. Thus, Φ is surjective. □

For the rest of this subsection, we consider torsors under algebraic groups in the category $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$ of crystalline representations of $\mathscr{G}_F$.

Definition 12. We define the finite part $H^1_f(F, G(\mathbb{Q}_p))$ of $H^1(F, G(\mathbb{Q}_p))$ by

$$H^1_f(F, G(\mathbb{Q}_p)) := \mathrm{Ker}\left(H^1(F, G(\mathbb{Q}_p)) \to H^1(F, G(B_{\mathrm{crys}}))\right).$$

Recall that there exists an equivalence of Tannakian categories

$$D_{\mathrm{crys}}\colon \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F) \xrightarrow{\sim} \mathrm{MF}^{\mathrm{ad}}_F(\varphi); V \mapsto D_{\mathrm{crys}}(V) := H^0(\mathscr{G}_F, V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})$$

which is a quasi-inverse of the functor V_{crys} (cf. [4]).

Proposition 3.8. *Let G be an algebraic group in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$. Then, the map Φ in Proposition 3.7 induces the canonical bijection*

$$\Phi\colon H^1_{\omega_{\mathbb{Q}_p}, D'_{\mathrm{crys}}}(\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F), G) \xrightarrow{\sim} H^1_f(F, G(\mathbb{Q}_p)),$$

where D'_{crys} is the composite $\omega_{F_0} \circ D_{\mathrm{crys}}\colon \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F) \to \mathrm{Vec}_{F_0}$.

Proof. Let X be a right G-torsor in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$ which is $\omega_{\mathbb{Q}_p}$-trivial. Note that $\Phi([X])$ is contained in the kernel of $H^1(F, G(\mathbb{Q}_p)) \to H^1(F, G(B_{\mathrm{crys}}))$ if and only if the $\mathscr{G}_F$-invariant part of $X(B_{\mathrm{crys}})$ is non-empty. In particular, we have $\mathcal{O}(X) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}} \cong \mathcal{O}(G) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}$ as $\mathscr{G}_F$-modules if $\Phi([X]) \in H^1_f(F, G(\mathbb{Q}_p))$. This implies that $\mathcal{O}(X)$ is ind-crystalline, namely, X is an affine scheme in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$.

Now, we suppose that X is a G-torsor in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$. By definition, we have the following equalities:

$$\begin{aligned}
X(B_{\mathrm{crys}})^{\mathscr{G}_F} &= \mathrm{Hom}_{\mathbb{Q}_p\text{-alg.}}(\mathcal{O}(X), B_{\mathrm{crys}})^{\mathscr{G}_F} \\
&= \mathrm{Hom}_{B_{\mathrm{crys}}\text{-alg.}}(\mathcal{O}(X) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}, B_{\mathrm{crys}})^{\mathscr{G}_F} \\
&= \mathrm{Hom}_{B_{\mathrm{crys}}\text{-alg.}}(D_{\mathrm{crys}}(\mathcal{O}(X)) \otimes_{F_0} B_{\mathrm{crys}}, B_{\mathrm{crys}})^{\mathscr{G}_F} \\
&= \mathrm{Hom}_{F_0\text{-alg.}}(D_{\mathrm{crys}}(\mathcal{O}(X)), B_{\mathrm{crys}})^{\mathscr{G}_F} \\
&= \mathrm{Hom}_{F_0\text{-alg.}}(D_{\mathrm{crys}}(\mathcal{O}(X)), F_0) \\
&= D_{\mathrm{crys}}(X)(F_0).
\end{aligned}$$

Therefore, $\Phi(X)$ is contained in $H^1_f(F, G(\mathbb{Q}_p))$ if and only if X is D'_{crys}-trivial. This completes the proof of the proposition. □

3.3. **Generalized exponential map and logarithmic map.** In this subsection, we define the generalized Bloch–Kato exponential map and logarithmic map assuming Proposition 3.3. We fix an algebraic group G in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$ and denote $D_{\mathrm{crys}}(G)$ by H.

Theorem 3.9. *We suppose that G is $\omega_{\mathbb{Q}_p}$-trivial and that H satisfies the conditions* (Non-neg) *and* (Bij).

(1) *If H satisfies one of the conditions* (a) *and* (b) *of Proposition* 3.3, *then the equivalence $V_{\mathrm{crys}}\colon \mathrm{MF}^{\mathrm{ad}}_F(\varphi) \xrightarrow{\sim} \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$ of Tannakian categories induces a natural injection*

$$F^0H_F(F)\backslash H(F) \hookrightarrow H^1_f(F, G(\mathbb{Q}_p)).$$

(2) *If H is ω_F-trivial, then the equivalence of Tannakian categories $D_{\mathrm{crys}}\colon \mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F) \xrightarrow{\sim} \mathrm{MF}^{\mathrm{ad}}_F(\varphi)$ induces an injection between pointed sets*

$$H^1_f(F, G(\mathbb{Q}_p)) \hookrightarrow F^0H_F(F)\backslash H(F).$$

(3) *If H satisfies both conditions of* (1) *and* (2), *then the injections in* (1) *and* (2) *are bijections and one of them is the inverse of the other one.*

Proof. By the assumption of the triviality, we have

$$H^1_{D'_{\mathrm{crys}}}(\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F), G) = H^1_{\omega_{\mathbb{Q}_p}, D'_{\mathrm{crys}}}(\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F), G). \tag{3.5}$$

Then, by Proposition 3.8, we have the following morphisms of pointed sets:

$$H^1_{\omega_{F_0}, \omega_F}(\mathrm{MF}^{\mathrm{ad}}_F(\varphi), H) \xrightarrow[\sim]{V_{\mathrm{crys}}} H^1_{D'_{\mathrm{crys}}, \omega_F \circ D_{\mathrm{crys}}}(\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F), G) \hookrightarrow H^1_f(F, G(\mathbb{Q}_p)). \tag{3.6}$$

First, we show the assertion (1). If H satisfies one of the conditions (a) and (b) of Proposition 3.3, then the natural injection $H^1_{\omega_{F_0}, \omega_F}(\mathrm{MF}^{\mathrm{ad}}_F(\varphi), H) \hookrightarrow F^0H_F(F)\backslash H(F)$ is bijective by Proposition 3.3. By composing the inverse of that bijection and the injection (3.6), we have the desired injection.

Next, we show the assertion (2). If H is ω_F-trivial, then G is $\omega_F \circ D_{\mathrm{crys}}$-trivial. Therefore, the injection (3.6) is bijective by Proposition 3.8 and the equality (3.5). Since D_{crys} is a quasi-inverse of V_{crys}, the inverse of the bijection (3.6) is induced by the functor V_{crys}. Then, by composing the inverse of (3.6) and the natural injection $H^1_{\omega_{F_0}, \omega_F}(\mathrm{MF}^{\mathrm{ad}}_F(\varphi), H) \hookrightarrow F^0H_F(F)\backslash H(F)$, we obtain the second inclusion.

The assertion (3) is easily proved by the constructions of two injections. □

Definition 13. We call the injection in Theorem 3.9 (1) *the generalized Bloch–Kato exponential map* for G and denote it by $\exp_G$. We also define *the generalized Bloch–Kato logarithmic map* $\log_G$ to be the injection in Theorem 3.9 (2).

The following theorem is deduced from the definition of the exponential maps and Remark 3.

Theorem 3.10. *Let* $1 \to G' \to G \to G'' \to 1$ *be an exact sequence of algebraic groups in* $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$ *and set* $H' := D_{\mathrm{crys}}(G'), H'' := D_{\mathrm{crys}}(G'')$. *We assume that whole the groups satisfy whole the conditions in Theorem* 3.9 (1), (2). *Then, we have the following commutative diagram of pointed sets with exact rows*:

$$\begin{array}{ccccc} F^0H'_F(F)\backslash H'(F) & \longrightarrow & F^0H_F(F)\backslash H(F) & \longrightarrow & F^0H''_F(F)\backslash H''(F) \\ \cong\downarrow \exp_{G'} & & \cong\downarrow \exp_{G} & & \cong\downarrow \exp_{G''} \\ H^1_f(F, G'(\mathbb{Q}_p)) & \longrightarrow & H^1_f(F, G(\mathbb{Q}_p)) & \longrightarrow & H^1_f(F, G''(\mathbb{Q}_p)). \end{array} \tag{3.7}$$

Here, the horizontal maps are defined by the pushforward of torsors. Further, if G' *is contained in the center of* G, *then* $H^1_f(F, G'(\mathbb{Q}_p))$ (*resp.* $F^0H'_F(F)\backslash H'(F)$) *acts on* $H^1_f(F, G(\mathbb{Q}_p))$ (*resp.* $F^0H_F(F)\backslash H(F)$) *and the diagram* (3.7) *is compatible with those actions.*

The last assertion of Theorem 3.10 follows from well-known facts about long exact sequence of non-abelian group cohomology for central extensions and from the construction of $\exp_G$.

4. Unipotent case

In this section, we recall an exponential map $\exp'_G$ defined in the previous works [15] and [20] when G satisfies (Unip). The main result Proposition 4.9 of this section states that $\exp'_G$ is the inverse map of the generalized Bloch–Kato logarithm $\log_G$ defined in Subsection 3.3. Then, we complete the proof of Proposition 3.3 when H satisfies the condition (b) of Proposition 3.3 as a consequence of Proposition 4.9.

4.1. Remarks on nilpotent Lie algebras and unipotent algebraic groups.

Let k be a field of characteristic 0 and $\mathfrak{g}$ a finite dimensional nilpotent Lie algebra over k. Then, for any k-algebra R, the Campbell–Hausdorff product (cf. Campbell–Hausdorff's formula [21, Chapter IV, Section 8, p. 27 line 30]) defines a canonical group structure on $\mathfrak{g} \otimes_k R$. We denote this group by $(\mathfrak{g} \otimes_k R)^{\mathrm{CH}}$ and can regard the functor

$$\mathfrak{g}^{\mathrm{CH}} \colon (k\text{-algebras}) \to (\text{groups}); R \mapsto (\mathfrak{g} \otimes_k R)^{\mathrm{CH}}$$

as a unipotent algebraic group over k because this functor is represented by the scheme $\mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}^\vee))$ where $\mathfrak{g}^\vee$ is the dual k-vector space of $\mathfrak{g}$. Let NilLie_k (resp. Unip_k) be the category of finite dimensional nilpotent Lie algebras over k (resp. unipotent algebraic groups over k). Then, according to [5, Chapter IV, Section 2, Proposition 4.1, Corollaire 4.5 (b)], the functor

$$\mathrm{NilLie}_k \to \mathrm{Unip}_k; \mathfrak{g} \mapsto \mathfrak{g}^{\mathrm{CH}}$$

is an equivalence of categories with a quasi-inverse

$$\mathrm{Lie}\colon \mathrm{Unip}_k \to \mathrm{NilLie}_k; G \mapsto \mathrm{Lie}(G).$$

The following proposition is easily checked by $\mathfrak{g}^{\mathrm{CH}} = \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g}^\vee))$.

Proposition 4.1. *The functor* Lie *induces an equivalence between the category of algebraic groups in* $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$ *satisfying* (Unip) *and the category of nilpotent Lie algebra objects in* $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$. *In particular, for an algebraic group G in* $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$ *satisfying* (Unip), *G is an algebraic group in* $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$ *if and only if* $\mathrm{Lie}(G)$ *is a crystalline representation of* $\mathscr{G}_F$.

Let $\mathfrak{g}$ be a nilpotent Lie algebra object in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$ and let $\mathfrak{h} := D_{\mathrm{crys}}(\mathfrak{g})$.

Lemma 4.2. *The algebraic group* $\mathfrak{h}^{\mathrm{CH}}$ *in* $\mathrm{MF}^{\mathrm{ad}}_F(\varphi)$ *satisfies the condition* (Non-neg) *if and only if any Hodge-Tate weight of* $\mathfrak{h}^\vee$ *is non-negative.*

Proof. Since $\mathfrak{h}^{\mathrm{CH}} = \mathrm{Spec}(\mathrm{Sym}(\mathfrak{h}^\vee))$, $\mathfrak{h}^\vee$ is a sub-admissible filtered φ-module of $\mathcal{O}(\mathfrak{h}^{\mathrm{CH}})$. Thus, if $\mathfrak{h}^{\mathrm{CH}}$ satisfies (Non-neg), then any Hodge-Tate weight of $\mathfrak{h}^\vee$ is non-negative. On the other hand, if any Hodge-Tate weight of $\mathfrak{h}^\vee$ is non-negative, then any Hodge-Tate weight of $\mathrm{Sym}^n(\mathfrak{h}^\vee)$ is also non-negative for each $n \in \mathbb{Z}_{\geq 0}$. Hence $\mathfrak{h}^{\mathrm{CH}}$ satisfies (Non-neg) and we have the conclusion. □

We define the central descending series $\Gamma^i\mathfrak{g}$ by $\Gamma^1\mathfrak{g} := \mathfrak{g}$ and $\Gamma^{i+1}\mathfrak{g} := [\Gamma^i\mathfrak{g}, \mathfrak{g}]$.

Lemma 4.3. *We put* $V_i := \Gamma^i\mathfrak{g}/\Gamma^{i+1}\mathfrak{g}$. *If φ^f-invariant parts of* $D_{\mathrm{crys}}(V_i)$ *are zero for all i, then the algebraic group* $\mathfrak{h}^{\mathrm{CH}}$ *in* $\mathrm{MF}^{\mathrm{ad}}_F(\varphi)$ *satisfies the condition* (Bij).

Proof. We prove this lemma by induction on the nilpotency n of $\mathfrak{g}$. If $n = 1$, then the assertion is easily checked. We assume that $n > 1$ and the assertion holds when we replace n by $n - 1$. Let H be $\mathfrak{h}^{\mathrm{CH}}$, Z the center of H, and $H' = H/Z$. Then, we obtain the following commutative diagram with exact

rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & H & \longrightarrow & H' & \longrightarrow & 1 \\ & & \cong\downarrow{\scriptstyle \varphi_Z^f-1} & & \downarrow{\scriptstyle \varphi_H^f-1} & & \cong\downarrow{\scriptstyle \varphi_{H'}^f-1} & & \\ 1 & \longrightarrow & Z & \longrightarrow & H & \longrightarrow & H' & \longrightarrow & 1. \end{array}$$

We note that $\varphi_H^f - 1$ is not a group homomorphism except the case $n = 1$. However, the diagram above is compatible with the action of Z on H defined by multiplications. Indeed, for any $z \in Z$ and $h \in H$, the equality $(\varphi_Z^f(z)z^{-1})(\varphi_H^f(h)h^{-1}) = \varphi_H^f(zh)(zh)^{-1}$ holds. Hence, by the snake lemma, we have the conclusion of the lemma. □

4.2. **The fundamental exact sequence.** Let us fix an algebraic group G in $\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F)$ satisfying the condition (Unip). We denote by $\mathfrak{g}$ the Lie algebra of G. The key of the construction of the exponential map in [15] and [20] is the following lemma:

Lemma 4.4 (The fundamental exact sequence, [20, Lemma 2.1], [15, Proof of Proposition 1.4])**.** *There exists the following $\mathscr{G}_F$-equivariant exact sequence of topological pointed sets*:

$$1 \to G(\mathbb{Q}_p) \xrightarrow{\alpha} G(B_e) \xrightarrow{\beta} G(B_{\mathrm{dR}}^+)\backslash G(B_{\mathrm{dR}}) \to 1. \tag{4.1}$$

Here, B_e is the φ-invariant part $B_{\mathrm{crys}}^{\varphi=1}$ of B_{crys}. Moreover, the map β has a set theoretical continuous section.

Recall that $\mathfrak{g}$ is the Lie algebra of G. Then, $D_{\mathrm{dR}}(\mathfrak{g}) := H^0(F, \mathfrak{g} \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})$ and $D_{\mathrm{dR}}^0(\mathfrak{g}) := H^0(F, \mathfrak{g} \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)$ have natural structures of nilpotent Lie algebras over F. We define the unipotent algebraic group $D_{\mathrm{dR}}(G)$ (resp. $D_{\mathrm{dR}}^0(G)$) over F to be $D_{\mathrm{dR}}(\mathfrak{g})^{\mathrm{CH}}$ (resp. $D_{\mathrm{dR}}^0(\mathfrak{g})^{\mathrm{CH}}$).

Lemma 4.5 ([20, Lemma 5.3])**.** *Assume that $\mathfrak{g}$ is a de Rham representation of $\mathscr{G}_F$. Then, we have a canonical isomorphism of pointed sets*

$$D_{\mathrm{dR}}^0(G)(F)\backslash D_{\mathrm{dR}}(G)(F) \xrightarrow{\sim} H^0(F, G(B_{\mathrm{dR}}^+)\backslash G(B_{\mathrm{dR}})).$$

Definition 14.

(1) We define *the exponential part* $H_e^1(F, G(\mathbb{Q}_p))$ of $H^1(F, G(\mathbb{Q}_p))$ to be the kernel of the canonical map of pointed sets $H^1(F, G(\mathbb{Q}_p)) \to H^1(F, G(B_e))$ induced by the homomorphism α in (4.1).

(2) Assume that $\mathrm{Lie}(G) = \mathfrak{g}$ is a de Rham representation of $\mathscr{G}_F$. Then, we define *the exponential map* $\exp'_{\mathfrak{g}}\colon D_{\mathrm{dR}}^0(G)(F)\backslash D_{\mathrm{dR}}(G)(F) \to H_e^1(F, G(\mathbb{Q}_p))$ to be the composite of the isomorphism of Lemma

4.5 and the connecting homomorphism of the fundamental exact sequence (4.1) in Lemma 4.4. We also denote $\exp'_{\mathfrak{g}}$ by $\exp'_G$.

We can prove the following proposition by the induction on the nilpotency of $\mathfrak{g}$.

Proposition 4.6 (cf. [20, Proposition 5.7])**.** *Assume that $\mathfrak{g}$ satisfies the following conditions*:

(a) *The $\mathscr{G}_F$-representation $\mathfrak{g} = \mathrm{Lie}(G)$ is a de Rham representation.*
(b) *The algebraic group $D_{\mathrm{crys}}(G) := D_{\mathrm{crys}}(\mathfrak{g})^{\mathrm{CH}}$ in $\mathrm{MF}_F^{\mathrm{ad}}(\varphi)$ satisfies the condition* (Bij)[1].

Then, the following assertions hold.

(1) *The exponential map $\exp'_G$ is bijective.*
(2) *If $\mathfrak{g}$ is crystalline, then $H^1_e(F, G(\mathbb{Q}_p))$ coincides with $H^1_f(F, G(\mathbb{Q}_p))$.*

Now, we fix a $\mathbb{Z}_p$-basis $\zeta_{p^\infty} = (\zeta_{p^n})_{n\geq 0}$ of $\mathbb{Z}_p(1) := \varprojlim_n \mu_{p^n}$ and let $\mathbb{Q}_p(r) := \mathbb{Z}_p(1)^{\otimes r} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$. We regard $\mathbb{Q}_p(r)$ as an abelian Lie algebra object in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$. Although it is hard to calculate $\exp'_G$ in general, there exists an explicit formula for $\exp'_{\mathbb{Q}_p(r)}$ due to Bloch and Kato. Recall that the one-dimensional F_0-vector space $D_{\mathrm{crys}}(\mathbb{Q}_p(r))$ is generated by $\zeta_{p^\infty}^{\otimes r}/t^r$ where $t \in B_{\mathrm{crys}}$ is the p-adic analog of "$2\pi\sqrt{-1}$" attached to ζ_{p^∞} (cf. [8]).

Theorem 4.7 (The explicit reciprocity law, [2, Theorem 2.1 (4.8.2)])**.** *Assume that F is unramified over $\mathbb{Q}_p$. Then, for each $a \in F$ and for each positive integer r greater than 1, the following formula holds*:

$$\exp'_{\mathbb{Q}_p(r)}\left(a\frac{\zeta_{p^\infty}^{\otimes r}}{t^r}\right) = \frac{1}{(r-1)!}\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\left\{\left(1-\frac{\sigma_F}{p^r}\right)a\right\}\phi^{CW}_{F,r}\right).$$

Here, $\sigma_F \in \mathrm{Gal}(F/\mathbb{Q}_p)$ is the arithmetic Frobenius automorphism of F and $\phi^{CW}_{F,r} \in \mathrm{Hom}_{\mathrm{Gal}(F(\mu_{p^\infty})/F)}(\mathrm{Gal}(\overline{F}/F(\mu_{p^\infty})), F(r)) \cong H^1(F, F(r))$ is the Coates–Wiles homomorphism.

We will use this theorem in Subsection 5.2.

4.3. **End of the proof of Proposition 3.3.** In this subsection, we assume that G is an algebraic group in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$ satisfying (Unip) and that $H := D_{\mathrm{crys}}(G)$ satisfies the conditions (Non-neg) and (Bij). We denote $D_{\mathrm{crys}}(\mathfrak{g})$ by $\mathfrak{h}$. Note that H is ω_F-trivial by Lemma 3.1. Therefore, the generalized Bloch–Kato logarithm $\log_G$ for G is defined by Theorem 3.9 (2).

[1]This condition is slightly different from the condition used in [20]. However, the same proof works.

Lemma 4.8. *The unipotent algebraic group $D^0_{\mathrm{dR}}(G)$ over F is canonically isomorphic to F^0H_F.*

Proof. We have the following canonical isomorphism:

$$H = D_{\mathrm{crys}}(G) \cong \mathrm{Spec}\left(\mathrm{Sym}(\mathfrak{h}^\vee)\right).$$

By definition, the ideal of $\mathrm{Sym}(\mathfrak{h}^\vee) \otimes_{F_0} F = \mathrm{Sym}(D_{\mathrm{dR}}(\mathfrak{g}^\vee))$ corresponding to the closed subgroup scheme F^0H_F of H_F is generated by $F^1\, D_{\mathrm{dR}}(\mathfrak{g}^\vee)$. Thus, the ring of regular functions on F^0H_F is canonically isomorphic to $\mathrm{Sym}\left(D_{\mathrm{dR}}(\mathfrak{g}^\vee)/F^1(D_{\mathrm{dR}}(\mathfrak{g}^\vee))\right)$. Hence, for each F-algebra R, we have canonical isomorphisms of sets

$$F^0H_F(R) \cong \left(D_{\mathrm{dR}}(\mathfrak{g}^\vee)/F^1(D_{\mathrm{dR}}(\mathfrak{g}^\vee))\right)^\vee \otimes_F R \cong D^0_{\mathrm{dR}}(\mathfrak{g}) \otimes_F R = \left(D^0_{\mathrm{dR}}(\mathfrak{g}) \otimes_F R\right)^{\mathrm{CH}}$$

functorial in R. Furthermore, the group structures of both hand sides coincide under the isomorphism above because they are subgroups of $H_F(R) = (D_{\mathrm{dR}}(\mathfrak{g}) \otimes_F R)^{\mathrm{CH}}$. This completes the proof of the lemma. □

Proposition 4.9. *Let G be an algebraic group in $\mathrm{Rep}^{\mathrm{crys}}_{\mathbb{Q}_p}(\mathscr{G}_F)$ satisfying* (Unip)*. Put $H := D_{\mathrm{crys}}(G)$. If H satisfies the conditions* (Non-neg) *and* (Bij)*, then $\log_G$ defined in Definition* 13 *is a bijection and*

$$\exp'_G\colon D^0_{\mathrm{dR}}(G)(F)\backslash D_{\mathrm{dR}}(G)(F) = F^0H_F(F)\backslash H(F) \to H^1_f(F, G(\mathbb{Q}_p))$$

is the inverse map of $\log_G$.

Proof. Since $\log_G$ is injective, it is sufficient to show that the composite $\log_G \circ \exp'_G$ is the identity map. Let us take $h \in H(F)$ and $b \in G(B_e)$ whose image under $G(B_e) \to G(B^+_{\mathrm{dR}})\backslash G(B_{\mathrm{dR}})$ coincides with the image of h. Then, by the definition of $\exp'_G$, the continuous 1-cocycle $c_b\colon \mathscr{G}_F \to G(\mathbb{Q}_p); \sigma \mapsto b\ {}^\sigma(b^{-1})$ represents the cohomology class $\exp'_G(h \mod F^0H_F(F))$. We put $X := G_{c_b}$ (cf. Proof of Proposition 3.7). By definition, the action of $\mathscr{G}_F$ on $X = G_{c_b}$ is defined by $g \mapsto c_b(\sigma)\ {}^\sigma g = b\ {}^\sigma(b^{-1}g)$. Therefore, the left multiplication b^{-1} on $G \times_{\mathrm{Spec}(\mathbb{Q}_p)} \mathrm{Spec}(B_e)$ defines a *$\mathscr{G}_F$-equivariant* isomorphism $X \times_{\mathrm{Spec}(\mathbb{Q}_p)} \mathrm{Spec}(B_e) \xrightarrow{\sim} G \times_{\mathrm{Spec}(\mathbb{Q}_p)} \mathrm{Spec}(B_e)$ of right $G \times_{\mathrm{Spec}(\mathbb{Q}_p)} \mathrm{Spec}(B_e)$-torsors. In other words, the left multiplication by b^{-1} induces the following $\mathscr{G}_F$-equivariant B_{crys}-algebra isomorphism:

$$b^{-1,\sharp}\colon \mathcal{O}(G) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}} \xrightarrow{\sim} \mathcal{O}(X) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}\,. \tag{4.2}$$

Since b is Frobenius invariant, the isomorphism (4.2) is compatible with Frobenius endomorphisms of both hand sides. Furthermore, because b coincides with h modulo $G(B^+_{\mathrm{dR}})$, the ring homomorphism $b^{-1,\sharp}$ induces the natural isomorphism

$$F^i(\mathcal{O}(H_h) \otimes_F B_{\mathrm{dR}}) = h^\sharp F^i(\mathcal{O}(G) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \xrightarrow{\sim} F^i(\mathcal{O}(X) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})$$

for each i. Therefore, by taking $\mathscr{G}_F$-invariant parts of (4.2), we have an isomorphism

$$\mathcal{O}(H_h) \xrightarrow{\sim} \mathcal{O}(D_{\mathrm{crys}}(X))$$

in $\mathrm{Ind}(\mathrm{MF}_F(\varphi))$. Hence, $\log_G$ is the left inverse of $\exp'_G$ that we want to prove. □

Proof of Proposition 3.3 *when H satisfies the condition* (b). Let us suppose that H is an algebraic group in $\mathrm{MF}_F^{\mathrm{ad}}(\varphi)$ satisfying (Non-neg), (Bij), and (Unip). Let Y be a right torsor under H in $\mathrm{MF}_F(\varphi)$ which is (ω_{F_0}, ω_F)-trivial. We will show that Y is an H-torsor *in* $\mathrm{MF}_F^{\mathrm{ad}}(\varphi)$.

Let G be an algebraic group in $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$ such that $D_{\mathrm{crys}}(G) = H$. Since G also satisfies (Unip), $\log_G$ is a bijection (cf. Proposition 4.9). Therefore, by Proposition 3.8 and the definition of $\log_G$, there exists an $(\omega_{\mathbb{Q}_p}, D'_{\mathrm{crys}})$-trivial right torsor X under G in $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$ such that $D_{\mathrm{crys}}(X) \cong Y$ as right H-torsors in $\mathrm{MF}_F(\varphi)$. Since D_{crys} is an equivalence between $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$ and $\mathrm{MF}_F^{\mathrm{ad}}(\varphi)$, $D_{\mathrm{crys}}(X)$ is an affine scheme in $\mathrm{MF}_F^{\mathrm{ad}}(\varphi)$. This implies that Y is also an affine scheme in $\mathrm{MF}_F^{\mathrm{ad}}(\varphi)$ and this completes the proof of the proposition. □

Proposition 4.9 states that $\exp_G$ defined in Definition 13 coincides with $\exp'_G$ when G satisfies (Unip). Hence we always identify these two exponential maps $\exp_G$ and $\exp'_G$ for the rest of the present paper.

5. Path torsors under fundamental groups of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

In this section, we give an analog of the Coleman–Ihara formula which is a different version of [17, Theorem 1.1]. We fix a positive integer N greater than 1.

5.1. **Polylogarithmic quotients.** We denote by π_1 the maximal pro-p quotient of the étale fundamental group $\pi_1^{\text{ét}}(\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}, \overrightarrow{01})$ of the algebraic curve $\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}$. The topological group π_1 has a natural continuous action of $\mathscr{G}_F$ defined by the base point $\overrightarrow{01}$. Let $x, y \in \pi_1$ be the standard generators of π_1 (cf. [23, Sectoion 8, Picture 4]). It is a well-known fact that the equality ${}^{\sigma}x = x^{\chi^{\mathrm{cyc}}(\sigma)}$ holds for each $\sigma \in \mathscr{G}_F$ where $\chi^{\mathrm{cyc}} \colon \mathscr{G}_F \to \mathbb{Z}_p^{\times}$ is the p-adic cyclotomic character. Let $\mathbf{p} \colon \pi_1 \to \mathbb{Z}_p(1)$ be the $\mathscr{G}_F$-equivariant group homomorphism defined by $\mathbf{p}(x) = \zeta_{p^\infty}$, $\mathbf{p}(y) = 1$. Then, we have a canonical $\mathscr{G}_F$-equivariant section $\mathbf{s} \colon \mathbb{Z}_p(1) \to \pi_1; \zeta_{p^\infty} \mapsto x$ of $\mathbf{p}$.

Definition 15.

(1) We define the pro-p group π_1^{pol} to be the quotient $\pi_1/[\mathrm{Ker}\,\mathbf{p}, \mathrm{Ker}\,\mathbf{p}]$ where $[\mathrm{Ker}\,\mathbf{p}, \mathrm{Ker}\,\mathbf{p}]$ is the closed subgroup of π_1 generated by commutators of $\mathrm{Ker}\,\mathbf{p}$. We also denote by $\mathbf{p}$ the homomorphism $\pi_1^{\mathrm{pol}} \to \mathbb{Z}_p(1)$ induced by $\mathbf{p}$ by abuse of notation.

(2) We define *the N-th p-adic étale polylogarithmic quotient* $\mathscr{P}_N^{\text{ét}}(\overrightarrow{01})$ to be the unipotent completion of $\pi_1^{\mathrm{pol}}/\Gamma^{N+1}\pi_1^{\mathrm{pol}}$ over $\mathbb{Q}_p$, where $\Gamma^n\pi_1^{\mathrm{pol}}$ is the central descending series of π_1^{pol} defined by $\Gamma^1\pi_1^{\mathrm{pol}} := \pi_1^{\mathrm{pol}}$ and by $\Gamma^{i+1}\pi_1^{\mathrm{pol}} := [\Gamma^i\pi_1^{\mathrm{pol}}, \pi_1^{\mathrm{pol}}]$.

Remark 8. By construction, $\mathscr{P}_N^{\text{ét}}(\overrightarrow{01})$ is a quotient of $\pi_1^{\text{un-ét}}(\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}, \overrightarrow{01})$ and the p-adic étale realization of the motivic fundamental group $U(\overrightarrow{01})^{(N)}$ in the sense of Deligne (cf. [6, (16.11.2)]).

We recall some fundamental properties of the p-adic étale polylogarithmic quotient.

Lemma 5.1 ([6, Paragraphe 16.11, 16.12])**.**

(1) *The section* $\mathbf{s}$ *induces a* $\mathscr{G}_F$*-equivariant isomorphism*

$$\pi_1^{\mathrm{pol}} \cong \mathbb{Z}_p(1) \ltimes \mathbb{Z}_p[\![\,\mathbb{Z}_p(1)\,]\!](1).$$

Here, the action of $\mathbb{Z}_p(1)$ *on* $\mathbb{Z}_p[\![\,\mathbb{Z}_p(1)\,]\!](1)$ *is induced by translations of* $\mathbb{Z}_p(1)$ *on itself.*

(2) *The Lie algebra of* $\mathscr{P}_N^{\text{ét}}(\overrightarrow{01})$ *is canonically isomorphic to the Lie algebra* $\mathbb{Q}_p(1) \ltimes \prod_{n=1}^N \mathbb{Q}_p(n)$*. Here, the abelian Lie algebra* $\mathbb{Q}_p(1)$ *acts on the abelian Lie algebra* $\prod_{n=1}^N \mathbb{Q}_p(n)$ *via the canonical homomorphism* $\mathbb{Q}_p(1) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n) \xrightarrow{\sim} \mathbb{Q}_p(n+1)$ *for* $1 \leq n \leq N-1$ *and annihilates* $\mathbb{Q}_p(N)$*.*

According to Proposition 4.1, $\mathscr{P}_N^{\text{ét}}(\overrightarrow{01})$ is an algebraic group in the category $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{crys}}(\mathscr{G}_F)$. We introduce the N-th crystalline polylogarithmic quotient.

Definition 16. For any positive integer N and for any finite extension F of $\mathbb{Q}_p$, we define *the crystalline N-th polylogarithmic quotient* $\mathscr{P}_N^{\mathrm{crys}}(\overrightarrow{01})$ to be $D_{\mathrm{crys}}(\mathscr{P}_N^{\text{ét}}(\overrightarrow{01}))$.

By definition, $\mathscr{P}_N^{\mathrm{crys}}(\overrightarrow{01})$ is an algebraic group in $\mathrm{MF}_F^{\mathrm{ord}}(\varphi)$ satisfying (Unip). Furthermore, by Proposition 4.1, Lemma 4.2, and Lemma 4.3, we see that $\mathscr{P}_N^{\mathrm{crys}}(\overrightarrow{01})$ satisfies the conditions (Non-neg) and (Bij). Let k be the residue field of F and let z be an element of $\mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathcal{O}_F)$. We

denote by $\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}; \overrightarrow{01})$ and by $\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}; z, \overrightarrow{01})$ the crystalline fundamental group and the crystalline path torsor, respectively (cf. [10, Definition 1.6]). It is well-known that there exist comparison isomorphisms of F-schemes

$$\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}; \overrightarrow{01}) \otimes_{F_0} F \cong \pi_1^{\mathrm{dR}}(\mathbb{P}^1_F \setminus \{0,1,\infty\}; \overrightarrow{01})$$

and

$$\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}; z, \overrightarrow{01}) \otimes_{F_0} F \cong \pi_1^{\mathrm{dR}}(\mathbb{P}^1_F \setminus \{0,1,\infty\}; z, \overrightarrow{01})$$

(cf. [10, Lemma 2.2]). Therefore, $\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}; \overrightarrow{01})$ has a natural structure of an affine group scheme in $\mathrm{MF}_F(\varphi)$ and $\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}; z, \overrightarrow{01})$ has a natural structure of a right $\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}; \overrightarrow{01})$-torsor in $\mathrm{MF}_F(\varphi)$. Meanwhile, the crystalline polylogarithmic quotient $\mathscr{P}_N^{\mathrm{crys}}(\overrightarrow{01})$ is a quotient of $\pi_1^{\mathrm{crys}}(\mathbb{P}^1_k \setminus \{0,1,\infty\}; \overrightarrow{01})$ by the explicit descriptions of the Frobenius action (cf. [10, Lemma 2.9]) and the Hodge filtration (cf. [6, Paragraphe 13.7]). For the symbol ? = un-ét or crys, we denote by $u_N^?$ the canonical surjective homomorphism $\pi_1^?(\mathbb{P}^1 \setminus \{0,1,\infty\}, \overrightarrow{01}) \twoheadrightarrow \mathscr{P}_N^?(\overrightarrow{01})$. We define the right $\mathscr{P}_N^?(\overrightarrow{01})$-torsor $\mathscr{P}_N^?(z; \overrightarrow{01})$ to be the pushforward of the path torsor $\pi_1^?(\mathbb{P}^1 \setminus \{0,1,\infty\}; z, \overrightarrow{01})$ by $u_N^?$. According to Corollary 1, we have the following proposition.

Proposition 5.2. *Let $\mathscr{MT}_F^{\mathrm{ad}}(\varphi)$ be the minimal Tannakian subcategory of $\mathrm{MF}_F^{\mathrm{ad}}(\varphi)$ containing $F_0\langle n\rangle$ for all $n \in \mathbb{Z}$ closed under extensions. Then, for each point $z \in \mathbb{P}^1 \setminus \{0,1,\infty\}(\mathcal{O}_F)$, $\mathscr{P}_N^{\mathrm{crys}}(z; \overrightarrow{01})$ is a right $\mathscr{P}_N^{\mathrm{crys}}(\overrightarrow{01})$-torsor in $\mathscr{MT}_F^{\mathrm{ad}}(\varphi)$.*

In the last of this subsection, we recall two unipotent Albanese maps defined by Minhyong Kim.

Definition 17 (cf. [14, Introduction])**.**

(1) We define *the étale Albanese map*

$$\mathrm{Alb}_{F,N}^{\text{ét}} \colon \mathbb{P}^1 \setminus \{0,1,\infty\}(F) \to H^1(\mathrm{Rep}_{\mathbb{Q}_p}(\mathscr{G}_F), \mathscr{P}_N^{\text{ét}}(\overrightarrow{01})) \cong H^1(F, \mathscr{P}_N^{\text{ét}}(\overrightarrow{01})(\mathbb{Q}_p))$$

by $\mathrm{Alb}_{F,N}^{\text{ét}}(z) := [\mathscr{P}_N^{\text{ét}}(z; \overrightarrow{01})]$.

(2) We define *the crystalline-de Rham Albanese map*

$$\mathrm{Alb}_{F,N}^{\text{cr-dR}} \colon \mathbb{P}^1 \setminus \{0,1,\infty\}(\mathcal{O}_F) \to H^1(\mathscr{MT}_F^{\mathrm{ad}}(\varphi), \mathscr{P}_N^{\mathrm{crys}}(\overrightarrow{01})) \cong \mathscr{P}_N^{\mathrm{crys}}(\overrightarrow{01})(F)$$

by $\mathrm{Alb}_{F,N}^{\text{cr-dR}}(z) := [\mathscr{P}_N^{\mathrm{crys}}(z; \overrightarrow{01})]$.

5.2. **Computations of Galois L-functions.** Let K be a subfield[2] of $\mathbb{C}$. In the paper [16], Nakamura and Wojtkowiak studied a measure valued function

$$\kappa_z \colon \mathscr{G}_K \to \mathrm{Meas}(\mathbb{Z}_p, \mathbb{Z}_p(1))(1) = \mathbb{Z}_p \, [\![\, \mathbb{Z}_p(1) \,]\!](1) \subset \pi_1^{\mathrm{pol}}$$

for each $z \in K$ (cf. [24, Proposition 2.3])[3]. In this subsection, we calculate integrations of this measure when K is a finite extension F of $\mathbb{Q}_p$. By the construction of κ_z, we have the following equality:

Lemma 5.3. *Let z be an element of $\mathbb{P}^1 \setminus \{0,1,\infty\}(\mathcal{O}_F)$. Then, for each $\sigma \in \mathscr{G}_{F(\zeta_{p^\infty})}$, we have the following equation in $\mathscr{P}_N^{\text{ét}}(\overrightarrow{01})(\mathbb{Q}_p) \cong \mathbb{Q}_p(1) \ltimes \prod_{n=1}^N \mathbb{Q}_p(n)$ up to inner automorphisms of $\mathscr{P}_N^{\text{ét}}(\overrightarrow{01})(\mathbb{Q}_p)$:*

(5.1)

$$\mathrm{Alb}_{F,N}^{\text{ét}}(z)(\sigma) = \left(\kappa_{0,z}(\sigma); \left(\frac{1}{(n-1)!} \left\{ \int_{\mathbb{Z}_p} x^{n-1} d\kappa_z(\sigma)(x) \right\} \zeta_{p^\infty}^{\otimes n} \right)_{n=1}^{N} \right)$$

where $\kappa_{0,z}$ is the Kummer character associated to z.

Next, we recall that $\mathrm{Alb}_{F,N}^{\text{cr-dR}}$ is calculated by using Coleman's p-adic polylogarithms $\mathrm{Li}_n^{p\text{-adic}}(z)$.

Theorem 5.4 (A special case of [10, Theorem 2.3])**.** *For each $\mathcal{O}_F$-valued point z of $\mathbb{P}^1 \setminus \{0,1,\infty\}$, we have the following equality in $\mathscr{P}_N^{\text{crys}}(\overrightarrow{01})(F) \cong F\frac{\zeta_{p^\infty}}{t} \ltimes \prod_{n=1}^N F\frac{\zeta_{p^\infty}^{\otimes n}}{t^n}$:*

$$\mathrm{Alb}_{F,N}^{\text{cr-dR}}(z) = \left(\log_p(z) \frac{\zeta_{p^\infty}}{t}; \left(-\mathrm{Li}_n^{p\text{-adic}}(z) \frac{\zeta_{p^\infty}^{\otimes n}}{t^n} \right)_{n=1}^{N} \right). \tag{5.2}$$

Here, $\log_p$ is the Iwasawa p-adic logarithm.

The following lemma is a direct consequence of Theorem 3.10.

Lemma 5.5. *We have the following exact sequence of pointed sets*:

(5.3)

$$1 \to H_f^1(F, \mathbb{Q}_p(N)) \to H_f^1(F, \mathscr{P}_N^{\text{ét}}(\overrightarrow{01})(\mathbb{Q}_p)) \xrightarrow{\mathrm{pr}_N} H_f^1(F, \mathscr{P}_{N-1}^{\text{ét}}(\overrightarrow{01})(\mathbb{Q}_p)) \to 1.$$

The abelian group $H_f^1(F, \mathbb{Q}_p(N))$ acts on $H_f^1(F, \mathscr{P}_N^{\text{ét}}(\overrightarrow{01})(\mathbb{Q}_p))$ whose action is compatible with the generalized exponential maps. We denote the action of $c \in H_f^1(F, \mathbb{Q}_p(N))$ on $c' \in H_f^1(F, \mathscr{P}_N^{\text{ét}}(\overrightarrow{01})(\mathbb{Q}_p))$ by $c * c'$. For all

[2] In the paper [16], they assumed that K is a number field. However, we can construct κ_z for an arbitrary $z \in K \subset \mathbb{C}$ by exactly the same way.

[3] More precisely, we need to fix an étale path γ from $\overrightarrow{01}$ to z and we should write $\kappa_{z,\gamma}$.

$c_1, c_2 \in H^1_f(F, \mathscr{P}^{\text{ét}}_N(\overrightarrow{01})(\mathbb{Q}_p))$ *satisfying* $\mathrm{pr}_N(c_1) = \mathrm{pr}_N(c_2)$*, there exists a unique element* $c_3 \in H^1_f(F, \mathbb{Q}_p(N))$ *such that* $c_1 = c_3 * c_2$.

Proof. Note that $F^0(\mathscr{P}^{\text{crys}}_n \times_{\mathrm{Spec}(F_0)} \mathrm{Spec}(F)) = 1$ and $D^0_{\mathrm{dR}}(\mathbb{Q}_p(n)) = 0$ for any $n \in \mathbb{Z}_{>0}$. Therefore, by Theorem 3.10, we have the following commutative diagram with exact rows:

(5.4)

$$\begin{array}{ccccc} D_{\mathrm{dR}}(\mathbb{Q}_p(N)) & \longrightarrow & \mathscr{P}^{\text{crys}}_N(\overrightarrow{01})(F) & \longrightarrow & \mathscr{P}^{\text{crys}}_{N-1}(\overrightarrow{01})(F) \\ \cong\downarrow{\scriptstyle \exp_{\mathbb{Q}_p(N)}} & & \cong\downarrow{\scriptstyle \exp_{\mathscr{P}^{\text{ét}}_N(\overrightarrow{01})}} & & \cong\downarrow{\scriptstyle \exp_{\mathscr{P}^{\text{ét}}_{N-1}(\overrightarrow{01})}} \\ H^1_f(F, \mathbb{Q}_p(N))) & \longrightarrow & H^1_f(F, \mathscr{P}^{\text{ét}}_N(\overrightarrow{01})(\mathbb{Q}_p)) & \longrightarrow & H^1_f(F, \mathscr{P}^{\text{ét}}_{N-1}(\overrightarrow{01})(\mathbb{Q}_p)). \end{array}$$

It is clear that the first (resp. the last) map in the upper sequence is injective (resp. surjective). Thus, we obtain the short exact sequence (5.3). The rest of assertions follow from the commutative diagram (5.4) and the compatibility of the actions of the abelian groups $D_{\mathrm{dR}}(\mathbb{Q}_p(N))$ and $H^1_f(F, \mathbb{Q}_p(N)))$ on $\mathscr{P}^{\text{crys}}_N(\overrightarrow{01})(F)$ and $H^1_f(F, \mathscr{P}^{\text{ét}}_N(\overrightarrow{01})(\mathbb{Q}_p))$, respectively (cf. Theorem 3.10). □

The following proposition is one of the key propositions of [17].

Proposition 5.6 ([17, Proposition 5.1 (2)])**.** *For each positive integer* N *and for each unramified extension* F *of* $\mathbb{Q}_p$*, the composite of two maps*

$$\exp_{\mathscr{P}^{\text{ét}}_N(\overrightarrow{01})} \circ \mathrm{Alb}^{\text{cr-dR}}_{F,N} \colon \mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathcal{O}_F) \to \mathscr{P}^{\text{crys}}_N(\overrightarrow{01})(F) \xrightarrow{\sim} H^1_f(F, \mathscr{P}^{\text{ét}}_N(\overrightarrow{01})(\mathbb{Q}_p))$$

coincides with $\mathrm{Alb}^{\text{ét}}_{F,N}$.

For describing the integrations of the measure κ_z, we introduce a concept of standard lifting of an element of $H^1_f(F, \mathscr{P}^{\text{ét}}_{N-1}(\overrightarrow{01})(\mathbb{Q}_p))$.

Definition 18.

(1) Let c be an element of the finite part $H^1_f(F, \mathscr{P}^{\text{ét}}_{N-1}(\overrightarrow{01})(\mathbb{Q}_p))$ and write $\log_{\mathscr{P}^{\text{ét}}_{N-1}(\overrightarrow{01})}(c) = (d_0, d_1, \ldots, d_{N-1}) \in \mathscr{P}^{\text{crys}}_{N-1}(\overrightarrow{01})(F)$. Then, we define the standard lifting of c to be $\exp_{\mathscr{P}^{\text{ét}}_N(\overrightarrow{01})}((d_0, d_1, \ldots, d_{N-1}, 0)) \in H^1_f(F, \mathscr{P}^{\text{ét}}_N(\overrightarrow{01})(\mathbb{Q}_p))$ and denote it by $\tilde{c}$.

(2) For each $z \in \mathbb{P}^1 \setminus \{0, 1, \infty\}(\mathcal{O}_F)$, we define the field $\widetilde{F}(z, N) \supset F$ to be the kernel field of the standard lifting of $\mathrm{pr}_N\left(\mathrm{Alb}^{\text{ét}}_{F,N}(z)\right)$.

Lemma 5.7. *Let* $c \in H^1_f(F, \mathscr{P}^{\text{ét}}_N(\overrightarrow{01})(\mathbb{Q}_p))$ *and we denote* $\log_{\mathscr{P}^{\text{ét}}_N(\overrightarrow{01})}(c)$ *by* $(d_0, \ldots, d_N)$*. Then, we have* $c = \exp_{\mathbb{Q}_p(N)}(d_N) * \widetilde{\mathrm{pr}_N(c)}$.

Proof. It follows from Lemma 5.5 and the definition of the standard lifting. □

Theorem 5.8. *Let F be a finite unramified extension of $\mathbb{Q}_p$ and let z be an element of $\mathbb{P}^1\setminus\{0,1,\infty\}(\mathcal{O}_F)$. Let N be a positive integer greater than 1. Then, for each $\sigma\in\mathscr{G}_{\widetilde{F}(z,N)}$, the following formula holds*:

$$\frac{1}{(N-1)!}\int_{\mathbb{Z}_p}x^{N-1}d\kappa_z(\sigma)(x)=\frac{-1}{(N-1)!}\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\left\{\left(1-\frac{\sigma_F}{p^N}\right)\mathrm{Li}_N^{p\text{-adic}}(z)\right\}\phi_{F,N}^{CW}(\sigma)\right).$$

Proof. According to Theorem 5.4 and Lemma 5.7, we have

$$\mathrm{Alb}_{F,N}^{\text{ét}}(z)=\exp_{\mathbb{Q}_p(N)}\left(-\mathrm{Li}_N^{p\text{-adic}}(z)\frac{\zeta_{p^\infty}^{\otimes N}}{t^N}\right)*\mathrm{pr}_N(\widetilde{\mathrm{Alb}_{F,N}^{\text{ét}}(z)}).$$

Since $\widetilde{F}(z,N)$ is the kernel filed of $\mathrm{pr}_N(\widetilde{\mathrm{Alb}_{F,N}^{\text{ét}}(z)})$, the formula

$$\mathrm{Alb}_{F,N}^{\text{ét}}(z)(\sigma)=\exp_{\mathbb{Q}_p(N)}\left(-\mathrm{Li}_N^{p\text{-adic}}(z)\frac{\zeta_{p^\infty}^{\otimes N}}{t^N}\right)(\sigma)$$

holds for any $\sigma\in\mathscr{G}_{\widetilde{F}(z,N)}$. Therefore, we see the conclusion of Theorem 5.8 by Theorem 4.7 and Lemma 5.3. □

Corollary 2. Under the same setting as in Theorem 5.8, we have the following formula for each $\sigma\in\mathscr{G}_{\widetilde{F}(z,N)}$:

$$\ell i_N(z)(\sigma)=\frac{-1}{(N-1)!}\mathrm{Tr}_{F/\mathbb{Q}_p}\left(\left\{\left(1-\frac{\sigma_F}{p^N}\right)\mathrm{Li}_N^{p\text{-adic}}(z)\right\}\phi_{F,N}^{CW}(\sigma)\right). \tag{5.5}$$

Here, $\ell i_N(z)$ is the N-th ℓ-adic polylogarithm (cf. [16]).

Proof. In the paper [16], the following formula was proved:

$$\ell i_N(z)(\sigma)=\frac{1}{(N-1)!}\int_{\mathbb{Z}_p}x^{N-1}\kappa_z(\sigma)(x),\ \text{for any }\sigma\in\mathscr{G}_{\widetilde{F}(z,N)}. \tag{5.6}$$

Hence, by Theorem 5.8, we obtain the formula (5.5). □

Acknowledgement. The author would like to thank Professor Hiroaki Nakamura for valuable discussions and encouragement. He would also like to thank Professor Seidai Yasuda for discussions about classifying spaces of torsors in $\mathrm{MF}_F^{\mathrm{ad}}(\varphi)$. He thank Doctor Sho Ogaki for careful reading of an earlier version of the present manuscript. Finally, he would like to express his gratitude to the referee for a careful reading and many helpful comments.

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(*Received March 18, 2015*)
(*Accepted June 29, 2016*)