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WHEN PARALLELS AND MERIDIANS ARE LIMIT CYCLES FOR POLYNOMIAL VECTOR FIELDS ON QUADRICS OF REVOLUTION IN $\mathbb{R}^3$

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Abstract. We study polynomial vector fields of arbitrary degree in $\mathbb{R}^3$ with an invariant quadric of revolution. We characterize all the possible configurations of invariant meridians and parallels that these vector fields can exhibit. Furthermore we analyze when these invariant meridians and parallels can be limit cycles.

1. Introduction and statement of the results

One of the more difficult object to control in the qualitative theory of ordinary differential equations in dimension two are the limit cycles, their maximum number and their distribution. This problem is related with the second part of the 16th Hilbert problem (see [3]) when the differential equations are polynomial.

When the limit cycles are algebraic, that is they leave on an algebraic curve, the previous questions on the number of limit cycles and their distribution for polynomial differential equations in the plane have been solved for the so called generic algebraic limit cycles, see [7, 8, 10, 12].

The main objective of this paper is to extend the study of algebraic limit cycles to polynomial differential equations on quadrics of revolution, that is to some other 2−dimensional surfaces different from the plane. For such kind of polynomial differential equations we solve completely the maximum number of algebraic limit cycles formed by meridians and parallels independently. In order to state our main results we start providing some definitions and notation.

As usual we denote by $\mathbb{K}[x, y, z]$ the ring of the polynomials in the variables x, y and z with coefficients in $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. By definition a polynomial differential system in $\mathbb{R}^3$ is a system of the form

$$\frac{dx}{dt} = P_1(x, y, z), \quad \frac{dy}{dt} = P_2(x, y, z), \quad \frac{dz}{dt} = P_3(x, y, z), \tag{1}$$

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where $P_i \in \mathbb{R}[x,y,z]$ for $i = 1,2,3$ and t is the independent variable. We denote by

$$\mathcal{X}(x,y,z) = P_1(x,y,z)\frac{\partial}{\partial x} + P_2(x,y,z)\frac{\partial}{\partial y} + P_3(x,y,z)\frac{\partial}{\partial z}, \tag{2}$$

or simply by

$$\mathcal{X}(x,y,z) = (P_1(x,y,z), P_2(x,y,z), P_3(x,y,z)),$$

the polynomial vector field associated to system (1). We say that $m = \max\{m_i\}$, where m_i is the degree of P_i, $i = 1,2,3$, is the degree of the polynomial differential system (1), or of the polynomial vector field (2).

An *invariant algebraic surface* for system (1) or for the vector field (2) is an algebraic surface $f^{-1}(0)$ with $f \in \mathbb{R}[x,y,z]$, such that for some polynomial $K \in \mathbb{R}[x,y,z]$ we have $\mathcal{X}f = Kf$. This implies that if a solution curve of system (1) has a point on the algebraic surface $f^{-1}(0)$, then the whole solution curve is contained in $f^{-1}(0)$. The polynomial K is called the *cofactor* of the invariant algebraic surface $f^{-1}(0)$. We remark that if the polynomial system has degree m, then any cofactor has at most degree $m-1$.

We consider polynomial vector fields $\mathcal{X}$ of degree $m > 1$ in $\mathbb{R}^3$ having a non–degenerate quadric of revolution

$$\mathbb{Q}^2 = \mathcal{G}^{-1}(0)$$

as an invariant algebraic surface, that is $\mathcal{X}\mathcal{G} = K\mathcal{G}$, where $\mathcal{G}$ defines one of the non–degenerate quadric of revolution that after an affine change of coordinates we can assume of the form (see Section 2):

Cone $\mathcal{G}(x,y,z) = x^2 + y^2 - z^2$,
Cylinder $\mathcal{G}(x,y,z) = x^2 + y^2 - 1$,
Hyperboloid of one sheet $\mathcal{G}(x,y,z) = x^2 + y^2 - z^2 - 1$,
Hyperboloid of two sheets $\mathcal{G}(x,y,z) = x^2 + y^2 - z^2 + 1$,
Paraboloid $\mathcal{G}(x,y,z) = x^2 + y^2 - z$,
Sphere $\mathcal{G}(x,y,z) = x^2 + y^2 + z^2 - 1$,

and K is a polynomial of degree at most $m-1$. Such vector fields will be called *polynomial vector fields of degree m on $\mathbb{Q}^2$*.

On $\mathbb{Q}^2$ we define *meridians* and *parallels* as the curves obtained by the intersection of $\mathbb{Q}^2$ with the planes containing the $z-$axis and the planes orthogonal to the $z-$axis, respectively. More precisely, the parallels are obtained intersecting the planes $z = k$ (for suitable $k \in \mathbb{R}$) with $\mathbb{Q}^2$, and the meridians and their number are obtained intersecting the planes $ax + by = 0$ (where $a, b \in \mathbb{R}$) with $\mathbb{Q}^2$, according to the following conventions. The intersections of a plane $ax + by = 0$ with $\mathbb{Q}^2$ give:

2 meridians formed by two concurrent straight lines if $\mathbb{Q}^2$ is a cone;

2 meridians formed by two parallel straight lines if $\mathbb{Q}^2$ is a cylinder;

2 meridians formed by two branches of a hyperbola if $\mathbb{Q}^2$ is either a hyperboloid of one sheet, or a hyperboloid of two sheets;

1 meridian formed by a parabola if $\mathbb{Q}^2$ is a paraboloid;

1 meridian formed by a circle if $\mathbb{Q}^2$ is a sphere.

Here we say that the *parallel* contained in the plane $z - z_0 = 0$ is *invariant* by the flow of the polynomial vector field $\mathcal{X}$ on $\mathbb{Q}^2$ if $\mathcal{X}(z - z_0) = K_{z_0}(z - z_0)$, for some $K_{z_0} \in \mathbb{R}[x,y,z]$. In a similar way we define that the *meridian(s)* contained in the plane $ax+by=0$ is (are) *invariant* by the flow of the polynomial vector field $\mathcal{X}$ on $\mathbb{Q}^2$ if $\mathcal{X}(ax+by)=K_{a,b}(ax+by)$, for some $K_{a,b}\in\mathbb{R}[x,y,z]$.

In this article we shall characterize all the possible configurations of invariant parallels and meridians that a polynomial vector field of degree $m>1$ on a quadric $\mathbb{Q}^2$ can exhibit. Additionally we shall consider when the invariant parallels or the invariant meridians can be limit cycles.

Our main results are the following.

Theorem 1. *Let $\mathcal{X}$ be a polynomial vector field of degree $m>1$ on the cone. Assume that $\mathcal{X}$ has finitely many invariant meridians and invariant parallels. The following statements hold.*

(a) *The number of invariant meridians of $\mathcal{X}$ is at most $2(m-1)$.*

(b) *The number of invariant parallels of $\mathcal{X}$ is at most $m-1$.*

(c) *If*

$$\mathcal{X}(x,y,z)=\left(-2yF_1(x,y)+xF_2(z),2xF_1(x,y)+yF_2(z),zF_2(z)\right), \tag{3}$$

with

$$F_1(x,y)=\prod_{i=1}^{m-1}(a_ix+b_iy),\quad a_i,b_i\in\mathbb{R},\quad a_i^2+b_i^2\neq 0,\quad (a_i,b_i)\neq(a_j,b_j),\ i\neq j,$$

$$F_2(z)=\prod_{i=1}^{m-1}(z-z_i),\quad z_i\neq z_j,\ i\neq j,\quad z_i\in\mathbb{R}\backslash\{0\},$$

then $\mathcal{X}$ has exactly $m-1$ invariant parallels and $2(m-1)$ invariant meridians.

(d) *Fix $1\leq k\leq m-1$ and consider the vector field*

$$\mathcal{X}(x,y,z)=\left(-2y+xF_3(z),2x+yF_3(z),zF_3(z)\right), \tag{4}$$

on the cone where

$$F_3(z)=z^{m-k-1}\prod_{i=1}^{k}(z-z_i),\quad z_i\neq z_j,\ i\neq j,\quad z_i\in\mathbb{R}\backslash\{0\}.$$

Then $\mathcal{X}$ has exactly k invariant parallels which are limit cycles. These limit cycles are stable or unstable alternately in each sheet of the cone.

Theorem 2. *Let $\mathcal{X}$ be a polynomial vector field of degree $m>1$ on the cylinder. Assume that $\mathcal{X}$ has finitely many invariant meridians and invariant parallels. The following statements hold.*

(a) *The number of invariant meridians of $\mathcal{X}$ is at most $2(m-1)$.*

(b) *The number of invariant parallels of $\mathcal{X}$ is at most m.*

(c) *If*

$$\mathcal{X}(x,y,z)=\left(2yF_1(x,y),-2xF_1(x,y),F_4(z)\right), \tag{5}$$

where F_1 is given in (3) and

$$F_4(z)=\prod_{i=1}^{m}(z-z_i),\quad z_i\neq z_j,\ i\neq j,\quad z_i\in\mathbb{R},$$

then $\mathcal{X}$ has exactly m invariant parallels and $2(m-1)$ invariant meridians.

(d) *Fix $1 \leq k \leq m$ and consider the vector field*

$$\mathcal{X}(x,y,z) = (2y, -2x, F_5(z)), \tag{6}$$

on the cylinder where

$$F_5(z) = \begin{cases} z^m & \text{if } k = 1, \\ z^{m-k+1} \displaystyle\prod_{i=1}^{k-1} (z - z_i), \ z_i \neq z_j, \ i \neq j, \ z_i \in \mathbb{R} \backslash \{0\} & \text{if } k > 1. \end{cases}$$

Then $\mathcal{X}$ has exactly k invariant parallels which are limit cycles. These limit cycles are stable or unstable alternately except the limit cycle determined by $z = 0$ that can be semi–stable.

Theorem 3. *Let $\mathcal{X}$ be a polynomial vector field of degree $m > 1$ on the paraboloid. Assume that $\mathcal{X}$ has finitely many invariant meridians and invariant parallels. The following statements hold.*

(a) *The number of invariant meridians of $\mathcal{X}$ is at most $m - 1$.*

(b) *The number of invariant parallels of $\mathcal{X}$ is at most $m - 1$.*

(c) *There is no a polynomial vector field of degree $m > 1$ on the paraboloid having exactly $m - 1$ invariant parallels and $m - 1$ invariant meridians.*

(d) *If*

$$\mathcal{X}(x,y,z) = \left(2yF_1(x,y) + xF_6(z), -2xF_1(x,y) + yF_6(z), 2(x^2 + y^2)F_6(z)\right), \tag{7}$$

where F_1 is given in (3) *and*

$$F_6(z) = \begin{cases} 1 & \text{if } m = 2, \\ \displaystyle\prod_{i=1}^{m-2} (z - z_i), \quad z_i \neq z_j, \ i \neq j, \quad z_i > 0 & \text{if } m > 2, \end{cases}$$

then $\mathcal{X}$ has exactly $m-2$ invariant parallels and $m-1$ invariant meridians.

(e) *Fix $1 \leq k \leq m - 1$ and consider the vector field*

$$\mathcal{X}(x,y,z) = (2y(x - 2k) + F_7(z), -2x(x - 2k), 2xF_7(z)), \tag{8}$$

on the paraboloid where

$$F_7(z) = \varepsilon z^{m-k-1} \prod_{i=1}^{k} (z - i), \quad \varepsilon > 0 \ small.$$

Then $\mathcal{X}$ has exactly k invariant parallels which are limit cycles. These limit cycles are stable or unstable alternately.

Theorem 4. *Let $\mathcal{X}$ be a polynomial vector field of degree $m > 1$ on the hyperboloid of one sheet. Assume that $\mathcal{X}$ has finitely many invariant meridians and invariant parallels. The following statements hold.*

(a) *The number of invariant meridians of $\mathcal{X}$ is at most $2(m - 1)$.*

(b) *The number of invariant parallels of $\mathcal{X}$ is at most $m - 1$.*

(c) *There is no a polynomial vector field of degree $m > 1$ on the hyperboloid of one sheet having exactly $m - 1$ invariant parallels and $2(m - 1)$ invariant meridians.*

(d) *If*

$$\mathcal{X}(x,y,z) = \left(-2yF_1(x,y) + 2xzF_8(z), 2xF_1(x,y) + 2yzF_8(z), 2(x^2+y^2)F_8(z)\right), \tag{9}$$

where F_1 *is given in* (3) *and*

$$F_8(z) = \begin{cases} 1 & \text{if } m = 2, \\ \displaystyle\prod_{i=1}^{m-2}(z - z_i), \quad z_i \neq z_j,\ i \neq j, \quad z_i \in \mathbb{R} & \text{if } m > 2, \end{cases}$$

then $\mathcal{X}$ *has exactly* $m-2$ *invariant parallels and* $2(m-1)$ *invariant meridians.*

(e) *Fix* $1 \leq k \leq m-1$ *and consider the vector field*

$$\mathcal{X}(x,y,z) = \left(-2y(x-2k) + 2zF_9(z), 2x(x-2k), 2xF_9(z)\right), \tag{10}$$

on the hyperboloid of one sheet where

$$F_9(z) = \begin{cases} \varepsilon z^{m-1} & \text{if } k = 1, \\ \varepsilon z^{m-k}\displaystyle\prod_{i=1}^{k-1}(z-i) & \text{if } k > 1, \end{cases}$$

and $\varepsilon > 0$ *small. Then* $\mathcal{X}$ *has exactly* k *invariant parallels which are limit cycles. These limit cycles are stable or unstable alternately, except the limit cycle determined by* $z = 0$ *that can be semi–stable.*

Theorem 5. *Let* $\mathcal{X}$ *be a polynomial vector field of degree* $m > 1$ *on the hyperboloid of two sheets. Assume that* $\mathcal{X}$ *has finitely many invariant meridians and invariant parallels. The following statements hold.*

(a) *The number of invariant meridians of* $\mathcal{X}$ *is at most* $2(m-1)$.

(b) *The number of invariant parallels of* $\mathcal{X}$ *is at most* $m-1$.

(c) *There is no a polynomial vector field of degree* $m > 1$ *on the hyperboloid of two sheets having exactly* $m-1$ *invariant parallels and* $2(m-1)$ *invariant meridians.*

(d) *If*

$$\mathcal{X}(x,y,z) = \left(-2yF_1(x,y) + 2xzF_{10}(z), 2xF_1(x,y) + 2yzF_{10}(z), 2(x^2+y^2)F_{10}(z)\right), \tag{11}$$

where F_1 *is given in* (3) *and*

$$F_{10}(z) = \begin{cases} 1 & \text{if } m = 2, \\ \displaystyle\prod_{i=1}^{m-2}(z - z_i), \quad z_i \neq z_j,\ i \neq j, \quad |z_i| > 1 & \text{if } m > 2, \end{cases}$$

then $\mathcal{X}$ *has exactly* $m-2$ *invariant parallels and* $2(m-1)$ *invariant meridians.*

(e) *Fix* $1 \leq k \leq m-1$ *and consider the vector field*

$$\mathcal{X}(x,y,z) = \left(-2y(x-2k) + 2zF_{11}(z), 2x(x-2k), 2xF_{11}(z)\right), \tag{12}$$

on the hyperboloid of two sheets where

$$F_{11}(z) = \varepsilon z^{m-k-1} \prod_{i=1}^{k} (z - 2i), \quad \varepsilon > 0 \textit{ small}.$$

Then $\mathcal{X}$ has exactly k invariant parallels which are limit cycles. These limit cycles are stable or unstable alternately.

Theorem 6. *Let $\mathcal{X}$ be a polynomial vector field of degree $m > 1$ on the sphere. Assume that $\mathcal{X}$ has finitely many invariant meridians and invariant parallels. The following statements hold.*

(a) *The number of invariant meridians of $\mathcal{X}$ is at most $m - 1$.*

(b) *The number of invariant parallels of $\mathcal{X}$ is at most $m - 1$.*

(c) *There is no a polynomial vector field of degree $m > 1$ on the sphere having exactly $m - 1$ invariant parallels and $m - 1$ invariant meridians.*

(d) *If*

$$\mathcal{X}(x, y, z) = \left(-yF_1(x, y) + xzF_{12}(z), xF_1(x, y) + yzF_{12}(z), -(x^2 + y^2)F_{12}(z)\right), \tag{13}$$

where F_1 is given in (3) *and*

$$F_{12}(z) = \begin{cases} 1 & \text{if } m = 2, \\ \displaystyle\prod_{i=1}^{m-2} (z - z_i), \quad z_i \neq z_j, \ i \neq j, \quad |z_i| < 1 & \text{if } m > 2, \end{cases}$$

then $\mathcal{X}$ has exactly $m-1$ invariant meridians and $m-2$ invariant parallels.

In the next theorem we study when the invariant meridians and parallels of Theorem 6 are limit cycles.

Theorem 7. *There are polynomial vector fields $\mathcal{X}$ of degree m on the sphere having exactly*

(a) *either* 1 *invariant meridian which is a limit cycle;*

(b) *or k invariant parallels which are limit cycles for $k = 1, 2, \ldots, m - 1$.*

Theorems 1 to 7 are proved in Section 3. All the theorems need for their proofs the notion of extactic polynomial which is defined in Section 2, where we also present some definitions and results about non–degenerate quadrics and averaging theory that we shall need for proving our results.

2. Preliminary results

2.1. **Vector fields on non–degenerate quadrics of revolution.** A quadric is a surface $\mathbb{Q}^2 = \mathcal{G}^{-1}(0)$ in $\mathbb{R}^3$ implicitly defined by an algebraic equation of degree two. The quadric $\mathcal{G}^{-1}(0)$ is non–degenerate if the polynomial $\mathcal{G}$ is irreducible in $\mathbb{C}[x, y, z]$. The non–degenerate quadrics are classified as ellipsoid or sphere, parabolic cylinder, hyperbolic cylinder, elliptic cylinder, elliptic paraboloid, hyperbolic paraboloid, hyperboloid of one sheet, hyperboloid of two sheets and cone. In this paper we only consider non–degenerate quadrics of revolution: sphere, cylinder, paraboloid, hyperboloid of one sheet, hyperboloid of two sheets and cone.

The proof of the following theorem can be found in [5].

Theorem 8. *Assume that a non–degenerate quadric $\mathbb{Q}^2$ is an invariant algebraic surface of the polynomial differential system* (1). *Then after an affine change of coordinates system* (1) *and the quadric $\mathbb{Q}^2$ can be written in one of the following six normal forms, where A, B, C, D, E, F, G and Q are arbitrary polynomials of $\mathbb{R}[x, y, z]$:*

(i) *If $\mathbb{Q}^2$ is a cone, then system* (1) *can be written as*

$$
\begin{aligned}
x' &= \mathcal{G}(x,y,z)A(x,y,z) - 2yD(x,y,z) - 2zE(x,y,z) + xG(x,y,z),\\
y' &= \mathcal{G}(x,y,z)B(x,y,z) + 2xD(x,y,z) + 2zF(x,y,z) + yG(x,y,z),\\
z' &= \mathcal{G}(x,y,z)C(x,y,z) - 2xE(x,y,z) + 2yF(x,y,z) + zG(x,y,z),
\end{aligned}
\tag{14}
$$

with $\mathcal{G}(x,y,z) = x^2 + y^2 - z^2$;

(ii) *If $\mathbb{Q}^2$ is a cylinder, then system* (1) *can be written as*

$$
\begin{aligned}
x' &= \mathcal{G}(x,y,z)A(x,y,z) + 2yE(x,y,z),\\
y' &= \mathcal{G}(x,y,z)B(x,y,z) - 2xE(x,y,z),\\
z' &= Q(x,y,z),
\end{aligned}
\tag{15}
$$

with $\mathcal{G}(x,y,z) = x^2 + y^2 - 1$;

(iii) *If $\mathbb{Q}^2$ is a paraboloid, then system* (1) *can be written as*

$$
\begin{aligned}
x' &= \mathcal{G}(x,y,z)A(x,y,z) + E(x,y,z) + 2yF(x,y,z),\\
y' &= \mathcal{G}(x,y,z)B(x,y,z) - D(x,y,z) - 2xF(x,y,z),\\
z' &= \mathcal{G}(x,y,z)C(x,y,z) - 2yD(x,y,z) + 2xE(x,y,z),
\end{aligned}
\tag{16}
$$

with $\mathcal{G}(x,y,z) = x^2 + y^2 - z$;

(iv) *If $\mathbb{Q}^2$ is a hyperboloid of one sheet, then system* (1) *can be written as*

$$
\begin{aligned}
x' &= \mathcal{G}(x,y,z)A(x,y,z) - 2yD(x,y,z) - 2zE(x,y,z),\\
y' &= \mathcal{G}(x,y,z)B(x,y,z) + 2xD(x,y,z) + 2zF(x,y,z),\\
z' &= \mathcal{G}(x,y,z)C(x,y,z) - 2xE(x,y,z) + 2yF(x,y,z),
\end{aligned}
\tag{17}
$$

with $\mathcal{G}(x,y,z) = x^2 + y^2 - z^2 - 1$;

(v) *If $\mathbb{Q}^2$ is a hyperboloid of two sheets, then system* (1) *can be written as*

$$
\begin{aligned}
x' &= \mathcal{G}(x,y,z)A(x,y,z) - 2yD(x,y,z) - 2zE(x,y,z),\\
y' &= \mathcal{G}(x,y,z)B(x,y,z) + 2xD(x,y,z) + 2zF(x,y,z),\\
z' &= \mathcal{G}(x,y,z)C(x,y,z) - 2xE(x,y,z) + 2yF(x,y,z),
\end{aligned}
\tag{18}
$$

with $\mathcal{G}(x,y,z) = x^2 + y^2 - z^2 + 1$;

(vi) *If $\mathbb{Q}^2$ is a sphere, then system* (1) *can be written as*

$$
\begin{aligned}
x' &= \mathcal{G}(x,y,z)A(x,y,z) - 2yD(x,y,z) + 2zE(x,y,z),\\
y' &= \mathcal{G}(x,y,z)B(x,y,z) + 2xD(x,y,z) - 2zF(x,y,z),\\
z' &= \mathcal{G}(x,y,z)C(x,y,z) - 2xE(x,y,z) + 2yF(x,y,z),
\end{aligned}
\tag{19}
$$

with $\mathcal{G}(x,y,z) = x^2 + y^2 + z^2 - 1$.

2.2. **The extactic polynomial.** Let $\mathcal{X}$ be a polynomial vector field on $\mathbb{R}^3$ and let W be a finite $\mathbb{R}$–vector subspace of $\mathbb{R}[x,y,z]$. The *extactic polynomial of $\mathcal{X}$ associated to W* is the polynomial

$$\mathcal{E}_W(\mathcal{X}) = \det \begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ \mathcal{X}(v_1) & \mathcal{X}(v_2) & \cdots & \mathcal{X}(v_l) \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{X}^{l-1}(v_1) & \mathcal{X}^{l-1}(v_2) & \cdots & \mathcal{X}^{l-1}(v_l) \end{pmatrix},$$

where $\{v_1, \ldots, v_l\}$ is a basis of W, $l = \dim(W)$ is the dimension of W, and $\mathcal{X}^j(v_i) = \mathcal{X}^{j-1}(\mathcal{X}(v_i))$. It is known due to the properties of the determinant and of the derivation that the definition of extactic polynomial is independent of the chosen basis of W.

The notion of the extactic polynomial goes back to the work of Lagutinskii (see [2] and references therein) and have been used in different papers, see for instance [1, 4, 6, 9].

We use the extactic polynomial in order to prove the main results of this article. We have the following proposition whose proof can be found in [9].

Proposition 9. *Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{R}^3$ and let W be a finite $\mathbb{R}$–vector subspace of $\mathbb{R}[x,y,z]$ with $\dim(W) > 1$. Then every algebraic invariant surface $f^{-1}(0)$ for the vector field $\mathcal{X}$, with $f \in W$, is a factor of $\mathcal{E}_W(\mathcal{X})$.*

For determining the invariant meridians of $\mathcal{X}$ we have to find the curves obtained by the intersection of $\mathbb{Q}^2$ and the planes of the form $ax+by=0$ such that they are invariant by the flow of $\mathcal{X}$. By Proposition 9 it is necessary that $f(x,y,z) = ax+by$ be a factor of the extactic polynomial $\mathcal{E}_{\{x,y\}}(\mathcal{X})$, which can be written as

$$\mathcal{E}_{\{x,y\}}(\mathcal{X}) = \det \begin{pmatrix} x & y \\ \mathcal{X}(x) & \mathcal{X}(y) \end{pmatrix} = \det \begin{pmatrix} x & y \\ P_1 & P_2 \end{pmatrix} = xP_2 - yP_1.$$

In order to study invariant parallels we must consider the intersection of the planes $z - z_0 = 0$ (for suitable z_0) with $\mathbb{Q}^2$ such that they are invariant by the flow of $\mathcal{X}$. By Proposition 9 it is necessary that $g(x,y,z) = z - z_0$ be a factor of the extactic polynomial $\mathcal{E}_{\{1,z\}}(\mathcal{X})$, which can be written as

$$\mathcal{E}_{\{1,z\}}(\mathcal{X}) = \det \begin{pmatrix} 1 & z \\ \mathcal{X}(1) & \mathcal{X}(z) \end{pmatrix} = \det \begin{pmatrix} 1 & z \\ 0 & P_3 \end{pmatrix} = P_3.$$

The proof of the following proposition can be found in [9].

Proposition 10. *Let $\mathcal{X}$ be the polynomial vector field* (2)*.*

(a) *If $f(x,y,z) = ax+by$ is a factor of the extactic polynomial $\mathcal{E}_{\{x,y\}}(\mathcal{X})$, then $f^{-1}(0)$ is an invariant plane of $\mathcal{X}$;*

(b) *If $g(x,y,z) = z - z_0$ is a factor of the extactic polynomial $\mathcal{E}_{\{1,z\}}(\mathcal{X})$, then $g^{-1}(0)$ is an invariant plane of $\mathcal{X}$.*

We remark that Propositions 9 and 10 transform the study of invariant meridians and parallels of a polynomial vector field $\mathcal{X} = (P_1, P_2, P_3)$ of degree $m > 1$ on $\mathbb{Q}^2$ into the study of factors of the form $f(x,y,z) = ax+by$ and $g(x,y,z) = z - z_0$ of the extactic polynomials $\mathcal{E}_{\{x,y\}}(\mathcal{X}) = xP_2 - yP_1$ and $\mathcal{E}_{\{1,z\}}(\mathcal{X}) = P_3$, respectively.

2.3. **Averaging theorem.** In order to prove that some invariant parallels and meridians are limit cycles we will use the averaging theorem of first order. In general, the averaging method gives a relation between the solutions of a non autonomous differential system and the solutions of an autonomous one, the averaged differential system. More details can be found in [11].

Theorem 11 (Averaging Theorem of First Order)**.** *Consider the system*

$$\dot{u}(t) = \varepsilon F(t, u(t)) + \varepsilon^2 R(t, u(t), \varepsilon) \tag{20}$$

and assume that the functions F, R, $D_u F$, $D_u^2 F$ *and* $D_u R$ *are continuous and bounded by a constant* M *(independent of* ε*) in* $[0, \infty) \times D \subset \mathbb{R} \times \mathbb{R}^n$ *with* $-\varepsilon_0 < \varepsilon < \varepsilon_0$*. Moreover, suppose that* F *and* R *are* T*–periodic in* t*, with* T *independent of* ε*.*

(a) *If* $a \in D$ *is a zero of the averaged function*

$$f(u) = \frac{1}{T} \int_0^T F(s, u) ds, \tag{21}$$

such that $\det(D_u f(a)) \neq 0$ *then, for* $|\varepsilon| > 0$ *sufficiently small, there exists a* T*–periodic solution* $u_\varepsilon(t)$ *of system* (20) *such that* $u_\varepsilon(0) \to a$ *when* $\varepsilon \to 0$.

(b) *If the real part of all the eigenvalues of* $D_u f(a)$ *are negative, then the periodic solution* $u_\varepsilon(t)$ *is stable, if the real part of some eigenvalue of* $D_u f(a)$ *is positive then the periodic solution is unstable.*

3. Proofs of the main results

In spherical coordinates $(x, y, z) = (\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)$ system (1) becomes

$$\begin{aligned} \rho' &= P_1 \sin\phi\cos\theta + P_2 \sin\phi \sin\theta + P_3 \cos\phi, \\ \theta' &= \frac{1}{\rho\sin\phi} \left(-P_1 \sin\theta + P_2\cos\theta\right), \\ \phi' &= \frac{1}{\rho}\left(P_1 \cos\phi\cos\theta + P_2\cos\phi\sin\theta - P_3 \sin\phi\right), \end{aligned}$$

where $P_i = P_i(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)$, $i = 1, 2, 3$.

In cylindrical coordinates $(x, y, z) = (r\cos\theta, r\sin\theta, z)$ system (1) becomes

$$\begin{aligned} r' &= P_1(r\cos\theta, r\sin\theta, z)\cos\theta + P_2(r\cos\theta, r\sin\theta, z)\sin\theta, \\ \theta' &= \frac{1}{r}\left(-P_1(r\cos\theta, r\sin\theta, z)\sin\theta + P_2(r\cos\theta, r\sin\theta, z)\cos\theta\right), \\ z' &= P_3(r\cos\theta, r\sin\theta, z). \end{aligned}$$

An invariant meridian of the vector field $\mathcal{X} = (P_1, P_2, P_3)$ is given by the intersection of a plane $ax + by = 0$ with $\mathbb{Q}^2$. In this case the polynomial $ax + by$ must be a factor of the extactic polynomial

$$\mathcal{E}_{\{x,y\}}(\mathcal{X}) = xP_2 - yP_1 = (x^2 + y^2)\,\theta',$$

where the last equation is obtained by using either spherical or cylindrical coordinates. Since the polynomial $xP_2 - yP_1$ has at most degree $m + 1$, the right hand side of the above equation is a polynomial of degree at most $m+1$. Thus the vector field $\mathcal{X}$ has at most $m - 1$ invariant planes of the form $ax + by = 0$. This implies that $\mathcal{X}$ has at most $m - 1$ invariant meridians in the cases where $\mathbb{Q}^2$ is either the paraboloid or the sphere and at most $2(m-1)$ invariant meridians in the cases where $\mathbb{Q}^2$ is either the cone, or the cylinder, or the hyperboloid of one sheet, or the hyperboloid of two sheets. This proves statements (a) of Theorems 1 to 6.

In order to determine the invariant parallels we must consider the intersection of the planes $z=k$, $k\in\mathbb{R}$ with $\mathbb{Q}^2$. By Proposition 9 it is necessary that $g(x,y,z)=z-k$ be a factor of the extactic polynomial $\mathcal{E}_{\{1,z\}}(\mathcal{X})=P_3$, where:

(i) $P_3(x,y,z)=-2xE(x,y,z)+2yF(x,y,z)+zG(x,y,z)$, if $\mathbb{Q}^2$ is the cone,

(ii) $P_3(x,y,z)=Q(x,y,z)$, if $\mathbb{Q}^2$ is the cylinder,

(iii) $P_3(x,y,z)=-2yD(x,y,z)+2xE(x,y,z)$, if $\mathbb{Q}^2$ is the paraboloid,

(iv) $P_3(x,y,z)=-2xE(x,y,z)+2yF(x,y,z)$, if $\mathbb{Q}^2$ is the hyperboloid of one sheet,

(v) $P_3(x,y,z)=-2xE(x,y,z)+2yF(x,y,z)$, if $\mathbb{Q}^2$ is the hyperboloid of two sheets,

(vi) $P_3(x,y,z)=-2xE(x,y,z)+2yF(x,y,z)$, if $\mathbb{Q}^2$ is the sphere.

In the case of the cylinder we have at most m factors of the form $z-k$ in the extactic polynomial. In all the other cases we have at most $m-1$ factors of the form $z-k$. This proves statements (b) of Theorems 1 to 6.

3.1. **Proof of Theorem 1.** In this subsection we prove statements (c) and (d) of Theorem 1.

Consider the polynomial vector field (3). It is easy to see that

$$\begin{aligned}\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z) &= xP_2(x,y,z)-yP_1(x,y,z)=2\left(x^2+y^2\right)F_1(x,y)\\ &= 2\left(x^2+y^2\right)\prod_{i=1}^{m-1}(a_ix+b_iy),\end{aligned}$$

where $a_i,b_i\in\mathbb{R}$, $a_i^2+b_i^2\neq 0$, $(a_i,b_i)\neq(a_j,b_j)$, $i\neq j$, and

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z)=P_3(x,y,z)=zF_2(z)=z\prod_{i=1}^{m-1}(z-z_i),$$

where $z_i\neq z_j$, $i\neq j$, $z_i\in\mathbb{R}\backslash\{0\}$. Therefore the vector field $\mathcal{X}$ has exactly $2(m-1)$ invariant meridians and $m-1$ invariant parallels. We have proved statement (c).

Now we prove statement (d). The polynomial vector field (4) has one singular point at the vertex of the cone. By a straightforward calculation we have

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z)=P_3(x,y,z)=zF_3(z)=z^{m-k}\prod_{i=1}^{k}(z-z_i),$$

where $z_i\neq z_j$, $\neq j$, $z_i\in\mathbb{R}\backslash\{0\}$. Thus, since on the cone there are no equilibria, for $1\leq k\leq m-1$ fixed, the k invariant parallels $z=z_i$, $z_i\in\mathbb{R}\backslash\{0\}$ are limit cycles because between two consecutive invariant parallels the sign of P_3 is fixed. So the limit cycles on the invariant parallels are stable or unstable alternately.

In short, we have proved Theorem 1.

3.2. **Proof of Theorem 2.** In this subsection we prove statements (c) and (d) of Theorem 2.

Consider the polynomial vector field (5). We have

$$\begin{aligned}\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z) &= xP_2(x,y,z) - yP_1(x,y,z) = -2\left(x^2+y^2\right)F_1(x,y)\\ &= -2\left(x^2+y^2\right)\prod_{i=1}^{m-1}(a_ix+b_iy),\end{aligned}$$

where $a_i, b_i \in \mathbb{R}$, $a_i^2+b_i^2 \neq 0$, $(a_i,b_i)\neq(a_j,b_j)$, $i\neq j$, and

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = P_3(x,y,z) = F_4(z) = \prod_{i=1}^{m}(z-z_i),$$

where $z_i \neq z_j$, $i \neq j$, $z_i \in \mathbb{R}$. Therefore the vector field $\mathcal{X}$ has exactly $2(m-1)$ invariant meridians and m invariant parallels. We have proved statement (c).

Now we prove statement (d). The polynomial vector field (6) has no equilibria. By a straightforward calculation we have

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = P_3(x,y,z) = F_5(z),$$

where

$$F_5(z) = \begin{cases} z^m & \text{if } k=1,\\ z^{m-k+1}\displaystyle\prod_{i=1}^{k-1}(z-z_i),\ \ z_i\neq z_j,\ i\neq j,\ \ z_i\in\mathbb{R}\backslash\{0\} & \text{if } k>1.\end{cases}$$

This implies that, for $1 \le k \le m$ fixed, the k invariant parallels are limit cycles. These limit cycles are stable or unstable alternately except the limit cycle determined by $z=0$ that can be semi–stable if either m is even and $k=1$, or $m-k+1$ is even and $k>1$.

In short, we have proved Theorem 2.

3.3. **Proof of Theorem 3.** In this subsection we give the proofs of statements (c), (d) and (e) of Theorem 3.

In order to prove statement (c) we need to study factors of the forms $f(x,y,z) = ax+by$ and $g(x,y,z) = z-z_0$ of the extactic polynomials

$$\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z) = -2(x^2+y^2)F(x,y,z) - xD(x,y,z) - yE(x,y,z)$$

and

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = 2xE(x,y,z) - 2yD(x,y,z),$$

respectively, where $\mathcal{X}$ is given in (16). Suppose

$$\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z) = K_1(x,y,z)\prod_{i=1}^{m-1}(a_ix+b_iy)$$

and

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = K_2(x,y,z)\prod_{i=1}^{m-1}(z-z_i).$$

From the expressions of $\mathcal{E}_{\{1,z\}}(\mathcal{X})$ we have

$$E(x,y,z)=\prod_{i=1}^{m-1}(z-z_i),\quad D(x,y,z)=\alpha E(x,y,z),\quad \alpha\in\mathbb{R}.$$

Substituting E and D into the expression of $\mathcal{E}_{\{x,y\}}(\mathcal{X})$ we obtain

$$\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z)=-2(x^2+y^2)F(x,y,z)-(\alpha x+y)\prod_{i=1}^{m-1}(z-z_i).$$

Since F has degree at most $m-1$ it is not possible to obtain a factor of the form

$$\prod_{i=1}^{m-1}(a_ix+b_iy)$$

in the extactic polynomial $\mathcal{E}_{\{x,y\}}(\mathcal{X})$. This proves statement (c).

Consider the polynomial vector field (7). We obtain

$$\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z)=-2\left(x^2+y^2\right)F_1(x,y)=-2\left(x^2+y^2\right)\prod_{i=1}^{m-1}(a_ix+b_iy),$$

where $a_i,b_i\in\mathbb{R}$, $a_i^2+b_i^2\neq 0$, $(a_i,b_i)\neq(a_j,b_j)$, $i\neq j$, and

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z)=P_3(x,y,z)=2(x^2+y^2)F_6(z),$$

that is

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z)=\begin{cases}2(x^2+y^2) & \text{if } m=2,\\[2mm] 2(x^2+y^2)\displaystyle\prod_{i=1}^{m-2}(z-z_i),\ \ z_i\neq z_j,\ i\neq j,\ \ z_i>0 & \text{if } m>2.\end{cases}$$

Therefore the vector field $\mathcal{X}$ has exactly $m-1$ invariant meridians and $m-2$ invariant parallels. We have proved statement (d).

Now we prove statement (e). Consider the polynomial vector field (8). By a calculation we obtain

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z)=P_3(x,y,z)=2xF_7(z)=2\varepsilon x z^{m-k-1}\prod_{i=1}^{k}(z-i).$$

This implies that, for $1\leq k\leq m-1$ fixed, vector field (8) has exactly k invariant parallels given by $z=i$, $i=1,\dots,k$. It is easy to see that this vector field has no equilibria on the invariant parallels. In order to complete the proof we need to show that these invariant parallels are limit cycles.

Note that the paraboloid $x^2+y^2-z=0$ can be written in the explicit form $z=x^2+y^2$. In the coordinates (x,y) vector field (8) has the form

$$\mathcal{X}^*(x,y)=\left(2y(x-2k)+F_7^*(x^2+y^2),-2x(x-2k)\right),$$

with

$$F_7^*(x^2+y^2)=\varepsilon(x^2+y^2)^{m-k-1}\prod_{i=1}^{k}(x^2+y^2-i),\quad \varepsilon>0\ \text{small}.$$

In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ the above vector field is equivalent to

$$\frac{dr}{d\theta} = \frac{\varepsilon r^{2(m-k-1)}\prod_{i=1}^{k}(r^2-i)\cos\theta}{-2(r\cos\theta-2k)-\varepsilon r^{2(m-k-1)-1}\prod_{i=1}^{k}(r^2-i)\sin\theta}.$$

Expanding $dr/d\theta$ in Taylor series with respect to ε at $\varepsilon = 0$ we have

$$\frac{dr}{d\theta} = \frac{r^{2(m-k-1)-1}\prod_{i=1}^{k}(r^2-i)\cos\theta}{-2(r\cos\theta-2k)}\,\varepsilon + O(\varepsilon^2). \tag{22}$$

Equation (22) satisfies the hypotheses of Theorem 11. The averaged function (21) can be written as

$$\begin{aligned} f(r) =& \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{r^{2(m-k-1)-1}\prod_{i=1}^{k}(r^2-i)\cos\theta}{-2(r\cos\theta-2k)}\right)d\theta \\ =& -\frac{1}{4\pi}\left((r^{2(m-k-1)-1}\prod_{i=1}^{k}(r^2-i)\right)\int_0^{2\pi}\frac{\cos\theta}{(r\cos\theta-2k)}d\theta \\ =& -\frac{1}{4\pi}\left(r^{2(m-k-1)-1}\prod_{i=1}^{k}(r^2-i)\right)g(r), \end{aligned}$$

where

$$g(r) = 2\pi\left(\frac{\sqrt{4k^2-r^2}-2k}{r\sqrt{4k^2-r^2}}\right).$$

It is easy to check that $g(r) < 0$ for $0 < r < k$. Therefore, for $0 < r < k$, the simple zeros of f are given by $r = \sqrt{i}$, for $i = 1,\dots,k$, which correspond to the k limit cycles of $\mathcal{X}^*$. The stability of each limit cycle is easily determined by the sign of the derivative of f at each simple zero.

In short, we have proved Theorem 3.

3.4. **Proof of Theorem 4.** In this subsection we give the proofs of statements (c), (d) and (e) of Theorem 4.

In order to prove the statement (c) we need to study factors of the forms $f(x,y,z) = ax+by$ and $g(x,y,z) = z-z_0$ of the extactic polynomials

$$\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z) = 2(x^2+y^2)D(x,y,z) - 2z(xF(x,y,z)+yE(x,y,z))$$

and

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = -2xE(x,y,z) + 2yF(x,y,z),$$

respectively, where $\mathcal{X}$ is given in (17). Suppose

$$\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z) = K_1(x,y,z)\prod_{i=1}^{m-1}(a_ix+b_iy)$$

and

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = K_2(x,y,z)\prod_{i=1}^{m-1}(z-z_i).$$

From the expressions of $\mathcal{E}_{\{1,z\}}(\mathcal{X})$ we have

$$E(x,y,z) = \prod_{i=1}^{m-1}(z-z_i), \quad F(x,y,z) = \alpha E(x,y,z), \quad \alpha\in\mathbb{R}.$$

Substituting E and F into the expression of $\mathcal{E}_{\{x,y\}}(\mathcal{X})$ we obtain

$$\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z) = 2(x^2+y^2)D(x,y,z) - 2z(\alpha x + y)\prod_{i=1}^{m-1}(z-z_i).$$

Since D has degree at most $m-1$ it is not possible to obtain a factor of the form

$$\prod_{i=1}^{m-1}(a_i x + b_i y)$$

in the extactic polynomial $\mathcal{E}_{\{x,y\}}(\mathcal{X})$. This proves statement (c).

Consider the polynomial vector field (9). We obtain

$$\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z) = 2\left(x^2+y^2\right)F_1(x,y) = 2\left(x^2+y^2\right)\prod_{i=1}^{m-1}(a_i x + b_i y),$$

where $a_i, b_i \in \mathbb{R}$, $a_i^2 + b_i^2 \neq 0$, $(a_i,b_i) \neq (a_j,b_j)$, $i \neq j$, and

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = P_3(x,y,z) = 2(x^2+y^2)F_8(z),$$

that is

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = \begin{cases} 2(x^2+y^2) & \text{if } m=2, \\ 2(x^2+y^2)\displaystyle\prod_{i=1}^{m-2}(z-z_i), \ z_i \neq z_j, \ i\neq j, \ z_i \in \mathbb{R} & \text{if } m>2. \end{cases}$$

Therefore the vector field $\mathcal{X}$ has exactly $2(m-1)$ invariant meridians and $m-2$ invariant parallels. We have proved statement (d).

Now we prove statement (e). Consider the polynomial vector field (10). By a straightforward calculation we have

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = P_3(x,y,z) = 2xF_9(z) = \begin{cases} 2\varepsilon x z^{m-1} & \text{if } k=1, \\ 2\varepsilon x z^{m-k}\displaystyle\prod_{i=1}^{k-1}(z-i) & \text{if } k>1, \end{cases}$$

where $\varepsilon > 0$ small. This implies that, for $1 \leq k \leq m-1$ fixed, the vector field (10) has exactly k invariant parallels at $z=i$, for $i=0,\ldots,k-1$. This vector field has no equilibria on the invariant parallels. In order to prove that these k invariant parallels are limit cycles we take the hyperboloid of one sheet as the graphs $z = \pm\sqrt{x^2+y^2-1}$ where $x^2+y^2 \geq 1$.

The hyperboloid of one sheet $x^2+y^2-z^2-1=0$ for $z \geq 0$ can be written in the explicit form $z = \sqrt{x^2+y^2-1}$ where $x^2+y^2 \geq 1$. In the coordinates (x,y) vector field (10) has the form

$$\mathcal{X}^*(x,y) = \left(-2y(x-2k) + 2\sqrt{x^2+y^2-1}\,F_9^*\left(\sqrt{x^2+y^2-1}\right), 2x(x-2k)\right)$$

where

$$F_9^*\left(\sqrt{x^2+y^2-1}\right) = \begin{cases} \varepsilon\left(\sqrt{x^2+y^2-1}\right)^{m-1} & \text{if } k=1, \\ \varepsilon\left(\sqrt{x^2+y^2-1}\right)^{m-k}\displaystyle\prod_{i=1}^{k-1}\left(\sqrt{x^2+y^2-1}-i\right) & \text{if } k>1, \end{cases}$$

and $\varepsilon > 0$ small. In polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ the above vector field is equivalent to

$$\frac{dr}{d\theta} = \begin{cases} -\dfrac{\varepsilon\left(r^2-1\right)^{m/2} r\cos\theta}{\varepsilon\left(r^2-1\right)^{m/2}\sin\theta - r^2\cos\theta + 2r} & \text{if } k=1,\\[2ex] \dfrac{\varepsilon\left(r^2-1\right)^{(m-k+1)/2} r\prod_{i=1}^{k-1}\left(\sqrt{r^2-1}-i\right)\cos\theta}{r^2\cos\theta - 2rk - \varepsilon\sin\theta\left(r^2-1\right)^{(m-k+1)/2}\prod_{i=1}^{k-1}\left(\sqrt{r^2-1}-i\right)} & \text{if } k>1. \end{cases}$$

Expanding $dr/d\theta$ in Taylor series with respect to ε at $\varepsilon = 0$ we obtain

$$\frac{dr}{d\theta} = \begin{cases} \varepsilon\dfrac{\cos\theta\left(r^2-1\right)^{m/2}}{r\cos\theta-2} + O(\varepsilon^2) & \text{if } k=1,\\[2ex] \varepsilon\dfrac{\cos\theta\left(r^2-1\right)^{(m-k+1)/2}\prod_{i=1}^{k-1}\left(\sqrt{r^2-1}-i\right)}{r\cos\theta-2k} + O(\varepsilon^2) & \text{if } k>1. \end{cases} \tag{23}$$

Equation (23) satisfies the hypotheses of Theorem 11. The averaged function (21) is given by

$$f(r) = \begin{cases} \left(r^2-1\right)^{m/2} g_1(r) & \text{if } k=1,\\[1ex] (r^2-1)^{(m-k+1)/2}\displaystyle\prod_{i=1}^{k-1}\left(\sqrt{r^2-1}-i\right) g_2(r) & \text{if } k>1, \end{cases}$$

where

$$g_1(r) = \frac{\sqrt{4-r^2}-2}{r\sqrt{4-r^2}}, \quad g_2(r) = \frac{\sqrt{4k^2-r^2}-2k}{r\sqrt{4k^2-r^2}}.$$

The hyperboloid of one sheet $x^2+y^2-z^2-1=0$ for $z \le 0$ can be written in the explicit form $z = -\sqrt{x^2+y^2-1}$ where $x^2+y^2 \ge 1$. Repeating the above calculations for this case we obtain for the averaged function (21)

$$f(r) = \begin{cases} (-1)^m\left(r^2-1\right)^{m/2} g_1(r) & \text{if } k=1,\\[1ex] (-1)^{m-k}(r^2-1)^{(m-k+1)/2}\displaystyle\prod_{i=1}^{k-1}\left(-\sqrt{r^2-1}-i\right) g_2(r) & \text{if } k>1, \end{cases}$$

where g_1 and g_2 are the same functions of the previous case.

We can see that $g_1(r) < 0$ for $1 \le r < 2$, and $g_2(r) < 0$ for $1 \le r < 2k$. Therefore the simple zeros of f are given by $r = 1$ when $k = 1$, and by $r = 1$ and $r = \sqrt{i^2+1}$ for $i = 1, \ldots, k-1$ when $k > 1$, which correspond to the k limit cycles of $\mathcal{X}^*$. The stability of a limit cycle at $r = \sqrt{i^2+1}$, for $i = 1, \ldots, k-1$ when $k > 1$ is easily determined. On the other hand, the stability of the limit cycle at $r = 1$ is as follows:

(i) When $k = 1$: stable if m is even and semi–stable if m is odd;

(ii) When $k > 1$: stable if m and k are odd, unstable if m and k are even and semi–stable if either m is odd and k is even, or m is even and k is odd.

In short, we have proved Theorem 4.

3.5. **Proof of Theorem 5.** The proofs of statements (c), (d) and (e) of Theorem 5 are similar to the ones of Theorems 3 and 4 and they will be omitted here.

3.6. **Proof of Theorem 6.** In this subsection we give the proof of statement (d) of Theorem 6. The proof of statement (c) of Theorem 6 is similar to the one of Theorems 3 and 4 and it will be omitted here.

Consider the polynomial vector field (13). For this vector field we obtain

$$\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z) = (x^2+y^2)F_1(x,y), \quad \mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = -(x^2+y^2)F_{12}(z).$$

Thus a polynomial of the form $ax+by$ only divides F_1 in the first equation. So $\mathcal{X}$ has $m-1$ invariant meridians on $\mathbb{S}^2$. A polynomial $z-z_i$ only divides F_{12} in the second equation, then $\mathcal{X}$ has $m-2$ invariant parallels. This concludes the proof of statement (d) and Theorem 6 is proved.

3.7. **Proof of Theorem 7.** In this subsection we give the proof of Theorem 7. We separate the proof of statement (a) in two cases.

Case 1. Consider the quadratic vector field

$$\mathcal{X}(x,y,z) = \left(2\varepsilon xy, -2\varepsilon x^2 - 2z(y+2), 2y(y+2)\right), \tag{24}$$

on $\mathbb{S}^2$ with $\varepsilon > 0$ small. For this $\mathcal{X}$ we get

$$\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z) = -2x\left(\varepsilon(x^2+y^2) + z(y+2)\right) = -2x\left(\varepsilon(1-z^2) + z(y+2)\right).$$

Since that $\varepsilon(1-z^2)+z(y+2)$ has no factors of the form $ax+by$, we get that $\mathcal{X}$ has $x=0$ as the unique invariant meridian on $\mathbb{S}^2$. It is easy to see that this vector field has no equilibria on the invariant meridian $x=0$. In order to complete the proof we need to show that this invariant meridian is a limit cycle.

Note that the sphere $x^2+y^2+z^2=1$ can be written in the explicit forms $x=\pm\sqrt{1-y^2-z^2}$ with $y^2+z^2 \le 1$. In the coordinates (y,z) vector field (24) has the form

$$\mathcal{X}^*(y,z) = \left(-2\varepsilon(1-y^2-z^2) - 2z(y+2), 2y(y+2)\right).$$

In polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$ the above vector field writes

$$\frac{dr}{d\theta} = \frac{\varepsilon r(r^2-1)\cos\theta}{\varepsilon\sin\theta(1-r^2) + r(2+r\cos\theta)}.$$

Expanding $dr/d\theta$ in Taylor series with respect to ε at $\varepsilon=0$ we have

$$\frac{dr}{d\theta} = \varepsilon\frac{(r^2-1)\cos\theta}{2+r\cos\theta} + O(\varepsilon^2). \tag{25}$$

Equation (25) satisfies the hypotheses of Theorem 11 and the correspondent averaged function is given by

$$\begin{aligned} f(r) =& \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2-1)\cos\theta}{2+r\cos\theta}d\theta \\ =& \frac{(r^2-1)}{2\pi}\int_0^{2\pi} \frac{\cos\theta}{2+r\cos\theta}d\theta = (r^2-1)g(r), \end{aligned}$$

where

$$g(r) = \frac{\sqrt{4-r^2}-2}{r\sqrt{4-r^2}}.$$

It is easy to check that $g(r) < 0$ for $0 < r \le 1$. Therefore the unique simple zero of f is given by $r=1$, which provides a stable limit cycle of $\mathcal{X}^*$.

Case 2. Consider now the polynomial vector field of degree $m > 2$

$$\mathcal{X}(x,y,z) = \left(-2y\left(xy^{m-2}+z\right) + 2z, 2\left(x^2+z^2\right)y^{m-2}, -2x + 2y\left(x - y^{m-2}z\right)\right)$$

on $\mathbb{S}^2$. For this vector field we obtain

$$\mathcal{E}_{\{x,y\}}(\mathcal{X})(x,y,z) = y\left(2x(x^2+y^2)y^{m-3} + 2y\left(xy^{m-2}+z\right) - 2z\right).$$

We can see that y is the unique factor of the form $ax+by$ in the above extactic polynomial. Thus $y=0$ is the unique invariant meridian on $\mathbb{S}^2$. The equilibria of $\mathcal{X}$ are $P_\pm = (0,\pm 1, 0)$ and the second component of $\mathcal{X}$ is $P_2(x,y,z) = 2(x^2+z^2)y^{m-2}$ whose sign depends only on the variable y and of the power $m-2$. Therefore $y=0$ is a limit cycle and its stability depends on the degree m of the vector field. This completes the proof of statement (a) of Theorem 7.

(b) For $1 \le k \le m-1$ fixed consider the polynomial vector field

$$\mathcal{X}(x,y,z) = \left(-2y(x-2) + 2zF_{13}(z), 2x(x-2) - 2zF_{13}(z), 2(y-x)F_{13}(z)\right),$$

on $\mathbb{S}^2$ where

$$F_{13}(z) = \begin{cases} \varepsilon z^{m-1} & \text{if } k=1, \\ \varepsilon z^{m-k}\displaystyle\prod_{i=1}^{k-1}(z-z_i), \quad z_i \in (-1,1), \quad z_i \neq 0 & \text{if } k>1, \end{cases}$$

with $\varepsilon > 0$ small. From a direct calculation we obtain

$$\mathcal{E}_{\{1,z\}}(\mathcal{X})(x,y,z) = 2(y-x)F_{13}(z).$$

This implies that, for $1 \le k \le m-1$ fixed, this vector field has exactly k invariant parallels given by $z=0$ if $k=1$, and $z=z_i$ and $z=0$ if $k>1$. It follows easily that the vector field has no equilibria on the invariant parallels. Similarly to the proof of Theorem 4 we have that the k invariant parallels are limit cycles. The limit cycle at $z=0$ can be stable, unstable or semi–stable depending on the choice of m, k and $z_i \in (-1,1)$, $z_i \neq 0$. For example, if $k=1$ then the limit cycle at $z=0$ is stable if m is even, and it is semi–stable if m is odd.

In short, Theorem 7 is proved.

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