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HOPF PERIODIC ORBITS FOR A RATIO–DEPENDENT PREDATOR–PREY MODEL WITH STAGE STRUCTURE

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Abstract. A ratio–dependent predator-prey model with stage structure for prey was investigated in [8]. There the authors mentioned that they were unable to show if such a model admits limit cycles when the unique equilibrium point E^* at the positive octant is unstable.

Here we characterize the existence of Hopf bifurcations for the systems. In particular we provide a positive answer to the above question showing for such models the existence of small–amplitude Hopf limit cycles being the equilibrium point E^* unstable.

1. **Introduction and statement of the main results.** Frequently it is assumed in the ratio–dependent predator-prey models that each individual prey admits the same risk to be attacked by a predator. This hypothesis is not always realistic for many species. Thus there are many species whose individuals along his life pass through two stages, immature and mature. Here the prey individuals are classified in immature or mature, and we assume that the immature ones cannot be attacked by the predators. This assumption is reasonable for many mammals, because the immature preys concealed in a mountain cave, are raised by their parents and they do not necessarily go out for seeking food, so the possibility of being attacked by the predators is negligible.

Stage structured models have been studied with attention in these last years. Thus a stage–structured model of single species growth with of immature and mature individuals was stated and analyzed in [1]. Later on in [2] it was also assumed that the time from immaturity to maturity is itself state dependent. More recently, in the articles [4, 5, 6, 7, 9] the authors considered predator–prey models with stage structure for prey or predator in order to analyze the influence of a stage structure for the prey or the predator. Xu, Chaplain and Davidson [8] studied the effect of

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stage structure for prey on the dynamics of ratio–dependent predator-prey system by considering the differential system

$$\begin{aligned} \dot{x} &= -(b+r_1)x + ay, \\ \dot{y} &= bx - b_1 y^2 - \frac{a_1 yz}{y+mz}, \\ \dot{z} &= z\left(-r + \frac{a_2 y}{y+mz}\right), \end{aligned} \tag{1}$$

where $x = x(t)$ represents the density of immature individual preys at time t, and $y = y(t)$ denotes the density of mature individual preys at time t, $z = z(t)$ represents the density of the predator at time t. As usual the eight parameters a, a_1, a_2, b, b_1, m, r and r_1 of the system are positive, see [8] for their meaning.

When $y^* > 0$ and $a_2 > r$ the differential system (1) has a unique equilibrium point

$$E^* = \left(\frac{ay^*}{b+r_1}, y^*, \frac{(a_2-r)y^*}{mr}\right) \quad \text{with} \quad y^* = \frac{aa_2bm - a_1(a_2-r)(b+r_1)}{a_2b_1m(b+r_1)},$$

in the positive octant, i.e., in $\{(x,y,z) \in \mathbb{R}^3 : x>0,\ y>0,\ z>0\}$.

In the conclusions of the paper [8] the authors said: *We would like to mention here that we are unable to show system* (1) *admits limit cycles when* E^* *exists and is unstable. We leave this for future work.* As far as we know this question remained open. The main difficulty for solving it was that the differential system (1) depends on 8 parameters.

Here we shall characterize the existence of Hopf bifurcations for systems (1). Moreover using the Hopf bifurcation for these systems we provide a positive answer to the above question restricted to the small–amplitude Hopf limit cycles.

The following result characterizes when the equilibrium point E^* has eigenvalues of the form ρ, $\varepsilon \pm \omega i$.

Proposition 1. *The equilibrium point* E^* *of the differential system* (1) *has eigenvalues of the form* ρ, $\varepsilon \pm \omega i$ *if and only if*

$$\begin{aligned} a &= \frac{\beta(b+r_1)(a_2^2r^2 - 2a_2r^3 + r^4 + 2a_2^2r\varepsilon - 2a_2r^2\varepsilon + a_2^2\varepsilon^2 + a_2r\varepsilon^2 + a_2^2\omega^2 + a_2r\omega^2)}{\alpha b}, \\ a_1 &= \frac{\beta a_2 m(a_2^2r^2 - 2a_2r^3 + r^4 + 2a_2^2r\varepsilon - 2a_2r^2\varepsilon + a_2^2\varepsilon^2 + a_2^2\omega^2)}{\alpha(a_2-r)}, \end{aligned}$$

where

$$\begin{aligned} \alpha = \ & -a_2^2br^2 + a_2b^2r^2 + 2a_2br^3 - b^2r^3 - br^4 - a_2^2r^2r_1 + 2a_2br^2r_1 + 2a_2r^3r_1 \\ & -2br^3r_1 - r^4r_1 + a_2r^2r_1^2 - r^3r_1^2 - 2a_2^2br\varepsilon + 4a_2br^2\varepsilon - 2br^3\varepsilon - 2a_2^2rr_1\varepsilon \\ & +4a_2r^2r_1\varepsilon - 2r^3r_1\varepsilon - a_2^2b\varepsilon^2 - a_2br\varepsilon^2 + 2a_2r^2\varepsilon^2 - 2r^3\varepsilon^2 - a_2^2r_1\varepsilon^2 \\ & -a_2rr_1\varepsilon^2 - a_2^2b\omega^2 - a_2br\omega^2 + 2a_2r^2\omega^2 - 2r^3\omega^2 - a_2^2r_1\omega^2 - a_2rr_1\omega^2, \\ \beta = \ & b^2 + 2br_1 + r_1^2 + 2b\varepsilon + 2r_1\varepsilon + 2\varepsilon^2 + 2\omega^2. \end{aligned}$$

Proposition 1 is proved in section 2.

Hypothesis. In the rest of the paper we assume that the parameters a and a_1 are the ones given in the statement of Proposition 1.

Since $a = a(b,m,r,r_1,\varepsilon,\omega)$ and $a_1 = a_1(a_2,b,m,r,r_1,\varepsilon,\omega)$, the parameters a_2, b, m, r, r_1, ε and ω must be chosen in such a way that $a > 0$ and $a_1 > 0$, later on we shall show that such election of the parameters exist. Note that in fact we have changed the parameters a and a_1 by the parameters ε and ω.

The linearization of the differential system (1) at E^* when $\varepsilon = 0$ has the pair of conjugate purely imaginary eigenvalues $\pm\omega i$, and if the other one eigenvalues ρ is non–zero this is the setting for a Hopf bifurcation. We can expect to see a small–amplitude limit cycle bifurcating from the equilibrium point E^*. It remains to compute the first Liapunov coefficient $\ell_1(E^*|_{\varepsilon=0})$ of (1) at E^* when $\varepsilon = 0$ and see that it is non–zero. When $\ell_1(E^*|_{\varepsilon=0}) < 0$ the point E^* is a weak focus on the corresponding two–dimensional central manifold of the system (1) and the limit cycle that borns from E^* is stable restricted to this central manifold, consequently the weak focus becomes unstable restricted to the central surface. Due to the fact that the eigenvalues of E^* are ρ and $\varepsilon \pm \omega i$ where the small–amplitude limit cycle which appears in the Hopf bifurcation is stable, it follows that $\varepsilon > 0$. In this case we say that the Hopf bifurcation is supercritical. When $\ell_1(E^*|_{\varepsilon=0}) > 0$ the point E^* is also a weak focus of system (1) but the limit cycle that borns from E^* is unstable, and the weak focus becomes stable, and consequently $\varepsilon < 0$. In this second case we say that the Hopf bifurcation is subcritical. Here we use the results on the Hopf bifurcation stated in Chapters 3 and 5 of the book of Kuznetsov [3] for computing $\ell_1(E^*|_{\varepsilon=0})$.

We have computed in function of all parameters of the system the Liapunov coefficient $\ell_1(E^*|_{\varepsilon=0})$, but his expression is so huge that we shall need tens of pages for writing it. All the steps of the computation of $\ell_1(E^*|_{\varepsilon=0})$ are described in section 2. These steps as we shall see can be used for computing effectively $\ell_1(E^*|_{\varepsilon=0})$ for a given set of parameters. Using them we have proved the following result.

Theorem 2. *The following system ratio–dependent predator–prey model with stage structure*

$$\begin{aligned} \dot{x} &= -7x + \frac{7y\left(\varepsilon^2+14\varepsilon+50\right)\left(6\varepsilon^2+4\varepsilon+7\right)}{-40\varepsilon^2-14\varepsilon+2}, \\ \dot{y} &= x - y^2 + \frac{2z\left(2\varepsilon^2+14\varepsilon+51\right)\left(4\varepsilon^2+4\varepsilon+5\right)y}{(y+z)\left(40\varepsilon^2+14\varepsilon-2\right)}, \\ \dot{z} &= z\left(-1+\frac{2y}{y+z}\right), \end{aligned} \tag{2}$$

has a Hopf bifurcation at the equilibrium point $E^* = (x^*, y^*, z^*)$ where

$$x^* = \frac{\left(-2\varepsilon^2+24\varepsilon+95\right)\left(\varepsilon^2+1\right)\left(\varepsilon^2+14\varepsilon+50\right)\left(6\varepsilon^2+4\varepsilon+7\right)}{4\left(20\varepsilon^2+7\varepsilon-1\right)^2} = \frac{16625}{2} + \mathcal{O}(\varepsilon),$$

$$y^* = z^* = \frac{\left(-2\varepsilon^2+24\varepsilon+95\right)\left(\varepsilon^2+1\right)}{2\left(-20\varepsilon^2-7\varepsilon+1\right)} = \frac{95}{2} + \mathcal{O}(\varepsilon),$$

when $\varepsilon = 0$. The eigenvalues at E^* are

$$-\frac{7\left(2\varepsilon^2-24\varepsilon-95\right)}{4\left(20\varepsilon^2+7\varepsilon-1\right)} = -\frac{665}{4} + \mathcal{O}(\varepsilon) \quad \text{and} \quad \varepsilon \pm i.$$

Since $\ell_1(E^*|_{\varepsilon=0}) = -5796656446586/592156593152863031275 < 0$ the Hopf bifurcation is supercritical and an stable Hopf limit cycle on the central manifold through E^* there exist for $\varepsilon > 0$ sufficiently small. So locally the equilibrium point E^* is unstable on the central manifold when it exhibits the small–amplitude Hopf limit cycle.

Theorem 2 is proved in Section 4.

We also have the following result.

Proposition 3. *Assume that a Hopf limit cycle borns at the equilibrium point E^* of the differential system (1), then the real eigenvalue at this equilibrium is negative.*

Proposition 3 is also proved in Section 4.

2. **Proof of Proposition 1.** The characteristic polynomial of the linear part of the differential system (1) at the equilibrium point E^* is

$$p(\lambda) = \lambda^3 + d_2\lambda^2 + d_1\lambda + d_0,$$

where

$$\begin{aligned} d_2 = \;& \frac{1}{a_2^2 m(b+r_1)}\big(-a_1a_2^2b + 2aa_2^2bm + a_2^2b^2m + a_2^2bmr + a_1br^2 - a_2bmr^2 \\ & -a_1a_2^2r_1 + 2a_2^2bmr_1 + a_2^2mrr_1 + a_1r^2r_1 - a_2mr^2r_1 + a_2^2mr_1^2\big), \\ d_1 = \;& \frac{1}{a_2^2 m(b+r_1)}\big(-a_1a_2^2b^2 + aa_2^2b^2m - a_1a_2^2br + 2aa_2^2bmr + a_2^2b^2mr \\ & +2a_1a_2br^2 + a_1b^2r^2 - 2aa_2bmr^2 - a_2b^2mr^2 - a_1br^3 - 2a_1a_2^2br_1 \\ & +aa_2^2bmr_1 - a_1a_2^2rr_1 + 2a_2^2bmrr_1 + 2a_1a_2r^2r_1 + 2a_1br^2r_1 - 2a_2bmr^2r_1 \\ & -a_1r^3r_1 - a_1a_2^2r_1^2 + a_2^2mrr_1^2 + a_1r^2r_1^2 - a_2mr^2r_1^2\big), \\ d_0 = \;& -\frac{r(-a_2+r)(-a_1a_2b + aa_2bm + a_1br - a_1a_2r_1 + a_1rr_1)}{a_2^2m}. \end{aligned}$$

Forcing that the characteristic polynomial $p(\lambda)$ be equal to $(\lambda-\rho)(\lambda-\varepsilon-\omega i)(\lambda-\varepsilon+\omega i)$, we get the expressions for a and a_1 given in the statement of Proposition 1. Moreover

$$\begin{aligned} \rho = \;& \frac{(a_2-r)r(b+r_1)}{\alpha a_2}\big(-a_2b^2r + a_2^2r^2 - 2a_2r^3 + r^4 - 2a_2brr_1 - a_2rr_1^2 \\ & +2a_2^2r\varepsilon - 2a_2br\varepsilon - 2a_2r^2\varepsilon - 2a_2rr_1\varepsilon + a_2^2\varepsilon^2 - a_2r\varepsilon^2 + a_2^2\omega^2 - a_2r\omega^2\big). \end{aligned}$$

Hence, the proposition follows.

3. **The computation of the Liapunov coefficient.** The following result comes from [3].

Theorem 4. *Let $\dot{x} = F(x)$ be a differential system in $\mathbb{R}^n$ having E^* as an equilibrium point. Consider the third order Taylor approximation of F around E^* given by $F(x) = Ax + \frac{1}{2!}B(x,x) + \frac{1}{3!}C(x,x,x) + \mathcal{O}(|x|^4)$. Assume that A has a pair of purely imaginary eigenvalues $\pm\omega i$. Let q be the eigenvector of A corresponding to the eigenvalue ωi, normalized so that $\overline{q}.q = 1$, where $\overline{q}$ is the conjugate vector of q. Let p be the adjoint eigenvector such that $A^Tp = -\omega ip$ and $\overline{p}.q = 1$. If I denotes the $n\times n$ identity matrix, then $\ell_1(E^*|_{\varepsilon=0})$ is equal to*

$$\frac{1}{2\omega}Re(\overline{p}\cdot C(q,q,\overline{q}) - 2\overline{p}\cdot B(q, A^{-1}B(q,\overline{q})) + \overline{p}\cdot B(\overline{q}, (2\omega iI - A)^{-1}B(q,q))). \quad (3)$$

The linear part of system (1) at the equilibrium E^* is

$$A = \begin{pmatrix} -b-r_1 & a & 0 \\ b & \dfrac{a_1(a_2^2-r^2)(b+r_1) - 2aa_2^2bm}{a_2^2m(b+r_1)} & -\dfrac{a_1r^2}{a_2^2} \\ 0 & \dfrac{(a_2-r)^2}{a_2m} & \dfrac{r(r-a_2)}{a_2} \end{pmatrix},$$

and its eigenvalues are $\varepsilon \pm \omega i$ and $\rho = -\delta(a_2 - r)r(b + r_1)/(\sigma a_2)$, where

$$\begin{aligned}
\delta = \; & r^4 - 2a_2r^3 + a_2^2r^2 - 2a_2\varepsilon r^2 - a_2b^2r - a_2r_1^2r - a_2\varepsilon^2r - a_2\omega^2r - 2a_2br_1r + \\
& 2a_2^2\varepsilon r - 2a_2b\varepsilon r - 2a_2r_1\varepsilon r + a_2^2\varepsilon^2 + a_2^2\omega^2, \\
\sigma = \; & br^4 + r_1r^4 + b^2r^3 + r_1^2r^3 + 2\varepsilon^2r^3 + 2\omega^2r^3 - 2a_2br^3 - 2a_2r_1r^3 + 2br_1r^3 \\
& +2b\varepsilon r^3 + 2r_1\varepsilon r^3 - a_2b^2r^2 - a_2r_1^2r^2 - 2a_2\varepsilon^2r^2 - 2a_2\omega^2r^2 + a_2^2br^2 + a_2^2r_1r^2 \\
& -2a_2br_1r^2 - 4a_2b\varepsilon r^2 - 4a_2r_1\varepsilon r^2 + a_2b\varepsilon^2r + a_2r_1\varepsilon^2r + a_2b\omega^2r + a_2r_1\omega^2r \\
& +2a_2^2b\varepsilon r + 2a_2^2r_1\varepsilon r + a_2^2b\varepsilon^2 + a_2^2r_1\varepsilon^2 + a_2^2b\omega^2 + a_2^2r_1\omega^2.
\end{aligned}$$

Now we compute the bi– and tri–linear functions B and C of Theorem 4, and we obtain

$$B(x_1, y_1, z_1, x_2, y_2, z_2) = \big(B_1, B_2, B_3\big),$$

where $B_i = B_i(x_1, y_1, z_1, x_2, y_2, z_2)$ are

$$\begin{aligned}
B_1 = \; & 0, \\
B_2 = \; & -\frac{b_1}{\eta a_2^2}\big(a_1a_2^3by_1y_2 - aa_2^3bmy_1y_2 - 2a_1a_2br^2y_1y_2 + a_1br^3y_1y_2 + a_1a_2^3r_1y_1y_2 \\
& -2a_1a_2r^2r_1y_1y_2 + a_1r^3r_1y_1y_2 - 2a_1a_2bmr^2y_2z_1 + 2a_1bmr^3y_2z_1 \\
& -2a_1a_2mr^2r_1y_2z_1 + 2a_1mr^3r_1y_2z_1 - 2a_1a_2bmr^2y_1z_2 + 2a_1bmr^3y_1z_2 \\
& -2a_1a_2mr^2r_1y_1z_2 + 2a_1mr^3r_1y_1z_2 + a_1bm^2r^3z_1z_2 + a_1m^2r^3r_1z_1z_2\big), \\
B_3 = \; & \frac{b_1r(b + r_1)}{\eta a_2}\big(a_2^2y_1y_2 - 2a_2ry_1y_2 + r^2y_1y_2 - 2a_2mry_2z_1 + 2mr^2y_2z_1 \\
& -2a_2mry_1z_2 + 2mr^2y_1z_2 + m^2r^2z_1z_2\big), \\
\eta = \; & a_1a_2b - aa_2bm - a_1br + a_1a_2r_1 - a_1rr_1;
\end{aligned}$$

and

$$C(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) = \big(C_1, C_2, C_3\big),$$

where $C_i = C_i(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3)$ are

$$\begin{aligned}
C_1 = \; & 0, \\
C_2 = \; & -\frac{a_1b_1^2m(a_2 - r)r^2(b + r_1)^2}{\eta^2a_2^2}\big(a_2y_1y_2y_3 - ry_1y_2y_3 + a_2my_2y_3z_1 \\
& -3mry_2y_3z_1 + a_2my_1y_3z_2 - 3mry_1y_3z_2 + a_2my_1y_2z_3 - 3mry_1y_2z_3\big), \\
C_3 = \; & \frac{a_2}{a_1}C_2.
\end{aligned}$$

Computing the normalized eigenvector q of A when $\varepsilon = 0$, associated to the eigenvalue ωi, we obtain $q = (q_1, q_2, q_3)$ where

$$\begin{aligned}
q_1 &= \frac{im(b + r_1)(ib + ir_1 + \omega)(a_2r - r^2 + ia_2\omega)(a_2^2r^2 - 2a_2r^3 + r^4 + a_2^2\omega^2 + a_2r\omega^2)}{-\alpha_0\mu b(a_2 - r)^2}, \\
q_2 &= \frac{m(a_2r - r^2 + ia_2\omega)}{\mu(a_2 - r)^2}, \\
q_3 &= \frac{a}{\mu},
\end{aligned}$$

where $\alpha_0 = \alpha|_{\varepsilon=0}$ and

$$\begin{aligned}
\mu = \; & \Big[1 + \frac{m^2(a_2^2r^2 - 2a_2r^3 + r^4 + a_2^2\omega^2)}{(a_2 - r)^4} + \frac{m^2(b + r_1)^2}{\alpha_0^2b^2(a_2 - r)^4}(b^2 + 2br_1 + r_1^2 + \omega^2) \\
& (a_2^2r^2 - 2a_2r^3 + r^4 + a_2^2\omega^2)(a_2^2r^2 - 2a_2r^3 + r^4 + a_2^2\omega^2 + a_2r\omega^2)^2\Big]^{1/2}.
\end{aligned}$$

The adjoint eigenvector $p = (p_1, p_2, p_3)$ is

$$p_1 = \frac{b\nu}{b + r_1 - i\omega},$$
$$p_2 = \nu,$$
$$p_3 = \frac{(mr^2(a_2 r - r^2 + ia_2\omega)(b^2 + 2br_1 + r_1^2 + 2\omega^2)\nu}{\alpha_0(a_2 - r)},$$

where

$$\nu = \frac{\alpha_0 \mu (a_2 - r)^2 (b + r_1 + i\omega)}{2m\omega\nu_1},$$

with

$$\begin{aligned}\nu_1 = \; & -ia_2^2b^3r^2 + ia_2^3br^3 + ia_2b^3r^3 - 3ia_2^2br^4 + 3ia_2br^5 - ibr^6 - 3ia_2^2b^2r^2r_1 \\ & +ia_2^3r^3r_1 + 3ia_2b^2r^3r_1 - 3ia_2^2r^4r_1 + 3ia_2r^5r_1 - ir^6r_1 - 3ia_2^2br^2r_1^2 \\ & +3ia_2br^3r_1^2 - ia_2^2r^2r_1^3 + ia_2r^3r_1^3 - a_2^3br^2\omega + a_2^2b^2r^2\omega + 2a_2^2br^3\omega \\ & -a_2b^2r^3\omega - a_2br^4\omega - a_2^3r^2r_1\omega + 2a_2^2br^2r_1\omega + 2a_2^2r^3r_1\omega - 2a_2br^3r_1\omega \\ & -a_2r^4r_1\omega + a_2^2r^2r_1^2\omega - a_2r^3r_1^2\omega + ia_2^3br\omega^2 - 2ia_2^2br^2\omega^2 + ia_2br^3\omega^2 \\ & +ia_2^3rr_1\omega^2 - 2ia_2^2r^2r_1\omega^2 + ia_2r^3r_1\omega^2 - a_2^3b\omega^3 - a_2^2br\omega^3 + 2a_2^2r^2\omega^3 \\ & -2a_2r^3\omega^3 - a_2^3r_1\omega^3 - a_2^2rr_1\omega^3.\end{aligned}$$

In short, all the elements appearing in Theorem 4 are given explicitly for the differential system (1), only remains to compute the Liapunov coefficient $\ell_1(E^*|_{\varepsilon=0})$ using formula (3). As we mentioned in the introduction we have computed explicitly $\ell_1(E^*|_{\varepsilon=0})$ in function of the coefficients of the system (1), using the algebraic manipulator mathematica, but this expression occupies tens of pages.

4. **Proofs of Theorem 2 and Proposition 3.**

Proof of Theorem 2. We shall use the notations and definitions introduced in section 3. Then, for the differential system (2) we have the matrix

$$A = \begin{pmatrix} -7 & -\dfrac{7\left(\varepsilon^2 + 14\varepsilon + 50\right)\left(6\varepsilon^2 + 4\varepsilon + 7\right)}{2\left(20\varepsilon^2 + 7\varepsilon - 1\right)} & 0 \\ 1 & \dfrac{160\varepsilon^3 + 642\varepsilon^2 + 370\varepsilon + 635}{4\left(20\varepsilon^2 + 7\varepsilon - 1\right)} & \dfrac{\left(2\varepsilon^2 + 14\varepsilon + 51\right)\left(4\varepsilon^2 + 4\varepsilon + 5\right)}{4\left(20\varepsilon^2 + 7\varepsilon - 1\right)} \\ 0 & \dfrac{1}{2} & -\dfrac{1}{2} \end{pmatrix},$$

having the eigenvalues given in the statement of Theorem 2.

The normalized eigenvector q of A when $\varepsilon = 0$ associated to the eigenvalue ωi is

$$q = \left(-\frac{441 + 637i}{\sqrt{600274}}, -\sqrt{\frac{2}{300137}}(1 + 2i), -\sqrt{\frac{2}{300137}} \right),$$

The adjoint eigenvector $p = (p_1, p_2, p_3)$ is

$$p_1 = \frac{4559 - 15988i}{11056025}\sqrt{\frac{300137}{2}},$$
$$p_2 = \frac{637 - 4659i}{442241}\sqrt{\frac{300137}{2}},$$
$$p_3 = \frac{-507705 + 172635i}{884482}\sqrt{\frac{300137}{2}}.$$

The bi–linear function $B(x_1, y_1, z_1, x_2, y_2, z_2) = (B_1, B_2, B_3)$ with $B_i = B_i(x_1, y_1, z_1, x_2, y_2, z_2)$ is given by

$$\begin{aligned} B_1 &= 0, \\ B_2 &= \frac{1}{2660}(-875y_1y_2 - 3570y_2z_1 - 3570y_1z_2 + 1785z_1z_2), \\ B_3 &= \frac{1}{190}(-y_1y_2 + 2y_2z_1 + 2y_1z_2 - z_1z_2); \end{aligned}$$

and the tri–linear function $C(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) = (C_1, C_2, C_3)$ with $C_i = C_i(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3)$ is given by

$$\begin{aligned} C_1 &= 0, \\ C_2 &= -\frac{51}{7220}(y_1y_2y_3 - y_2y_3z_1 - y_1y_3z_2 - y_1y_2z_3), \\ C_3 &= \frac{1}{18050}(y_1y_2y_3 - y_2y_3z_1 - y_1y_3z_2 - y_1y_2z_3). \end{aligned}$$

According to Theorem 4 for computing $\ell_1(E^*|_{\varepsilon=0})$, we need to compute first A^{-1} and a $(2iI - A)^{-1}$. Note that now $\omega = 1$. We have that

$$A^{-1} = \begin{pmatrix} -\dfrac{89}{133} & -\dfrac{70}{19} & \dfrac{8925}{19} \\ -\dfrac{2}{665} & -\dfrac{2}{95} & \dfrac{51}{19} \\ -\dfrac{2}{665} & -\dfrac{2}{95} & \dfrac{13}{19} \end{pmatrix},$$

and

$$(2iI - A)^{-1} = \begin{pmatrix} 7 + 2i & -1225 & 0 \\ -1 & \dfrac{635}{4} + 2i & \dfrac{255}{4} \\ 0 & -\dfrac{1}{2} & \dfrac{1}{2} + 2i \end{pmatrix}.$$

The first, second and third terms of $\ell_1(E^*|_{\varepsilon=0})$ given in Theorem 4 are

$$\begin{aligned} \overline{p} \cdot C(q, q, \overline{q}) &= \frac{950436 + 129948i}{239582861065685}, \\ -2\overline{p} \cdot B(q, A^{-1}B(q, \overline{q})) &= \frac{-152367568 + 104825176i}{4552074360248015}, \\ \overline{p} \cdot B(\overline{q}, (2iI - A)^{-1}B(q, q)) &= \frac{928154260416 + 1952607677788i}{934984094451888996 75}. \end{aligned}$$

Consequently we get

$$\ell_1(E^*|_{\varepsilon=0}) = -\frac{5796656446586}{592156593152863031275}.$$

This completes the proof of the theorem. □

Proof of Proposition 3. From the expression of ρ, given in section 2, and for $|\varepsilon|$ sufficiently small it follows that $\rho > 0$ if and only if

$$\alpha_0(-a_2b^2r + a_2^2r^2 - 2a_2r^3 + r^4 - 2a_2brr_1 - a_2rr_1^2 + a_2^2\omega^2 - a_2r\omega^2) > 0. \quad (4)$$

Here we have used that $a_2 > r$, recall that this is due to the fact that the equilibrium point E^* must be in the positive octant.

By Proposition 1 and for $|\varepsilon|$ sufficiently small we get that $a > 0$ if and only if

$$\alpha_0(a_2^2r^2 - 2a_2r^3 + r^4 + a_2^2\omega^2 + a_2r\omega^2) > 0. \tag{5}$$

Note that $\beta|_{\varepsilon=0} > 0$, see its definition in the statement of Proposition 1.

On the other hand, the coordinate x of the equilibrium point E^* is positive if and only

$$\begin{aligned} &-(-a_2b^2r + a_2^2r^2 - 2a_2r^3 + r^4 - 2a_2brr_1 - a_2rr_1^2 + a_2^2\omega^2 - a_2r\omega^2)\cdot \\ &(a_2^2r^2 - 2a_2r^3 + r^4 + a_2^2\omega^2 + a_2r\omega^2) > 0. \end{aligned} \tag{6}$$

Now, since the x–coordinate of E^* and the parameter a are positive, the inequalities (5) and (6) hold. Then, from (4) we obtain that $\rho < 0$. That is, the real eigenvalue of the equilibrium point E^* is negative for $|\varepsilon|$ sufficiently small. Hence the proposition is proved. □

5. **Conclusions.** Xu, Chaplain and Davidson [8] studied the effect of stage structure for prey on the dynamics of ratio–dependent predator-prey system by considering the differential system

$$\begin{aligned} \dot{x} &= -(b + r_1)x + ay, \\ \dot{y} &= bx - b_1y^2 - \frac{a_1yz}{y + mz}, \\ \dot{z} &= z\left(-r + \frac{a_2y}{y + mz}\right), \end{aligned} \tag{7}$$

where $x = x(t)$ represents the density of immature individual preys at time t, and $y = y(t)$ denotes the density of mature individual preys at time t, $z = z(t)$ represents the density of the predator at time t. As usual the eight parameters a, a_1, a_2, b, b_1, m, r and r_1 of the system are positive. It is verified that when $y^* > 0$ and $a_2 > r$ the differential system (7) has a unique equilibrium point

$$E^* = \left(\frac{ay^*}{b + r_1}, y^*, \frac{(a_2 - r)y^*}{mr}\right) \quad \text{with} \quad y^* = \frac{aa_2bm - a_1(a_2 - r)(b + r_1)}{a_2b_1m(b + r_1)},$$

in the positive octant, i.e., in $\{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0\}$. In the conclusions of the paper [8] the authors said: *We would like to mention here that we are unable to show system* (1) *admits limit cycles when* E^* *exists and is unstable. We leave this for future work.* As far as we know this question remained open. In this work we answer affirmatively this question, see Theorem 2. The main difficulty for solving it was that the differential system (7) depends on 8 parameters, our main result show the existence of a Hopf bifurcation at the equilibrium point E^* for appropriate values of the parameters.

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