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Kernel-based estimation of $P(X > Y)$ in ranked set sampling

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Abstract

This article is directed at the problem of reliability estimation using ranked set sampling. A non-parametric estimator based on kernel density estimation is developed. The estimator is shown to be superior to its analog in simple random sampling. Monte Carlo simulations are employed to assess performance of the proposed estimator. Two real data sets are analysed for illustration.

MSC: 62G30, 62N05.

Keywords: Bandwidth selection, Judgment ranking, Stress-strength model.

1. Introduction

Ranked set sampling (RSS) is a cost-efficient alternative to simple random sampling (SRS) in situations where exact measurements of sample units are difficult or expensive to obtain but (judgment) ranking of them according to the variable of interest is relatively easy and cheap. A variety of methods can be used to implement the ranking, including visual inspection, expert opinion, or through the use of auxiliary variables, but it cannot entail actual measurements on the selected units. The RSS was first introduced by McIntyre (1952) in an agricultural experiment for estimating the mean pasture yield. Since then, it has been well adopted to environmental, ecological and health studies. The reader is referred to Chen (2007) for some novel applications in areas such as clinical trials and genetic quantitative trait loci mappings.

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The RSS procedure can be described as follows. First, m^2 units are collected as independent and identically distributed draws from the population. These units are randomly partitioned into m sets, each of size m. In the first set, the response judged to be smallest is taken for full quantification; in the second set, the response judged to be second smallest is taken; and so on, until in the last set, the response judged to be largest is taken. These measured values, along with the associated ranks form a ranked set sample of size m. The parameter m is called set size, which should be kept small to facilitate the judgment ranking process. Let $X_{[i]}$ $(i = 1, \ldots, m)$ be the ith judgement order statistic from the ith set; then the resulting sample is denoted by $X_{[1]}, \ldots, X_{[m]}$. Here, the square bracket is used to indicate that the judgement ranks may not be correct. If our ranking is accurate, then we replace the square brackets with the round ones, and $X_{(i)}$ becomes the ith true order statistic from the ith set. If a larger sample size is needed the above procedure may be repeated k times (cycles). So a ranked set sample, in its general setup, may be represented by $\{X_{[i]r} : i = 1, \ldots, m; r = 1, \ldots, k\}$, where $X_{[i]r}$ is the ith judgement order statistic in the rth cycle.

A ranked set sample contains more information than a simple random sample of comparable size because it contains not only information carried by quantified observations but also information provided by the ranking process. Thus, it is expected that statistical procedures based on RSS tend to be superior to their SRS analogues. For a good review of RSS and its applications, see Chen et el. (2004). The interested reader is also referred to Wolfe (2004, 2010) and the references therein. Mahdizadeh and Arghami (2013), and Tahmasebi and Jafari (2014) are examples of recently published papers on RSS methods.

The stress-strength model, in its simplest form, defines the reliability of a component as the probability that the strength of the unit (X) is greater than the stress (Y) imposed on it. The quantity $\theta = P(X > Y)$ is referred to as the reliability parameter. Although the use of stress-strength models was originally motivated by problems in physics and engineering, it is not limited to these contexts. It is worth mentioning that θ provides a general measure of the difference between two populations, and has found applications in different fields such as economics, quality control, psychology, medicine and clinical trials. For instance, if Y is the response of a control group, and X is that of a treatment group, then θ is a measure of the treatment effect. This situation is exemplified in Section 5.

There has been continuous interest in the problem of estimating θ when X and Y are independent variables, and belong to the same family of distributions. A comprehensive account of this topic appears in Kotz et al. (2003). The reliability estimation under RSS has also drawn some attention. Muttlak et al. (2010) derived estimators for θ using RSS in the case of the exponential distribution. Sengupta and Mukhuti (2008) studied unbiased estimation of θ using RSS in nonparametric setting based on the empirical distribution function. They showed that the proposed estimator is more efficient than its SRS counterpart, even in the presence of ranking errors. In this work, the kernel density estimator is used to suggest a new estimator.

Section 2 presents the estimator along with some notions and results which will be used in the sequel. Theoretical properties are studied in Section 3. Results from simulation experiments appear in Section 4. Two applications are provided in Section 5. A summary and concluding remarks are given in Section 6. Proofs are postponed to an appendix.

2. The proposed estimator

Let $X_1,\ldots,X_m$ and $Y_1,\ldots,Y_n$ be independent random samples from two continuous populations with density functions f and g, respectively. The corresponding distribution functions are denoted by F and G. The standard nonparametric estimator of θ is

$$\tilde{\theta}_{\mathrm{SRS}} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} I(X_i > Y_j). \tag{1}$$

Under the assumptions of independence, it is possible to write

$$\theta = P(X > Y) = E\left(I(X > Y)\right) = \int\int I(u > v) f(u) g(v)\, \mathrm{d}u\, \mathrm{d}v, \tag{2}$$

where $I(.)$ is the usual indicator function. An alternative estimator of θ can be made by replacing f and g in (2) with some estimates. To this end, the kernel density estimators may be utilized which are given by

$$\hat{f}(u) = \frac{1}{mh_1}\sum_{i=1}^{m} K\left(\frac{u - X_i}{h_1}\right)$$

and

$$\hat{g}(v) = \frac{1}{nh_2}\sum_{j=1}^{n} K\left(\frac{v - Y_j}{h_2}\right),$$

where the kernel K is a symmetric probability density, and the smoothing parameters h_1 and h_2 are known as the bandwidths.

Incorporating $\hat{f}$ and $\hat{g}$ in (2), we have

$$\begin{aligned}\hat{\theta}_{\mathrm{SRS}} &= \int\int I(u>v)\hat{f}(u)\hat{g}(v)\,\mathrm{d}u\,\mathrm{d}v \\ &= \int\int I(u>v)\left[\frac{1}{mh_1}\sum_{i=1}^{m}K\left(\frac{u-X_i}{h_1}\right)\right]\left[\frac{1}{nh_2}\sum_{j=1}^{n}K\left(\frac{v-Y_j}{h_2}\right)\right]\mathrm{d}u\,\mathrm{d}v \\ &= \int\int\int\int I(u>v)\left[\frac{1}{h_1}K\left(\frac{u-x}{h_1}\right)\right]\left[\frac{1}{h_2}K\left(\frac{v-y}{h_2}\right)\right]\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}\hat{F}(x)\,\mathrm{d}\hat{G}(y), \quad (3)\end{aligned}$$

where $\hat{F}$ and $\hat{G}$ are the empirical distribution functions. Using the change of variables $r=(u-x)/h_1$ and $s=(v-y)/h_2$ in (3), it follows that

$$\begin{aligned}\hat{\theta}_{\mathrm{SRS}} &= \int\int\left[\int\int I(h_1r+x>h_2s+y)K(r)K(s)\,\mathrm{d}r\,\mathrm{d}s\right]\mathrm{d}\hat{F}(x)\,\mathrm{d}\hat{G}(y) \\ &= \int\int H(x-y)\,\mathrm{d}\hat{F}(x)\,\mathrm{d}\hat{G}(y),\end{aligned}$$

where H is the distribution function of h_2S-h_1R and R and S are independent random variables with common density K. If K is the standard normal density, then H is the distribution function of a normal random variable with mean 0 and standard deviation $t=\sqrt{h_1^2+h_2^2}$. In this case, $\hat{\theta}_{\mathrm{SRS}}$ takes the form

$$\hat{\theta}_{\mathrm{SRS}} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Phi\left(\frac{X_i-Y_j}{t}\right), \quad (4)$$

where $\Phi(.)$ is the standard normal distribution function. Baklizi and Eidous (2006) used the above estimator to construct confidence intervals for θ.

Proceeding in the same way, we arrive at the RSS analogue of (4) defined as

$$\hat{\theta}_{\mathrm{RSS}} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n}\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right), \quad (5)$$

where $X_{[1]},\ldots,X_{[m]}$ and $Y_{[1]},\ldots,Y_{[n]}$ are ranked set samples drawn from f and g (with a single cycle), respectively. In the next section, properties of this estimator are studied. The results can be extended for other choices of the kernel function.

The success of RSS procedures hinges on how well the within-set rankings to select the units for measurement can be achieved. Although perfect rankings are the ideal case for any RSS-based method, it is unlikely to be feasible. Thus it is worth in practice to evaluate the effect of imperfect rankings on our procedures. The proper way to this would be using statistical models designed to capture possible errors in the ranking process. A number of such imperfect ranking models can be found in the literature. We build on a model introduced by Bohn and Wolfe (1994). They consider the distributions of the judgment order statistics to be mixtures of distributions of the true order statistics. The model is now set forth for our two-sample problem.

The density functions of the ith true and judgement order statistic of a random sample of size m from f are denoted by $f_{(i)}$ and $f_{[i]}$, respectively. Similar notations are used for a random sample of size n from G. We postulate an imperfect ranking model M_X under which $X_{[i]}$'s are assumed to be independently distributed as

$$P\left(X_{[i]} = X_{(r)}\right) = p_{ir}, \quad (r = 1, \ldots, m),$$

where p_{ir} is the probability that the rth order statistic is judged to have rank i, and thus $\sum_{r=1}^{m} p_{ir} = 1$. It is further assumed that $\sum_{i=1}^{m} p_{ir} = 1$. Obviously, this is true in the perfect ranking scenario, i.e. when $p_{ii} = 1$ and $p_{ir} = 0\,(r \neq i)$. Similarly, we postulate an imperfect ranking model M_Y under which $Y_{[j]}$'s are assumed to be independently distributed as

$$P\left(Y_{[j]} = Y_{(s)}\right) = q_{js}, \quad (s = 1, \ldots, n),$$

where q_{js} is the probability that the sth order statistic is judged to have rank j, and therefore $\sum_{s=1}^{n} q_{js} = 1$. Moreover, it is assumed that $\sum_{j=1}^{n} q_{js} = 1$. The model considering M_X and M_Y together is referred to as M. Also, misplacement probability matrices are denoted by $\mathbf{P} = [p_{ir}]_{m\times m}$ and $\mathbf{Q} = [q_{js}]_{n\times n}$.

According to a basic identity in RSS, which simply follows from the binomial expansion, we have

$$\frac{1}{m}\sum_{i=1}^{m} f_{(i)}(x) = f(x), \quad \frac{1}{n}\sum_{j=1}^{n} g_{(j)}(y) = g(y). \tag{6}$$

For details, see Chen et al. (2004, Chapter 2). It is easy to verify that these equations also hold under the model M, i.e.

$$\frac{1}{m}\sum_{i=1}^{m} f_{[i]}(x) = f(x), \quad \frac{1}{n}\sum_{j=1}^{n} g_{[j]}(y) = g(y). \tag{7}$$

These identities are repeatedly used in the sequel.

3. Main results

Theoretical properties of the suggested estimator are studied in this section. It can be seen that (4) and (5) are both biased and their expectations are $E\left\{\Phi\left(\frac{X-Y}{t}\right)\right\}$. The next proposition presents variance expression for the two estimators.

Proposition 1 *The variances of $\hat{\theta}_{SRS}$ and $\hat{\theta}_{RSS}$ are given by*

$$m^2n^2\,\mathrm{Var}(\hat{\theta}_{SRS}) = mn(n-1)EE^2\left\{\Phi\left(\frac{X-Y}{t}\right)\Big|X\right\} + nm(m-1)EE^2\left\{\Phi\left(\frac{X-Y}{t}\right)\Big|Y\right\} + mnE\left\{\Phi^2\left(\frac{X-Y}{t}\right)\right\} + \left(mn - m^2n - n^2m\right)E^2\left\{\Phi\left(\frac{X-Y}{t}\right)\right\} \tag{8}$$

and

$$m^2n^2\,\mathrm{Var}(\hat{\theta}_{RSS}) = mE\left(n^2E^2\left\{\Phi\left(\frac{X-Y}{t}\right)\Big|X\right\} - \sum_{j=1}^{n}E^2\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|X\right\}\right) + E\left(m^2\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2 - \sum_{i=1}^{m}\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2\right) + mnE\left\{\Phi^2\left(\frac{X-Y}{t}\right)\right\} - m^2n^2E^2\left\{\Phi\left(\frac{X-Y}{t}\right)\right\}. \tag{9}$$

The variances of $\hat{\theta}_{\mathrm{SRS}}$ and $\hat{\theta}_{\mathrm{RSS}}$ are compared in the next proposition.

Proposition 2 *Under model M,* $\mathrm{Var}(\hat{\theta}_{RSS}) \le \mathrm{Var}(\hat{\theta}_{SRS})$, *and the equality holds if $f_{[i]} = f\,(i = 1,\ldots,m)$ and $g_{[j]} = g\,(j = 1,\ldots,n)$. The latter happens when $p_{ir} = 1/m\,(i,r = 1,\ldots,m)$ and $q_{js} = 1/n\,(j,s = 1,\ldots,n)$.*

The RSS-based procedures tend to outperform their SRS analogues as long as the judgment ranking is not random. In the case of estimating θ, this was formally shown (under model M) in the previous proposition. The maximum efficiency is expected to happen in the perfect ranking setup. We now give a result confirming this property. It should be mentioned that the approach adopted in proof is distinctly different from that of similar result in Sengupta and Mukhuti (2008).

Proposition 3 *Under model M, the variance of $\hat{\theta}_{RSS}$ is minimized in the absence of ranking errors.*

It is worth noting that the case of perfect rankings is not the only one where the minimum variance is achieved. It would also be attained in the case where model M holds, and $\mathbf{P}$ and $\mathbf{Q}$ are permutation matrices. In addition, there are cases in the RSS literature where an appropriately chosen imperfect rankings scheme can lead to more efficient estimation than is possible with perfect rankings. To put it another way, Proposition 3 may not hold under other imperfect ranking models.

We close this section by some remarks on a general form of our proposed estimator. The estimator (5) is defined for the case where RSS is done with a single cycle. The ranked set sample size, however, is increased not by increasing the set size, but by increasing the number of cycles. It is therefore important to study the multi-cycle case as well. In this setup, the estimator is given by

$$\hat{\theta}_{\text{RSS}} = \frac{1}{mkn\ell} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{r=1}^{k} \sum_{s=1}^{\ell} \Phi\left(\frac{X_{[i]r} - Y_{[j]s}}{t}\right), \tag{10}$$

where $\{X_{[i]r} : i = 1, \ldots, m; r = 1, \ldots, k\}$ and $\{Y_{[j]s} : j = 1, \ldots, n; s = 1, \ldots, \ell\}$ are ranked set samples of size mk and $n\ell$ drawn from f and g, respectively. The above estimator can be represented as

$$\hat{\theta}_{\text{RSS}} = \frac{1}{mn} \sum_{i=1}^{m} \sum_{j=1}^{n} h\left(X_{[i]}, Y_{[j]}\right),$$

where

$$h\left(X_{[i]}, Y_{[j]}\right) = \frac{1}{k\ell} \sum_{r=1}^{k} \sum_{s=1}^{\ell} \Phi\left(\frac{X_{[i]r} - Y_{[j]s}}{t}\right).$$

Now, one can proceed with proving analogues of Propositions 1-3. The steps are similar to current proofs in which

$$h\left(X_{[i]}, Y_{[j]}\right) = \Phi\left(\frac{X_{[i]} - Y_{[j]}}{t}\right).$$

As a reviewer pointed out, the estimator (10) was also studied by Yin et al. (2016). The authors, however, build on the theory of U-statistics in computing the variance expression. Moreover, they only show that this estimator is asymptotically more efficient than its counterpart in SRS. And last but not least, no theoretical result in the imperfect ranking setup is provided in the aforesaid paper.

For simplicity, we consider the estimator (5) in the next section. But the data analysis in Section 5 is based on the estimator (10).

4. Simulation study

This section contains results of simulation studies performed to evaluate behaviours of $\hat{\theta}_{\text{SRS}}$ and $\hat{\theta}_{\text{RSS}}$. It is assumed that both populations follow normal, exponential or uniform distribution. Suppose X and $Y-\mu$ are standard normal random variables. Then, it is simply shown that

$$\theta = \Phi\left(\frac{-\mu}{\sqrt{2}}\right),$$

where $\Phi(.)$ is the distribution function of X. Similarly, for standard exponential random variables X and Y/α, we have

$$\theta = \frac{1}{1+\alpha}.$$

Finally, let X and Y/β be uniformly distributed on the unit interval. Then, it follows that

$$\theta = \begin{cases} 1-\beta/2 & 0<\beta<1 \\ 1/(2\beta) & \beta \geq 1 \end{cases}.$$

Under each parent distribution, five values were assigned to the associated parameter so as to produce $\theta = 0.1, 0.3, 0.5, 0.7, 0.9$. The appropriate parameter values are given in Table 1. Also, set sizes $(m,n) = (3,3), (3,7), (5,5), (10,10)$ were selected.

Table 1: *Parameter values corresponding to different reliability parameters.*

	θ				
Parameter	0.1	0.3	0.5	0.7	0.9
μ	1.812388	0.7416143	0	−0.7416143	−1.812388
α	9	7/3	1	3/7	1/9
β	5	5/3	1	3/5	1/5

We first consider the perfect ranking situation. For each combination of distributions and sample sizes, 10,000 pairs of samples were generated in SRS and RSS settings. The two estimators were computed from each pair of samples in the corresponding designs, and their mean squared errors (MSEs) were determined. The relative efficiency (RE) is defined as the ratio of $\widehat{MSE}(\hat{\theta}_{\text{SRS}})$ to $\widehat{MSE}(\hat{\theta}_{\text{RSS}})$. The RE values larger than one indicate that $\hat{\theta}_{\text{RSS}}$ is more efficient than $\hat{\theta}_{\text{SRS}}$. Tables 2 and 3 display the results (to save space, tables for the uniform distribution are provided as supplementary material), where RE1-RE4 are based on the following four methods for bandwidth selection, re-

Table 2: *Estimated REs under normal distribution (RE1, RE2, RE3 and RE4 are based on bandwidth selection using AMISE, UCV, BCV and PI methods, respectively).*

(m,n)	θ	RE1	RE2	RE3	RE4
(3,3)	0.1	1.00 (0.38)	1.02 (0.40)	1.00 (0.38)	1.56 (1.17)
	0.3	1.48 (1.74)	1.53 (1.76)	1.48 (1.74)	1.95 (1.34)
	0.5	2.38 (4.35)	2.36 (4.02)	2.38 (4.35)	2.04 (1.36)
	0.7	1.44 (1.71)	1.49 (1.73)	1.44 (1.71)	1.91 (1.32)
	0.9	0.99 (0.38)	1.00 (0.41)	0.99 (0.38)	1.51 (1.18)
(3,7)	0.1	1.00 (0.25)	1.04 (0.32)	1.00 (0.25)	1.68 (1.00)
	0.3	1.45 (1.23)	1.64 (1.30)	1.45 (1.23)	2.23 (1.29)
	0.5	2.71 (4.03)	2.72 (3.07)	2.71 (4.03)	2.42 (1.38)
	0.7	1.43 (1.25)	1.61 (1.32)	1.43 (1.25)	2.18 (1.31)
	0.9	0.99 (0.25)	1.01 (0.32)	0.99 (0.25)	1.59 (1.00)
(5,5)	0.1	1.01 (0.20)	1.09 (0.30)	1.01 (0.20)	1.81 (0.91)
	0.3	1.46 (0.96)	1.83 (1.13)	1.46 (0.95)	2.73 (1.30)
	0.5	3.51 (4.27)	3.52 (2.93)	3.51 (4.28)	3.14 (1.43)
	0.7	1.48 (0.95)	1.87 (1.14)	1.48 (0.95)	2.77 (1.31)
	0.9	1.02 (0.20)	1.11 (0.30)	1.02 (0.20)	1.84 (0.92)
(10,10)	0.1	1.02 (0.08)	1.19 (0.19)	1.02 (0.08)	2.14 (0.60)
	0.3	1.43 (0.37)	2.30 (0.64)	1.43 (0.37)	4.13 (1.06)
	0.5	5.78 (3.42)	6.12 (2.18)	5.78 (3.42)	5.81 (1.47)
	0.7	1.44 (0.38)	2.28 (0.65)	1.44 (0.38)	4.09 (1.07)
	0.9	1.02 (0.09)	1.18 (0.19)	1.02 (0.09)	2.11 (0.61)

spectively. Minimizing asymptotic mean integrated squared error (AMISE) of the kernel density estimator is a basic scheme. Rudemo (1982) and Bowman (1984) proposed unbiased (least-squares) cross-validation (UCV) method. Biased cross-validation (BCV) was studied by Scott and George (1987). A plug-in (PI) method was suggested by Sheather and Jones (1991). All these techniques are developed for SRS, and more details on them can be found in Sheather (2004). The methods can be implemented in R statistical software using the `kedd` and `KernSmooth` packages. In the RSS setup, we treat data as if collected by SRS to choose bandwidth.

Sengupta and Mukhuti (2008) introduced the RSS competitor of (1) defined as

$$\tilde{\theta}_{\mathrm{RSS}} = \frac{1}{mn}\sum_{i=1}^{m}\sum_{j=1}^{n} I(X_{[i]} > Y_{[j]}). \tag{11}$$

The entries of Tables 2–4 appearing in parentheses show efficiency of $\hat{\theta}_{\mathrm{RSS}}$ relative to $\tilde{\theta}_{\mathrm{RSS}}$.

Table 3: *Estimated REs under exponential distribution (RE1, RE2, RE3 and RE4 are based on bandwidth selection using AMISE, UCV, BCV and PI methods, respectively).*

(m,n)	θ	RE1	RE2	RE3	RE4
(3,3)	0.1	0.96 (0.33)	0.97 (0.35)	0.96 (0.33)	1.37 (1.19)
	0.3	1.51 (1.93)	1.56 (1.96)	1.51 (1.93)	1.99 (1.51)
	0.5	2.23 (4.11)	2.21 (3.79)	2.23 (4.11)	2.04 (1.38)
	0.7	1.50 (1.90)	1.55 (1.94)	1.50 (1.90)	2.00 (1.50)
	0.9	0.97 (0.33)	0.97 (0.35)	0.97 (0.33)	1.35 (1.20)
(3,7)	0.1	0.93 (0.13)	1.04 (0.33)	0.93 (0.13)	1.27 (0.71)
	0.3	1.39 (1.33)	1.88 (1.58)	1.39 (1.33)	2.44 (1.52)
	0.5	2.45 (3.76)	2.43 (2.55)	2.45 (3.76)	2.41 (1.37)
	0.7	1.45 (1.34)	1.64 (1.47)	1.45 (1.34)	2.21 (1.43)
	0.9	0.93 (0.26)	0.93 (0.28)	0.93 (0.26)	1.26 (0.98)
(5,5)	0.1	0.92 (0.14)	1.02 (0.28)	0.92 (0.14)	1.32 (0.74)
	0.3	1.44 (1.07)	2.13 (1.48)	1.44 (1.07)	2.83 (1.50)
	0.5	2.94 (3.78)	3.00 (2.38)	2.94 (3.78)	2.96 (1.40)
	0.7	1.43 (1.08)	2.07 (1.47)	1.43 (1.08)	2.78 (1.50)
	0.9	0.93 (0.15)	1.01 (0.28)	0.93 (0.15)	1.31 (0.75)
(10,10)	0.1	0.93 (0.05)	1.05 (0.19)	0.93 (0.05)	1.22 (0.32)
	0.3	1.30 (0.38)	3.42 (1.14)	1.30 (0.38)	4.33 (1.32)
	0.5	4.77 (3.14)	5.33 (1.67)	4.77 (3.14)	5.50 (1.41)
	0.7	1.27 (0.37)	3.34 (1.12)	1.27 (0.37)	4.30 (1.30)
	0.9	0.93 (0.05)	1.04 (0.19)	0.93 (0.05)	1.23 (0.33)

It is observed that $\hat{\theta}_{\text{SRS}}$ is outperformed by $\hat{\theta}_{\text{RSS}}$, using one of the bandwidths, at least. Moreover, the results from AMISE and BCV methods are in close agreement. For each pair of sample sizes, the RE values are generally larger when the reliability parameter is 0.5. Also, the PI method works better than the others for $\theta \neq 0.5$. It is to be noted that in the case of $\theta = 0.1, 0.9$, only RE4 values exceed unity markedly. Given a total sample size, the efficiency gain is generally larger for equal sample sizes setup. Compare similar REs for $(m,n) = (3,7),(5,5)$ under different parent distributions.

As to the comparison of the suggested estimator with its rival based on empirical distribution function, the following conclusions can be made (based on entries given in parentheses in Tables 2 and 3). Again, the RE values are generally larger for the reliability parameter 0.5, given a pair of sample sizes. There are cases that $\hat{\theta}_{\text{RSS}}$ is less efficient than $\tilde{\theta}_{\text{RSS}}$. For example, see the results when $(m,n) = (5,5),(10,10)$ and $\theta = 0.1, 0.9$. Sometimes the REs from AMISE, UCV and BCV methods fall much below unity. In such instances, the PI method is still the best one.

Table 4: *Estimated REs for $(m,n) = (3,3)$ (upper panel) and $(m,n) = (5,5)$ (lower panel) with imperfect ranking (RE1, RE2, RE3 and RE4 are based on bandwidth selection using AMISE, UCV, BCV and PI methods, respectively).*

Dist.	θ	RE1	RE2	RE3	RE4
Normal	0.1	1.03 (0.42)	1.05 (0.45)	1.03 (0.42)	1.47 (1.18)
	0.3	1.41 (1.89)	1.44 (1.90)	1.41 (1.89)	1.65 (1.31)
	0.5	1.94 (4.10)	1.92 (3.78)	1.94 (4.10)	1.70 (1.31)
	0.7	1.34 (1.82)	1.37 (1.82)	1.34 (1.82)	1.63 (1.29)
	0.9	0.99 (0.43)	1.00 (0.46)	0.99 (0.43)	1.37 (1.18)
Exponential	0.1	0.98 (0.38)	0.98 (0.40)	0.98 (0.38)	1.30 (1.26)
	0.3	1.40 (2.03)	1.43 (2.05)	1.40 (2.03)	1.71 (1.46)
	0.5	1.82 (4.01)	1.81 (3.70)	1.82 (4.01)	1.69 (1.36)
	0.7	1.36 (2.01)	1.39 (2.03)	1.36 (2.01)	1.66 (1.45)
	0.9	0.96 (0.37)	0.96 (0.39)	0.96 (0.37)	1.21 (1.22)
Normal	0.1	1.02 (0.22)	1.10 (0.33)	1.02 (0.22)	1.65 (0.94)
	0.3	1.42 (1.08)	1.70 (1.24)	1.42 (1.08)	2.31 (1.30)
	0.5	2.81 (4.07)	2.80 (2.79)	2.81 (4.07)	2.58 (1.40)
	0.7	1.39 (1.07)	1.70 (1.23)	1.39 (1.07)	2.33 (1.30)
	0.9	1.00 (0.23)	1.07 (0.34)	1.00 (0.23)	1.60 (0.95)
Exponential	0.1	0.94 (0.16)	1.00 (0.30)	0.94 (0.16)	1.26 (0.79)
	0.3	1.39 (1.19)	1.91 (1.55)	1.39 (1.19)	2.41 (1.49)
	0.5	2.59 (3.82)	2.63 (2.38)	2.59 (3.82)	2.58 (1.39)
	0.7	1.42 (1.16)	1.99 (1.52)	1.42 (1.16)	2.57 (1.49)
	0.9	0.94 (0.16)	1.02 (0.30)	0.94 (0.16)	1.29 (0.78)

As mentioned before, although perfect rankings are ideal case for any RSS-based method, it is unlikely to be feasible. Let **P** and **Q** be misplacement probability matrices defined in Section 2. The perfect ranking setup corresponds to the case that **P** and **Q** are the identity matrices. We conducted a partial simulation study to assess performance of the suggested estimator in the presence of ranking errors. To do so, the REs were estimated when $(m,n) = (3,3), (5,5)$ and the matrices **P** and **Q** are selected to be

$$\mathbf{P} = \mathbf{Q} = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.9 \end{bmatrix}$$

and

$$\mathbf{P} = \mathbf{Q} = \begin{bmatrix} 0.9 & 0.1 & 0 & 0 & 0 \\ 0.1 & 0.8 & 0.1 & 0 & 0 \\ 0 & 0.1 & 0.8 & 0.1 & 0 \\ 0 & 0 & 0.1 & 0.8 & 0.1 \\ 0 & 0 & 0 & 0.1 & 0.9 \end{bmatrix},$$

respectively. The results are given in Table 4. The entries outside parentheses are generally smaller than similar entries in Tables 2 and 3. We note, however, that these REs still exceed the unity, and this is consistent with our theoretical results. There is not a uniform trend for entries inside parentheses as compared with analogous ones under perfect ranking assumption. It is to be mentioned that these REs are associated with $\hat{\theta}_{\text{RSS}}$ and $\tilde{\theta}_{\text{RSS}}$ that both are affected by ranking errors.

All simulation studies in this work are programmed using R statistical software, and the corresponding code is available from the first author.

5. Application

The RSS is applicable in the following situations: (i) the ranking of a set of sampling units can be done easily by judgment relating to their latent values of the variable of interest through visual inspection, expert opinion, etc. (ii) there are certain easily accessible concomitant variables. We now illustrate the proposed procedure using some real data from two different fields.

5.1. Agriculture

Murray et al. (2000) conducted an experiment in which apple trees are sprayed with chemical containing fluorescent tracer, Tinopal CBS-X, at 2% concentration level in water. Two nine-tree plots were chosen for spraying. One plot was sprayed at high volume, using coarse nozzles on the sprayer to give a large average droplet size. The other plot was sprayed at low volume, using fine nozzles to give a small average droplet size. Fifty sets of five leaves were identified from the central five trees of each plot, and used to draw a ranked set sample with set size 5 and cycle size 10, from each plot. The variable of interest is the percentage of area covered by the spray on the surface of the leaves. The formal measurement entails chemical analysis of the solution collected from the surface of the leaves, and thereby is a time-consuming and expensive process. The judgment ranking within each set is based on the visual appearance of the spray deposits on the leaf surfaces when viewed under ultraviolet light. Clearly, the latter method is cheap, and fairly accurate if implemented by an expert observer.

The data are given in Table 5, where measurements obtained from the plot sprayed at high (low) volume constitute the control (treatment) group. The interest centres on knowing whether the sprayer settings affect the percentage area coverage. If X (Y) denotes the response variable from treatment (control) group, then $\hat{\theta}_{\text{RSS}}$ and $\tilde{\theta}_{\text{RSS}}$ can serve as measures of the treatment effect.

Let $\breve{\theta}$ be either $\hat{\theta}_{\text{RSS}}$ or $\tilde{\theta}_{\text{RSS}}$. Then the bootstrap method, introduced by Efron (1979), can be used to estimate the variance of $\breve{\theta}$, and to construct confidence interval. Modarres et al. (2006) suggested three bootstrap algorithms in RSS design. Bootstrap ranked set sampling (BRSS) and bootstrap RSS by rows (BRSSR) are the most efficient methods

Table 5: *Ranked set sample data for the percentage area covered on the surface of the leaves of apple trees.*

Group	Cycle	Rank 1	Rank 2	Rank 3	Rank 4	Rank 5
Control	1	0.003	0.028	0.244	0.057	0.143
	2	0.039	0.119	0.126	0.105	0.565
	3	0.034	0.118	0.130	0.218	0.296
	4	0.051	0.104	0.193	0.210	0.150
	5	0.032	0.141	0.130	0.250	0.229
	6	0.069	0.070	0.260	0.225	0.285
	7	0.100	0.091	0.244	0.130	0.347
	8	0.012	0.096	0.069	0.373	0.133
	9	0.046	0.117	0.126	0.223	0.273
	10	0.028	0.083	0.108	0.212	0.261
Treatment	1	0.036	0.137	0.183	0.270	0.487
	2	0.250	0.181	0.290	0.328	0.715
	3	0.089	0.032	0.269	0.419	0.315
	4	0.180	0.111	0.130	0.194	0.742
	5	0.100	0.009	0.184	0.277	0.122
	6	0.042	0.089	0.199	0.269	0.395
	7	0.044	0.083	0.227	0.177	0.742
	8	0.044	0.171	0.067	0.192	0.336
	9	0.009	0.017	0.217	0.438	0.544
	10	0.071	0.132	0.310	0.343	0.379

which are used here. Suppose B pairs of bootstrap samples are drawn from the two ranked set samples by either of the algorithms. If $\breve{\theta}_b$ is value of the estimator based on data in the b th ($b = 1, \ldots, B$) replication, then the bootstrap variance estimator is given by

$$\widehat{\text{Var}}_{\text{boot}}(\breve{\theta}) = \frac{1}{B-1} \sum_{b=1}^{B} \left(\breve{\theta}_b - \bar{\theta}\right)^2, \tag{12}$$

where $\bar{\theta} = \sum_{b=1}^{B} \breve{\theta}_b / B$. An approximate $(1-\alpha)$ normal interval for θ is then constructed as

$$\left(\breve{\theta} - z_{\alpha/2}\sqrt{\widehat{\text{Var}}_{\text{boot}}(\breve{\theta})}, \breve{\theta} + z_{\alpha/2}\sqrt{\widehat{\text{Var}}_{\text{boot}}(\breve{\theta})}\right), \tag{13}$$

where $z_{\alpha/2}$ is the $(1-\alpha/2)$ quantile of the standard normal distribution. We may alternatively use $(1-\alpha)$ bootstrap percentile interval defined as

$$\left(\breve{\theta}_{\alpha/2}, \breve{\theta}_{1-\alpha/2}\right), \tag{14}$$

where $\breve{\theta}_\beta$ is the β quantile of $\breve{\theta}_1, \ldots, \breve{\theta}_B$.

Table 6: *Estimates of θ along with their estimated variances, and the corresponding 0.95 confidence intervals.*

Estimator	Value	Estimated variance	Normal interval	Bootstrap interval
$\hat{\theta}_{\text{RSS}}$ (AMISE)	0.5903	0.000456	(0.548, 0.632)	(0.550, 0.633)
		0.000468	*(0.548, 0.633)*	*(0.549, 0.634)*
$\hat{\theta}_{\text{RSS}}$ (UCV)	0.6161	0.001021	(0.553, 0.679)	(0.557, 0.680)
		0.001052	*(0.553, 0.680)*	*(0.556, 0.684)*
$\hat{\theta}_{\text{RSS}}$ (BCV)	0.6118	0.000748	(0.558, 0.665)	(0.558, 0.664)
		0.000789	*(0.557, 0.667)*	*(0.557, 0.667)*
$\hat{\theta}_{\text{RSS}}$ (PI)	0.6168	0.000927	(0.557, 0.676)	(0.559, 0.678)
		0.000964	*(0.556, 0.678)*	*(0.558, 0.680)*
$\tilde{\theta}_{\text{RSS}}$	0.6184	0.001163	(0.552, 0.685)	(0.553, 0.685)
		0.001224	*(0.550, 0.687)*	*(0.552, 0.688)*

Table 6 displays the estimates along with their estimated variances computed using (12). Two 0.95 intervals (13) and (14) are also reported. The number of bootstrap replications is chosen to be 5000, and entries associated with BRSSR method are in italic. Clearly, the kernel-based estimators have smaller estimated variances as expected. It is concluded that the treatment effect is significant at 0.05 level as none of the intervals contain 0.5.

5.2. Medicine

The RSS can be used in studying certain medical measures, which usually involves expensive laboratory tests. Samawi et al. (2009) employed this design in comparing bilirubin level between male and female jaundice babies. To this end, blood sample must be taken from the sampled babies and tested in a laboratory. But, on the other hand, the ranking of the bilirubin levels of a small number of babies can be done by observing whether their face, chest, lower parts of the body and the terminal parts of the whole body are yellowish. The yellowish color goes from face to the terminal parts of the whole body, the level of bilirubin in blood goes higher.

Table 7 shows the results of 15 measurements for male/female babies collected by RSS with set size 3 and cycle size 5. Assume that X and Y represent the response variable for male and female babies, respectively. Then $\hat{\theta}_{\text{RSS}}$ and $\tilde{\theta}_{\text{RSS}}$ can be used to decide whether male babies are more likely to experience jaundice. Table 8 displays the estimates along with their estimated variances. The corresponding 0.95 confidence intervals are also provided. Again, the kernel-based estimators have smaller estimated variances. All the intervals contain 0.5, and the null hypothesis that male and female babies are equally likely to experience jaundice is not rejected, at 0.05 level.

Table 7: *Ranked set sample data of bilirubin level in jaundice babies.*

Group	Cycle	Rank 1	Rank 2	Rank 3
Male	1	7.50	10.50	7.30
	2	7.50	15.00	8.60
	3	8.90	14.60	13.53
	4	7.00	11.90	15.70
	5	10.24	13.18	18.47
Female	1	1.20	8.94	15.00
	2	7.50	12.82	10.80
	3	8.00	8.82	10.70
	4	8.90	8.94	14.59
	5	8.53	8.20	18.29

Table 8: *Estimates of θ along with their estimated variances, and the corresponding 0.95 confidence intervals.*

Estimator	Value	Estimated variance	Normal interval	Bootstrap interval
$\hat{\theta}_{\mathrm{RSS}}$ (AMISE)	0.5549	0.002183	(0.463, 0.646)	(0.464, 0.649)
		0.001813	*(0.471, 0.638)*	*(0.474, 0.638)*
$\hat{\theta}_{\mathrm{RSS}}$ (UCV)	0.5753	0.005465	(0.430, 0.720)	(0.409, 0.704)
		0.004715	*(0.441, 0.710)*	*(0.421, 0.687)*
$\hat{\theta}_{\mathrm{RSS}}$ (BCV)	0.5576	0.002398	(0.462, 0.654)	(0.464, 0.657)
		0.002016	*(0.470, 0.646)*	*(0.473, 0.647)*
$\hat{\theta}_{\mathrm{RSS}}$ (PI)	0.5774	0.005067	(0.438, 0.717)	(0.434, 0.717)
		0.004172	*(0.451, 0.704)*	*(0.450, 0.700)*
$\tilde{\theta}_{\mathrm{RSS}}$	0.5467	0.006564	(0.388, 0.705)	(0.382, 0.707)
		0.005709	*(0.399, 0.695)*	*(0.400, 0.689)*

6. Conclusion

The RSS design employs ranking of the characteristic of interest via auxiliary information to improve estimation of population attributes. The rankings can be performed through subjective judgment, concomitant variable, or a combination of them. These preparatory rankings are made before any actual measurements on the variable of interest, and are utilized to select more informative units to include in our sample for measurement.

In this article, a nonparametric reliability estimator based on kernel density estimation is suggested. Some theoretical results are presented under an imperfect ranking model. The perfect ranking setup is treated separately. Monte Carlo simulations are used to compare the estimator with its SRS competitor, and the RSS analogue based on empirical distribution function. The results confirm preference of the new estimator in many situations. In a subsequent work, we plan to study interval estimation of the reliability parameter under the RSS scheme.

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References

Baklizi, A. and Eidous, O. (2006). Nonparametric estimation of $P(X < Y)$ using kernel methods. *Metron*, LXIV, 47–60.

Bohn, L.L. and Wolfe, D.A. (1994). The effect of imperfect judgment rankings on properties of procedures based on the ranked-set samples analog of the Mann-Whitney-Wilcoxon statistic. *Journal of the American Statistical Association*, 89, 168–176.

Bowman, A.W. (1984). An alternative method of cross-validation for the smoothing of kernel density estimates. *Biometrika*, 71, 353–360.

Chen, Z. (2007). Ranked set sampling: its essence and some new applications. *Environmental and Ecological Statistics*, 14, 355–363.

Chen, Z., Bai, Z. and Sinha, B.K. (2004). *Ranked set sampling: Theory and Applications*. Springer, New York.

Datta, B.N. (2010). *Numerical Linear Algebra and Applications*, Second Edition. SIAM.

Efron, B. (1979). Bootstrap methods: Another look at the jackknife. *Annals of Statistics*, 7, 1–26.

Kotz, S., Lumelskii, Y. and Pensky, M. (2003). The stress-strength model and its generalizations. *Theory and applications. World Scientific*, Singapore.

Mahdizadeh, M. and Arghami, N.R. (2013). Improved entropy based test of uniformity using ranked set sample. *SORT*, 37, 3–18.

McIntyre, G.A. (1952). A method of unbiased selective sampling using ranked sets. *Australian Journal of Agricultural Research*, 3, 385–390.

Modarres, R., Hui, T.P. and Zheng, G. (2006). Resampling methods for ranked set samples. *Computational Statistics and Data Analysis*, 51, 1039–1050.

Murray, R.A., Ridout, M.S. and Cross, J.V. (2000). The use of ranked set sampling in spray deposit assessment. *Aspects of Applied Biology*, 57, 141–146.

Muttlak, H.A., Abu-Dayyeh, W.A., Al-Saleh, M.F. and Al-Sawi, E. (2010). Estimating $P(Y < X)$ using ranked set sampling in case of the exponential distribution. *Communications in Statistics: Theory and Methods*, 39, 1855–1868.

Rudemo, M. (1982). Empirical choice of histograms and kernel density estimators. *Scandinavian Journal of Statistics*, 9, 65–78.

Samawi, H.M., Al-Saleh, M.F. and Al-Saidy, O. (2009). The matched pair sign test using bivariate ranked set sampling for different ranking based schemes. *Statistical Methodology*, 6, 397–407.

Scott, D.W. and George, R.T. (1987). Biased and unbiased cross-validation in density estimation. *Journal of the American Statistical Association*, 82, 1131–1146.

Sengupta, S. and Mukhuti, S. (2008). Unbiased estimation of $P(X>Y)$ using ranked set sample data. *Statistics*, 42, 223–230.

Sheather, S.J. (2004). Density estimation. *Statistical Science*, 19, 588–597.

Sheather, S.J. and Jones, M.C. (1991). A reliable data-based bandwidth selection method for kernel density estimation. *Journal of the Royal Statistical Society*, Series B 53, 683–690.

Tahmasebi, S. and Jafari, A.A. (2014). Estimators for the parameter mean of Morgenstern type bivariate generalized exponential distribution using ranked set sampling. *SORT*, 38, 161–180.

Wolfe, D.A. (2004). Ranked set sampling: An approach to more efficient data collection. *Statistical Science*, 19, 636–643.

Wolfe, D.A. (2010). Ranked set sampling. Wiley interdisciplinary reviews. *Computational Statistics*, 2, 460–466.

Yin, J., Hao, Y., Samawi, H. and Rochani, H. (2016). Rank-based kernel estimation of the area under the ROC curve. *Statistical Methodology*, 32, 91–106.

Appendix

Proof of Proposition 1. It is easy to show that

$$m^2n^2E(\hat{\theta}^2_{\text{SRS}}) = E(A_1+A_2+A_3+A_4), \tag{15}$$

where

$$\begin{aligned} E(A_1) &= E\left\{\sum_{i\neq i'=1}^{m}\sum_{j\neq j'=1}^{n}\Phi\left(\frac{X_i-Y_j}{t}\right)\Phi\left(\frac{X_{i'}-Y_{j'}}{t}\right)\right\} \\ &= m(m-1)n(n-1)E^2\left\{\Phi\left(\frac{X-Y}{t}\right)\right\}, \end{aligned} \tag{16}$$

$$\begin{aligned} E(A_2) &= E\left\{\sum_{i=1}^{m}\sum_{j\neq j'=1}^{n}\Phi\left(\frac{X_i-Y_j}{t}\right)\Phi\left(\frac{X_i-Y_{j'}}{t}\right)\right\} \\ &= mn(n-1)EE^2\left\{\Phi\left(\frac{X-Y}{t}\right)\Big|X\right\}, \end{aligned} \tag{17}$$

$$\begin{aligned} E(A_3) &= E\left\{\sum_{j=1}^{n}\sum_{i\neq i'=1}^{m}\Phi\left(\frac{X_i-Y_j}{t}\right)\Phi\left(\frac{X_{i'}-Y_j}{t}\right)\right\} \\ &= nm(m-1)EE^2\left\{\Phi\left(\frac{X-Y}{t}\right)\Big|Y\right\} \end{aligned} \tag{18}$$

and

$$E(A_4)=E\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\Phi^2\left(\frac{X_i-Y_j}{t}\right)\right\}=mnE\left\{\Phi^2\left(\frac{X-Y}{t}\right)\right\}. \tag{19}$$

From (15)-(19) and the expectation of $\hat{\theta}_{\text{SRS}}$, the proof of the first part is complete. Similarly,

$$m^2n^2E(\hat{\theta}^2_{\text{RSS}})=E(B_1+B_2+B_3), \tag{20}$$

where

$$\begin{aligned}
E(B_1)=&E\left\{\sum_{i\neq i'=1}^{m}\sum_{j\neq j'=1}^{n}\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Phi\left(\frac{X_{[i']}-Y_{[j']}}{t}\right)\right.\\
&\left.+\sum_{j=1}^{n}\sum_{i\neq i'=1}^{m}\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Phi\left(\frac{X_{[i']}-Y_{[j]}}{t}\right)\right\}\\
=&E\left(\sum_{i\neq i'=1}^{m}\sum_{j\neq j'=1}^{n}E\left\{\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}E\left\{\Phi\left(\frac{X_{[i']}-Y_{[j']}}{t}\right)\Big|Y_{[j']}\right\}\right.\\
&\left.+\sum_{j=1}^{n}\sum_{i\neq i'=1}^{m}E\left\{\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}E\left\{\Phi\left(\frac{X_{[i']}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right)\\
=&E\left(\left[\sum_{i=1}^{m}\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2-\sum_{i=1}^{m}\sum_{j=1}^{n}E^2\left\{\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right.\\
&\left.-\sum_{i=1}^{m}\sum_{j\neq j'=1}^{n}E\left\{\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}E\left\{\Phi\left(\frac{X_{[i]}-Y_{[j']}}{t}\right)\Big|Y_{[j']}\right\}\right)\\
=&E\left(\left[\sum_{i=1}^{m}\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2-\sum_{i=1}^{m}\left[\sum_{j=1}^{n}E^2\left\{\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right.\right.\\
&\left.\left.+\sum_{j\neq j'=1}^{n}E\left\{\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}E\left\{\Phi\left(\frac{X_{[i]}-Y_{[j']}}{t}\right)\Big|Y_{[j']}\right\}\right]\right)\\
=&E\left(m^2\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2-\sum_{i=1}^{m}\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2\right),
\end{aligned} \tag{21}$$

$$E(B_2) = E\left\{ \sum_{i=1}^{m} \sum_{j\neq j'=1}^{n} \Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right) \Phi\left(\frac{X_{[i]}-Y_{[j']}}{t}\right) \right\}$$

$$= mE\left\{ \sum_{j\neq j'=1}^{n} \Phi\left(\frac{X-Y_{[j]}}{t}\right) \Phi\left(\frac{X-Y_{[j']}}{t}\right) \right\}$$

$$= mE\left(\sum_{j\neq j'=1}^{n} E\left\{ \Phi\left(\frac{X-Y_{[j]}}{t}\right) \Big| X \right\} E\left\{ \Phi\left(\frac{X-Y_{[j']}}{t}\right) \Big| X \right\} \right)$$

$$= mE\left(\left[\sum_{j=1}^{n} E\left\{ \Phi\left(\frac{X-Y_{[j]}}{t}\right) \Big| X \right\} \right]^2 - \sum_{j=1}^{n} E^2\left\{ \Phi\left(\frac{X-Y_{[j]}}{t}\right) \Big| X \right\} \right)$$

$$= mE\left(n^2 E^2\left\{ \Phi\left(\frac{X-Y}{t}\right) \Big| X \right\} - \sum_{j=1}^{n} E^2\left\{ \Phi\left(\frac{X-Y_{[j]}}{t}\right) \Big| X \right\} \right), \tag{22}$$

and

$$E(B_3) = E\left\{ \sum_{i=1}^{m} \sum_{j=1}^{n} \Phi^2\left(\frac{X_{[i]}-Y_{[j]}}{t}\right) \right\} = mnE\left\{ \Phi^2\left(\frac{X-Y}{t}\right) \right\}. \tag{23}$$

Now the second part follows from (20)-(23) and the expectation of $\hat{\theta}_{\text{RSS}}$. ■

Proof of Proposition 2. Using equations (8) and (9), it can be shown

$$m^2 n^2 \left[\text{Var}(\hat{\theta}_{\text{SRS}}) - \text{Var}(\hat{\theta}_{\text{RSS}}) \right] = \Delta_1 + \Delta_2 + \Delta_3, \tag{24}$$

where

$$\Delta_1 = E\left(\sum_{i=1}^{m} \left[\sum_{j=1}^{n} E\left\{ \Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right) \Big| Y_{[j]} \right\} \right]^2 - m \left[\sum_{j=1}^{n} E\left\{ \Phi\left(\frac{X-Y_{[j]}}{t}\right) \Big| Y_{[j]} \right\} \right]^2 \right)$$

$$= E\left(\sum_{i=1}^{m} \left[\sum_{j=1}^{n} E\left\{ \Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right) \Big| Y_{[j]} \right\} - \sum_{j=1}^{n} E\left\{ \Phi\left(\frac{X-Y_{[j]}}{t}\right) \Big| Y_{[j]} \right\} \right]^2 \right), \tag{25}$$

$$\begin{aligned}
\Delta_2 &= mn(n-1)EE^2\left\{\Phi\left(\frac{X-Y}{t}\right)\Big|X\right\}\\
&\quad - mE\left(n^2E^2\left\{\Phi\left(\frac{X-Y}{t}\right)\Big|X\right\} - \sum_{j=1}^{n}E^2\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|X\right\}\right)\\
&= mE\left(\sum_{j=1}^{n}E^2\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|X\right\} - nE^2\left\{\Phi\left(\frac{X-Y}{t}\right)\Big|X\right\}\right)\\
&= mE\left(\sum_{j=1}^{n}\left[E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|X\right\} - E\left\{\Phi\left(\frac{X-Y}{t}\right)\Big|X\right\}\right]^2\right) \qquad (26)
\end{aligned}$$

and

$$\begin{aligned}
\Delta_3 &= m(m-1)n(n-1)E^2\left\{\Phi\left(\frac{X-Y}{t}\right)\right\} + nm(m-1)EE^2\left\{\Phi\left(\frac{X-Y}{t}\right)\Big|Y\right\}\\
&\quad - m(m-1)E\left(\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2\right)\\
&= m(m-1)\left[n(n-1)E^2\left\{\Phi\left(\frac{X-Y}{t}\right)\right\}\right.\\
&\quad \left. + E\left(\sum_{j=1}^{n}E^2\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\} - \left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2\right)\right]\\
&= m(m-1)\left[\left(1-\frac{1}{n}\right)\left(\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\right\}\right)^2\right.\\
&\quad \left. - \sum_{j\neq j'=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\right\}E\left\{\Phi\left(\frac{X-Y_{[j']}}{t}\right)\right\}\right]\\
&= m(m-1)\left[\sum_{j=1}^{n}E^2\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\right\} - \frac{1}{n}\left(\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\right\}\right)^2\right]\\
&= m(m-1)\sum_{j=1}^{n}E^2\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right) - \Phi\left(\frac{X-Y}{t}\right)\right\}. \qquad (27)
\end{aligned}$$

Clearly, $\Delta_i \geq 0\,(i=1,2,3)$, as was to be shown. Proof of the next part is straightforward, and is omitted. ■

The next lemma paves the way for Proposition 3.

Lemma 1 *If* $\ell = \sum_{j=1}^{n} E\left\{\Phi\left(\frac{X-Y_{(j)}}{t}\right)\Big|Y_{(j)}\right\}$ *and* $L = \sum_{j=1}^{n} E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}$, *then* $\text{Var}(\ell) \leq \text{Var}(L)$.

Proof of Lemma 1. Using conditional variance formula, we have

$$\begin{aligned}\text{Var}(L) &= \sum_{j=1}^{n} \text{Var}\left(E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right)\\ &\geq \sum_{j=1}^{n}\sum_{k=1}^{n} q_{jk}\text{Var}\left(E\left\{\Phi\left(\frac{X-Y_{(k)}}{t}\right)\Big|Y_{(k)}\right\}\right)\\ &= \sum_{k=1}^{n} \text{Var}\left(E\left\{\Phi\left(\frac{X-Y_{(k)}}{t}\right)\Big|Y_{(k)}\right\}\right) = \text{Var}(\ell),\end{aligned}$$

as was asserted. ■

Proof of Proposition 3. First, some necessary notions and results from matrix algebra are provided.

The L_1, L_∞ and L_2 norms for an $r \times c$ matrix $\mathbf{A} = [a_{ij}]$ are defined as

$$\|\mathbf{A}\|_1 = \max_{j=1,\ldots,c} \sum_{i=1}^{r} a_{ij},$$

$$\|\mathbf{A}\|_\infty = \max_{i=1,\ldots,r} \sum_{j=1}^{c} a_{ij}$$

and

$$\|\mathbf{A}\|_2 = \sqrt{\lambda_{\max}(\mathbf{A}'\mathbf{A})},$$

where $\lambda_{\max}(\mathbf{A}'\mathbf{A})$ is the largest eigenvalue of $\mathbf{A}'\mathbf{A}$ matrix. If the product of matrices $\mathbf{A}$ and $\mathbf{B}$ is defined, then

$$\|\mathbf{AB}\|_2 \leq \|\mathbf{A}\|_2 \|\mathbf{B}\|_2 \tag{28}$$

and

$$\|\mathbf{A}\|_2^2 \leq \|\mathbf{A}\|_1 \|\mathbf{A}\|_\infty. \tag{29}$$

See Datta (2010) for more details.

In view of (9), it suffices to show that

$$E\left(m^2\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{(j)}}{t}\right)\Big|Y_{(j)}\right\}\right]^2-\sum_{i=1}^{m}\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{(j)}}{t}\right)\Big|Y_{(j)}\right\}\right]^2\right)\le$$
$$E\left(m^2\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2-\sum_{i=1}^{m}\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2\right)\quad(30)$$

and

$$E\left(\sum_{j=1}^{n}E^2\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|X\right\}\right)\le E\left(\sum_{j=1}^{n}E^2\left\{\Phi\left(\frac{X-Y_{(j)}}{t}\right)\Big|X\right\}\right).\quad(31)$$

We begin with proving the first inequality. Assume that $Z_{(i)}=\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X_{(i)}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}$ and $Z_{[i]}=\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}$. Then one can write

$$Z_{[i]}=\sum_{j=1}^{n}\sum_{k=1}^{m}p_{ik}E\left\{\Phi\left(\frac{X_{(k)}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}=\sum_{k=1}^{m}p_{ik}Z_{(k)}.\quad(32)$$

Let Ω_Y be the sample space on which Y is defined. If $\mathbf{P}=[p_{ir}]_{m\times m}$ and $\mathbf{Z}'=\left(Z_{(1)}(\vartheta),\ldots,Z_{(m)}(\vartheta)\right)$ given a fixed $\vartheta\in\Omega_Y$, then using (28), (29) and (32) it follows that

$$\begin{aligned}\sum_{i=1}^{m}Z_{[i]}^2(\vartheta)=\sum_{i=1}^{m}\left(\sum_{k=1}^{m}p_{ik}Z_{(k)}(\vartheta)\right)^2&=\|\mathbf{PZ}\|_2^2\le\|\mathbf{P}\|_2^2\;\|\mathbf{Z}\|_2^2\\&\le\|\mathbf{P}\|_1\;\|\mathbf{P}\|_\infty\sum_{i=1}^{m}Z_{(i)}^2(\vartheta)\\&=\sum_{i=1}^{m}Z_{(i)}^2(\vartheta).\end{aligned}$$

The last equality holds because $\sum_{i=1}^{m}p_{ik}=\sum_{k=1}^{m}p_{ik}=1$. Hence,

$$E\left(m^2\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2-\sum_{i=1}^{m}\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X_{(i)}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2\right)\le$$
$$E\left(m^2\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2-\sum_{i=1}^{m}\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X_{[i]}-Y_{[j]}}{t}\right)\Big|Y_{[j]}\right\}\right]^2\right).$$

Now, (30) is deduced if

$$E\left(m^2\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{(j)}}{t}\right)\middle|Y_{(j)}\right\}\right]^2-\sum_{i=1}^{m}\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X_{(i)}-Y_{(j)}}{t}\right)\middle|Y_{(j)}\right\}\right]^2\right)\leq$$
$$E\left(m^2\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\middle|Y_{[j]}\right\}\right]^2-\sum_{i=1}^{m}\left[\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X_{(i)}-Y_{[j]}}{t}\right)\middle|Y_{[j]}\right\}\right]^2\right) \quad (33)$$

For $i=1,\ldots,m$, suppose $\ell_{(i)}=\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X_{(i)}-Y_{(j)}}{t}\right)\middle|Y_{(j)}\right\}$ and ℓ be as in Lemma 1. We note that $\ell_{(1)}<\cdots<\ell_{(m)}$ are order statistics from a sample of size m. Therefore,

$$\sum_{i=1}^{m}E(\ell_{(i)}^2)=\sum_{i=1}^{m}\int t^2 f_{\ell_{(i)}}(t)\,\mathrm{d}t=m\int t^2 f_{\ell}(t)\,\mathrm{d}t=mE(\ell^2), \quad (34)$$

where $f_{\ell_{(i)}}$ and f_ℓ denote the density function of $\ell_{(i)}$ and ℓ, respectively. Similarly, one can define $L_{(i)}=\sum_{j=1}^{n}E\left\{\Phi\left(\frac{X_{(i)}-Y_{[j]}}{t}\right)\middle|Y_{[j]}\right\}$, and conclude that

$$\sum_{i=1}^{m}E(L_{(i)}^2)=mE(L^2), \quad (35)$$

where L is as in Lemma 1. From (34) and (35), (33) reduces to $E(\ell^2)\leq E(L^2)$. This is equivalent to $\mathrm{Var}(\ell)\leq\mathrm{Var}(L)$ which holds thanks to Lemma 1.

Assume that $W_{(j)}=E\left\{\Phi\left(\frac{X-Y_{(j)}}{t}\right)\middle|X\right\}$ and $W_{[j]}=E\left\{\Phi\left(\frac{X-Y_{[j]}}{t}\right)\middle|X\right\}$. Then, it can be shown that $W_{[j]}=\sum_{k=1}^{n}q_{jk}W_{(j)}$. Let Ω_X be the sample space on which X is defined. If $\mathbf{Q}=[q_{js}]_{n\times n}$ and $\mathbf{W}^\intercal=\left(W_{(1)}(\eta),\ldots,W_{(n)}(\eta)\right)$ for each fixed $\eta\in\Omega_X$, then applying (28) and (29) using $\mathbf{Q}$ and $\mathbf{W}$ yields

$$\begin{aligned}\sum_{j=1}^{n}W_{[j]}^2(\eta)&=\sum_{j=1}^{n}\left(\sum_{k=1}^{m}q_{jk}W_{(k)}(\eta)\right)^2=\|\mathbf{QW}\|_2^2\leq\|\mathbf{Q}\|_2^2\,\|\mathbf{W}\|_2^2\\&\leq\|\mathbf{Q}\|_1\,\|\mathbf{Q}\|_\infty\sum_{i=1}^{m}W_{(i)}^2(\eta)\\&=\sum_{i=1}^{m}W_{(i)}^2(\eta).\end{aligned}$$

This completes the proof of (31). ■