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ON THE POLYNOMIAL INTEGRABILITY OF A SYSTEM MOTIVATED BY THE RIEMANN ELLIPSOID PROBLEM

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Abstract. We consider differential systems obtained by coupling two Euler–Poinsot systems. The motivation to consider such systems can be traced back to the Riemann ellipsoid problem. We provide new cases for which these systems are completely integrable. We also prove that these systems either are completely integrable or have at most four functionally independent polynomial first integrals.

1. Introduction and statement of the main results

Consider the following system of differential equations

$$
\begin{aligned}
\frac{dx}{dt} &= \nabla_x G(x,y) \wedge x,\\
\frac{dy}{dt} &= \nabla_y G(x,y) \wedge y,
\end{aligned}
\tag{1}
$$

where $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ and G is the quadratic form

$$G = \frac{1}{2}\sum_{i=1}^{3}(a_i(x_i^2 + y_i^2) + 2b_i x_i y_i).$$

The a_i, b_i, $i = 1, 2, 3$ are real constants. To avoid the trivial cases, at least one of the coupling constants b_i's is assumed to be different from zero. Of course, $x = 0$ (or $y = 0$) is an invariant subspace and here system (1) reduces to the Euler–Poinsot equations. The motivation to consider such systems can be traced back to the Riemann ellipsoid problem, see for more details [3, 5, 6].

Expanding the notation, system (1) writes as

$$
\begin{aligned}
\dot{x}_1 &= (a_2 - a_3)x_2x_3 + b_2x_3y_2 - b_3x_2y_3 = P_1(x_1, x_2, x_3, y_1, y_2, y_3),\\
\dot{x}_2 &= (a_3 - a_1)x_1x_3 + b_3x_1y_3 - b_1x_3y_1 = P_2(x_1, x_2, x_3, y_1, y_2, y_3),\\
\dot{x}_3 &= (a_1 - a_2)x_1x_2 + b_1x_2y_1 - b_2x_1y_2 = P_3(x_1, x_2, x_3, y_1, y_2, y_3),\\
\dot{y}_1 &= b_2x_2y_3 - b_3x_3y_2 + (a_2 - a_3)y_2y_3 = P_4(x_1, x_2, x_3, y_1, y_2, y_3),\\
\dot{y}_2 &= b_3x_3y_1 - b_1x_1y_3 + (a_3 - a_1)y_1y_3 = P_5(x_1, x_2, x_3, y_1, y_2, y_3),\\
\dot{y}_3 &= b_1x_1y_2 - b_2x_2y_1 + (a_1 - a_2)y_1y_2 = P_6(x_1, x_2, x_3, y_1, y_2, y_3),
\end{aligned}
\tag{2}
$$

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where $(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6$ and $a_i, b_i \in \mathbb{R}$ for $i = 1, 2, 3$ such that at least one b_i is assumed to be different from zero.

It is immediate to verify that system (2) has the following polynomial three first integrals

$$H_1 = \sum_{i=1}^{3} x_i^2, \quad H_2 = \sum_{i=1}^{3} y_i^2, \quad H_3 = \sum_{i=1}^{3} [a_i(x_i^2 + y_i^2) + 2b_i x_i y_i].$$

which are functionally independent. We recall that given U an open set of $\mathbb{R}^6$ such that $\mathbb{R}^6 \setminus U$ has zero Lebesgue measure, we say that a real function $H = H(x_1, x_2, x_3, y_1, y_2, y_3) \colon U \subset \mathbb{R}^6 \to \mathbb{R}$ is a *first integral* if H is constant for all values of a solution $(x_1(t), x_2(t), x_3(t), y_1(t), y_2(t), y_3(t))$ of system (2) contained in U, i.e. H is a first integral in U if and only if

$$\sum_{i=1}^{3} \Big(\frac{\partial H}{\partial x_i} P_i(x_1, x_2, x_3, y_1, y_2, y_3) + \frac{\partial H}{\partial y_i} P_{i+3}(x_1, x_2, x_3, y_1, y_2, y_3) \Big) = 0$$

on the points of U. Moreover, the first integrals $H_1, \ldots, H_r$ are *functionally independent* if the $r \times 6$ matrix

$$\begin{pmatrix} \partial H_1/\partial x_1 & \cdots & \partial H_1/\partial y_3 \\ \vdots & \cdots & \vdots \\ \partial H_r/\partial x_1 & \cdots & \partial H_r/\partial y_3 \end{pmatrix} (x_1, x_2, x_3, y_1, y_2, y_3)$$

has rank r at all points $(x_1, x_2, x_3, y_1, y_2, y_3) \in \mathbb{R}^6$ where they are defined except perhaps in a zero Lebesgue measure set.

We are interested in finding additional polynomial first integrals which are functionally independent with H_1, H_2 and H_3.

We know that since system (2) has zero divergence it follows from Theorem 2.7 of [2] that if it has 4 functionally independent analytic first integrals then the system is completely integrable, i.e. it has 5 first integrals functionally independent first integrals.

We note that system (2) is invariant under the diffeomorphism

$$\tau(x_1, x_2, x_3, y_1, y_2, y_3, a_1, a_2, a_3, b_1, b_2, b_3) \to (x_2, x_3, x_1, y_2, y_3, y_1, a_2, a_3, a_1, b_2, b_3, b_1)$$

First we obtain some polynomial first integrals.

Theorem 1. *The differential systems* (2) *have a fourth polynomial first integral H_4 functionally independent with H_1, H_2 and H_3 if*

(a) $b_1 = \pm b_2$ *and* $a_1 = a_2$, *then* $H_4 = \pm x_3 + y_3$;
(b) $b_1 = \pm b_3$ *and* $a_1 = a_3$, *then* $H_4 = \pm x_2 + y_2$;
(c) $b_2 = \pm b_3$ *and* $a_2 = a_3$, *then* $H_4 = \pm x_1 + y_1$;
(d) $b_1 = a_2 - a_1 + b_2$, $b_3 = a_3 - a_2 - b_2$, $a_1 \neq a_2$ *and* $a_3 \neq a_2$, *then* $H_4 = (x_1 + y_1)^2 + (x_2 + y_2)^2 + (x_3 - y_3)^2$;
(e) $b_1 = a_1 - a_2 + b_2$, $b_3 = a_3 - a_2 + b_2$, $a_1 \neq a_2$ *and* $a_3 \neq a_2$, *then* $H_4 = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2$;

(f) $b_1 = a_2 - a_1 - b_2$, $b_3 = a_2 - a_3 - b_2$, $a_1 \neq a_2$ *and* $a_3 \neq a_2$, *then* $H_4 = (x_1+y_1)^2 + (x_2-y_2)^2 + (x_3+y_3)^2$;

(g) $b_1 = a_1 - a_2 - b_2$, $b_3 = a_2 - a_3 + b_2$, $a_1 \neq a_2$ *and* $a_2 \neq a_3$, *then* $H_4 = (x_1-y_1)^2 + (x_2+y_2)^2 + (x_3+y_3)^2$;

(h) $b_1 = a_1 - a_2 + b_2$, $b_3 = a_2 - a_3 - b_2$, $a_1 \neq a_2$ *and* $a_3 \neq a_2$, *then* $H_4 = \dfrac{a_2-a_3-b_3}{a_2-a_3}(x_1-y_1)^2 + \dfrac{a_1-a_3-b_3}{a_1-a_3}(x_2-y_2)^2 + (x_3-y_3)^2$;

(i) $b_1 = a_2 - a_1 - b_2$, $b_3 = a_3 - a_2 + b_2$, $a_1 \neq a_2$ *and* $a_3 \neq a_2$, *then* $H_4 = -\dfrac{a_1-a_2+b_1}{a_2-a_3}(x_1+y_1)^2 - \dfrac{b_1}{a_1-a_3}(x_2-y_2)^2 + (x_3+y_3)^2$;

(j) $b_1 = a_2 - a_1 + b_2$, $b_3 = a_2 - a_3 + b_2$, $a_1 \neq a_2$ *and* $a_2 \neq a_3$, *then* $H_4 = -\dfrac{b_2}{a_2-a_3}(x_1+y_1)^2 + \dfrac{a_1-a_2-b_2}{a_1-a_3}(x_2+y_2)^2 + (x_3-y_3)^2$;

(k) $b_1 = a_1 - a_2 - b_2$, $b_3 = a_3 - a_2 - b_2$, $a_1 \neq a_2$ *and* $a_2 \neq a_3$, *then* $H_4 = -\dfrac{b_2}{a_2-a_3}(x_1-y_1)^2 + \dfrac{a_1-a_2-b_2}{a_1-a_3}(x_2+y_2)^2 + (x_3+y_3)^2$.

The cases of integrability of Theorem 1 were already known by Negrini (see Theorems 2 and 3 of [6]), but he did not know that the fourth functionally independent first integral of systems (2) of the statements (a), (b) and (c) of Theorem 1 can be polynomial.

Theorem 1 can be checked easily by direct computations.

Corollary 2. *The differential systems* (2) *satisfying the conditions of Theorem 1 are completely integrable.*

Corollary 2 is proved in section 2, but it was also known by Negrini in [6].

Theorem 3. *The differential systems* (2) *either satisfy the conditions of Theorem 1, or have at most four functionally independent polynomial first integrals.*

Theorem 3 is proved in section 3.

In [6] the author also gaves conditions for the existence or nonexistence of meromorphic first integrals for system (2).

2. Proof of Corollary 2

The following result is due to Jacobi. For a proof in a more general setting see Theorem 2.7 of [2].

Theorem 4. *Consider an analytic differential system in* $\mathbb{R}^n$ *of the form*

$$\frac{dx}{dt} = \dot{x} = P(x), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n \tag{3}$$

with $P(x) = (P_1(x), \ldots, P_n(x))$. *Assume that*

$$\sum_{i=1}^{n} \frac{\partial P_i}{\partial x_i} = 0 \quad \textit{(i.e. it has zero divergence)}$$

and that it admits $n-2$ first integrals, $I_i(x) = c_i$ with $i = 1, \ldots, n-2$ functionally independent. These integrals define, up to a relabeling of the variables, an invertible transformation mapping from $(x_1, \ldots, x_n)$ to $(c_1, \ldots, c_{n-2}, x_{n-1}, x_n)$ given by

$$y_i = I_i(x), \quad i = 1, \ldots, n-2, \quad y_{n-1} = x_{n-1}, \quad y_n = x_n.$$

Let Δ be the Jacobian of the transformation

$$\Delta = \det \begin{pmatrix} \partial_{x_1} I_1 & \partial_{x_2} I_1 & \cdots & \partial_{x_{n-2}} I_1 \\ \partial_{x_1} I_2 & \partial_{x_2} I_2 & \cdots & \partial_{x_{n-2}} I_2 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{x_1} I_{n-2} & \partial_{x_2} I_{n-2} & \cdots & \partial_{x_{n-2}} I_{n-2} \end{pmatrix}.$$

Then system (3) admits an extra first integral given by

$$I_{n-1} = \int \frac{1}{\tilde{\Delta}} (\tilde{P}_n \, dx_{n-1} - \tilde{P}_{n-1} \, dx_n),$$

where the tilde denotes the quantities expressed in the variables $(c_1, \ldots, c_{n-2}, x_{n-1}, x_n)$. Moreover this first integral is functionally independent with the previous $n-2$ first integrals, that is, the system is completely integrable.

Proof of Theorem 2. It is immediate to verify that the differential systems (2) in $\mathbb{R}^6$ have zero divergence because every P_i does not depend on x_i for $i = 1, 2, 3$, and P_i does not depend on y_{i-3} for $i = 4, 5, 6$. In the case of the conditions given in Theorem 1 the differential systems (2) have $4 = 6-2$ first integrals functionally independent. So in this case they satisfy the assumptions of Theorem 4. Therefore this case is completely integrable. □

3. Proof of Theorem 3

We denote by $\mathbb{Z}_+$ the set of non–negative integers. The following result, due to Zhang [7], will be used in a strong way in the proof of Theorem 3.

Theorem 5. *For an analytic vector field $\mathcal{X}$ defined in a neighborhood of the origin in $\mathbb{R}^n$ associated to system (3) with $P(0) = 0$, let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $DP(0)$. Set*

$$G = \left\{ (k_1, \ldots, k_n) \in (\mathbb{Z}_+)^n : \sum_{i=1}^{n} k_i \lambda_i = 0, \sum_{i=1}^{n} k_i > 0 \right\}.$$

Assume that system (3) has $r < n$ functionally independent analytic first integrals $\Phi_1(x), \ldots, \Phi_r(x)$ in a neighborhood of the origin. If the $\mathbb{Z}$-linear space generated by G has dimension r, then any nontrivial analytic first integral of system (3) in a neighborhood of the origin is an analytic function of $\Phi_1(x), \ldots, \Phi_r(x)$.

Extensions of Theorem 5 can be found in [1, 4].

We call each element $(k_1, \ldots, k_n) \in G$ a *resonant lattice* of the eigenvalues $\lambda_1, \ldots, \lambda_n$.

Direct calculations show that the differential systems (2) have seven planes of singularities, but we only use for proving our result two of these planes of singularities.

At the singularity $S_1 = (0, x_2, 0, 0, y_2, 0)$, the 6-tuple of eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_6)$ of the linear part of the differential systems (2) are

$$\lambda = \left(0, 0, -\sqrt{\frac{A_1 - \sqrt{B_1}}{2}}, \sqrt{\frac{A_1 - \sqrt{B_1}}{2}}, -\sqrt{\frac{A_1 + \sqrt{B_1}}{2}}, \sqrt{\frac{A_1 + \sqrt{B_1}}{2}}\right), \tag{4}$$

where

$$A_1 = ((a_1 - a_2)(a_2 - a_3) - b_2^2)(x_2^2 + y_2^2) + 2((a_1 - 2a_2 + a_3)b_2 - b_1 b_3)x_2 y_2,$$
$$B_1 = A_1^2 - 4\Delta_1,$$

with

$$\begin{aligned}\Delta_1 =& ((a_2 - a_1)b_2(x_2^2 + y_2^2) + ((a_1 - a_2)^2 + b_2^2 - b_1^2)x_2 y_2)\\ & ((a_2 - a_3)b_2(x_2^2 + y_2^2) + ((a_2 - a_3)^2 + b_2^2 - b_3^2)x_2 y_2).\end{aligned}$$

From Theorem 5 we know that the number of functionally independent analytic first integrals of the differential systems (2) in a neighborhood of the singularities S is at most the number of linearly independent elements of the set

$$G_1 = \left\{(k_1, \ldots, k_6) \in (\mathbb{Z}_+)^6 : \sum_{i=1}^{6} k_i \lambda_i = 0, \sum_{i=1}^{6} k_i > 0\right\}.$$

According to the eigenvalues (4) the resonant lattices satisfy

$$\sqrt{A_1 - \sqrt{B_1}}(k_4 - k_3) + \sqrt{A_1 + \sqrt{B_1}}(k_6 - k_5) = 0. \tag{5}$$

This last equation has the following linearly independent non–negative solutions $(k_1, \ldots, k_6)$:

$$(1, 0, 0, 0, 0, 0),\ (0, 1, 0, 0, 0, 0),\ (0, 0, 1, 1, 0, 0) \text{ and } (0, 0, 0, 0, 1, 1).$$

In order that equation (5) has an additional linearly independent non–negative integer solutions different from the above list, we must have

(i) either $(A_1 - \sqrt{B_1})(A_1 + \sqrt{B_1}) = 0$;

(ii) or $(A_1 - \sqrt{B_1})(A_1 + \sqrt{B_1}) \neq 0$ and $\sqrt{A_1 - \sqrt{B_1}}/\sqrt{A_1 + \sqrt{B_1}}$ is a rational number. Then $\Delta_1 \neq 0$ and $A_1 \neq 0$ (otherwise $\sqrt{-\sqrt{B_1}}/\sqrt{\sqrt{B_1}}$ cannot be a rational number). Set

$$\frac{\sqrt{A_1 - \sqrt{B_1}}}{\sqrt{A_1 + \sqrt{B_1}}} = \frac{m}{n}, \quad m, n \in \mathbb{Z} \setminus \{0\} \text{ coprime}.$$

This last equality can be written in an equivalent way as

$$\frac{\Delta_1}{A_1^2} = \frac{m^2 n^2}{(m^2 + n^2)^2},$$

where we have used the fact that $B_1 = A_1^2 - 4\Delta_1$.

In case (i) we obtain the following independent conditions:

$$
\begin{aligned}
&b_1 = \pm b_2,\ a_1 = a_2;\\
&b_2 = \pm b_3,\ a_2 = a_3;\\
&b_1 = \pm(a_1 - a_2),\ b_2 = 0;\\
&b_3 = \pm(a_2 - a_3),\ b_2 = 0.
\end{aligned}
\tag{6}
$$

In the first four cases we are inside the conditions of Theorem 1. Now we shall consider the last four cases. We denote them by

$$s_{1,2} = \{b_1 = \pm(a_1 - a_2),\ b_2 = 0\}, \quad s_{3,4} = \{b_3 = \pm(a_2 - a_3),\ b_2 = 0\}.$$

Lemma 6. *The differential systems* (2) *under one of the conditions* s_1, s_2, s_3 *or* s_4*, either satisfy the conditions of Theorem 1, or the eigenvalues of the singularity* $S_2 = (x_1, 0, 0, y_1, 0, 0)$ *do not have a fifth linearly independent resonant lattice.*

Proof. At the singularity $S_2 = (x_1, 0, 0, y_1, 0, 0)$, the 6-tuple of eigenvalues of the linear part of the differential systems (2) are given by

$$\lambda = \left(0, 0, -\sqrt{\frac{A_2 - \sqrt{B_2}}{2}}, \sqrt{\frac{A_2 - \sqrt{B_2}}{2}}, -\sqrt{\frac{A_2 + \sqrt{B_2}}{2}}, \sqrt{\frac{A_2 + \sqrt{B_2}}{2}}\right), \tag{7}$$

where

$$
\begin{aligned}
A_2 &= -((a_1 - a_2)(a_1 - a_3) + b_1^2)(x_1^2 + y_1^2) + 2((a_2 - 2a_1 + a_3)b_1 - b_2 b_3)x_1 y_1,\\
B_2 &= A_2^2 - 4\Delta_2,
\end{aligned}
$$

with

$$
\begin{aligned}
\Delta_2 =&((a_1 - a_2)b_1(x_1^2 + y_1^2) + ((a_1 - a_2)^2 + b_1^2 - b_2^2)x_1 y_1)\\
&((a_1 - a_3)b_1(x_1^2 + y_1^2)((a_1 - a_3)^2 + b_1^2 - b_3^2)x_1 y_1).
\end{aligned}
$$

Now direct calculations show that under one of the conditions s_1, s_2, s_3, s_4, the equation $\Delta_2 = 0$ yields that either $b_1 = \pm b_3$, $a_1 = a_3$ and $b_2 = 0$; or $b_2 = b_1 = 0$ and $a_1 = a_2$; or $b_1 = b_2 = 0$ and $b_3 = \pm(a_2 - a_3) = \pm(a_1 - a_3)$. This last condition in fact splits into four different conditions. In all the cases we are under the conditions of Theorem 1. Then, under one of the conditions s_1, s_2, s_3 or s_4, either $\Delta_2 = 0$ and then we are under the conditions of Theorem 1, or $\Delta_2 \neq 0$. Now, working in a similar way as we did for the singularities S_1 for studying if there is a fifth linearly independent resonant lattice at S_1, we need to check if $\sqrt{A_2 - \sqrt{B_2}}/\sqrt{A_2 + \sqrt{B_2}} \neq 0$ is a rational number. For S_2 under one of the conditions s_1, s_2, s_3 or s_4 we write

$$\frac{\sqrt{A_2 - \sqrt{B_2}}}{\sqrt{A_2 + \sqrt{B_2}}} = \frac{m_2}{n_2}, \quad m_2, n_2 \in \mathbb{Z} \setminus \{0\} \text{ coprime.}$$

This last equation can be written as

$$\frac{\Delta_2}{A_2^2} = \frac{m_2^2 n_2^2}{(n_2 + m_2)^2}. \tag{8}$$

Clearly we have that $A_2 \neq 0$, otherwise $\sqrt{A_2 - \sqrt{B_2}}/\sqrt{A_2 + \sqrt{B_2}}$ is not a rational number. Δ_2 should be a square of $(a_1 - a_2)b_1(x_1^2 + y_1^2) + ((a_1 - a_2)^2 + b_1^2 - b_2^2)x_1 y_1$ or of $(a_1 - a_3)b_1(x_1^2 + y_1^2) + ((a_1 - a_3)^2 + b_1^2 - b_3^2)x_1 y_1$. Without loss of generality we can write it as

$$
\begin{aligned}
&(a_1 - a_2)b_1(x_1^2 + y_1^2) + ((a_1 - a_2)^2 + b_1^2 - b_2^2)x_1 y_1 \\
&= L_2^2((a_1 - a_3)b_1(x_1^2 + y_1^2) + ((a_1 - a_3)^2 + b_1^2 - b_3^2)x_1 y_1),
\end{aligned} \tag{9}
$$

and it is easy to check that $A_2/((a_1 - a_3)b_1(x_1^2 + y_1^2) + ((a_1 - a_3)^2 + b_1^2 - b_3^2)x_1 y_1)$ is a constant. Set

$$
A_2 = K_2((a_1 - a_3)b_1(x_1^2 + y_1^2) + ((a_1 - a_3)^2 + b_1^2 - b_3^2)x_1 y_1). \tag{10}
$$

Then, from (9) and (10) equating to zero the coefficients of the monomials in the variables x_1 and y_1 we have

$$
\begin{aligned}
-b_1(-a_1 + a_2 + L_2^2(a_1 - a_3)) &= 0, \\
(a_1 - a_2)^2 + b_1^2 - b_2^2 - L_2^2((a_1 - a_3)^2 + b_1^2 - b_3^2) &= 0, \\
(a_1 - a_3)(a_2 - a_1) - b_1^2 + b_1 K_2(a_3 - a_1) &= 0, \\
2b_1(a_2 + a_3 - 2a_1) - 2b_2 b_3 - K_2((a_1 - a_3)^2 + b_1^2 - b_3^2) &= 0,
\end{aligned} \tag{11}
$$

where $L_2/K_2 = m_2 n_2/(n_2^2 + m_2^2) \neq 0$. For the conditions s_1 and s_2 we have that the solutions of (11) are

$$
\begin{aligned}
&b_3 = \pm(a_1 - a_3), \quad a_1 = a_2; \\
&b_3 = \pm\frac{L_2^2 - 1}{L_2^2}(a_1 - a_2), \quad K = \pm(1 + L_2^2), \quad a_3 = a_1 + \frac{a_2 - a_1}{L_2^2},
\end{aligned}
$$

where the last condition corresponds in fact to four conditions. In all cases we are under the assumptions of Theorem 1 (note that in the last four cases we have in fact that $b_3 = \pm(a_2 - a_3)$). Finally, for the conditions s_3 and s_4 we have that the solutions of (11) are

$$
\begin{aligned}
&a_2 = a_1, \quad b_1 = 0; \\
&b_1 = \pm(a_1 - a_2) = \pm(a_1 - a_3), \quad K_2^2 = 4, \quad L_2^2 = 1; \\
&b_1 = \pm(a_1 - a_2), \quad K_2 = \mp(1 + L_2^2), \quad a_3 = a_1 \mp \frac{b_1}{L_2^2},
\end{aligned}
$$

where every one of the two last conditions correspond in fact to two conditions. In all cases we are under the assumptions of Theorem 1 (note that in the last four cases we also have that $b_3 = \pm(a_2 - a_3)$). This ends the proof of the lemma. □

From Theorem 5 and Lemma 6 we have proved that in the case (i) the differential systems (2) either satisfy the conditions of Theorem 1, or have at most four functionally independent polynomial first integrals. Next we consider the case (ii).

In case (ii) Δ_1/A_1^2 has the form $m_1^2 n_1^2/(n_1 + m_1)^2$ with $m, n \in \mathbb{Z} \setminus \{0\}$ coprime. So it follows from the expressions of Δ_1 and A_1 that Δ_1 should be a square of

$(a_2-a_1)b_2(x_2^2+y_2^2)+((a_1-a_2)^2+b_2^2-b_1^2)x_2y_2$ or of $(a_2-a_3)b_2(x_2^2+y_2^2)+((a_2-a_3)^2+b_2^2-b_3^2)x_2y_2$. Without loss of generality we can write it as

$$\begin{aligned}(a_2-a_1)b_2(x_2^2+y_2^2)+((a_1-a_2)^2+b_2^2-b_1^2)x_2y_2\\ =L_1^2((a_2-a_3)b_2(x_2^2+y_2^2)+((a_2-a_3)^2+b_2^2-b_3^2)x_2y_2),\end{aligned} \tag{12}$$

and it is easy to check that $A_1/((a_2-a_3)b_2(x_2^2+y_2^2)+((a_2-a_3)^2+b_2^2-b_3^2)x_2y_2)$ is a constant. Set

$$A_1=K_1((a_2-a_3)b_2(x_2^2+y_2^2)+((a_2-a_3)^2+b_2^2-b_3^2)x_2y_2). \tag{13}$$

Then, from (12) and (13) equating to zero the coefficients of the monomials in the variables x_2 and y_2 we have

$$\begin{aligned}-b_2(a_1-a_2+L_1^2(a_2-a_3))=0,\\ (a_1-a_2)^2-b_1^2+b_2^2-L_1^2((a_2-a_3)^2+b_2^2-b_3^2)=0,\\ (a_1-a_2)(a_3-a_2)+b_2^2+b_2K_1(a_2-a_3)=0,\\ -2b_2(a_1+a_3-2a_2)+2b_1b_3+K_1((a_2-a_3)^2+b_2^2-b_3^2)=0,\end{aligned} \tag{14}$$

where $L_1/K_1=m_1n_1/(n_1^2+m_1^2)\neq 0$.

Subcase (ii.1): If $K_1^2=4L_1^2$, solving (14), using that $(b_1,b_2,b_3)\neq(0,0,0)$ and $b_i,a_i\in\mathbb{R}$ for $i=1,2,3$ we obtain

$$\begin{aligned}&b_1=\pm(a_2-a_1),\quad b_3=0,\quad b_2=0,\quad a_2=a_3;\\ &b_1=0,\quad b_2=0,\quad a_1=a_2,\quad b_3=\pm(a_3-a_2),\\ &b_1=b_3L_1,\quad a_1=a_2+b_2L_1,\quad a_3=a_2+\frac{b_2}{L_1}.\end{aligned}$$

Note that the first four conditions are inside the conditions of Theorem 1. Now we consider the last condition.

Lemma 7. *The differential systems* (2) *under condition*

$$s_5=\Big\{b_1=b_3L_1,\ a_1=a_2+b_2L_1,\ a_3=a_2+\frac{b_2}{L_1}\Big\}$$

either satisfy the conditions of Theorem 1, or the eigenvalues of the singularity S_2 do not have a fifth linearly independent resonant lattice.

Proof. At the singularities S_2, the 6-tuple of eigenvalues of the linear part of the differential systems (2) are given in (7). Direct calculations show that under the condition s_5, the equation $\Delta_2=0$ yields that either $b_2=b_3=0$, which is not possible since otherwise $b_i=0$ for $i=1,2,3$, or $L_1^2=1$ and then $b_1=b_3$, $b_2=a_1-a_2=a_3-a_2$. Hence $a_1=a_3$. Thus, we are under the assumptions of Theorem 1. Then, under condition s_5 either $\Delta_2=0$ and we are under the assumptions of Theorem 1, or $\Delta_2\neq 0$. Now, working in a similar way as we did for the singularities S_1 for studying if there is a fifth linearly independent resonant lattice at S_1, we need to check if

$\sqrt{A_2-\sqrt{B_2}}/\sqrt{A_2+\sqrt{B_2}} \neq 0$ is a rational number. For S_2 under the condition s_5 we have that

$$\frac{\Delta_2}{A_2^2} = \frac{(L_1^2-1)(b_3^2L_1^2x_1y_1 + b_2^2(L_1^2-1)x_1y_1 + b_2b_3L_1^2(x_1^2+y_1^2))^2}{L_1^2(4b_2b_3L_1^2x_1y_1 + b_3^2L_1^2(x_1^2+y_1^2) + b_2^2(L_1^2-1)(x_1^2+y_1^2))^2}.$$

Since $L_1 \neq 0$ this shows that there always exist infinitely many singularities S_2 which cannot satisfy condition (8). At these singularities S_2 the eigenvalues do not have a fifth linearly independent resonant lattice. □

From Theorem 5 and Lemma 7 we have proved that in the case (ii.1), the differential systems (2) either satisfy the conditions in Theorem 1 or have at most four functionally independent polynomial first integrals.

Subcase (ii.2): If $K_1^2 \neq 4L_1^2$, solving (14), using that $(b_1,b_2,b_3) \neq (0,0,0)$ and $b_i, a_i \in \mathbb{R}$ for $i = 1,2,3$ we obtain (note that $K_1^2-4L_1^2 = K_1^2(m_1-n_1)^2(m_1+n_1)^2/(m_1^2+n_1^2)^2 > 0$),

$$\begin{aligned}
&b_1 = \frac{b_2}{2}(2\sigma_1 + K_1\sigma_2 + \sigma_3\sqrt{K_1^2-4L_1^2}), \quad a_1 = a_2 + \frac{b_2}{2}(K_1 + \sigma_4\sqrt{K_1^2-4L_1^2}),\\
&b_3 = \frac{b_2}{2L_1^2}(\sigma_5K_1 + 2\sigma_6L_1^2 + \sigma_7\sqrt{K_1^2-4L_1^2}), \quad a_3 = a_2 + \frac{b_2}{2L_1^2}(K_1 + \sigma_8\sqrt{K_1^2-4L_1^2});\\
&b_1 = \mp\frac{2L_1^2}{\sqrt{K_1^2-4L_1^2}}(a_2-a_3), \quad a_1 = a_2, \quad b_3 = \mp\frac{K_1}{\sqrt{K_1^2-4L_1^2}}(a_2-a_3), \quad b_2 = 0;\\
&b_1 = \frac{b_3K_1}{2}, \quad a_1 = a_2 \mp \frac{b_3}{2}\sqrt{K_1^2-4L_1^2}, \quad a_3 = a_2, \quad b_2 = 0;
\end{aligned}$$

with $(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6,\sigma_7,\sigma_8)$ equal to

$$\begin{aligned}
&(1,-1,1,-1,1,-1,-1,-1), \quad (1,1,-1,-1,1,1,-1,-1),\\
&(1,1,1,1,1,1,1,1), \quad (-1,-1,-1,1,-1,-1,-1,1),\\
&(-1,1,1,1,-1,1,-1,1), \quad (-1,-1,1,-1,-1,-1,1,-1),\\
&(1,-1,-1,1,1,-1,1,1), \quad (-1,1,-1,-1,-1,1,1,-1).
\end{aligned}$$

We remark that the second case is in fact four cases. We note that the first eight cases can be written, in particular, as one of the following four conditions

$$\begin{aligned}
b_1 &= a_2 - a_1 + b_2 \quad \text{and} \quad b_3 = a_3 - a_2 - b_2,\\
b_1 &= a_1 - a_2 + b_2 \quad \text{and} \quad b_3 = a_3 - a_2 + b_2,\\
b_1 &= a_2 - a_1 - b_2 \quad \text{and} \quad b_3 = a_2 - a_3 - b_2,\\
b_1 &= a_1 - a_2 - b_2 \quad \text{and} \quad b_3 = a_2 - a_3 + b_2.
\end{aligned}$$

These cases are all inside the conditions of Theorem 1. Now we consider the six last conditions and we set

$$
\begin{aligned}
s_{6,7} &= \Big\{b_1 = \mp\frac{2L_1^2(a_2-a_3)}{\sqrt{K_1^2-4L_1^2}}, \quad a_1=a_2, \quad b_3 = -\frac{K_1(a_2-a_3)}{\sqrt{K_1^2-4L_1^2}}, \quad b_2=0\Big\},\\
s_{8,9} &= \Big\{b_1 = \mp\frac{2L_1^2(a_2-a_3)}{\sqrt{K_1^2-4L_1^2}}, \quad a_1=a_2, \quad b_3 = \frac{K_1(a_2-a_3)}{\sqrt{K_1^2-4L_1^2}}, \quad b_2=0\Big\},\\
s_{10,11} &= \Big\{b_1 = \frac{b_3K_1}{2}, \quad a_1 = a_2 \mp \frac{b_3}{2}\sqrt{K_1^2-4L_1^2}, \quad a_3=a_2, \quad b_2=0\Big\}.
\end{aligned} \tag{15}
$$

Lemma 8. *The differential systems (2) under one of the conditions $s_6,\dots,s_9$ or s_{10}, s_{11}, either satisfy the conditions of Theorem 1, or the eigenvalues of the singularity S_2 do not have a fifth linearly independent resonant lattice.*

Proof. At the singularities S_2, the 6-tuple of eigenvalues of the linear part of the differential systems (2) are given in (7). Now direct calculations show that under one of the conditions $s_6,\dots,s_9$, the equation $\Delta_2=0$ yields $a_2=a_3$, which is not possible since otherwise $b_i=0$ for $i=1,2,3$. Then, under one of the conditions $s_6,\dots,s_9$, we have $\Delta_2\neq 0$. Now working in a similar way as we did for the singularities S_1 for studying if there is a fifth linearly independent resonant lattice at S_2, we need to check if $\sqrt{A_2-\sqrt{B_2}}/\sqrt{A_2+\sqrt{B_2}}\neq 0$ is a rational number. Since on either $s_6,\dots,s_9$ we have

$$
\left|\frac{\Delta_2}{A_2^2}\right| = \frac{L_1^2(K_1^2-4L_1^2)\big(\sqrt{K_1^2-4L_1^2}(x_1^2+y_1^2)+2(L_1^2-1)x_1y_1\big)}{2\big(-K_1^2x_1y_1+L_1^2(\sqrt{K_1^2-4L_1^2}(x_1^2+y_1^2)+4x_1y_1)\big)^2}x_1y_1,
$$

and since $L_1^2(K_1^2-4L_1^2)\neq 0$ this shows that there always exist infinitely many singularities S_2 which cannot satisfy condition (8). At these singularities S_2 the eigenvalues do not have a fifth linearly independent resonant lattice.

Now direct calculations show that under one of the conditions s_{10}, s_{11} using that $K_1^2\neq 4L_1^2$, the equation $\Delta_2=0$ yields $b_3=0$ which is not possible since otherwise $b_i=0$ for $i=1,2,3$. Then, under one the conditions s_{10}, s_{11} we have $\Delta_2\neq 0$. Now working in a similar way as we did for the cases $s_6,\dots,s_9$, we must have that under one of the conditions s_{10}, s_{11}, condition (8) must hold. However, since $\Delta_2/A_2^2 = N_1N_2/D_1$ with

$$
\begin{aligned}
N_1 &= K_1\sqrt{K_1^2-4L_1^2}(x_1^2+y_1^2)+2(K_1^2-2L_1^2)x_1y_1,\\
N_2 &= K_1\sqrt{K_1^2-4L_1^2}(x_1^2+y_1^2)+2(K_1^2-2L_1^2-2)x_1y_1,\\
D_1 &= 4\big((K_1^2-2L_1^2)(x_1^2+y_1^2)+2K_1\sqrt{K_1^2-4L_1^2}x_1y_1\big)^2,
\end{aligned}
$$

and since $L_1K_1(K_1^2-4L_1^2)\neq 0$ this shows again that there always exist infinitely many singularities S_2 which cannot satisfy condition (8). At these singularities S_2 the eigenvalues do not have a fifth linearly independent resonant lattice. □

From Theorem 5 and Lemma 8 we have proved that in the case (ii.2), the differential systems (2) either satisfy the conditions in Theorem 1 or have at most four functionally independent polynomial first integrals.

In short, if cases (i) or (ii) hold then the conditions given in Theorem 1 are satisfied and by Corollary 2 the differential systems (2) are completely integrable. If cases (i) and (ii) do not hold then by Theorem 5 the differential systems (2) can have at most four analytic integrals in a neighborhood of a point of S. Consequently the differential systems (2) have at most four functionally independent polynomial first integrals. This completes the proof of Theorem 3

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