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GLOBAL PHASE PORTRAITS OF UNIFORM ISOCHRONOUS CENTERS WITH QUARTIC HOMOGENEOUS POLYNOMIAL NONLINEARITIES

JACKSON ITIKAWA[1] AND JAUME LLIBRE[1]

Abstract. We classify the global phase portraits in the Poincaré disc of the differential systems $\dot{x} = -y + xf(x, y)$, $\dot{y} = x + yf(x, y)$, where $f(x, y)$ is a homogeneous polynomial of degree 3. These systems have a uniform isochronous center at the origin. This paper together with the results presented in [9] completes the classification of the global phase portraits in the Poincaré disc of all quartic polynomial differential systems with a uniform isochronous center at the origin.

1. Introduction and statement of the main results

Isochronicity appears in many physical phenomena and it has a close relation to the stability theory and to the uniqueness and existence of solutions for some boundary value and bifurcation problems, for further information on these topics see [3]. The first researches on isochronous systems date back at least to the XVII century, even before the development of differential calculus, with the investigation of the cycloidal pendulum by Christian Huygens, see [6].

In the last decades the study of isochronicity, specially in the case of polynomial differential systems, has been boosted because of the proliferation of more powerful methods of computerized analysis, see for instance [1, 2, 4, 8, 12] and the bibliography therein.

Let $p \in \mathbb{R}^2$ be a center of a planar differential polynomial system. Without loss of generality we can assume that p is at the origin of coordinates. Then p is an *isochronous center* if it is a center having a neighborhood such that all the periodic orbits in this neighborhood have the same period. Moreover p is a *uniform isochronous center* if the system, in polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, is of the form

$$\dot{r} = G(\theta, r), \quad \dot{\theta} = k, \tag{1}$$

with $k \in \mathbb{R} \setminus \{0\}$. For further information see Conti [4].

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In this paper we classify the global phase portraits in the Poincaré disc of the uniform isochronous centers with quartic homogeneous polynomial nonlinearities. Our results together with those presented in [9] completes the classification of the global phase portraits in the Poincaré disc of all quartic polynomial differential systems with a uniform isochronous center at the origin.

Our main result is the following.

Theorem 1. *Let*

$$\dot{x} = -y + xf(x,y), \quad \dot{y} = x + yf(x,y), \tag{2}$$

be a polynomial differential system of degree 4, such that $f(x,y)$ is a cubic homogeneous polynomial. Then any quartic polynomial differential system which can be written into the form (2) has a uniform isochronous center at the origin and its global phase portrait is topologically equivalent to one of the 3 phase portraits of Figure 1.

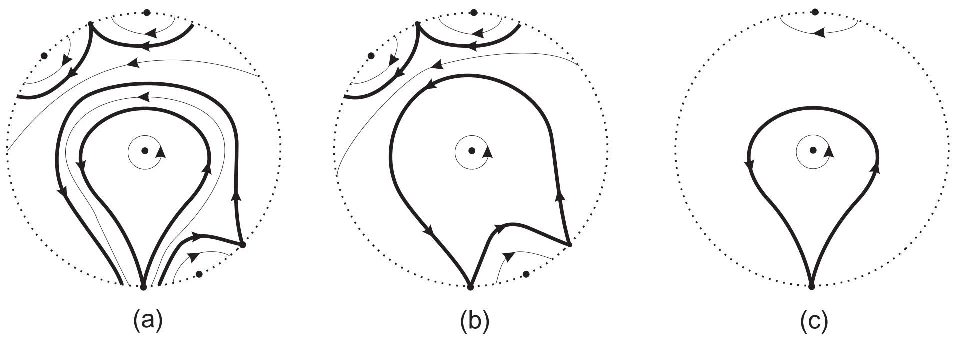

FIGURE 1. Phase portraits of the uniform isochronous centers with quartic homogeneous polynomial nonlinearities.

Our results have been checked with the software *P4*, see for further information on this software the Chapters 9 and 10 of [5].

The rest of the paper is organized as follows. In section 2 we present some concepts and results used in our study. In section 3 we prove Theorem 1.

2. Preliminary results

In this section we present some results we shall use in our study.

2.1. **Poincaré compactification** Consider $\mathcal{X}$ a planar polynomial vector field of degree n. The *Poincaré compactified vector field $p(\mathcal{X})$ corresponding to $\mathcal{X}$* is an analytic vector field induced on $\mathbb{S}^2$ as follows, for further details see for instance [7], or Chapter 5 of [5]. Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the so called *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at the point y. Moreover, consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \to \mathbb{S}^2$. This map defines 2 copies of $\mathcal{X}$, one in the northern hemisphere and the other in the southern one. Denote by $\mathcal{X}'$ the vector field $Df \circ \mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1 = \{y \in \mathbb{S}^2 : y_3 = 0\}$. Note that $\mathbb{S}^1$ is identified to the *infinity* of $\mathbb{R}^2$. Then $p(\mathcal{X})$ is the only analytic extension of $y_3^{n-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2\backslash\mathbb{S}^1$ there are two symmetric copies of $\mathcal{X}$, and studying the behavior of $p(\mathcal{X})$ around $\mathbb{S}^1$, we obtain the behavior of $\mathcal{X}$ at infinity. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$ is known as the *Poincaré disc,* and it is denoted by $\mathbb{D}^2$. One important property of the Poincaré compactification is that $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

Since $\mathbb{S}^2$ is a differentiable manifold we consider the six local charts $U_i = \{y \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y \in \mathbb{S}^2 : y_i < 0\}$ where $i = 1, 2, 3$ for computing the expression for $p(\mathcal{X})$. The diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1, 2, 3$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, and $(0, 0, -1)$ respectively. We denote by (u, v) the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$. Note that (u, v) represents different things according to the local charts under consideration.

In the local chart (U_1, F_1), $p(\mathcal{X})$ is written as

$$\dot{u} = v^n \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right)\right], \quad \dot{v} = -v^{n+1}P\left(\frac{1}{v}, \frac{u}{v}\right),$$

and the expression for $p(\mathcal{X})$ in the local chart (U_2, F_2) is

$$\dot{u} = v^n \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right)\right], \quad \dot{v} = -v^{n+1}Q\left(\frac{u}{v}, \frac{1}{v}\right),$$

and finally for (U_3, F_3) it is

$$\dot{u} = P(u, v), \quad \dot{v} = Q(u, v).$$

The expression for $p(\mathcal{X})$ in each chart (V_i, G_i) is the same as in the chart (U_i, F_i), multiplied by $(-1)^{n-1}$, $i = 1, 2, 3$. The points of $\mathbb{S}^1$ in any chart have $v = 0$. Therefore we have a polynomial vector field in each local chart.

The unique singular points at infinity which cannot be contained into the charts $U_1 \cup V_1$ are the origins of U_2 and V_2. Then, when we study the infinite singular points on the charts $U_2 \cup V_2$, we only have to verify if the origin of these charts are singularities.

2.2. **Topological equivalence** Two polynomial vector fields X and Y on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ which preserves the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $p(X)$ into orbits of the flow induced by $p(Y)$, preserving or reversing simultaneously the sense of all orbits.

A *separatrix* of $p(X)$ is an orbit which is either a singular point, or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a finite or infinity singular point.

We denote by $\text{Sep}(p(X))$ the set formed by all separatrices of $p(X)$. The set $\text{Sep}(p(X))$ is closed, see [11]. Each open connected component of $\mathbb{S}^2 \setminus \text{Sep}(p(X))$ is called a *canonical region* of $p(X)$. A *separatrix configuration* is a union of $\text{Sep}(p(X))$ plus one representative solution chosen from each canonical region. Moreover, $\text{Sep}(p(X))$ and $\text{Sep}(p(Y))$ are *equivalent* if there exists a homeomorphism in $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of $\text{Sep}(p(X))$ into orbits of $\text{Sep}(p(Y))$, preserving or reversing simultaneously the sense of all orbits.

The next result is due to Neumann [11] and characterizes the topologically equivalence between two Poincaré compactified vector fields.

Theorem 2. *Let X and Y be two polynomial vector fields in $\mathbb{R}^2$. If $p(X)$ and $p(Y)$ have finitely many separatrices, then $p(X)$ and $p(Y)$ are topologically equivalent if and only if their separatrix configurations are equivalent.*

Theorem 2 implies that, to obtain the global phase portrait of a polynomial vector field $p(X)$ with finitely many separatrices, we need to determine the separatrices of $p(X)$ and one orbit in each canonical region.

Using the arguments of the proof of Theorem 2 the next result follows.

Theorem 3. *Let X and Y be two polynomial vector fields in $\mathbb{R}^2$. If $p(X)$ and $p(Y)$ have the infinity filled of singular points and finitely many separatrices in $\mathbb{R}^2$, then $p(X)$ and $p(Y)$ are topologically equivalent if and only if their separatrix configurations are equivalent.*

According to Theorem 3, in order to have the global phase portrait of a polynomial vector field X with the infinity filled of singular points and finitely many separatrices in $\mathbb{R}^2$, we need to determine the separatrices of $p(X)$ and one orbit in each canonical region.

2.3. **Some results on the uniform isochronous centers**

Proposition 4. *Assume that a planar differential polynomial system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ of degree n has a center at the origin of coordinates. Then this center is uniform isochronous if and only if by doing a linear change of variables and a rescaling of time it can be written as*

$$\dot{x} = -y + x\, f(x, y), \quad \dot{y} = x + y\, f(x, y), \tag{3}$$

with $f(x, y)$ a polynomial in x and y of degree $n-1$, $f(0, 0) = 0$.

See for instance [9] for a proof of Proposition 4.

In the case of homogeneous uniform isochronous centers, Conti provided the following result in Theorem 2.1 of [4].

Theorem 5. *Let* $f(x,y) = \sum_{i+j=n-1} f_{i,j} x^i y^j$ *be a homogeneous polynomial of degree* $n-1$. *Then system* (3) *has a uniform isochronous center at the origin if either* n *is even, or if* n *is odd and*
$$\sum_{\nu=0}^{n-1} \left[f_{n-1-\nu,\nu} \int_0^{2\pi} \cos^{n-1-\nu}\theta \sin^{\nu}\theta \, d\theta \right] = 0.$$

We remark that by Theorem 5 all the homogeneous quartic polynomial systems of the form (3) have a uniform isochronous center at the origin.

The following result and its proof can be found in Theorem 3.1 of [4].

Theorem 6. *Consider the differential system* (3) *of degree* n. *If* $n > 1$ *and the origin is a uniform isochronous center then it cannot be a global center.*

3. Proof of Theorem 1

The fact that any quartic polynomial differential system of the form (2) has a uniform isochronous center at the origin is a direct consequence of Theorem 5.

In order to provide all possible phase portraits in the Poincaré disc for the uniform isochronous system (2) with homogeneous nonlinearities of degree 4, we shall study the finite and infinite singular points of such systems.

3.1. **Finite singular points** In polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ the differential system (2) is written as (1). It follows that the origin is the unique finite singular point of (2).

More generally, if a planar polynomial differential system has a uniform isochronous center at the origin, this system can be written as (1). Thus no polynomial differential system can have more than one uniformly isochronous center. On the other hand, there are differential systems with more than one isochronous center, see for instance [10].

Theorem 6 together with the fact that the origin is the unique finite singular point in system (2) imply that the boundary of the period annulus of the uniform isochronous center at the origin is a graphic formed by infinite singular points and their separatrices.

3.2. **Infinite singular points** In the chart U_1 the differential system (2) when $f(x,y)$ is a homogeneous polynomial of degree 3 becomes

$$\begin{aligned} \dot{u} &= v^3(1+u^2), \\ \dot{v} &= v(uv^4 - f(1,u)). \end{aligned} \tag{4}$$

Therefore all the points $(u,0)$, for all $u \in \mathbb{R}$ are infinite singular points in U_1. In order to obtain the local phase portraits near the infinity, we rescale system (4) doing $ds = vdt$ and we obtain

$$
\begin{aligned}
u' &= v^3(1+u^2),\\
v' &= v(uv^4 - f(1,u)),
\end{aligned}
\tag{5}
$$

where the prime denotes derivative with respect to s.

Now the infinite singular points of system (5) are $(u^*,0)$ with u^* a zero of $f(1,u)$. So at the chart U_1 we have at most 3 infinite singular points.

Since at most one more additional infinite singular point can appear, which is the origin of the chart U_2, without loss of generality we can assume that all the infinite singular points of system (2) after the rescaling $ds = vdt$ are in the local chart U_1, otherwise doing a rotation in the coordinates (x,y) this would be the case. So in what follows we do not need to study whether the origin of the chart U_2 is an infinite singular point.

To investigate the infinite singular points of system (2), we need to split our study into several cases, according to the cubic homogeneous polynomial $f(x,y)$. Taking into account that we can assume that all the infinite singular points after the rescaling $ds = vdt$ are in the local chart U_1, the polynomial $f(x,y)$ must have one of the following expressions, with $a \neq 0$

$f_1 = a(y - r_1x)(y - r_2x)(y - r_3x)$, $r_1 < r_2 < r_3$,

$f_2 = a(y - r_1x)^2(y - r_2x)$, $r_1 < r_2$,

$f_3 = a(y - r_1x)^3$,

$f_4 = (\alpha x^2 + \beta xy + \gamma y^2)(y - r_1x)$, $\beta^2 - 4\alpha\gamma < 0$,

$f_5 = (\alpha x^2 + \beta xy + \gamma y^2)y$, $\beta^2 - 4\alpha\gamma < 0$.

In the polynomial f_k, $k = 1,2,3$ we can assume that $a = 1$ in system (2) by doing the rescaling $(x,y) \mapsto (x/\sqrt[3]{a}, y/\sqrt[3]{a})$, and otherwise we can assume $\gamma = 1$ applying the rescaling $(x,y) \mapsto (x/\sqrt[3]{\gamma}, y/\sqrt[3]{\gamma})$.

In short we must study the phase portraits of the uniform isochronous system (2) with $f(x,y)$ in one of the following cases.

Case I: $f(x,y) = (y - r_1x)(y - r_2x)(y - r_3x)$, $r_1 < r_2 < r_3$;

Case II: $f(x,y) = (y - r_1x)^2(y - r_2x)$, $r_1 < r_2$;

Case III: $f(x,y) = (y - r_1x)^3$;

Case IV: $f(x,y) = (\alpha x^2 + \beta xy + y^2)(y - r_1x)$, with $\beta^2 - 4\alpha < 0$;

Case V: $f(x,y) = (\alpha x^2 + \beta xy + y^2)y$, with $\beta^2 - 4\alpha < 0$.

Except for the Cases **I** and **IV**, in which system (2) depends of 3 parameters, in all other cases it depends at most of 2 parameters.

For the characterization of each local phase portrait, we shall apply the well known results for the hyperbolic and nilpotent singular points, see for instance Theorems 2.15 and 3.5 of [5]. In what follows we study each case in detail.

Case I. In the chart U_1 the differential system (2) becomes

$$\begin{aligned} \dot{u} &= (1+u^2)v^3, \\ \dot{v} &= v[r_1r_2r_3 - (r_1r_2 + r_1r_3 + r_2r_3)u + (r_1+r_2+r_3)u^2 - u^3 + uv^3], \end{aligned} \tag{6}$$

Performing the rescaling of time $ds = vdt$ system (6) writes as

$$\begin{aligned} u' &= (1+u^2)v^2, \\ v' &= r_1r_2r_3 - (r_1r_2 + r_1r_3 + r_2r_3)u + (r_1+r_2+r_3)u^2 - u^3 + uv^3. \end{aligned} \tag{7}$$

The singular points at infinity are $p_1 = (r_1, 0)$, $p_2 = (r_2, 0)$ and $p_3 = (r_3, 0)$. The linear parts of system (7) at each of these points are respectively

$$\begin{pmatrix} 0 & 0 \\ (r_2 - r_1)(r_1 - r_3) & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ (r_1 - r_2)(r_2 - r_3) & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 \\ (r_1 - r_3)(r_3 - r_2) & 0 \end{pmatrix}.$$

Since $r_1 < r_2 < r_3$ the terms $r_i - r_j$ for $i \neq j$, $i, j = 1, 2, 3$ never vanish. Consequently the corresponding linear parts of system (7) at p_1, p_2 and p_3 are never identically zero and thus they are nilpotent singular points. For each singular point, we perform appropriate translations and rescalings of time to have system (7) under the normal form necessary to apply Theorem 3.5 of [5]. Taking into account the hypothesis $r_1 < r_2 < r_3$, we conclude that each one of these 3 singular points is a cusp. Therefore modulus a translation to the origin and undoing the rescaling of time $ds = vdt$, the local phase portrait for each singular point of system (6) might be one of the two shown in Figure 2.

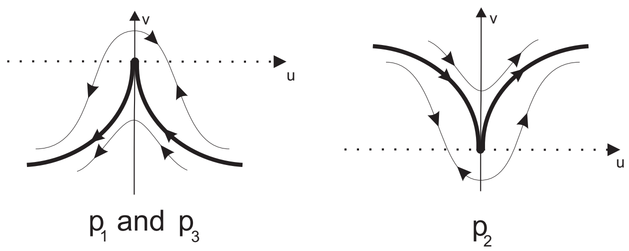

FIGURE 2. Local phase portraits at p_1, p_2 and p_3 of system (6). The horizontal axis is filled of singular points.

Each global phase portrait for this case is obtained taking into account: all the local phase portraits of the finite and infinite singular points; the existence and uniqueness theorem for the solutions of a differential system; the boundary of the Poincar'e disc consists entirely of singular points; and that the graphic at the boundary of the period annulus of the uniform isochronous center at the origin is formed by separatrices of infinite singular points. Hence the global phase portrait for Case I is topologically equivalent to the ones of Figure 1(a) or (b) of Theorem 1. We remark that the two configurations are possible by setting convenient values to the real parameters r_1, r_2 and r_3.

Case II. In the chart U_1 system (2) is written as

$$\begin{aligned} \dot{u} &= (1+u^2)v^3, \\ \dot{v} &= v[r_1^2 r_2 - (r_1^2 + 2r_1 r_2)u + (2r_1 + r_2)u^2 - u^3 + uv^3], \end{aligned} \tag{8}$$

and after the rescaling of time $ds = vdt$ system (8) becomes

$$\begin{aligned} u' &= (1+u^2)v^2, \\ v' &= r_1^2 r_2 - (r_1^2 + 2r_1 r_2)u + (2r_1 + r_2)u^2 - u^3 + uv^3. \end{aligned} \tag{9}$$

The singular points at infinity are $p_1 = (r_1, 0)$ and $p_2 = (r_2, 0)$. We first analyze p_1. The corresponding linear part of system (9) at this singular point is identically zero. Thus it is necessary to apply a directional blow up $(u, v) \mapsto (u, w)$ where $v = uw$, obtaining the following system, modulo a translation of p_1 to the origin

$$\begin{aligned} \dot{u} &= u^2(1 + r_1^2 + 2r_1 u + u^2)w^2, \\ \dot{w} &= u[r_2 - r_1 - u - (1 + r_1^2)w^3 - r_1 uw^3]. \end{aligned} \tag{10}$$

Performing a change of the independent variable of the form $dT = u\, ds$ in (10), we get the system

$$\begin{aligned} u' &= u(1 + r_1^2 + 2r_1 u + u^2)w^2, \\ w' &= r_2 - r_1 - u - (1 + r_1^2)w^3 - r_1 uw^3, \end{aligned} \tag{11}$$

where the prime now denotes derivative with respect to T. On the axis $u = 0$ there is a unique singularity $q_1 = (0, \sqrt[3]{(r_2 - r_1)/(1 + r_1^2)})$. The corresponding linear part of system (11) at q_1 is

$$\begin{pmatrix} -3(1 + r_1^2)^{1/3}(r_2 - r_1)^{2/3} & 0 \\ 0 & (1 + r_1^2)^{1/3}(r_2 - r_1)^{2/3} \end{pmatrix}.$$

Applying Theorem 2.15 of [5] and the hypothesis $r_1 < r_2$ we conclude that q_1 is a saddle. The local phase portrait at q_1 for system (11) and system (10) are shown in Figures 3 and 4, respectively

Going back through the blow up we get the local phase portrait at the origin of system (9), see Figure 5. Finally, taking into account the rescaling of time $ds = vdt$, we obtain that the phase portrait at the origin of system (8) is topologically equivalent to the one of Figure 6.

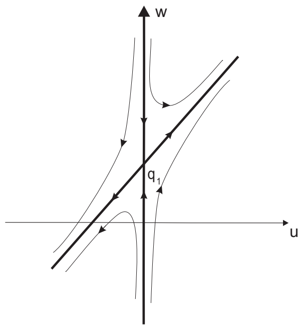

FIGURE 3. Phase portrait of system (11).

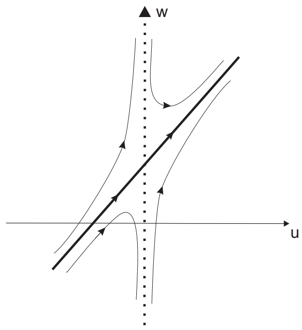

FIGURE 4. Phase portrait of system (10). The vertical axis is filled of singular points.

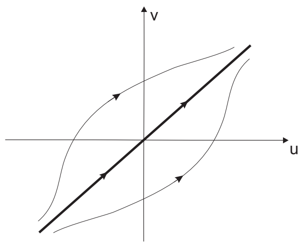

FIGURE 5. Phase portrait of system (9).

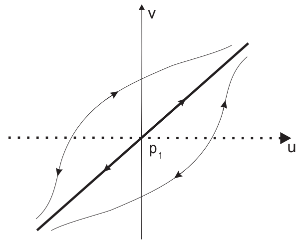

FIGURE 6. Phase portrait of system (6). The horizontal axis is filled of singular points.

Now we perform the study for p_2. The corresponding linear part of system (9) at this singular point is

$$\begin{pmatrix} 0 & 0 \\ -(r_1 - r_2)^2 & 0 \end{pmatrix}.$$

Since $r_1 < r_2$ by hypothesis, $(r_1 - r_2)$ never vanishes. Therefore p_2 is a nilpotent singular point. By performing convenient translation and rescaling of time to have system (9) under the normal form necessary to apply Theorem 3.5 of [5], and taking into account the hypothesis $r_1 < r_2$, we conclude that this singular point is a cusp. Therefore modulus a translation to the origin and undoing the rescaling of time $ds = vdt$, the local phase portrait of system (8) for p_2 is topologically equivalent to the picture on the left side of Figure 2. The global phase portrait for Case II is topologically equivalent to the one of Figure 1(c) of Theorem 1.

Case III. In the chart U_1 system (2) becomes

$$\begin{aligned} \dot{u} &= (1+u^2)v^3, \\ \dot{v} &= v(r_1^3 - 3r_1^2 u + 3r_1 u^2 - u^3 + uv^3), \end{aligned} \tag{12}$$

and after the rescaling of time $ds = vdt$ system (12) is written as

$$\begin{aligned} u' &= (1+u^2)v^2, \\ v' &= r_1^3 - 3r_1^2 u + 3r_1 u^2 - u^3 + uv^3. \end{aligned} \tag{13}$$

The only singular point at infinity is $p_1 = (r_1, 0)$. The corresponding linear part of system (13) at this singular point is identically zero. Thus it is necessary to apply a directional blow up $(u, v) \mapsto (u, w)$ where $v = uw$, obtaining the following system, after performing a translation of p_1 to the origin

$$\begin{aligned} \dot{u} &= u^2(1 + r_1^2 + 2r_1 u + u^2)w^2, \\ \dot{w} &= -u[u + (1+r_1^2)w^3 + r_1 u w^3]. \end{aligned} \tag{14}$$

Performing a change of the independent variable of the form $dT = u\,ds$ in (14), we get the system

$$\begin{aligned} u' &= u(1 + r_1^2 + 2r_1 u + u^2)w^2, \\ w' &= -(u + (1+r_1^2)w^3 + r_1 u w^3). \end{aligned} \tag{15}$$

On the axis $u = 0$ there is a unique singularity $q_1 = (0, 0)$. The corresponding linear part of system (15) at q_1 is

$$\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Applying Theorem 3.5 of [5] we conclude that q_1 is a saddle. The local phase portrait at q_1 for system (15) and system (14) are shown in Figures 7 and 8, respectively

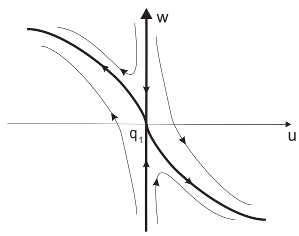

FIGURE 7. Phase portrait of system (15).

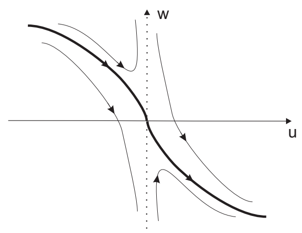

FIGURE 8. Phase portrait of system (14). The vertical axis is filled of singular points.

Going back through the blow up we get the local phase portrait at the origin of system (13), see Figure 9. Finally, we obtain that the phase portrait at the origin of system (12) is topologically equivalent to the one of Figure 10. Thus the global phase portrait for this case is topologically equivalent to the one of Figure 1(c) of Theorem 1.

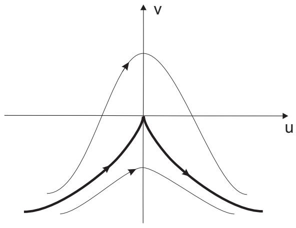

FIGURE 9. Phase portrait of system (13).

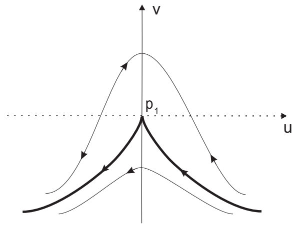

FIGURE 10. Phase portrait of system (12). The horizontal axis is filled of singular points.

Case IV. In the chart U_1 system (2) is

$$\begin{aligned}\dot u &= (1+u^2)v^3,\\ \dot v &= v[r_1\alpha + (r_1\beta-\alpha)u + (r_1-\beta)u^2 - u^3 + uv^3].\end{aligned} \tag{16}$$

We perform the rescaling of time $ds = vdt$ to obtain

$$\begin{aligned}u' &= (1+u^2)v^2,\\ v' &= r_1\alpha + (r_1\beta-\alpha)u + (r_1-\beta)u^2 - u^3 + uv^3.\end{aligned} \tag{17}$$

The unique singular point at infinity is $p_1 = (r_1, 0)$ and the corresponding linear part of system (17) at p_1 is

$$\begin{pmatrix} 0 & 0 \\ -\alpha - r_1(r_1+\beta) & 0 \end{pmatrix}.$$

Due to the hypothesis $\beta^2 - 4\alpha < 0$, the expression $-\alpha - r_1(r_1+\beta)$ never vanishes. In fact, if $\alpha = -r_1(r_1+\beta)$ then by the hypothesis we would have $(\beta+2r_1)^2 < 0$ which is obviously a contradiction. Thus p_1 is a nilpotent singular point. Applying Theorem 3.5 of [5] we conclude that the resulting local phase portrait at the origin of system (16) is topologically equivalent to the one on the left of Figure 2. This local phase portrait is obtained using a similar method applied in the previous cases. The global phase portrait for Case IV is topologically equivalent to the one of Figure 1(c) of Theorem 1.

Case V. In the chart U_1 system (2) is written as

$$\begin{aligned} \dot{u} &= (1+u^2)v^3, \\ \dot{v} &= uv(-\alpha - \beta u - u^2 + v^3). \end{aligned} \tag{18}$$

We perform the rescaling of time $ds = vdt$ to obtain

$$\begin{aligned} u' &= (1+u^2)v^2, \\ v' &= u(-\alpha - \beta u - u^2 + v^3). \end{aligned} \tag{19}$$

The origin is the unique singular point at the infinity and the linear part of system (19) at $(0,0)$ is

$$\begin{pmatrix} 0 & 0 \\ -\alpha & 0 \end{pmatrix}.$$

Since $\alpha > 0$, due to the hypothesis $\beta^2 - 4\alpha < 0$, the linear part of system (19) at $(0,0)$ is never identically zero and therefore the origin is a nilpotent singular point. Applying Theorem 3.5 of [5] and a similar procedure as those applied in the previous cases, we conclude that the resulting local phase portrait at the origin of system (18) is topologically equivalent to the one on the left of Figure 2. The global phase portrait for this case is topologically equivalent to the one of Figure 1(c) of Theorem 1.

Remark 7. *From the proof of Theorem 1 it follows that the global phase portrait of any quartic polynomial differential system which can be written into the form* (2) *is topologically equivalent to the phase portrait (a) or (b) of Figure 1 if we are in Case I, and to the phase portrait (c) of Figure 1 otherwise.*

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain. Fax +34 935812790. Phone +34 93 5811303
E-mail address: `itikawa@mat.uab.cat, jllibre@mat.uab.cat`