ON THE PERIODIC SOLUTIONS OF PERTURBED 4D NON–RESONANT SYSTEMS

JAUME LLIBRE[1], DOUGLAS D. NOVAES[2] AND MARCO ANTONIO TEIXEIRA[2]

ABSTRACT. We provide sufficient conditions for the existence of periodic solutions of a 4D non–resonant system perturbed by smooth or non–smooth functions. We apply these results to study the small amplitude periodic solutions of the non–linear planar double pendulum perturbed by smooth or non–smooth function.

1. INTRODUCTION

A major problem in general perturbation theory is to detect how persistent are some given properties. In other words we want to translate some dynamical properties from the unperturbed system to the perturbed one. Often the unperturbed system is linear and the objects to be continued to the perturbed system are equilibrium points, periodic orbits or invariant torus. In this direction, the aim of this paper, is to deal with the periodic solutions of the following kind of differential systems

$$
\begin{aligned}
x'(t) &= \omega_1\, y + \varepsilon P_1(t, x, y, z, w) + \varepsilon^2 R_1(t, x, y, z, w, \varepsilon), \\
(1) \quad y'(t) &= -\omega_1\, x + \varepsilon P_2(t, x, y, z, w) + \varepsilon^2 R_2(t, x, y, z, w, \varepsilon), \\
z'(t) &= \omega_2\, w + \varepsilon P_3(t, x, y, z, w) + \varepsilon^2 R_3(t, x, y, z, w, \varepsilon), \\
w'(t) &= -\omega_2\, z + \varepsilon P_4(t, x, y, z, w) + \varepsilon^2 R_4(t, x, y, z, w, \varepsilon),
\end{aligned}
$$

where

$$
P_i(t, x, y, z, w) = P_i^1(t, x, y, z, w) + \operatorname{sign}(h_i(t, x, y, z, w)) P_i^2(t, x, y, z, w), \text{ and}
$$

$$
R_i(t, x, y, z, w, \varepsilon) = R_i^1(t, x, y, z, w, \varepsilon) + \operatorname{sign}(h_i(t, x, y, z, w)) R_i^2(t, x, y, z, w, \varepsilon).
$$

Here $P_i^j : \mathbb{R} \times D \to \mathbb{R}$, $R_i^j : \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ for $i = 1, 2, 3, 4$ and $j = 1, 2$ are smooth functions T–periodic in the variable t being T the period of some periodic solution of (1) when $\varepsilon = 0$, and D an open subset of $\mathbb{R}^4$. The functions h_i for $i = 1, 2, 3, 4$ are smooth having 0 as a regular value. The prime in (1) denotes derivative with respect to time t. We denote by $\Sigma = \{(t, x, y, z, w) \in \mathbb{R} \times D : (h_1 h_2 h_3 h_4)(t, x, y, z) = 0\}$ the set of discontinuity. Furthermore

Corresponding author Douglas D. Novaes: Departamento de Matematica, Universidade Estadual de Campinas, Caixa Postal 6065, 13083–859, Campinas, SP, Brazil. Tel. +55 19 91939832, Fax. +55 19 35216094, email: ddnovaes@gmail.com

we assume that $h_1 = h_2$ and $h_3 = h_4$. The function $\text{sgn}(z)$ denotes the sign function, i.e.

$$\text{sgn}(z) = \begin{cases} -1 & \text{if } z < 0, \\ 0 & \text{if } z = 0, \\ 1 & \text{if } z > 0. \end{cases}$$

Observe that system (1) with $\varepsilon = 0$ is simply the unperturbed system. Otherwise we have the perturbed system.

Our goal is to provide an algorithm, via tools in advanced averaging theory, for detecting the periodic orbits of the perturbed system which emerge from the set of periodic orbits of the unperturbed system. It is to be noted that a very similar scenario holds for a model of the double pendulum in which our main results will be applied.

1.1. **Background and historical facts.** The first serious proofs in the averaging theory for differential systems can be traced back to the works of Fatou [6] in 1928, Krylov and Bogoliubov [4] in the 1930s and Bogoliubov [3] in 1945. For a modern point of view on this theory see the book of Sanders and Verhulst [13].

The method of averaging plays an important role in the study of nonlinear systems related to complex behavior patterns such as bifurcation and stability of the periodic solutions of such systems. Here we need to deal with systems in the normal form of the averaging theory given by

$$(2) \qquad \dot{\mathbf{x}}(t) = G_0(t, \mathbf{x}) + \varepsilon G_1(t, \mathbf{x}) + \varepsilon^2 G_2(t, \mathbf{x}, \varepsilon),$$

see the basic results on this theory in section 3. One of the basic problems for applying the averaging theory is to write the system under study in the normal form (23).

Another tool used in this paper is the *regularization process* for discontinuous differential systems introduced by Sotomayor and Teixeira in [14]. In this process a discontinuous vector field $Z(t, x)$ is approximated by an one–parameter family of continuous vector fields $Z_\delta(t, x)$.

As far as we know, the two methods cited above has been firstly used together in [10] by Llibre and Teixeixa where it was studied the stable limit cycle of a weight–driven pendulum clock. In this paper we use the ideas presented in [10] to study the periodic solutions of (1).

1.2. **Setting the problem.** The objective of this paper is to provide a system of equations whose simple zeros provide periodic solutions of (1). In order to present our results we need some preliminary definitions and notations.

The unperturbed system (1) has the origin as its unique singular point with eigenvalues $\pm\omega_1 i, \pm\omega_2 i$. If the *non–resonant* condition $\omega_1/\omega_2 \in \mathbb{R}\backslash\mathbb{Q}$ is satisfied then this system has, in the phase space (x, y, z, w), two planes filled with periodic solutions except the origin. The periods of such periodic orbits are

$$T_1 = \frac{2\pi}{\omega_1} \quad \text{or} \quad T_2 = \frac{2\pi}{\omega_2}.$$

These periodic orbits live into the planes associated to the eigenvectors with eigenvalues $\pm\omega_1 i$ or $\pm\omega_2 i$, respectively. We shall study which of these periodic solutions persist for the perturbed system (1) when the parameter ε is sufficiently small and the functions of perturbation are either T_1 or T_2 periodic.

Let $\varphi_1(t, u, v)$ be the periodic function

$$(3) \qquad \varphi_1(t, u, v) = \Big(u\cos(\omega_1 t) + v\sin(\omega_1 t), v\cos(\omega_1 t) - u\sin(\omega_1 t), 0, 0\Big).$$

Define $\mathcal{F}_1(x, y)$:

$$
(4) \quad
\begin{aligned}
&\int_0^{T_1} \Big(\cos(\omega_1 t) P_1^1(t, \varphi_1(t, x, y)) - \sin(\omega_1 t) P_2^1(t, \varphi_1(t, x, y)) \Big) dt \\
&+ \int_0^{T_1} \Big(\cos(\omega_1 t) P_1^2(t, \varphi_1(t, x, y)) - \sin(\omega_1 t) P_2^2(t, \varphi_1(t, x, y)) \Big) \operatorname{sign}(h_1(t, \varphi_1(t, x, y))) dt,
\end{aligned}
$$

and $\mathcal{F}_2(x, y)$:

$$
(5) \quad
\begin{aligned}
&\int_0^{T_1} \Big(\cos(\omega_1 t) P_2^1(t, \varphi_1(t, x, y)) + \sin(\omega_1 t) P_1^1(t, \varphi_1(t, x, y)) \Big) dt \\
&+ \int_0^{T_1} \Big(\cos(\omega_1 t) P_2^2(t, \varphi_1(t, x, y)) - \sin(\omega_1 t) P_1^2(t, \varphi_1(t, x, y)) \Big) \operatorname{sign}(h_1(t, \varphi_1(t, x, y))) dt.
\end{aligned}
$$

Now let $\varphi_2(t, u, v)$ be the periodic function

$$
(6) \quad \varphi_2(t, u, v) = \Big(0, 0, u \cos(\omega_2 t) + v \sin(\omega_2 t), v \cos(\omega_2 t) - u \sin(\omega_2 t) \Big).
$$

Define $\mathcal{F}^1(z, w)$:

$$
(7) \quad
\begin{aligned}
&\int_0^{T_2} \Big(\cos(\omega_2 t) P_3^1(t, \varphi_2(t, x, y)) - \sin(\omega_2 t) P_4^1(t, \varphi_1(t, x, y)) \Big) dt \\
&+ \int_0^{T_2} \Big(\cos(\omega_2 t) P_3^2(t, \varphi_2(t, x, y)) - \sin(\omega_2 t) P_4^2(t, \varphi_1(t, x, y)) \Big) \operatorname{sign}(h_3(t, \varphi_2(t, x, y))) dt,
\end{aligned}
$$

and $\mathcal{F}^2(z, w)$:

$$
(8) \quad
\begin{aligned}
&\int_0^{T_2} \Big(\cos(\omega_2 t) P_4^1(t, \varphi_2(t, x, y)) + \sin(\omega_2 t) P_3^1(t, \varphi_2(t, x, y)) \Big) dt \\
&+ \int_0^{T_2} \Big(\cos(\omega_2 t) P_4^2(t, \varphi_2(t, x, y)) - \sin(\omega_2 t) P_3^2(t, \varphi_2(t, x, y)) \Big) \operatorname{sign}(h_3(t, \varphi_2(t, x, y))) dt.
\end{aligned}
$$

A zero (x^*, y^*) of the system of functions

$$
(9) \quad \mathcal{F}_1(x, y) = 0, \quad \mathcal{F}_2(x, y) = 0,
$$

such that

$$
\det \left(\left. \frac{\partial(\mathcal{F}_1, \mathcal{F}_2)}{\partial(x, y)} \right|_{(x,y)=(x^*, y^*)} \right) \neq 0,
$$

is called a *simple zero* of (9). Similarly, we define a simple zero of the system of functions

$$
(10) \quad \mathcal{F}^1(z, w) = 0, \quad \mathcal{F}^2(z, w) = 0.
$$

Remark 1. *For $p \in D$, let $\varphi(t, p)$ be the solution of (1) such that $\varphi(0, p) = p$. We say that the Crossing Hypothesis is satisfied if there exists a compact set $V \subset D$ such that the curve $t \mapsto (t, \varphi(t, p))$ reaches the set Σ at points of crossing regions (see Appendix) for every $p \in V$ and $t \in [0, T]$.*

1.3. **Statement of results.** Our main result on the periodic solutions of the perturbed system (1) which bifurcate from the periodic solutions of the unperturbed system with period T_1 is the following.

Theorem A. *Assume that the functions P_i^j and R_i^j of (1) are T_1–periodic in the variable t. Also assume that the Crossing Hypothesis (see Remark 1) is satisfied for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ and $(x, y, z, w) \in V$. Then for $\varepsilon > 0$ sufficiently small and for every simple zero (x^*, y^*) of (9) such that $(x^*, y^*, 0, 0) \in V$, the perturbed system (1) has a T_1–periodic solution $\varphi(t, \varepsilon)$ such that $\varphi(0, \varepsilon) \to (x^*, y^*, 0, 0)$ when $\varepsilon \to 0$.*

For $\varepsilon > 0$ sufficiently small the solution $\varphi(t, \varepsilon)$ of Theorem A is close to the plane defined by the eigenvectors of the eigenvalues $\pm i\omega_1$.

With a change of variables we can also state a result on the periodic solutions of perturbed system (1) which bifurcate from the periodic solutions of the unperturbed system with period T_2.

Corollary 1. *Assume that the functions P_i^j and R_i^j of (1) are T_2–periodic in the variable t. Also assume that the Crossing Hypothesis (see Remark 1) is satisfied for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$. and $(x, y, z, w) \in V$. Then for $\varepsilon > 0$ sufficiently small and for every simple zero (z^*, w^*) of (10) such that $(0, 0, z^*, w^*) \in V$, the perturbed system (1) has a T_2–periodic solution $\varphi(t, \varepsilon)$ such that $\varphi(0, \varepsilon) \to (0, 0, z^*, w^*)$ when $\varepsilon \to 0$.*

Again, for $\varepsilon > 0$ sufficiently small the solution $\varphi(t, \varepsilon)$ of Corollary 1 is close to the plane defined by the eigenvectors of the eigenvalues $\pm i\omega_2$.

Theorem A and Corollary 1 are proved in Section 4. Its proof is based in the averaging theory for computing periodic solutions, see Section 3.

Remark 2. *The theory could be developed for a more general system by considering the smooth functions P_i^j and R_i^j, $T_{P_i^j}$–periodic and $T_{R_i^j}$–periodic in the variable t and respectively in reso-nance $p_{P_i^j}{:}q_{P_i^j}$ and $p_{R_i^j}{:}q_{R_i^j}$ with some of the periodic solutions of the unperturbed system (1), being p and q relatively prime positive integers for $p = p_{P_i^j}, p_{R_i^j}$, $q = q_{P_i^j}, q_{R_i^j}$, $i = 1, 2, 3, 4$ and $j = 1, 2$. However, in this case we may assume that the functions P_i^j and R_i^j for $i = 1, 2, 3, 4$ and $j = 1, 2$ are kT–periodic being T the period of the periodic solution of the unperturbed system (1) for which the functions are resonant. Indeed, if we take k the least common multiple among $p_{P_i^j}$ and $p_{R_i^j}$ for $i = 1, 2, 3, 4$ and $j = 1, 2$, then there exists integers $n_{P_i^j}$ and $n_{R_i^j}$ such that $k = n_{P_i^j} p_{P_i^j} = n_{R_i^j} p_{R_i^j}$. Hence*

$$kT = n_{P_i^j} q_{P_i^j} \frac{p_{P_i^j}}{q_{P_i^j}} T = n_{R_i^j} q_{R_i^j} \frac{p_{R_i^j}}{q_{R_i^j}} T,$$

for $i = 1, 2, 3, 4$, and $j = 1, 2$. Here to make the notation simpler we assume that all functions have the same period.

2. Application: Double pendulum model

The planar *double pendulum* model consists in a system of two point masses m_1 and m_2 moving in a fixed plane, in which the distance between a point P (called pivot) and m_1 and the distance between m_1 and m_2 are fixed, and equal to l_1 and l_2 respectively. We assume the masses do not interact. We allow gravity to act on the masses m_1 and m_2.

The position of the double pendulum is determined by the two angles ϕ_1 and ϕ_2 shown in Figure 1. The corresponding Lagrange equations of motion are

$$(m_1 + m_2)l_1\ddot{\phi}_1 + m_2 l_2 \ddot{\phi}_2 \cos(\phi_1 - \phi_2) + (m_1 + m_2)g \sin(\phi_1)$$

(11)
$$+ m_2 l_2 \dot{\phi}_2^2 \sin(\phi_1 - \phi_2) = 0,$$

$$m_2 l_1 \ddot{\phi}_1 \cos(\phi_1 - \phi_2) + m_2 l_2 \ddot{\phi}_2 + m_2 g \sin(\phi 2) + m_2 l_1 \dot{\phi}_1^2 \sin(\phi_1 - \phi_2) = 0,$$

where g is the acceleration of the gravity. For more details on these equations of motion see [8]. Here the dot denotes derivative with respect to the time τ.

The authors in [9] have studied, in the vicinity of the equilibrium $\phi_1 = \phi_2 = 0$, the persistence of periodic solutions of system (11) perturbed smoothly in the particular case when $m_1 = m_2$ and $l_1 = l_2$. In this paper stronger generalizations are considered.
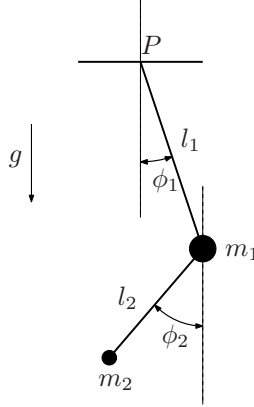
FIGURE 1. The planar double pendulum.

Denote the expressions for $\ddot{\phi}_1$ and $\ddot{\phi}_2$ in (11) respectively by $H_1(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2)$ and $H_2(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2)$. In this section we consider the following perturbed problem:

$$
\begin{aligned}
\ddot{\phi}_1(\tau) = {} & H_1(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \varepsilon \left(F_1(\tau, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \mathrm{sgn}(\phi_1) F_2(\tau, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \right) \\
& + \varepsilon^2 \left(G_1(\tau, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \mathrm{sgn}(\phi_1) G_2(\tau, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \right) + \mathcal{O}(\varepsilon^3), \\
\ddot{\phi}_2(\tau) = {} & H_2(\phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \varepsilon \left(F_3(\tau, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \mathrm{sgn}(\phi_2) F_4(\tau, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \right) \\
& + \varepsilon^2 \left(G_3(\tau, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) + \mathrm{sgn}(\phi_2) G_4(\tau, \phi_1, \dot{\phi}_1, \phi_2, \dot{\phi}_2) \right) + \mathcal{O}(\varepsilon^3).
\end{aligned}
\tag{12}
$$

The smooth functions F_i and G_i for $i = 1, 2, 3, 4$ define the perturbation. These functions are respectively T–periodic in the variable τ, being T the period of some periodic solution of (12) when $\varepsilon = 0$, if there exists. We also assume that $F_i(\tau, 0, 0, 0, 0) = 0$ for $i = 1, 2, 3, 4$.

Roughly speaking, the functions F_i and G_i for $i = 1, 2, 3, 4$, can be taken in a certain way arbitrary. It makes us able to provide, in a physical context, the real meaning of these functions. In our case, since we are working with discontinuities in the variables ϕ_1 and ϕ_2, the functions F_1, F_2, G_1 and G_2 could model the escapement for the particle m_1, and the functions F_3, F_4, G_3 and G_4 could model the escapement for the particle m_2. If discontinuities in the variables $\dot{\phi}_1$ and $\dot{\phi}_2$ are considered instead discontinuities in the variables ϕ_1 and ϕ_2, the respective functions could model the Coulomb Friction. We can also work by composing these two phenomena. For more details on physical systems with discontinuous models see, for instance, [1] and [2].

The objective of this section is to provide a system of equations whose simple zeros provide periodic solutions (see Figure 2) of the perturbed planar double pendulum (12). In order to present our results we need some preliminary definitions and notations.

In what follows we define the real parameters

$$
\gamma = \sqrt{\frac{l_1 m_1}{g m_2}}, \quad a = \frac{m_1 + m_2}{m_2} > 1, \quad \text{and} \quad b = \frac{l_1(m_1 + m_2)}{l_2 m_2} > 0,
\tag{13}
$$

the frequencies

$$
\omega_1 = \frac{\sqrt{a + b - \sqrt{\Delta}}}{\sqrt{2}} \quad \text{and} \quad \omega_2 = \frac{\sqrt{a + b + \sqrt{\Delta}}}{\sqrt{2}},
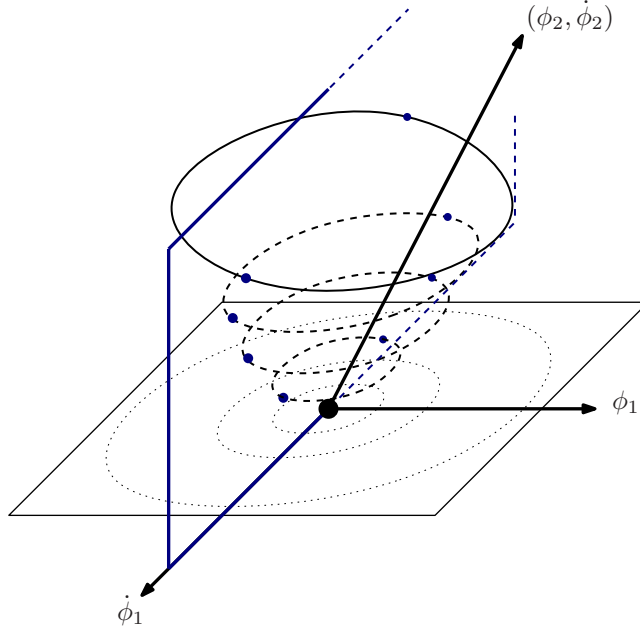\tag{14}
$$

FIGURE 2.   Periodic solution of the perturbed system (12) converging to the origin when $\varepsilon \to 0$.

with $\Delta = (a - b)^2 + 4b > 0$, and the periods

(15)
$$T_1 = \frac{2\pi}{\omega_1} \quad \text{and} \quad T_2 = \frac{2\pi}{\omega_2}.$$

We shall study the persistence of periodic solutions for the perturbed system (12) when the parameter ε is sufficiently small and the functions F_i and G_i for $i = 1, 2, 3, 4$ have period either γT_1, or γT_2.

Now let

(16)
$$\tilde{F}_i(t, u_1, u_2, u_3, u_4) = d_i^1(t)u_1 + d_i^2(t)u_2 + d_i^3(t)u_3 + d_i^4(t)u_4,$$

with

$$d_i^1(t) = \gamma^2 \frac{\partial F_i}{\partial \phi_1}(\gamma t, 0, 0, 0, 0), \quad d_i^2(t) = \gamma \frac{\partial F_i}{\partial \dot{\phi}_1}(\gamma t, 0, 0, 0, 0),$$
$$d_i^3(t) = \gamma^2 \frac{\partial F_i}{\partial \phi_2}(\gamma t, 0, 0, 0, 0), \quad d_i^4(t) = \gamma \frac{\partial F_i}{\partial \dot{\phi}_2}(\gamma t, 0, 0, 0, 0),$$

and let

(17)
$$\tilde{G}_i(t) = \gamma^2 G_i(\gamma t, 0, 0, 0, 0).$$

We define the functions:

$$\mathcal{F}_1(x,y) = \int_0^{T_1} \sin(\omega_1 t)\left(2b(\underline{F}_1 + \tilde{G}_1(t)) + (\underline{F}_3 + \tilde{G}_3(t))\left(a - b + \sqrt{\Delta}\right)\right) dt$$

$$+ \int_0^{T_1} \sin(\omega_1 t)\left(2b(\underline{F}_2 + \tilde{G}_2(t))\right.$$

$$\left. + (\underline{F}_4 + \tilde{G}_4(t))\left(a - b + \sqrt{\Delta}\right)\right) \operatorname{sgn}(\pi_1 \circ \varphi_1(t,x,y)) dt,$$

(18)

$$\mathcal{F}_2(x,y) = \int_0^{T_1} \cos(\omega_1 t)\left(2b(\underline{F}_1 + \tilde{G}_1(t)) + (\underline{F}_3 + \tilde{G}_3(t))\left(a - b + \sqrt{\Delta}\right)\right) dt$$

$$+ \int_0^{T_1} \cos(\omega_1 t)\left(2b(\underline{F}_2 + \tilde{G}_2(t))\right.$$

$$\left. + (\underline{F}_4 + \tilde{G}_4(t))\left(a - b + \sqrt{\Delta}\right)\right) \operatorname{sgn}(\pi_1 \circ \varphi_1(t,x,y)) dt$$

where

(19)
$$\underline{F}_i = \tilde{F}_i(t, A_1, B_1, C_1, D_1)$$

for $i = 1, 2, 3, 4$ with

$$A_1 = \frac{\left(-a + b + \sqrt{\Delta}\right)}{2\,b\,\omega_1}\left(x\cos(\omega_1 t) + y\sin(\omega_1 t)\right),$$

$$B_1 = \frac{\left(-a + b + \sqrt{\Delta}\right)}{2b}\left(y\cos(\omega_1 t) - x\sin(\omega_1 t)\right),$$

$$C_1 = \frac{1}{\omega_1}\left(x\cos(\omega_1 t) + y\sin(\omega_1 t)\right),$$

$$D_1 = y\cos(\omega_1 t) - x\sin(\omega_1 t),$$

and π_1 is the projection onto the first coordinate.

We also define the functions:

$$\mathcal{F}^1(z,w) = \int_0^{T_2} \sin(\omega_2 t)\left(-2b(\overline{F}_1 + \tilde{G}_1(t)) + (\overline{F}_3 + \tilde{G}_3(t))\left(-a + b + \sqrt{\Delta}\right)\right) dt$$

$$+ \int_0^{T_2} \sin(\omega_2 t)\left(2b(\overline{F}_2 + \tilde{G}_2(t))\right.$$

$$\left. + (\overline{F}_4 + \tilde{G}_4(t))\left(-a + b + \sqrt{\Delta}\right)\right) \operatorname{sgn}(\pi_3 \circ \varphi_2(t,z,w)) dt,$$

(20)

$$\mathcal{F}^2(z,w) = \int_0^{T_2} \cos(\omega_2 t)\left(-2b(\overline{F}_1 + \tilde{G}_1(t)) + (\overline{F}_3 + \tilde{G}_3(t))\left(a - b + \sqrt{\Delta}\right)\right) dt$$

$$+ \int_0^{T_2} \cos(\omega_2 t)\left(2b(\overline{F}_2 + \tilde{G}_2(t))\right.$$

$$\left. + (\overline{F}_4 + \tilde{G}_4(t))\left(a - b + \sqrt{\Delta}\right)\right) \operatorname{sgn}(\pi_3 \circ \varphi_2(t,z,w)) dt.$$

where

$$\overline{F}_i = \tilde{F}_i(t, A_2, B_2, C_2, D_2)$$

for $i = 1, 2, 3, 4$ with

$$A_2 = -\frac{\left(a - b + \sqrt{\Delta}\right)}{2\,b\,\omega_2}\left(z\cos\left(\omega_2\,t\right) + w\sin\left(\omega_2\,t\right)\right),$$

$$B_2 = -\frac{\left(a - b + \sqrt{\Delta}\right)}{2b}\left(w\cos\left(\omega_2\,t\right) - z\sin\left(\omega_2\,t\right)\right),$$

$$C_2 = \frac{1}{\omega_2}\left(z\cos\left(\omega_2\,t\right) + w\sin\left(\omega_2\,t\right)\right),$$

$$D_2 = w\cos\left(\omega_2\,t\right) - z\sin\left(\omega_2\,t\right),$$

and π_3 is the projection onto the third coordinate.

Consider the systems

$$(21) \qquad\qquad \mathcal{F}_1(x, y) = 0, \quad \mathcal{F}_2(x, y) = 0,$$

and

$$(22) \qquad\qquad \mathcal{F}^1(z, w) = 0, \quad \mathcal{F}^2(z, w) = 0.$$

In the next proposition we state a result on the existence of periodic solutions of the non-smooth perturbed double pendulum (12).

Proposition 2. *Assume that F_i and G_i of (12) are γT_1–periodic in the variable t. Also assume that the crossing hypothesis is satisfied for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ and $(x, y, z, w) \in V$. Then for $\varepsilon > 0$ sufficiently small and for every simple zero $(x^*, y^*, 0, 0) \in V$ of the non-smooth system (21) such that the orbits pass by D, the non-smooth perturbed double pendulum (12) has a γT_1–periodic solution $\varphi(t, \varepsilon)$ such that $\varphi(0, \varepsilon) \to (0, 0, 0, 0)$ when $\varepsilon \to 0$.*

In a similar way we can also state a result on the periodic solutions of the non–smooth perturbed double pendulum (12) which bifurcate from the periodic solutions of the unperturbed one with period T_2.

Proposition 3. *Assume that F_i and G_i of (12) are γT_2–periodic in the variable t. Also assume that the crossing hypothesis is satisfied for $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 > 0$ and $(x, y, z, w) \in V$. Then for $\varepsilon > 0$ sufficiently small and for every simple zero $(0, 0, z^*, w^*) \in V$ of the non-smooth system (22) such that the orbits pass by D, the non-smooth perturbed double pendulum (12) has a γT_2–periodic solution $\varphi(t, \varepsilon)$ such that $\varphi(0, \varepsilon) \to (0, 0, 0, 0)$ when $\varepsilon \to 0$.*

We provide now an application of Propositions 2 and 3.

Corollary 4. *Suppose that $F_1 = (k_1/\gamma^2)z + f_1$, $F_3 = (k_3/\gamma^2)x + f_3$, $G_2 = 1/\gamma^2 + g_2$, and $G_4 = 1/\gamma^2 + g_4$, where*

$$k_1 = \frac{-\omega_1^2}{2b\pi}, \quad and \quad k_3 = \frac{-4b\omega_1^2}{\pi\left((a - b)^2 - \Delta\right)},$$

with f_1, f_3, g_2, g_4, F_2, F_4, G_1, and G_3 being γT_i–periodic functions in the variable τ having no linear terms and no constant terms in relation with the spatial variables. Then the differential system (12) for $|\varepsilon| > 0$ sufficiently small has two γT_i–periodic solution bifurcating from the origin. Here $i = 1, 2$.

Propositions 2 and 3, and Corollary 4 are proved in section 5.

3. BASIC RESULTS ON AVERAGING THEORY

For proving the main results of this paper we present the basic result from the averaging theory that we need here.

Consider the differential systems of the form

$$(23) \qquad \dot{\mathbf{x}}(t) = G_0(t, \mathbf{x}) + \varepsilon G_1(t, \mathbf{x}) + \varepsilon^2 G_2(t, \mathbf{x}, \varepsilon),$$

with $|\varepsilon| > 0$ sufficiently small, where the functions $G_0, G_1 : \mathbb{R} \times \Lambda \to \mathbb{R}^n$ and $G_2 : \mathbb{R} \times \Lambda \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the variable t, and Λ is an open subset of $\mathbb{R}^n$. Assume that the T–periodic solutions of the unperturbed system

$$(24) \qquad \dot{\mathbf{x}}(t) = G_0(t, \mathbf{x}),$$

form a submanifold $\mathcal{Z}$ of dimension k in $\mathbb{R}^n$. In the study of the periodic solutions the objective of the averaging theory is to detect which periodic solutions of the unperturbed system (24) persist as periodic solutions in the perturbed differential system (23) for $|\varepsilon| > 0$ sufficiently small.

We denote by $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ the solution of system (24) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. The first variational equation of system (24) on the periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ is

$$(25) \qquad \dot{\mathbf{y}} = D_{\mathbf{x}} G_0(t, \mathbf{x}(t, \mathbf{z}, 0)) \mathbf{y},$$

where $\mathbf{y}$ is a $n \times n$ matrix. From now on let $M_{\mathbf{z}}(t)$ be the fundamental matrix of system (25) such tat $M_{\mathbf{z}}(0) =$ Id. Let $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ be the projection $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

The T–periodic solutions of the perturbed system (24) coming from the unperturbed system (23) for $|\varepsilon| > 0$ sufficiently small can be computed using the following result.

Theorem 5. *We denote by V an open and bounded subset of $\mathbb{R}^k$ containing the submanifold $\mathcal{Z}$, and we denote by $\beta : \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ a $\mathcal{C}^2$ function such that $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)), \ \alpha \in \mathrm{Cl}(V)\} \subset \Lambda$, where for every $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of (24) is T–periodic. Let $M_{\mathbf{z}_\alpha}(t)$ be fundamental matrix of (25) associated to the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$. Assume that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the lower right corner a $(n-k) \times (n-k)$ matrix A_α with $\det(A_\alpha) \neq 0$, and in the upper right corner the $k \times (n-k)$ zero matrix. Let $\mathcal{G} : \mathrm{Cl}(V) \to \mathbb{R}^k$ be the function defined by*

$$(26) \qquad \mathcal{G}(\alpha) = \xi \left(\frac{1}{T} \int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) G_1(t, \mathbf{x}(t, \mathbf{z}_\alpha)) dt \right).$$

If there is $a \in V$ such that $\mathcal{G}(a) = 0$ and $\det\left((d\mathcal{G}/d\alpha)(a)\right) \neq 0$, therefore there exists a T–periodic solution $\varphi(t, \varepsilon)$ of system (23) satisfying $\varphi(0, \varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 5 was proved by Malkin [11] and Roseau [12], for a shorter and new proof see [5].

4. PROOFS OF THEOREM A AND COROLLARY 1

The averaging theory we shall use here (see Appendix) deals with smooth system. So, first of all, instead of working with the discontinuous differential system (1) we shall work with the smooth differential system

$$(27) \qquad \begin{aligned} x'(t) &= \omega_1 \, y + \varepsilon P_{1\delta}(t, x, y, z, w) + \varepsilon^2 R_{1\delta}(t, x, y, z, w, \varepsilon), \\ y'(t) &= -\omega_1 \, x + \varepsilon P_{2\delta}(t, x, y, z, w) + \varepsilon^2 R_{2\delta}(t, x, y, z, w, \varepsilon), \\ z'(t) &= \omega_2 \, w + \varepsilon P_{3\delta}(t, x, y, z, w) + \varepsilon^2 R_{3\delta}(t, x, y, z, w, \varepsilon), \\ w'(t) &= -\omega_2 \, z + \varepsilon P_{4\delta}(t, x, y, z, w) + \varepsilon^2 R_{4\delta}(t, x, y, z, w, \varepsilon), \end{aligned}$$

where

$$P_{i\delta}(t,x,y,z,w) = P_i^1(t,x,y,z,w) + s_\delta(h_i(t,x,y,z,w))P_i^2(t,x,y,z,w),$$

$$R_{i\delta}(t,x,y,z,w,\varepsilon) = F_i^1(t,x,y,z,w,\varepsilon) + s_\delta(h_i(t,x,y,z,w))F_i^2(t,x,y,z,w,\varepsilon).$$

where $s_\delta(x)$ is the smooth function defined in Figure 5, such that
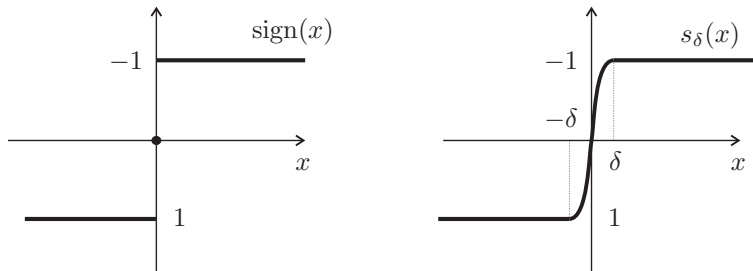
$$\lim_{\delta \to 0} s_\delta(x) = \mathrm{sgn}(x).$$

FIGURE 3.   The functions $\mathrm{sign}(x)$ and $s_\delta(x)$.

Lemma 6. *The functions* (3) *and* (6) *are periodic solutions of the unperturbed system* (27) *respectively with periods T_1 and T_2.*

Proof. Since system (27) with $\varepsilon = 0$ is a linear differential system, the proof follows easily. □

Proof of Theorem A. It is well known that a Poincaré map defined in a smooth differential system is smooth. So the Poincaré maps associated to the periodic orbits of the differential system (27) are smooth.

The Poincaré maps, restricted at V, associated to the periodic solutions of the non-smooth differential system (1), which are perturbations of the periodic solutions (3) are also smooth, indeed, since the orbits starting in V reach the discontinuity set only at points of crossing region (see Appendix), such Poincaré maps are compositions of smooth Poincaré maps. In a similar way it follows that the Poincaré maps, restricted at V, associated to the periodic solutions of the non-smooth differential system (1), which are perturbations of the periodic solutions (6) are also smooth.

We can use Theorem 5 (see Section 3) for computing some periodic solutions of the smooth systems. The periodic solutions are zeros of the displacement function, which is the Poincaré map associated to periodic solutions minus the identity. In fact, the non-linear function (26) whose zeros can provide periodic solutions, is the first term of order ε of the displacement function. See for more details the proof of Theorem 5 in [5].

Since the Poincaré maps associated to periodic solutions of system (1), coming from the perturbed periodic solutions (3) or (6), are smooth and these Poincaré maps are the limit of the Poincaré maps associated to the smooth system (27), for which we can use Theorem 5, it follows that we also can use Theorem 5 for computing some of the periodic solutions of the non-smooth system (1). In other words, we can apply Theorem 5 to the smooth systems (27) and then pass to the limit, when $\delta \to 0$, the function (26) for obtaining a function whose zeros can give periodic solutions of the non-smooth system (1).

We note that system (27) can be written as system (23) taking

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}, \quad t = \tau, \quad G_0(t, \mathbf{x}) = \begin{pmatrix} \omega_1 \, y \\ -\omega_1 \, x \\ \omega_2 \, w \\ -\omega_2 \, z \end{pmatrix}, G_1(t, \mathbf{x}) = \begin{pmatrix} P_{1\delta}(t, \mathbf{x}) \\ P_{2\delta}(t, \mathbf{x}) \\ P_{3\delta}(t, \mathbf{x}) \\ P_{4\delta}(t, \mathbf{x}) \end{pmatrix}$$

$$\text{and} \quad G_2(t, \mathbf{x}, \varepsilon) = \begin{pmatrix} R_{1\delta}(t, \mathbf{x}, \varepsilon) \\ R_{2\delta}(t, \mathbf{x}, \varepsilon) \\ R_{3\delta}(t, \mathbf{x}, \varepsilon) \\ R_{4\delta}(t, \mathbf{x}, \varepsilon) \end{pmatrix}.$$

We shall describe the different elements which appear in the statement of Theorem 5 in the particular case of the differential system (27). Thus we have that $\Lambda = \mathbb{R}^4$, $k = 2$ and $n = 4$. Let $r_1 > 0$ be arbitrarily small and let $r_2 > 0$ be arbitrarily large. We take the open and bounded subset V of the plane $z = w = 0$ as

$$V = \{(x, y, 0, 0) \in \mathbb{R}^4 : r_1 < \sqrt{x^2 + y^2} < r_2\}.$$

As usual $\mathrm{Cl}(V)$ denotes the closure of V. If $\alpha = (x, y)$, then we can identify V with the set

$$\{\alpha \in \mathbb{R}^2 : r_1 < ||\alpha|| < r_2\},$$

here $|| \cdot ||$ denotes the Euclidean norm of $\mathbb{R}^2$. The function $\beta : \mathrm{Cl}(V) \to \mathbb{R}^2$ is $\beta(\alpha) = (0, 0)$. Therefore, in our case the set

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)), \ \alpha \in \mathrm{Cl}(V)\} = \{(x, y, 0, 0) \in \mathbb{R}^4 : r_1 \leq \sqrt{x^2 + y^2} \leq r_2\}.$$

Clearly for each $\mathbf{z}_\alpha \in \mathcal{Z}$ we can consider the periodic solution $\mathbf{x}(t, \mathbf{z}_\alpha) = \varphi_1(t, x, y)$ given by (3) with period T_1.

Computing the fundamental matrix $M_{\mathbf{z}_\alpha}(\tau)$ of the linear differential system (27) with $\varepsilon = 0$ associated to the T–periodic solution $\mathbf{z}_\alpha = (x, y, 0, 0)$ such that $M_{\mathbf{z}_\alpha}(0)$ be the identity of $\mathbb{R}^4$, we get that $M(\tau) = M_{\mathbf{z}_\alpha}(\tau)$ is equal to

$$\begin{pmatrix} \cos(\omega_1 t) & \sin(\omega_1 t) & 0 & 0 \\ -\sin(\omega_1 t) & \cos(\omega_1 t) & 0 & 0 \\ 0 & 0 & \cos(\omega_2 t) & \sin(\omega_2 t) \\ 0 & 0 & -\sin(\omega_2 t) & \cos(\omega_2 t) \end{pmatrix}.$$

Note that the matrix $M_{\mathbf{z}_\alpha}(\tau)$ does not depend on the particular periodic solution $\mathbf{x}(\tau, \mathbf{z}_\alpha)$. Since the matrix

$$
M^{-1}(0) - M^{-1}(pT_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2\sin^2\left(\dfrac{p\pi\,\omega_2}{\omega_1}\right) & \sin\left(\dfrac{2p\pi\,\omega_2}{\omega_1}\right) \\ 0 & 0 & -\sin\left(\dfrac{2p\pi\,\omega_2}{\omega_1}\right) & 2\sin^2\left(\dfrac{p\pi\,\omega_2}{\omega_1}\right) \end{pmatrix},
$$

satisfies the assumptions of statement (ii) of Theorem 5. Indeed

$$
\left| \begin{matrix} 2\sin^2\left(\dfrac{p\pi\,\omega_2}{\omega_1}\right) & \sin\left(\dfrac{2p\pi\,\omega_2}{\omega_1}\right) \\ -\sin\left(\dfrac{2p\pi\,\omega_2}{\omega_1}\right) & 2\sin^2\left(\dfrac{p\pi\,\omega_2}{\omega_1}\right) \end{matrix} \right| = 4\sin^2\left(\dfrac{p\pi\,\omega_2}{\omega_1}\right) \neq 0.
$$

because $p\omega_2/\omega_1 \notin \mathbb{Z}$. So we can apply Theorem 5 to system (27).

Now $\xi : \mathbb{R}^4 \to \mathbb{R}^2$ is $\xi(x, y, z, w) = (x, y)$. We calculate, when $\delta \to 0$, the function

$$
\mathcal{G}(x, y) = \mathcal{G}(\alpha) = \xi\left(\frac{1}{T_1} \int_0^{T_1} M_{\mathbf{z}_\alpha}^{-1}(t) G_1(t, \mathbf{x}(t, \mathbf{z}_\alpha)) dt \right),
$$

and we obtain the function $\mathcal{G}_1(x, y)$

(28)
$$
\frac{1}{T_1} \int_0^{T_1} \Big(\cos(\omega_1 t) P_1^1(t, \varphi_1(t, x, y)) - \sin(\omega_1 t) P_2^1(t, \varphi_1(t, x, y)) \Big) dt
$$
$$
+ \frac{1}{T_1} \int_0^{T_1} \Big(\cos(\omega_1 t) P_1^2(t, \varphi_1(t, x, y)) - \sin(\omega_1 t) P_2^2(t, \varphi_1(t, x, y)) \Big) \operatorname{sign}(h_1(t, \varphi_1(t, x, y))) dt,
$$

and the function $\mathcal{G}_2(x, y)$

(29)
$$
\frac{1}{T_1} \int_0^{T_1} \Big(\cos(\omega_1 t) P_2^1(t, \varphi_1(t, x, y)) + \sin(\omega_1 t) P_1^1(t, \varphi_1(t, x, y)) \Big) dt
$$
$$
+ \frac{1}{T_1} \int_0^{T_1} \Big(\cos(\omega_1 t) P_2^2(t, \varphi_1(t, x, y)) - \sin(\omega_1 t) P_1^2(t, \varphi_1(t, x, y)) \Big) \operatorname{sign}(h_1(t, \varphi_1(t, x, y))) dt.
$$

Then, by Theorem 5 we have that for every simple zero $(x^*, y^*) \in V$ of the system

(30) $$ \mathcal{G}_1(x, y) = 0 \quad , \quad \mathcal{G}_2(x, y) = 0, $$

we have a periodic solution $\varphi(t, \varepsilon)$ of system (27) such that

$$
\varphi(0, \varepsilon) \to (x^*, y^*, 0, 0) \quad \text{as } \varepsilon \to 0.
$$

Note that system (30) is equivalent to system (9), because both equations only differs in a non–zero multiplicative constant. Hence Theorem A is proved. $\qquad\square$

Proof of Corollary 1. This proof follows immediately from Theorem A. $\qquad\square$

5. PROOFS OF PROPOSITIONS 2 AND 3, AND COROLLARY 4

5.1. **Proofs of Propositions.** In order to apply Theorem A to system (12) we have to write this system in the standard form (1). For this purpose firstly we follow the steps:

(i) proceed with the change of variable $\phi_1 = \varepsilon\theta_1$ and $\phi_2 = \varepsilon\theta_2$;

$$
\begin{aligned}
\ddot{\theta}_1(\tau) =\ & \frac{1}{\varepsilon}H_1(\varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + F_1(\tau, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + \mathrm{sgn}(\theta_1)F_2(\tau, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) \\
&+ \varepsilon\left(G_1(\tau, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + \mathrm{sgn}(\theta_1)G_2(\tau, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2)\right) + \mathcal{O}(\varepsilon^3), \\
\ddot{\theta}_2(\tau) =\ & \frac{1}{\varepsilon}H_2(\varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + F_3(\tau, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + \mathrm{sgn}(\theta_2)F_4(\tau, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) \\
&+ \varepsilon\left(G_1(\tau, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2) + \mathrm{sgn}(\theta_2)G_2(\tau, \varepsilon\theta_1, \varepsilon\dot{\theta}_1, \varepsilon\theta_2, \varepsilon\dot{\theta}_2)\right) + \mathcal{O}(\varepsilon^3).
\end{aligned}
$$

(ii) expand in Taylor series, for $\varepsilon = 0$, the expressions of $\ddot{\theta}_1$ and $\ddot{\theta}_2$;
(iii) take a new time t given by the rescaling $\tau = \gamma t$, with $\gamma = \sqrt{l_1 m_1/(g m_2)}$;
(iv) and finally, denote

$$
a = \frac{m_1 + m_2}{m_2} > 1 \quad \text{and} \quad b = \frac{l_1(m_1 + m_2)}{l_2 m_2} > 0.
$$

Hence, we obtain the following equations of motion for the double pendulum

$$
\begin{aligned}
\theta_1''(t) =\ & -a\theta_1 + \theta_2 + \varepsilon\left(\tilde{F}_1(t, \theta_1, \theta_1', \theta_2, \theta_2') + \tilde{G}_1(t)\right) \\
&+ \varepsilon\mathrm{sgn}(\theta_1)\left(\tilde{F}_2(t, \theta_1, \theta_1', \theta_2, \theta_2') + \tilde{G}_2(t)\right) + \mathcal{O}(\varepsilon^2), \\
(31) \\
\theta_2''(t) =\ & b\theta_1 - b\theta_2 + \varepsilon\left(\tilde{F}_3(t, \theta_1, \theta_1', \theta_2, \theta_2') + \tilde{G}_3(t)\right) \\
&+ \varepsilon\mathrm{sgn}(\theta_2)\left(\tilde{F}_4(t, \theta_1, \theta_1', \theta_2, \theta_2') + \tilde{G}_4(t)\right) + \mathcal{O}(\varepsilon^2),
\end{aligned}
$$

where now the prime denotes derivative with respect to the new time t. Here the functions $\tilde{F}_i$, and $\tilde{G}_i$ for $i = 1, 2, 3, 4$ are given in (16) and (17). It is worth to say that the parameters γ, a and b defined above are the same defined in (13).

Introducing the variables $(X, Y, Z, W) = (\theta_1, \theta_1', \theta_2, \theta_2')$ we write the differential system of the non-smooth perturbed double pendulum (31) as a first–order differential system defined in $\mathbb{R}^4$. Thus we have the differential system

$$
\begin{aligned}
X'(t) =\ & Y, \\
Y'(t) =\ & -aX + Z + \varepsilon\left(\tilde{F}_1(t, X, Y, Z, W) + \tilde{G}_1(t)\right) \\
&+ \varepsilon\mathrm{sign}(X)\left(\tilde{F}_2(t, X, Y, Z, W) + \tilde{G}_2(t)\right) + \mathcal{O}(\varepsilon^2), \\
(32) \\
Z'(t) =\ & W, \\
W'(t) =\ & bX - bZ + \varepsilon\left(\tilde{F}_3(t, X, Y, Z, W) + \tilde{G}_3(t)\right) \\
&+ \varepsilon\mathrm{sign}(Z)\left(\tilde{F}_4(t, X, Y, Z, W) + \tilde{G}_4(t)\right) + \mathcal{O}(\varepsilon^2).
\end{aligned}
$$

Finally, we write system (32) in such a way that the linear part at the origin of the unperturbed system will be in its real normal Jordan form. Then, doing the change of variables

$(t, X, Y, Z, W) \to (t, x, y, z, w)$ given by

$$(33) \quad \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} \dfrac{b\omega_1}{\sqrt{\Delta}} & 0 & \dfrac{\omega_1\left(a - b + \sqrt{\Delta}\right)}{2\sqrt{\Delta}} & 0 \\ 0 & \dfrac{b}{\sqrt{\Delta}} & 0 & \dfrac{a - b + \sqrt{\Delta}}{2\sqrt{\Delta}} \\ -\dfrac{b\omega_2}{\sqrt{\Delta}} & 0 & \dfrac{\omega_2\left(-a + b + \sqrt{\Delta}\right)}{2\sqrt{\Delta}} & 0 \\ 0 & -\dfrac{b}{\sqrt{\Delta}} & 0 & \dfrac{-a + b + \sqrt{\Delta}}{2\sqrt{\Delta}} \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix},$$

the differential system (32) becomes

$$(34) \quad \begin{aligned} x' =\ & \omega_1 y, \\ y' =\ & -\omega_1 x + \varepsilon \frac{1}{2\sqrt{\Delta}} \left(2b\left(\check{F}_1 + \tilde{G}_1(t) + \left(\check{F}_2 + \tilde{G}_2(t)\right)\operatorname{sign}(\mathcal{A})\right)\right) \\ & + \varepsilon \frac{1}{2\sqrt{\Delta}} \left(\left(a - b + \sqrt{\Delta}\right)\left(\check{F}_3 + \tilde{G}_3(t) + \left(\check{F}_4 + \tilde{G}_4(t)\right)\operatorname{sign}(\mathcal{C})\right)\right) + \mathcal{O}(\varepsilon^2), \\ z' =\ & \omega_2 w, \\ w' =\ & -\omega_2 z + \varepsilon \frac{1}{2\sqrt{\Delta}} \left(-2b\left(\check{F}_1 + \tilde{G}_1(t) + \left(\check{F}_2 + \tilde{G}_2(t)\right)\operatorname{sign}(\mathcal{A})\right)\right) \\ & + \varepsilon \frac{1}{2\sqrt{\Delta}} \left(\left(-a + b + \sqrt{\Delta}\right)\left(\check{F}_3 + \tilde{G}_3(t) + \left(\check{F}_4 + \tilde{G}_4(t)\right)\operatorname{sign}(\mathcal{C})\right)\right) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

where $\check{F}_i(t, x, y, z, w) = \tilde{F}_i(t, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$ for $i = 1, 2, 3, 4$ with

$$\mathcal{A} = \frac{\left(-a + b + \sqrt{\Delta}\right)}{2\,b\,\omega_1} x - \frac{\left(a - b + \sqrt{\Delta}\right)}{2\,b\,\omega_2} z,$$

$$\mathcal{B} = \frac{\left(-a + b + \sqrt{\Delta}\right)}{2b} y - \frac{\left(a - b + \sqrt{\Delta}\right)}{2b} w,$$

$$\mathcal{C} = \frac{1}{\omega_1} x + \frac{1}{\omega_2} z,$$

$$\mathcal{D} = y + w.$$

Proof of Proposition 2. Computing the functions (4) and (5) for the differential system (34) we obtain the functions given in (18). Consequently, the system of functions (9) is equivalent to the system of functions (21). Then, by Theorem A we have that for every simple zero (x^*, y^*) of the system (21) there exists a periodic solution $(x, y, z, w)(t, \varepsilon)$ of (34) such that

$$(x, y, z, w)(0, \varepsilon) \to (x^*, y^*, 0, 0) \quad \text{as } \varepsilon \to 0.$$

Going back through the change of coordinates (33) we get a periodic solution $(X, Y, Z, W)(t, \varepsilon)$ of system (32) such that

$$
\begin{pmatrix} X(t, \varepsilon) \\ Y(t, \varepsilon) \\ Z(t, \varepsilon) \\ W(t, \varepsilon) \end{pmatrix} \rightarrow \begin{pmatrix} \dfrac{-a + b + \sqrt{\Delta}}{2\,b\,\omega_1} \left(x^* \cos(\omega_1 t) + y^* \sin(\omega_1 t) \right) \\ \dfrac{-a + b + \sqrt{\Delta}}{2b} \left(y^* \cos(\omega_1 t) - x^* \sin(\omega_1 t) \right) \\ \dfrac{1}{\omega_1} \left(x^* \cos(\omega_1 t) + y^* \sin(\omega_1 t) \right) \\ y^* \cos(\omega_1 t) - x^* \sin(\omega_1 t) \end{pmatrix}
$$

as $\varepsilon \to 0$. Consequently we obtain a periodic solution $\varphi(\tau, \varepsilon)$ of system (12) such that

$$
\varphi(\tau, \varepsilon) = \left(\varepsilon X \left(\frac{\tau}{\gamma}, \varepsilon \right), \frac{\varepsilon}{\gamma} Y \left(\frac{\tau}{\gamma}, \varepsilon \right), \varepsilon Z \left(\frac{\tau}{\gamma}, \varepsilon \right), \frac{\varepsilon}{\gamma} W \left(\frac{\tau}{\gamma}, \varepsilon \right) \right),
$$

which is clearly γT_1–periodic. Moreover $\varphi(0, \varepsilon) \to (0, 0, 0, 0)$ when $\varepsilon \to 0$. Hence Theorem A is proved. $\qquad \square$

Proof of Proposition 3. Computing the functions (7) and (8) to the differential system (34) we obtain the functions given in (20). Consequently, the system of functions (10) is equivalent to the system of functions (22). Then, by Corollary 1 we have that for every simple zero (z^*, w^*) of the system of functions (22) there exists a periodic solution $(x, y, z, w)(t, \varepsilon)$ of (34) such that

$$
(x, y, z, w)(0, \varepsilon) \to (0, 0, z^*, w^*) \quad \text{as } \varepsilon \to 0.
$$

From here, the proof follows analogously to the proof of Proposition 2. $\qquad \square$

5.2. **Proof of Corollary.** To obtain the expression of the functions given in (18) and (20) we have to study the changes of sign of the functions $\pi_1 \circ \varphi_1(t, x, y)$ (defined in (3)) and $\pi_3 \circ \varphi_2(t, z, w)$ (defined in (6)) respectively for $t \in [0, T_1]$ and $t \in [0, T_2]$.

Note that $\pi_1 \circ \varphi_1(t, x, y) = 0$ only for

$$
t_n = \frac{1}{\omega_1} \left(\pi n - \arctan \left(\frac{x}{y} \right) \right).
$$

So, if $xy < 0$, then $t_n \in [0, T_1]$ only for $n = 0, 1$; and if $xy > 0$, then $t_n \in [0, T_1]$ only for $n = 1, 2$. We know that for all $t \in [t_n, t_{n+1}]$ the function $\pi_1 \circ \varphi_1(t, x, y)$ has the same sign and different sign of any $t \in [t_{n-1}, t_n]$, thus the integral can be computed using the partitions $\{0, t_0, t_1, T_1\}$ when $xy < 0$ and $\{0, t_1, t_2, T_1\}$ when $xy > 0$ as the limits of integration.

The study of changes of the sign of the function $\pi_3 \circ \varphi_2(t, z, w)$ for $t \in [0, T_2]$ and $zw \neq 0$ is completely analogous.

Proof of Corollary 4. Firstly, we have to check the *crossing hypothesis* for the system (12) or equivalently for the system (32). Note that we have four different vector fields defined in four different regions (see Figure 4).

In the region $\mathcal{R}_1 = \{X > 0 \text{ and } Z > 0\}$ we have

$$
\mathcal{X}_1 = \begin{pmatrix} Y \\ -aX + Z + \varepsilon(1 + k_1 Z) \\ W \\ bX - bZ + \varepsilon(1 + k_3 Z), \end{pmatrix}.
$$

In the region $\mathcal{R}_2 = \{X < 0 \text{ and } Z > 0\}$ we have

$$\mathcal{X}_2 = \begin{pmatrix} Y \\ -aX + Z - \varepsilon(1 - k_1\, Z) \\ W \\ bX - bZ + \varepsilon(1 + k_3\, Z), \end{pmatrix}.$$

In the region $\mathcal{R}_3 = \{X < 0 \text{ and } Z < 0\}$ we have

$$\mathcal{X}_3 = \begin{pmatrix} Y \\ -aX + Z - \varepsilon(1 - k_1\, Z) \\ W \\ bX - bZ - \varepsilon(1 - k_3\, Z), \end{pmatrix}.$$

Finally, in the region $\mathcal{R}_4 = \{X > 0 \text{ and } Z < 0\}$ we have

$$\mathcal{X}_4 = \begin{pmatrix} Y \\ -aX + Z + \varepsilon(1 + k_1\, Z) \\ W \\ bX - bZ - \varepsilon(1 - k_3\, Z), \end{pmatrix}.$$
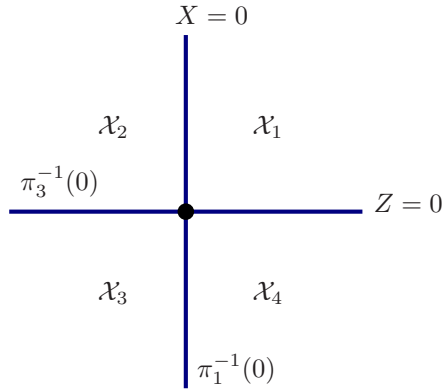

FIGURE 4.   Four different vector fields.

To study the types of the sets $\mathcal{M}_{ij}$ (see Appendix), we have to compute Lie derivative of the functions π_1 and π_3 with respect to the vector fields $\mathcal{X}_i$ for $i = 1, 2, 3, 4$, i.e.

$$(\mathcal{L}_{\mathcal{X}_i})(\pi_j)(X, Y, Z, W) = \langle \nabla \pi_j, \mathcal{X}_i \rangle (X, Y, Z, W),$$

where π_j is the projection onto the jth coordinate.

Proceeding with these calculations we have

$$(\mathcal{L}_{\mathcal{X}_1})(\pi_1)(X, Y, Z, W) = (\mathcal{L}_{\mathcal{X}_2})(\pi_1)(X, Y, Z, W) = \quad Y,$$

$$(\mathcal{L}_{\mathcal{X}_2})(\pi_2)(X, Y, Z, W) = (\mathcal{L}_{\mathcal{X}_3})(\pi_2)(X, Y, Z, W) = \quad W,$$

$$(\mathcal{L}_{\mathcal{X}_3})(\pi_1)(X, Y, Z, W) = (\mathcal{L}_{\mathcal{X}_4})(\pi_1)(X, Y, Z, W) = \quad Y,$$

$$(\mathcal{L}_{\mathcal{X}_1})(\pi_2)(X, Y, Z, W) = (\mathcal{L}_{\mathcal{X}_4})(\pi_2)(X, Y, Z, W) = \quad W.$$

Hence we can conclude that in the set

$$\mathcal{T} = \{(X, Y, 0, 0)\} \bigcup \{(0, 0, Z, W)\},$$

the flow is tangent to the discontinuous set, and in any other point the flow cross the set of discontinuity.

In the coordinates defined in (33), we have that

$$\mathcal{T} = \left\{ \left(x, y, -\frac{\omega_2}{\omega_1} x, -y \right) \right\} \bigcup \left\{ \left(x, y, \beta \frac{\omega_2}{\omega_1} x, \beta y \right) \right\},$$

where

$$\beta = -\frac{a - b - \sqrt{\Delta}}{a - b + \sqrt{\Delta}}.$$

Observe that the periodic orbits given by Lemma 6 filling the planes $\{(x, y, 0, 0)\}$ and $\{(0, 0, z, w)\}$, except the origin, do not reach the set $\mathcal{T}$. Thus, for $|\varepsilon| > 0$ sufficiently small, there exists a neighborhood of $\{(x, y, 0, 0)\} \backslash (0, 0, 0, 0)$ and $\{(0, 0, z, w)\} \backslash (0, 0, 0, 0)$ such that the orbits cross the set of discontinuity. In other words, the crossing hypothesis is satisfied for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for some $\varepsilon_0 > 0$.

Now assume that the function are γT_1–periodic in the variable τ. By studying the changes of the sign of the function $\pi_1 \circ \varphi_1(t, x, y)$ for $t \in [0, T_1]$ we conclude that the functions (18) and (20) are given by

$$\mathcal{F}_1(x, y) = \begin{cases} y + \dfrac{4(a + b + \sqrt{\Delta})}{\omega_1 \sqrt{1 + \dfrac{x^2}{y^2}}} & \text{if } y > 0, \\[4ex] y - \dfrac{4(a + b + \sqrt{\Delta})}{\omega_1 \sqrt{1 + \dfrac{x^2}{y^2}}} & \text{if } y < 0, \end{cases}$$

$$\mathcal{F}_2(x, y) = \begin{cases} x + \dfrac{4x(a + b + \sqrt{\Delta})}{\omega_1 y \sqrt{1 + \dfrac{x^2}{y^2}}} & \text{if } y > 0, \\[4ex] x - \dfrac{4x(a + b + \sqrt{\Delta})}{\omega_1 y \sqrt{1 + \dfrac{x^2}{y^2}}} & \text{if } y < 0, \end{cases}$$

So the system $\mathcal{F}_1(x, y) = 0$ and $\mathcal{F}_2(x, y) = 0$ has the following simple solutions:

$$(x_1^*, y_1^*) = \left(0, 4 \frac{a + b + \sqrt{\Delta}}{\omega_1} \right),$$

and

$$(x_2^*, y_2^*) = \left(0 \,,\, -4\frac{a + b + \sqrt{\Delta}}{\omega_1} \right).$$

Hence, by Theorem A we have two γT_1–periodic solution of the non-smooth perturbed double pendulum. The argument in the case when the functions are γT_2–periodic is completely analogous. So we have conclude the proof of corollary.   □

APPENDIX: BASIC CONCEPTS ON FILIPPOV SYSTEMS

We say that a vector field $X : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is *piecewise continuous* if its domain of definition D can be partitioned in a finite collection of connected, open and disjoint sets D_i, $i = 1, \cdots, k$, such that $\cup \overline{D_i} = D$, and the vector field $X\big|_{\overline{D_i}}$ is continuous for $i = 1, \cdots, k$.

We denote by $S_X \subset \partial D_1 \cup \cdots \cup \partial D_k$ the set of points where the vector field X is discontinuous. By assumptions, the set S_X has measure zero.

If $\Sigma \subset S_X$ is a manifold of codimension one, then Σ can be decomposed as the union of the closure of the following three kind of regions (see Figure 5):

$$\Sigma^c = \{ x \in \Sigma : (Xh)(Yh)(x) > 0 \};$$
$$\Sigma^e = \{ x \in \Sigma : (Xh)(x) > 0 \text{ e } (Yh)(x) < 0 \};$$
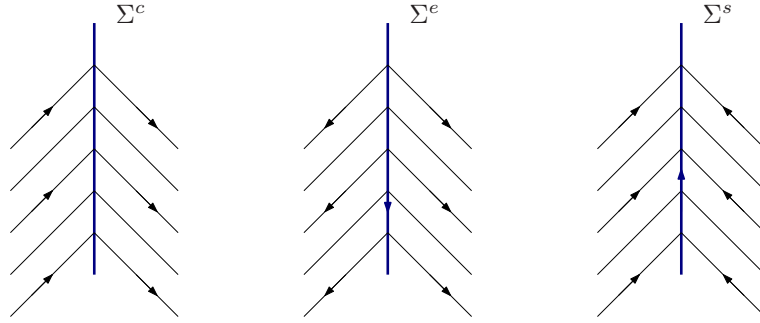$$\Sigma^s = \{ x \in \Sigma : (Xh)(x) < 0 \text{ e } (Yh)(x) > 0 \}.$$

FIGURE 5.   Crossing region (Σ^c), escaping region (Σ^e) and sliding region (Σ^s).

For $p \in \Sigma^e \cup \Sigma^s$ we define the *Sliding Vector Field* as

$$(35) \qquad Z_s(p) = \frac{1}{(Yh)(p) - (Xh)(p)} \left((Yh)(Xh)(p) - (Xh)(Yh)(p) \right).$$

Consider the following equation

$$(36) \qquad\qquad\qquad \dot{x} = X(x),$$

where $X : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is a piecewise continuous vector field. The local solution of the equation (36) passing through a point $p \in \Sigma$ is given by the Filippov convention:

(i) for $p \in \Sigma^c$ such that $(Xh)(p), (Yh)(p) > 0$ and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t, p) = \varphi_Y(t, p)$ for $t \in I_p \cap \{t < 0\}$ and $\varphi_Z(t, p) = \varphi_X(t, p)$ for $t \in I_p \cap \{t > 0\}$. For the case $(Xh)(p), (Yh)(p) < 0$ the definition is the same reversing time;

(i) for $p \in \Sigma^e \cup \Sigma^s$ such that $Z_s(p) \neq 0$, $\varphi_Z(t,p) = \varphi_{Z_s}(t,p)$ for $t \in I_p \subset \mathbb{R}$.

Here φ_W denotes the flow of a vector field W.

For more details about discontinuous differential equation see Filippov's book [7].

Acknowledgements

References

[1] A. A. Andronov, A.A. Vitt e S. E. Khaikin, *Theory of oscillators*, International Series of Monographs In Physics **4**, Pergamon Press, 1966.

[2] M. di Bernardo et al., *Bifurcation in nonsmooth dynamical systems*, Publications of the Bristol Centre for Applied Nonlinear Mathematics **4**, (2005), 629–701.

[3] N. N. Bogoliubov, *On some statistical methods in mathematical physics*, Izv. vo Akad. Nauk Ukr. SSR, Kiev, 1945.

[4] N. N. Bogoliubov and N. Krylov, *The application of methods of nonlinear mechanics in the theory of stationary oscillations*, Publ. 8 of the Ukrainian Acad. Sci. Kiev, 1934.

[5] A. Buică, J.P. Françoise and J. Llibre, *Periodic solutions of nonlinear periodic differential systems with a small parameter*, Communication on Pure and Applied Analysis **6** (2007), 103–111.

[6] P. Fatou, *Sur le mouvement d'un systàme soumis à des forces à courte période*, Bull. Soc. Math. France **56** (1928), 98–139.

[7] A. F. Filippov, *Differential equations with discontinuous righthand side*, Mathematics and Its Applications, Kluwer Academic Publishers, Dordrecht, 1988.

[8] H. Iro, *A modern approach to classical mechanics*, World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[9] J. Llibre, D.D. Novaes and M.A. Teixeira, *On the periodic solutions of a perturbed double pendulum*, São Paulo J. Math Sciences IME–USP **5** (2011), 317–330.

[10] J. Llibre and M.A. Teixeira, *On the stable limit cycle of a weight–driven pendulum clock*, Eur. J. Phys. **31** (2010), 1249–1254.

[11] I.G. Malkin, *Some problems of the theory of nonlinear oscillations*, (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1956.

[12] M. Roseau, *Vibrations non linéaires et théorie de la stabilité*, (French) Springer Tracts in Natural Philosophy, Vol.**8** Springer–Verlag, Berlin–New York, 1966.

[13] J. A. Sanders F. Verhulst and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, Second edition, Applied Mathematical Sciences **59**, Springer, New York, 2007.

[14] J. Sotomayor and M.A. Teixeira, *Regularization of Discontinuous Vector Field*, International Conference on Differential Equation, Lisboa, 1995, World Sci. Publ., River Edge, NJ, (1998), pp 207223.

[1] Departament de Matematiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`

[2] Departamento de Matematica, Universidade Estadual de Campinas, Caixa Postal 6065, 13083–859, Campinas, SP, Brazil

E-mail address: `ddnovaes@gmail.com, teixeira@ime.unicamp.br`