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THE KATO SQUARE ROOT PROBLEM FOLLOWS FROM AN EXTRAPOLATION PROPERTY OF THE LAPLACIAN

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Abstract: On a domain $\Omega \subseteq \mathbb{R}^d$ we consider second-order elliptic systems in divergence-form with bounded complex coefficients, realized via a sesquilinear form with domain $\mathrm{H}^1_0(\Omega) \subseteq \mathcal{V} \subseteq \mathrm{H}^1(\Omega)$. Under very mild assumptions on Ω and $\mathcal{V}$ we show that the solution to the Kato Square Root Problem for such systems can be deduced from a regularity result for the fractional powers of the negative Laplacian in the same geometric setting. This extends earlier results of McIntosh [**25**] and Axelsson–Keith–McIntosh [**6**] to non-smooth coefficients and domains.

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1. Introduction

We consider a second-order $m \times m$ elliptic system

$$Au = -\sum_{\alpha,\beta=1}^{d} \partial_\alpha(a_{\alpha,\beta}\partial_\beta u)$$

in divergence-form with bounded $\mathbb{C}^{m\times m}$-valued coefficients $a_{\alpha,\beta}$ on a domain $\Omega \subseteq \mathbb{R}^d$. As usual, A is interpreted as a maximal accretive operator on $\mathrm{L}^2(\Omega)$ via a sesquilinear form defined on some closed subset $\mathcal{V}$ of $\mathrm{H}^1(\Omega)$ that contains $\mathrm{H}^1_0(\Omega)$. A fundamental question due to Kato [**23**] and refined by Lions [**24**], having made history as the *Kato Square Root Problem*, is whether A has the *square root property* $\mathcal{D}(\sqrt{A}) = \mathcal{V}$, i.e. whether the domain of the maximal accretive square root of A coincides with the form domain.

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Whereas for self-adjoint A this is immediate from abstract form theory [**22**], the full problem remained open for almost 40 years. It were Auscher, Hofmann, Lacey, McIntosh, and Tchamitchian, who eventually gave a proof on $\Omega = \mathbb{R}^d$ exploiting the full strength of harmonic analysis [**2, 3**]. Shortly after, Auscher and Tchamitchian used localization techniques to solve the Kato Square Root Problem on strongly Lipschitz domains Ω complemented by either pure Dirichlet or pure Neumann boundary conditions [**5**]. These refer to the cases $\mathcal{V} = \mathrm{H}^1_0(\Omega)$ and $\mathcal{V} = \mathrm{H}^1(\Omega)$. For a survey we refer to [**2, 27**] and the references therein.

A milestone toward general form domains has then been set by Axelsson, Keith, and McIntosh [**6, 7**], who introduced an operator theoretic framework that allows to cast the Kato Square Root Problem for almost arbitrary Ω and $\mathcal{V}$ as an abstract first-order problem. By these means they gave a solution if Ω is a smooth domain, D is a smooth part of the boundary $\partial\Omega$, and $\mathcal{V}$ is the subspace of $\mathrm{H}^1(\Omega)$ containing those functions that vanish on D – and moreover for global bi-Lipschitz images of these configurations [**6**].

Much earlier, in 1985 McIntosh revealed another profound structural aspect of the Kato Square Root Problem: Assuming some smoothness on the coefficients and the domain Ω, he proved that on arbitrary form domains $\mathcal{V}$ the affirmative answer to Kato's problem follows if the square root property for the easiest elliptic differential operator – the self-adjoint negative Laplacian – can be extrapolated to fractional powers of exponent slightly above $\frac{1}{2}$, cf. [**25**]. A similar approach has been pursued in [**6**].

Our main result is a reduction theorem in this spirit for second-order elliptic systems whose coefficients are merely bounded. We do so under significantly weaker geometric assumptions than in [**6**] and [**25**] but in contrast to [**25**] we have to assume that the form domain is invariant under multiplication by smooth functions. As an application we have obtained an extension of previous results on the Kato Square Root Problem for mixed boundary conditions [**15**]. The key technique is a Π_B-*type theorem* in the spirit of [**7**], which we state as our second main result and which allows for further applications, e.g. to prove well-posedness of boundary value problems on cylindrical domains, see [**1**].

The paper is organized as follows. After introducing some notation and the geometric setup in Section 2, we state our main results in Section 3. The hypotheses underlying our Π_B-theorem are discussed in Section 5. In Section 6 we deduce our main result from the Π_B-theorem. For the reader's convenience, necessary tools from functional calculus are recalled beforehand in Section 4. In the remaining sections we develop the proof of the Π_B-theorem. Our argument builds upon the techniques being introduced in [**7**] as did many other square root type results before [**6, 8, 9, 28**], but as a novelty allows the presence of a non-smooth boundary. We suggest to keep a copy of [**7**] handy as duplicated arguments with this paper are omitted.

2. Notation and general assumptions

Most of our notation is standard. Throughout, the dimension $d \geq 2$ of the underlying Euclidean space is fixed. The open ball in $\mathbb{R}^d$ with center x and radius $r > 0$ is denoted by $B(x, r)$. For abuse of notation we use the symbol $|\cdot|$ for both the Euclidean norm of vectors in $\mathbb{C}^n$, $n \geq 1$, as well as for the d-dimensional Lebesgue measure. For $z \in \mathbb{C}$ we put $\langle z \rangle := 1 + |z|$. The Euclidean distance between subsets E and F of $\mathbb{R}^d$ is $\mathrm{d}(E, F)$. If $E = \{x\}$, then the abbreviation $\mathrm{d}(x, E)$ is used. The complex logarithm log is always defined on its principal branch $\mathbb{C} \setminus (-\infty, 0]$. The indicator function of a set $E \subseteq \mathbb{R}^d$ is $\mathbf{1}_E$ and for convenience we abbreviate the maps $z \mapsto 1$ and $z \mapsto z$ by 1 and z, respectively. For average integrals the symbol $⨍$ is used.

We allow ourselves the freedom to write $a \lesssim b$ if there exists $C > 0$ not depending on the parameters at stake such that $a \leq Cb$ holds. Likewise, we use the symbol $\gtrsim$ and we write $a \simeq b$ if both $a \lesssim b$ and $b \lesssim a$ hold.

2.1. Function spaces. The Hilbert space of square integrable, $\mathbb{C}^n$-valued functions on a Borel set $\Xi \subseteq \mathbb{R}^d$ is $\mathrm{L}^2(\Xi; \mathbb{C}^n)$. If Ξ is open, then $\mathrm{H}^1(\Xi; \mathbb{C}^n)$ is the associated first-order Sobolev space with its usual Hilbertian norm and $\mathrm{H}^1_0(\Omega; \mathbb{C}^n)$ denotes the H^1-closure of $\mathrm{C}^\infty_c(\Xi; \mathbb{C}^n)$, the space of smooth functions with compact support in Ξ. The *Bessel potential spaces* with differentiability $s > 0$ and integrability 2 are $\mathrm{H}^{s,2}(\mathbb{R}^d; \mathbb{C}^n)$, see [**31**, Sec. 2.3.3] and $\mathrm{H}^{s,2}(\Xi; \mathbb{C}^n) := \{u|_\Xi : u \in \mathrm{H}^{s,2}(\mathbb{R}^d; \mathbb{C}^n)\}$ is equipped with the quotient norm

$$\|u\|_{\mathrm{H}^{s,2}(\Xi;\mathbb{C}^n)} := \inf\{\|v\|_{\mathrm{H}^{s,2}(\mathbb{R}^d;\mathbb{C}^n)} : v = u \text{ a.e. on } \Xi\}.$$

2.2. Operators on Hilbert spaces. Any Hilbert space $\mathcal{H}$ under consideration is taken over the complex numbers. Concerning linear operators we follow standard notation. If B_1 and B_2 are operators in $\mathcal{H}$ then $B_1 + B_2$ and $B_1 B_2$ are defined on their natural domains

$$\mathcal{D}(B_1 + B_2) := \mathcal{D}(B_1) \cap \mathcal{D}(B_2) \qquad \text{and}$$
$$\mathcal{D}(B_1 B_2) := \{u \in \mathcal{D}(B_2) : B_2 u \in \mathcal{D}(B_1)\}.$$

Their *commutator* is $[B_1, B_2] := B_1 B_2 - B_2 B_1$.

2.3. Geometric setup and the elliptic operator. In this section we define the elliptic operator $Au = -\sum_{\alpha,\beta=1}^{d} \partial_\alpha(a_{\alpha,\beta}\partial_\beta u)$ under consideration properly by means of Kato's form method [**22**]. Starting from now, the codimension $m \geq 1$ – the number of "equations" – is fixed.

Throughout this work we assume the following geometric setup.

Assumption 2.1. (Ω) We assume that $\Omega \subseteq \mathbb{R}^d$ is a *d-set* in the sense of Jonsson–Wallin [**21**], i.e. that it satisfies the *d-Ahlfors* or *measure density condition*

$$|\Omega \cap B(x,r)| \simeq r^d \quad (x \in \Omega,\ 0 < r \leq 1).$$

($\partial\Omega$) We assume that $\partial\Omega$ is a $(d-1)$*-set* in the sense of Jonsson–Wallin [**21**], i.e. that it satisfies the *Ahlfors–David condition*

$$\mathrm{m}_{d-1}(\partial\Omega \cap B(x,r)) \simeq r^{d-1} \quad (x \in \partial\Omega,\ 0 < r \leq 1),$$

where here and throughout m_{d-1} denotes the $(d-1)$-dimensional Hausdorff measure.

($\mathcal{V}$) We assume that $\mathcal{V}$ is a closed subspace of $\mathrm{H}^1(\Omega;\mathbb{C}^m)$ that contains $\mathrm{H}^1_0(\Omega;\mathbb{C}^m)$ and is stable under multiplication by smooth functions in the sense

$$\varphi\mathcal{V} \subseteq \mathcal{V} \quad (\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d;\mathbb{C})).$$

Moreover, we assume that $\mathcal{V}$ has the H^1*-extension property*, i.e. there exists a bounded operator $\mathfrak{E}\colon \mathcal{V} \to \mathrm{H}^1(\mathbb{R}^d;\mathbb{C}^m)$ such that $\mathfrak{E}u = u$ a.e. on Ω for each $u \in \mathcal{V}$.

(α) We assume that for some $\alpha \in (0,1)$ the complex interpolation space $[\mathrm{L}^2(\Omega;\mathbb{C}^m), \mathcal{V}]_\alpha$ coincides with $\mathrm{H}^{\alpha,2}(\Omega;\mathbb{C}^m)$ and that their norms are equivalent.

Let us comment on these assumptions.

Remark 2.2. (i) The stability assumption on $\mathcal{V}$ is satisfied e.g. for the usual choices of $\mathcal{V}$ modeling (mixed) Dirichlet and Neumann boundary conditions [**15, 29**].

(ii) The H^1 extension property for $\mathcal{V}$ is trivially satisfied if Ω admits a bounded Sobolev extension operator $\mathfrak{E}\colon \mathrm{H}^1(\Omega;\mathbb{C}^m) \to \mathrm{H}^1(\mathbb{R}^d;\mathbb{C}^m)$. In this case also ($\Omega$) holds [**20**, Thm. 2].

(iii) Assumption $(d-1)$ is common in the treatment of boundary value problems, being among the weakest geometric conditions that allow to define boundary traces, cf. [**21**].

(iv) Assumption (α) should be considered as a geometric one. A common way to force its validity is to assume that Ω is a Sobolev extension domain and that

$$\text{(Mc)} \qquad \big[\mathrm{L}^2(\Omega;\mathbb{C}^m), \mathrm{H}^1_0(\Omega;\mathbb{C}^m)\big]_\alpha = \big[\mathrm{L}^2(\Omega;\mathbb{C}^m), \mathrm{H}^1(\Omega;\mathbb{C}^m)\big]_\alpha$$

holds up to equivalent norms. Indeed, (α) then follows from $\mathrm{H}_0^1(\Omega;\mathbb{C}^m) \subseteq \mathcal{V} \subseteq \mathrm{H}^1(\Omega;\mathbb{C}^m)$ and standard interpolation results [**31**, Sec. 1.2.4/2.4.2]. The condition (Mc) has been introduced in this context by McIntosh [**25**].

(v) Among the vast variety of Sobolev extension domains satisfying ($\partial\Omega$) and McIntosh's condition for all $\alpha \in (0, \frac{1}{2})$ are the whole space $\mathbb{R}^d$ [**31**, Sec. 2.4.1], the upper half space $\mathbb{R}^d_+$ [**31**, Sec. 2.10] from which the result for special Lipschitz domains can be deduced, as well as bounded Lipschitz domains [**18**, Thm. 3.1], [**31**, Sec. 4.3.1]. Assumption 2.1 then reduces to the stability assumption on $\mathcal{V}$. However, configurations in which Ω is not a Sobolev extension domain though (Ω), ($\partial\Omega$), ($\mathcal{V}$), and (α) are satisfied, naturally occur in the treatment of mixed boundary value problems, cf. [**15**] and the references therein.

Concerning the coefficients of A we make the following standard assumption.

Assumption 2.3. We assume $a_{\alpha,\beta} \in \mathrm{L}^\infty(\Omega;\mathbb{C}^{m\times m})$ for all $1 \leq \alpha, \beta \leq d$ and that the associated sesquilinear form

$$\mathfrak{a}\colon \mathcal{V} \times \mathcal{V} \to \mathbb{C}, \quad \mathfrak{a}(u,v) = \sum_{\alpha,\beta=1}^{d} \int_\Omega a_{\alpha,\beta}(x)\partial_\beta u(x) \cdot \overline{\partial_\alpha v(x)} \,\mathrm{d}x$$

is elliptic in the sense that for some $\lambda > 0$ it satisfies the *Gårding inequality*

$$\mathrm{Re}(\mathfrak{a}(u,u)) \geq \lambda \|\nabla u\|^2_{\mathrm{L}^2(\Omega;\mathbb{C}^{dm})} \quad (u \in \mathcal{V}). \tag{2.1}$$

Since $\mathcal{V}$ is dense in $\mathrm{L}^2(\Omega;\mathbb{C}^m)$ and $\mathfrak{a}$ is elliptic, classical form theory [**22**, Ch. VI] yields that the associated operator A on $\mathrm{L}^2(\Omega;\mathbb{C}^m)$ given by

$$\mathfrak{a}(u,v) = \langle Au, v\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^m)} \quad (u \in \mathcal{D}(A),\, v \in \mathcal{V})$$

on

$$\mathcal{D}(A) := \big\{u \in \mathcal{V} : \mathfrak{a}(u,\cdot) \text{ boundedly extends to } \mathrm{L}^2(\Omega;\mathbb{C}^m)\big\}$$

is *maximal accretive*. By this we mean that A is closed and for z in the open left complex halfplane the operator $z - A$ is invertible with $\|(z-A)^{-1}\|_{\mathcal{L}(\mathrm{L}^2(\Omega;\mathbb{C}^m))} \leq |\mathrm{Re}(z)|^{-1}$. The choice $a_{\alpha,\beta} = \delta_{\alpha,\beta}\,\mathrm{Id}_{\mathbb{C}^{m\times m}}$, where δ is Kronecker's delta, yields the negative of the (coordinatewise) *weak Laplacian* $\Delta_\mathcal{V}$ with form domain $\mathcal{V}$.

Maximal accretivity allows to define *fractional powers* $(\varepsilon + A)^\alpha$ for all $\alpha, \varepsilon \geq 0$ by means of the functional calculus for sectorial operators, see Section 4. The so-defined *square root* $\sqrt{A}$ of A is the unique maximal accretive operator such that $\sqrt{A}\sqrt{A} = A$ holds, cf. [**22**, Thm. V.3.35] and [**19**, Cor. 7.1.13].

3. Main results

The main result we want to prove in this paper is the following.

Theorem 3.1. *Let Assumptions 2.1 and 2.3 be satisfied and let $\Delta_{\mathcal{V}}$ be the weak Laplacian with form domain $\mathcal{V}$. If for the* same α *as in Assumption 2.1*

$$\text{(E)} \qquad \mathcal{D}((1-\Delta_{\mathcal{V}})^{1/2+\alpha/2}) \subseteq \mathrm{H}^{1+\alpha,2}(\Omega;\mathbb{C}^m)$$

with continuous inclusion, then A has the square root property

$$\mathcal{D}(\sqrt{A}) = \mathcal{D}(\sqrt{1+A}) = \mathcal{V} \quad \textit{with} \quad \|(\sqrt{1+A})u\|_{\mathrm{L}^2(\Omega;\mathbb{C}^m)} \simeq \|u\|_{\mathcal{V}} \quad (u \in \mathcal{V}).$$

By a classical result on operators on Hilbert spaces [**22**, Thm. VI. 2.23] the self-adjoint operator $1-\Delta_{\mathcal{V}}$ has the square root property $\mathcal{D}(\sqrt{1-\Delta_{\mathcal{V}}}) = \mathcal{V} \subseteq \mathrm{H}^1(\Omega;\mathbb{C}^m)$. Hence, our main result may informally be stated as follows:

> *If the square root property for the negative Laplacian with form domain $\mathcal{V}$ extrapolates to fractional powers with exponent slightly above $\frac{1}{2}$, then every elliptic differential operator in divergence form with form domain $\mathcal{V}$ has the square root property.*

Remark 3.2. (i) The conditions (Mc) and (E) are those imposed by McIntosh [**25**] to solve the Kato Square Root Problem for operators A with Hölder continuous coefficients.

(ii) In applications it usually suffices that (α) and (E) hold for different choices of α since then, by interpolation, both conditions can be met simultaneously for some possibly smaller value of α, cf. [**15**].

In Section 5 we will deduce Theorem 3.1 from the following Π_B*-theorem.* In fact, Theorem 3.3 is a generalization of the main result in [**6**] to non-smooth domains. For the notion of bisectorial operators see Section 4. Corollary 3.4 is discussed in more detail at the end of Section 4.

Theorem 3.3. *Let $k \in \mathbb{N}$ and $N = km$. On the Hilbert space $\mathcal{H} := (\mathrm{L}^2(\Omega;\mathbb{C}^m))^k$ consider operators Γ, B_1, and B_2 satisfying* (H1)–(H7), *see Section 5. Then the* perturbed Dirac type operator $\Pi_B := \Gamma + B_1\Gamma^* B_2$ *is bisectorial of some angle $\omega \in (0, \frac{\pi}{2})$ and satisfies quadratic estimates*

$$\text{(3.1)} \qquad \int_0^\infty \|t\Pi_B(1+t^2\Pi_B^2)^{-1}u\|_{\mathcal{H}}^2 \frac{\mathrm{d}t}{t} \simeq \|u\|_{\mathcal{H}}^2 \quad (u \in \overline{\mathcal{R}(\Pi_B)}).$$

Moreover, implicit constants depend on B_1 and B_2 only through the constants quantified in (H2).

Corollary 3.4. *The part of* Π_B *in* $\overline{\mathcal{R}(\Pi_B)}$ *is an injective bisectorial operator of angle* ω *with a bounded* $\mathrm{H}^\infty(\mathrm{S}_\psi)$*-calculus for each* $\psi \in (\omega, \frac{\pi}{2})$*. In particular, it shares the* Kato square root type estimate

$$\mathcal{D}(\sqrt{\Pi_B^2}) = \mathcal{D}(\Pi_B) \quad \textit{with} \quad \|\sqrt{\Pi_B^2} u\|_{\mathcal{H}} \simeq \|\Pi_B u\|_{\mathcal{H}} \quad (u \in \mathcal{D}(\Pi_B)).$$

4. Functional calculi

We recall the functional calculi for sectorial and bisectorial operators. For sectorial operators we follow the treatment in [**19**, Ch. 2]. Good references for the bisectorial case are [**12, 13**], see also [**14**, Ch. 3].

Given $\varphi \in (0, \pi)$, denote by $\mathrm{S}_\varphi^+ := \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \varphi\}$ the open *sector* with vertex 0 and opening angle 2φ symmetric around the positive real axis. If $\varphi \in (0, \frac{\pi}{2})$ then $\mathrm{S}_\varphi := \mathrm{S}_\varphi^+ \cup (-\mathrm{S}_\varphi^+)$ is the corresponding open *bisector*. An operator B on a Hilbert space $\mathcal{H}$ is *sectorial* of *angle* $\varphi \in (0, \pi)$ if its spectrum is contained in $\overline{\mathrm{S}_\varphi^+}$ and

$$\sup\bigl\{\|\lambda(\lambda - B)^{-1}\|_{\mathcal{L}(\mathcal{H})} : \lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\psi^+}\bigr\} < \infty \quad (\psi \in (\varphi, \pi)).$$

Likewise, B is *bisectorial* of *angle* $\varphi \in (0, \frac{\pi}{2})$ if $\sigma(B) \subseteq \overline{\mathrm{S}_\varphi}$ and

$$\sup\bigl\{\|\lambda(\lambda - B)^{-1}\|_{\mathcal{L}(\mathcal{H})} : \lambda \in \mathbb{C} \setminus \overline{\mathrm{S}_\psi}\bigr\} < \infty \quad (\psi \in (\varphi, \tfrac{\pi}{2})).$$

A sectorial or bisectorial operator B on $\mathcal{H}$ necessarily is densely defined and induces a topological decomposition $\mathcal{H} = \mathcal{N}(B) \oplus \overline{\mathcal{R}(B)}$, see [**19**, Prop. 2.1.1] or [**14**, Prop. 3.2.2].

4.1. Construction of the functional calculi.

For an open set $U \subseteq \mathbb{C}$ denote by $\mathrm{H}^\infty(U)$ the Banach algebra of bounded holomorphic functions on U equipped with the supremum norm $\|\cdot\|_{\infty,U}$ and let

$$\mathrm{H}_0^\infty(U) := \bigl\{g \in \mathrm{H}^\infty(U) \mid \exists\, C, s > 0 \;\forall\, z \in U : |g(z)| \le C \min\{|z|^s, |z|^{-s}\}\bigr\}$$

be the subalgebra of *regularly decaying* functions.

The holomorphic functional calculus for a sectorial operator B of angle $\varphi \in (0, \pi)$ on a Hilbert space $\mathcal{H}$ is defined as follows. For $\psi \in (\varphi, \pi)$ and $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi^+)$ define $f(B) \in \mathcal{L}(\mathcal{H})$ via the Cauchy integral

$$f(B) := \frac{1}{2\pi \mathrm{i}} \int_{\partial \mathrm{S}_\nu^+} f(z)(z - B)^{-1} \,\mathrm{d}z,$$

where $\nu \in (\varphi, \psi)$ and the boundary curve $\partial \mathrm{S}_\nu^+$ surrounds $\sigma(B)$ counterclockwise. This integral converges absolutely and is independent of the particular choice of ν due to Cauchy's theorem. Furthermore, define

$g(B) := f(B) + c(1+B)^{-1} + d$ if g is of the form $g = f + c(1+z)^{-1} + d$ for $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi^+)$ and $c, d \in \mathbb{C}$. This yields an algebra homomorphism

$$\mathcal{E}(\mathrm{S}_\psi^+) := \mathrm{H}_0^\infty(\mathrm{S}_\psi^+) \oplus \langle (1+z)^{-1} \rangle \oplus \langle \mathbf{1} \rangle \to \mathcal{L}(X), \quad g \mapsto g(B),$$

the *primary holomorphic functional calculus* for the sectorial operator B. It can be extended to a larger class of holomorphic functions by *regularization* [**19**, Sec. 1.2]: If f is a holomorphic function on S_ψ^+ for which there exists an $e \in \mathcal{E}(\mathrm{S}_\psi^+)$ such that $ef \in \mathcal{E}(\mathrm{S}_\psi^+)$ and $e(B)$ is injective, define $f(B) := e(B)^{-1}(ef)(B)$. This yields a closed and (in general) unbounded operator on $\mathcal{H}$ and the definition is independent of the particular regularizer e. If holomorphic functions $f, g\colon \mathrm{S}_\psi^+ \to \mathbb{C}$ can be regularized, then the composition rules

$$f(B) + g(B) \subseteq (f+g)(B) \quad \text{and} \quad f(B)g(B) \subseteq (fg)(B) \tag{4.1}$$

hold true and $\mathcal{D}(f(B)g(B)) = \mathcal{D}((fg)(B)) \cap \mathcal{D}(g(B))$, cf. [**19**, Prop. 1.2.2].

In particular, for each $\alpha > 0$ and each $\varepsilon \geq 0$ the function $(\varepsilon + z)^\alpha$ is regularizable by $(1+z)^{-k}$ for k a natural number larger than α and yields the *fractional power* $(\varepsilon + B)^\alpha$. The domain of $(\varepsilon + B)^\alpha$ is independent of $\varepsilon \geq 0$. Many rules for fractional powers of complex numbers remain valid for these operators, see [**19**, Sec. 3.1] for details. If B is injective, then each $f \in \mathrm{H}^\infty(\mathrm{S}_\psi^+)$ is regularizable by $z(1+z)^{-2}$ yielding the $\mathrm{H}^\infty(\mathrm{S}_\psi^+)$*-calculus* for B.

The holomorphic functional calculus for bisectorial operators can be set up in exactly the same manner by replacing sectors S_ψ^+ by the respective bisectors S_ψ and resolvents $(1+B)^{-1}$ by $(\mathrm{i} + B)^{-1}$. It shares all properties of the sectorial calculus listed above.

If B is bisectorial of angle $\varphi \in (0, \frac{\pi}{2})$, then B^2 is sectorial of angle 2φ. We remark that this correspondence is compatible with the respective functional calculi.

Lemma 4.1. *Let B be a bisectorial operator of angle $\varphi \in (0, \frac{\pi}{2})$ on a Hilbert space $\mathcal{H}$, let $\psi \in (\varphi, \frac{\pi}{2})$, and let $f \in \mathrm{H}_0^\infty(\mathrm{S}_{2\psi}^+)$. Then $f(z^2)(B)$ and $f(B^2)$ defined via the holomorphic functional calculi for the bisectorial operator B and the sectorial operator B^2 respectively, coincide.*

Proof: Note that z^2 maps the bisector S_ψ onto the sector $\mathrm{S}_{2\psi}^+$. Hence, $g := f(z^2) \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$ and the claim follows by a straightforward transformation of the defining Cauchy integrals. □

Corollary 4.2. *Suppose the setting of Lemma 4.1 and let $\beta > 0$. Then $(z^2)^\beta(B) = z^\beta(B^2)$.*

Proof: Let $k \in \mathbb{N}$ be larger than β. It suffices to remark that $e := (1+z)^{-k}$ regularizes z^β in the functional calculus for B^2 and that $e(z^2)$ regularizes $(z^2)^\beta$ in the functional calculus for B. □

4.2. Boundedness of the H^∞-calculus for bisectorial operators. Given an injective bisectorial operator B of angle $\varphi \in (0, \frac{\pi}{2})$ on a Hilbert space $\mathcal{H}$ and some angle $\psi \in (\varphi, \frac{\pi}{2})$, the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for B is said to be *bounded* with *bound* $C_\psi > 0$ if

$$\|f(B)\|_{\mathcal{L}(\mathcal{H})} \leq C_\psi \|f\|_{\infty,\mathrm{S}_\psi} \quad (f \in \mathrm{H}^\infty(\mathrm{S}_\psi)).$$

It is convenient that boundedness of the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus follows from a uniform bound for the $\mathrm{H}_0^\infty(\mathrm{S}_\psi)$-calculus. For a proof see [**19**, Sec. 5.3.4] or [**14**, Cor. 3.3.6].

Proposition 4.3. *Let B be an injective bisectorial operator of angle $\varphi \in (0, \frac{\pi}{2})$ on a Hilbert space $\mathcal{H}$ and let $\psi \in (\varphi, \frac{\pi}{2})$. If there exists a constant $C_\psi > 0$ such that*

$$\|f(B)\|_{\mathcal{L}(\mathcal{H})} \leq C_\psi \|f\|_{\infty,\mathrm{S}_\psi} \quad (f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)),$$

then the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for B is bounded with bound C_ψ.

On Hilbert spaces boundedness of the H^∞-calculus is equivalent to certain *quadratic estimates*, see e.g. [**11, 14**] and [**26**].

Proposition 4.4. *Let B be an injective bisectorial operator of angle $\varphi \in (0, \frac{\pi}{2})$ on a Hilbert space $\mathcal{H}$. If B satisfies quadratic estimates*

$$\int_0^\infty \|tB(1+t^2B^2)^{-1}u\|_{\mathcal{H}}^2 \frac{\mathrm{d}t}{t} \simeq \|u\|_{\mathcal{H}}^2 \quad (u \in \mathcal{H}),$$

then the $\mathrm{H}^\infty(\mathrm{S}_\psi)$-calculus for B is bounded for each $\psi \in (\varphi, \frac{\pi}{2})$.

For later references we include the classical proof drawing upon the following lemma.

Lemma 4.5 ([**7**, p. 473])**.** *If B is a bisectorial operator on a Hilbert space $\mathcal{H}$, then*

$$\lim_{\substack{r\to 0\\ R\to\infty}} \int_r^R (tB(1+t^2B^2)^{-1})^2 u \frac{\mathrm{d}t}{t} = \frac{1}{2}u \quad (u \in \overline{\mathcal{R}(B)}).$$

Proof of Proposition 4.4: We appeal to Proposition 4.3. Fix $\psi \in (\varphi, \frac{\pi}{2})$ and $f \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$. For $t > 0$ put $\Psi_t := tz(1+t^2z^2)^{-1} \in \mathrm{H}_0^\infty(\mathrm{S}_\psi)$. The most direct estimate on the defining Cauchy integral gives

$$\begin{aligned} \|\Psi_t(B)f(B)\Psi_s(B)\|_{\mathcal{L}(\mathcal{H})} &\lesssim \|f\|_{\infty,\mathrm{S}_\psi} \int_0^\infty \frac{ts^{-1}r}{(1+(ts^{-1}r)^2)(1+r^2)}\,\mathrm{d}r \\ &=: \|f\|_{\infty,\mathrm{S}_\psi} \zeta(ts^{-1}) \end{aligned} \tag{4.2}$$

for all $s, t > 0$ and an implicit constant depending only on ψ. Here, $\zeta \in \mathrm{L}^1(0,\infty; \mathrm{d}r/r)$. Recall $\mathcal{H} = \mathcal{N}(B) \oplus \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(B)}$ as B is injective. For $u \in \mathcal{H}$ apply the quadratic estimate to $f(B)u$ and then use Lemma 4.5 for u to find

$$\begin{aligned} \|f(B)u\|_{\mathcal{H}}^2 &\lesssim \int_0^\infty \|\Psi_t(B)f(B)u\|_{\mathcal{H}}^2 \frac{\mathrm{d}t}{t} \\ &\lesssim \int_0^\infty \bigg(\int_0^\infty \|\Psi_t(B)f(B)\Psi_s(B)\Psi_s(B)u\|_{\mathcal{H}} \frac{\mathrm{d}s}{s}\bigg)^2 \frac{\mathrm{d}t}{t}. \end{aligned}$$

By (4.2) and Hölder's inequality,

$$\lesssim \|f\|_{\infty,\mathrm{S}_\psi}^2 \int_0^\infty \bigg(\int_0^\infty \zeta(ts^{-1}) \frac{\mathrm{d}s}{s}\bigg)\bigg(\int_0^\infty \zeta(ts^{-1}) \|\Psi_s(B)u\|_{\mathcal{H}}^2 \frac{\mathrm{d}s}{s}\bigg) \frac{\mathrm{d}t}{t}.$$

The right-hand side is bounded by $\|f\|_{\infty,\mathrm{S}_\psi}^2 \|\zeta\|_{\mathrm{L}^1(0,\infty;\mathrm{d}r/r)}^2 \|u\|_{\mathcal{H}}^2$. □

Remark 4.6. Suppose that B is a self-adjoint (and hence bisectorial) operator on a Hilbert space $\mathcal{H}$. Then $\Psi_t(B) = tB(1+t^2B^2)^{-1}$ is self-adjoint for each $t > 0$ and Lemma 4.5 yields

$$\int_0^\infty \|\Psi_t(B)u\|_{\mathcal{H}}^2 \frac{\mathrm{d}t}{t} = \lim_{\substack{r \to 0 \\ R \to \infty}} \bigg\langle \int_r^R \Psi_t(B)^2 u \frac{\mathrm{d}t}{t}, u \bigg\rangle_{\mathcal{H}} = \frac{1}{2}\|u\|_{\mathcal{H}}^2 \quad (u \in \overline{\mathcal{R}(B)}).$$

The proof of Proposition 4.4 reveals the following: If $\{T_t\}_{t>0} \subseteq \mathcal{L}(\mathcal{H})$ is a family of operators for which there is $\zeta \in \mathrm{L}^1(0,\infty;\mathrm{d}r/r)$ such that $\|T_t\Psi_s(B)\|_{\mathcal{L}(\mathcal{H})} \lesssim \zeta(ts^{-1})$ for all $s, t > 0$, then

$$\int_0^\infty \|T_t u\|_{\mathcal{H}}^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|_{\mathcal{H}}^2 \quad (u \in \overline{\mathcal{R}(B)}).$$

This is usually called a *Schur type estimate.* In the proof of Proposition 4.4, $T_t = \Psi_t(B)f(B)$.

For completeness we add a short proof of Corollary 3.4.

Proof of Corollary 3.4: The first part is due to $\mathcal{H} = \mathcal{N}(\Pi_B) \oplus \overline{\mathcal{R}(\Pi_B)}$ and Proposition 4.4. Put $T := \Pi_B|_{\overline{\mathcal{R}(\Pi_B)}}$. As $\frac{z}{\sqrt{z^2}}, \frac{\sqrt{z^2}}{z} \in \mathrm{H}^\infty(\mathrm{S}_\psi)$, the composition rules (4.1) yield

$$\mathcal{D}(\sqrt{\Pi_B^2}) \cap \overline{\mathcal{R}(\Pi_B)} = \mathcal{D}(\Pi_B) \cap \overline{\mathcal{R}(\Pi_B)}$$

with

$$\|\sqrt{\Pi_B^2} u\|_{\mathcal{H}} \simeq \|\Pi_B u\|_{\mathcal{H}} \quad (u \in \mathcal{D}(\Pi_B) \cap \overline{\mathcal{R}(\Pi_B)}).$$

We used $\mathcal{D}(T) = \mathcal{D}(\Pi_B) \cap \overline{\mathcal{R}(\Pi_B)}$ and $\mathcal{D}(\sqrt{T^2}) = \mathcal{D}(\sqrt{\Pi_B^2}) \cap \overline{\mathcal{R}(\Pi_B)}$. The Kato square root type estimate follows from $\mathcal{N}(\Pi_B) \subseteq \mathcal{N}(\sqrt{\Pi_B^2})$. Proofs of these three properties of functional calculi are found e.g. in [**14, 19**]. □

5. The hypotheses underlying Theorem 3.3

In this section we introduce the hypotheses (H1)–(H7) underlying Theorem 3.3 and summarize their well-established operator theoretic consequences. The first four of our hypotheses are:

(H1) The operator Γ is *nilpotent*, i.e. closed, densely defined, and satisfies $\mathcal{R}(\Gamma) \subseteq \mathcal{N}(\Gamma)$. In particular $\Gamma^2 = 0$ on $\mathcal{D}(\Gamma)$.

(H2) The operators B_1 and B_2 are defined everywhere on $\mathcal{H}$. There exist $\kappa_1, \kappa_2 > 0$ such that they satisfy the *accretivity conditions*

$$\mathrm{Re}\langle B_1 u, u\rangle_{\mathcal{H}} \geq \kappa_1 \|u\|_{\mathcal{H}}^2 \quad (u \in \mathcal{R}(\Gamma^*)),$$
$$\mathrm{Re}\langle B_2 u, u\rangle_{\mathcal{H}} \geq \kappa_2 \|u\|_{\mathcal{H}}^2 \quad (u \in \mathcal{R}(\Gamma)),$$

and there exist K_1, K_2 such that they satisfy the *boundedness conditions*

$$\|B_1 u\|_{\mathcal{H}} \leq K_1 \|u\|_{\mathcal{H}} \quad \text{and} \quad \|B_2 u\|_{\mathcal{H}} \leq K_2 \|u\|_{\mathcal{H}} \quad (u \in \mathcal{H}).$$

(H3) The operator $B_2 B_1$ maps $\mathcal{R}(\Gamma^*)$ into $\mathcal{N}(\Gamma^*)$ and the operator $B_1 B_2$ maps $\mathcal{R}(\Gamma)$ into $\mathcal{N}(\Gamma)$. In particular, $\Gamma^* B_2 B_1 \Gamma^* = 0$ on $\mathcal{D}(\Gamma^*)$ and $\Gamma B_1 B_2 \Gamma = 0$ on $\mathcal{D}(\Gamma)$.

(H4) The operators B_1, B_2 are multiplication operators induced by $\mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^N))$-functions.

We define the *Dirac type operator* $\Pi := \Gamma + \Gamma^*$ and the perturbed operators $\Gamma_B^* := B_1 \Gamma^* B_2$ and $\Pi_B := \Gamma + \Gamma_B^*$. The first three hypotheses trace out the classical setup for perturbed Dirac type operators introduced in [**6**]. They have the following consequences. Firstly, (H1) implies that Γ^* is nilpotent and so is Γ_B^*, cf. [**7**, Lem. 4.1]. The operator Π_B induces the algebraic and topological *Hodge decomposition*

$$\mathcal{H} = \mathcal{N}(\Pi_B) \oplus \overline{\mathcal{R}(\Gamma_B^*)} \oplus \overline{\mathcal{R}(\Gamma)} \tag{5.1}$$

and in particular

$$\mathcal{N}(\Pi_B) = \mathcal{N}(\Gamma_B^*) \cap \mathcal{N}(\Gamma) \quad \text{and} \quad \overline{\mathcal{R}(\Pi_B)} = \overline{\mathcal{R}(\Gamma_B^*)} \oplus \overline{\mathcal{R}(\Gamma)} \tag{5.2}$$

hold [**7**, Prop. 2.2]. Moreover, Π_B is bisectorial of angle $\omega \in (0, \frac{\pi}{2})$, cf. [**7**, Prop. 2.5]. Consequently, Π_B^2 is sectorial of angle 2ω. The unperturbed operator Π is self-adjoint [**7**, Cor. 4.3] and thus satisfies quadratic estimates, cf. Remark 4.6. In particular, $\mathcal{D}(\sqrt{\Pi^2}) = \mathcal{D}(\Pi)$ with equivalence of the homogeneous graph norms as in Corollary 3.4. Finally, if Γ satisfies (H1), then (H2) and (H3) are always satisfied for $B_1 = B_2 = \mathrm{Id}$ and hence the results above remain true in the *unperturbed setting* when $\Gamma_B^* = \Gamma^*$ and $\Pi_B = \Pi$.

Remark 5.1. In all results from [**7**] implicit constants depend on the perturbations B_1 and B_2 only through the constants $\kappa_{1,2}$, $K_{1,2}$ quantified in (H2). This has already been stated in [**7**, Sec. 2] and has been worked out in greatest details in the master's thesis of one of the authors [**30**].

Similar to [**6, 7**] the set of hypotheses is completed by localization and coercivity assumptions on the unperturbed operators. The slight difference between (H7) and the corresponding hypothesis in [**6**] stresses that no further knowledge on the occurring interpolation spaces is necessary.

(H5) For every $\varphi \in \mathrm{C}_c^\infty(\mathbb{R}^d; \mathbb{C})$ the associated multiplication operator M_φ maps $\mathcal{D}(\Gamma)$ into itself and $[\Gamma, M_\varphi] = \Gamma M_\varphi - M_\varphi \Gamma$ with domain $\mathcal{D}([\Gamma, M_\varphi]) = \mathcal{D}(\Gamma)$ acts as a multiplication operator induced by some $c_\varphi \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^N))$ with entries

$$|c_\varphi^{i,j}(x)| \lesssim |\nabla \varphi(x)| \quad (x \in \Omega,\, 1 \leq i, j \leq N)$$

for an implicit constant independent of φ.

(H6) For every open ball B centered in Ω, and for all $u \in \mathcal{D}(\Gamma)$ and $v \in \mathcal{D}(\Gamma^*)$ with compact support in $B \cap \Omega$ it holds

$$\left| \int_\Omega \Gamma u \,\mathrm{d}x \right| \lesssim |B|^{\frac{1}{2}} \|u\|_{\mathcal{H}} \quad \text{and} \quad \left| \int_\Omega \Gamma^* v \,\mathrm{d}x \right| \lesssim |B|^{\frac{1}{2}} \|v\|_{\mathcal{H}}.$$

(H7) There exist $\beta_1, \beta_2 \in (0, 1]$ such that the fractional powers of Π^2 satisfy

$$\|u\|_{[\mathcal{H}, \mathcal{V}^k]_{\beta_1}} \lesssim \|(\Pi^2)^{\beta_1/2} u\|_{\mathcal{H}} \quad \text{and}$$
$$\|v\|_{[\mathcal{H}, \mathcal{V}^k]_{\beta_2}} \lesssim \|(\Pi^2)^{\beta_2/2} v\|_{\mathcal{H}}$$

for all $u \in \mathcal{R}(\Gamma^*) \cap \mathcal{D}(\Pi^2)$ and all $v \in \mathcal{R}(\Gamma) \cap \mathcal{D}(\Pi^2)$.

Remark 5.2. It is straightforward to check that if the triple of operators $\{\Gamma, B_1, B_2\}$ satisfies (H1)–(H7), then so do the triples $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$, and $\{\Gamma, B_1^*, B_2^*\}$.

6. The proof of Theorem 3.1

In this section we deduce Theorem 3.1 from Theorem 3.3 applied on $\mathcal{H} := \mathrm{L}^2(\Omega; \mathbb{C}^m) \times \mathrm{L}^2(\Omega; \mathbb{C}^m) \times (\mathrm{L}^2(\Omega; \mathbb{C}^m))^d$. The argument is similar to [**6**].

Recall that $\mathfrak{a}\colon \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is the sesquilinear form corresponding to $Au = -\sum_{\alpha,\beta=1}^d \partial_\alpha(a_{\alpha,\beta}\partial_\beta u)$ and let $\mathfrak{A}$ be the multiplication operator corresponding to the coefficient tensor $(a_{\alpha,\beta})_{1\leq\alpha,\beta\leq d} \in \mathrm{L}^\infty(\Omega; \mathcal{L}(\mathbb{C}^{dm}))$. Define $\nabla_\mathcal{V} u := \nabla u$ on $\mathcal{D}(\nabla_\mathcal{V}) := \mathcal{V}$ and put

$$\Gamma := \begin{bmatrix} 0 & 0 & 0 \\ \mathrm{Id} & 0 & 0 \\ \nabla_\mathcal{V} & 0 & 0 \end{bmatrix}, \quad B_1 := \begin{bmatrix} \mathrm{Id} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B_2 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ 0 & 0 & \mathfrak{A} \end{bmatrix}$$

on their natural domains. By these choices

$$\Pi_B = \begin{bmatrix} 0 & \mathrm{Id} & (\nabla_\mathcal{V})^*\mathfrak{A} \\ \mathrm{Id} & 0 & 0 \\ \nabla_\mathcal{V} & 0 & 0 \end{bmatrix} \quad \text{and} \quad \Pi_B^2 = \begin{bmatrix} 1 + A & 0 & 0 \\ 0 & \mathrm{Id} & (\nabla_\mathcal{V})^*\mathfrak{A} \\ 0 & \nabla_\mathcal{V} & \nabla_\mathcal{V}(\nabla_\mathcal{V})^*\mathfrak{A} \end{bmatrix}.$$

The corresponding unperturbed operators Π and Π^2 are obtained by replacing $\mathfrak{A}$ by Id and A by $-\Delta_\mathcal{V}$. Upon restricting to the first component of $\mathcal{H}$, these representations show that Theorem 3.1 follows from $\mathcal{D}(\sqrt{\Pi_B^2}) = \mathcal{D}(\Pi_B)$ with equivalences of the homogeneous graph norms, cf. Corollary 3.4. So, to complete the proof of Theorem 3.1 it remains to verify (H1)–(H7) for these particular choices of operators.

6.1. Verification of (H1)–(H7). It is obvious that (H1), (H3), and (H4) hold. Also (H2) is immediate for B_1 and for B_2 it follows from Assumption 2.3. The validity of (H5) is a consequence of $(\mathcal{V})$ in Assumption 2.1 and the product rule.

Since the integral over the gradient of a compactly supported function vanishes, the estimate for u in (H6) follows from Hölder's inequality. For v take $\varphi \in \mathrm{C}_c^\infty(\Omega; \mathbb{R})$ with $\varphi \equiv 1$ on $\mathrm{supp}(v)$ and denote by $\{e_j\}_{j=1}^{(d+2)m}$ the standard basis of $\mathbb{C}^{(d+2)m}$. Note $\mathrm{supp}(\Gamma^* v) \subseteq \mathrm{supp}(v)$ by (H5) for Γ^* in place of Γ, cf. Remark 5.2. As

$$\varphi e_j \in \mathrm{H}_0^1(\Omega; \mathbb{C}^m)^{d+2} \subseteq \mathcal{V}^{d+2} \subseteq \mathcal{D}(\Gamma)$$

for each j by Assumption 2.1, it follows

$$\left|\int_\Omega \Gamma^* v \,\mathrm{d}x\right| \simeq \sum_{j=1}^{(d+2)m} \left|\int_\Omega \langle \varphi e_j, \Gamma^* v\rangle \,\mathrm{d}x\right| = \sum_{j=1}^{(d+2)m} \left|\int_\Omega \langle \Gamma(\varphi e_j), v\rangle \,\mathrm{d}x\right|.$$

Since $|\Gamma(\varphi e_j)| \leq 1$ a.e. on $\operatorname{supp}(v)$, the required estimate is obtained by Hölder's inequality.

For the first part of (H7) take $\beta_1 = 1$ and note

$$\|u\|_{\mathcal{V}^{d+2}} = \|u\|_{\mathrm{H}^1(\Omega;\mathbb{C}^m)^{d+2}} = \|\Gamma u\|_{\mathcal{H}} = \|\Pi u\|_{\mathcal{H}} \simeq \|\sqrt{\Pi^2} u\|_{\mathcal{H}}.$$

For the second part take $\beta_2 = \alpha$ as in Assumption 2.1. Fix $v = (0, w, \nabla_{\mathcal{V}} w) \in \mathcal{R}(\Gamma) \cap \mathcal{D}(\Pi^2)$. Then $w \in \mathcal{D}((\nabla_{\mathcal{V}})^* \nabla_{\mathcal{V}}) = \mathcal{D}(1 - \Delta_{\mathcal{V}})$ so that by Assumption (E) of Theorem 3.1,

$$\begin{aligned}
\|v\|_{[\mathcal{H},\mathcal{V}^{d+2}]_\alpha} &\simeq \|w\|_{\mathrm{H}^{\alpha,2}(\Omega;\mathbb{C}^m)} + \|\nabla w\|_{\mathrm{H}^{\alpha,2}(\Omega;\mathbb{C}^m)^d} \\
&\lesssim \|w\|_{\mathrm{H}^{1+\alpha,2}(\Omega;\mathbb{C}^m)} \\
&\lesssim \|(1 - \Delta_{\mathcal{V}})^{1/2+\alpha/2} w\|_{\mathrm{L}^2(\Omega;\mathbb{C}^m)}.
\end{aligned}$$

However, $(1 - \Delta_{\mathcal{V}})^{1/2+\alpha/2} w = (\Pi^2)^{1/2+\alpha/2} \widehat{w}$, $\widehat{w} = (w, 0, 0) \in \mathcal{D}(\Pi)$. Thus, Corollary 4.2 and the composition rules (4.1) for the functional calculus for Π yield

$$\|(\Pi^2)^{1/2}(\Pi^2)^{\alpha/2}\widehat{w}\|_{\mathcal{H}} \simeq \|\Pi(\Pi^2)^{\alpha/2}\widehat{w}\|_{\mathcal{H}} = \|(\Pi^2)^{\alpha/2}\Pi\widehat{w}\|_{\mathcal{H}} = \|(\Pi^2)^{\alpha/2} v\|_{\mathcal{H}}$$

as required.

7. The proof of Theorem 3.3: Preliminaries

In this and the following two sections we develop the proof of Theorem 3.3. Throughout we assume that Γ, B_1, and B_2 are operators on $\mathcal{H}$ satisfying (H1)–(H7). We shall stick to the notions introduced in Section 5 but simply write $\|\cdot\|$ instead of $\|\cdot\|_{\mathcal{H}}$ as long as no misunderstandings are expected. We shall use the discussed properties of Γ, Γ^*, Π, Γ_B^*, and Π_B^* without further referencing. We also introduce the following bounded operators on $\mathcal{H}$:

$$\begin{aligned}
&R_t^B := (1 + \mathrm{i}t\Pi_B)^{-1}, && P_t^B := (1 + t^2\Pi_B^2)^{-1}, \\
&Q_t^B = t\Pi_B P_t^B, \quad \text{and} && \Theta_t^B := t\Gamma_B^* P_t^B \qquad (t \in \mathbb{R}).
\end{aligned}$$

In the unperturbed case, i.e. if $B_1 = B_2 = \mathrm{Id}$, we simply write R_t, P_t, Q_t, and Θ_t.

In order to carry out correctly the dependence of the implicit constants on the perturbations B_1 and B_2, we make the following

Agreement 7.1. In the proof of Theorem 3.3 the symbols $\lesssim$, $\gtrsim$, and $\simeq$ are reserved for estimates invoking implicit constants whose dependence on B_1 and B_2 is only through the constants quantified in (H2).

Lemma 7.2. *For each $t \in \mathbb{R}$ it holds $P_t^B = \frac{1}{2}(R_t^B + R_{-t}^B) = R_t^B R_{-t}^B$ and $Q_t^B = \frac{1}{2\mathrm{i}}(R_{-t}^B - R_t^B)$. Moreover,*

$$\|R_t^B\|_{\mathcal{L}(\mathcal{H})} + \|P_t^B\|_{\mathcal{L}(\mathcal{H})} + \|Q_t^B\|_{\mathcal{L}(\mathcal{H})} + \|\Theta_t^B\|_{\mathcal{L}(\mathcal{H})} \lesssim 1 \quad (t \in \mathbb{R}).$$

Proof: Checking the identities is a straightforward calculation. The boundedness of $\{R_t^B\}_{t\in\mathbb{R}}$, $\{P_t^B\}_{t\in\mathbb{R}}$, and $\{Q_t^B\}_{t\in\mathbb{R}}$ then follows by bisectoriality of Π_B. Finally, $\|\Theta_t^B\|_{\mathcal{L}(\mathcal{H})} \lesssim \|Q_t^B\|_{\mathcal{L}(\mathcal{H})}$ holds for all $t \in \mathbb{R}$ due to the topological decomposition $\overline{\mathcal{R}(\Pi_B)} = \overline{\mathcal{R}(\Gamma_B^*)} \oplus \overline{\mathcal{R}(\Gamma)}$, cf. (5.2). □

In [**7**, Prop. 4.8] Axelsson, Keith, and McIntosh reveal that (H1)–(H3) already imply

$$\int_0^\infty \|\Theta_t^B(1-P_t)u\|^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \quad (u \in \mathcal{R}(\Gamma)), \tag{7.1}$$

and that a sufficient condition for the quadratic estimate (3.1) for Π_B is

$$\int_0^\infty \|\Theta_t^B P_t u\|^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \quad (u \in \mathcal{R}(\Gamma)) \tag{7.2}$$

and the three analogous estimates obtained by replacing $\{\Gamma, B_1, B_2\}$ by $\{\Gamma^*, B_2, B_1\}$, $\{\Gamma^*, B_2^*, B_1^*\}$, and $\{\Gamma, B_1^*, B_2^*\}$. In fact, owing to Remark 5.2, it suffices to prove (7.2). In this section we shall take care of the integral over $t \geq 1$ and decompose the remaining finite time integral into three pieces that will be handled later on.

Lemma 7.3 (Reduction to finite time)**.** *It holds*

$$\int_1^\infty \|\Theta_t^B P_t u\|^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \quad (u \in \mathcal{R}(\Gamma)).$$

Proof: Fix $u = \Gamma w \in \mathcal{R}(\Gamma)$. By nilpotence of Γ and Γ^* one readily checks

$$\begin{aligned} P_t u &= (1+t^2\Pi^2)^{-1}\Gamma(1+t^2\Pi^2)(1+t^2\Pi^2)^{-1} w \\ &= \Gamma(1+t^2\Pi^2)^{-1} w = \Gamma P_t w \qquad\qquad (t \in \mathbb{R}\setminus\{0\}). \end{aligned} \tag{7.3}$$

Hence, the second part of (H7) applies to $v = P_t u$. Lemma 7.2 and the continuous inclusion $[\mathcal{H}, \mathcal{V}^k]_{\beta_2} \subseteq \mathcal{H}$, yield

$$\begin{aligned} \int_1^\infty \|\Theta_t^B P_t u\|^2 \frac{\mathrm{d}t}{t} &\lesssim \int_1^\infty \|P_t u\|^2_{[\mathcal{H},\mathcal{V}^k]_{\beta_2}} \frac{\mathrm{d}t}{t} \\ &\lesssim \int_1^\infty \|t^{\beta_2}(\Pi^2)^{\beta_2/2} P_t u\|^2 \frac{\mathrm{d}t}{t^{1+2\beta_2}}. \end{aligned}$$

Define holomorphic functions $f_t := (t^2 z)^{\beta_2/2}(1 + t^2 z)^{-1}$. A direct estimate on the defining Cauchy integral yields a bound for $\|f_t(\Pi^2)\|_{\mathcal{L}(\mathcal{H})}$ uniformly in $t \geq 1$. Thus,

$$
\begin{aligned}
&= \int_1^\infty \|f_t(\Pi^2)u\|^2 \frac{\mathrm{d}t}{t^{1+2\beta_2}} \\
&\lesssim \int_1^\infty \|u\|^2 \frac{\mathrm{d}t}{t^{1+2\beta_2}} = \frac{1}{2\beta_2}\|u\|^2. \qquad \square
\end{aligned}
$$

To proceed further, we introduce a slightly modified version of Christ's dyadic decomposition for doubling metric measure spaces [**10**, Thm. 11]. In fact, if one aims only at a *truncated dyadic cube structure* with a common bound for the diameter of all dyadic cubes, then Christ's argument literally applies to locally doubling metric measure spaces. This has been previously noticed e.g. by Morris [**28**]. Here, a metric measure space X with metric ρ and positive Borel measure μ is *doubling* if there is a constant $C > 0$ such that

$$\mu(\{x \in X : \rho(x, x_0) < 2r\}) \leq C\mu(\{x \in X : \rho(x, x_0) < r\}) \quad (x_0 \in X,\ r > 0)$$

and it is *locally doubling* if the inequality above holds for all $x_0 \in X$ and all $r \in (0, 1]$. Note that (Ω) of Assumption 2.1 entails that Ω equipped with the restricted Euclidean metric and the restricted Lebesgue measure is locally doubling.

Theorem 7.4 (Christ). *Under Assumption 2.1(Ω) there exists a collection of open subsets $\{Q_\alpha^k \subseteq \Omega : k \in \mathbb{N}_0,\ \alpha \in I_k\}$, where I_k are index sets, and constants $\delta \in (0,1)$ and $a_0, \widehat{\eta}, C_1, \widehat{C}_2 > 0$ such that:*

(i) $|\Omega \setminus \bigcup_{\alpha \in I_k} Q_\alpha^k| = 0$ *for each* $k \in \mathbb{N}_0$.

(ii) *If $l \geq k$, then for each $\alpha \in I_k$ and each $\beta \in I_l$ either $Q_\beta^l \subseteq Q_\alpha^k$ or $Q_\beta^l \cap Q_\alpha^k = \emptyset$ holds.*

(iii) *If $l \leq k$, then for each $\alpha \in I_k$ there is a unique $\beta \in I_l$ such that $Q_\alpha^k \subseteq Q_\beta^l$.*

(iv) *It holds* $\operatorname{diam}(Q_\alpha^k) \leq C_1 \delta^k$ *for each $k \in \mathbb{N}_0$ and each $\alpha \in I_k$.*

(v) *For each Q_α^k, $k \in \mathbb{N}_0$, $\alpha \in I_k$, there exists $z_\alpha^k \in \Omega$ such that $B(z_\alpha^k, a_0\delta^k) \cap \Omega \subseteq Q_\alpha^k$.*

(vi) *If $k \in \mathbb{N}_0$, $\alpha \in I_k$, and $t > 0$, then $|\{x \in Q_\alpha^k : d(x, \Omega \setminus Q_\alpha^k) \leq t\delta^k\}| \leq \widehat{C}_2 t^{\widehat{\eta}} |Q_\alpha^k|$.*

By a slight abuse of notation we refer to the Q_α^k as *dyadic cubes*. We denote the family of all dyadic cubes by Δ and each family of fixed *step size* δ^k by $\Delta_{\delta^k} := \{Q_\alpha^k : \alpha \in I_k\}$. Moreover, if $k \in \mathbb{N}_0$ and $t \in (\delta^{k+1}, \delta^k]$, then the family of dyadic cubes of step size t is $\Delta_t := \Delta_{\delta^k}$. The *sidelength* of $Q \in \Delta_{\delta^k}$ is $l(Q) := \delta^k$.

Remark 7.5. (i) Assumption 2.1(Ω) in combination with (iv) and (v) of Theorem 7.4 imply $|Q| \simeq l(Q)^d$ for all $Q \in \Delta$.

(ii) Since the dyadic cubes are open, for each $t \in (0, 1]$ the family Δ_t is countable.

(iii) The first item of Theorem 7.4 implies that there exists a nullset $\mathfrak{N} \subseteq \Omega$ such that for each $t \in (0, 1]$ and each $x \in \Omega \setminus \mathfrak{N}$ there exists a unique cube $Q \in \Delta_t$ that contains x.

A substantial drawback of Theorem 7.4 is that part (vi) gives an estimate for the inner boundary strips of dyadic cubes only near their relative boundary with respect to Ω. This of course is a relict of the very construction. The Ahlfors–David condition is an appropriate measure-theoretic assumption on $\partial\Omega$ allowing to control the measure of the complete inner boundary strip.

Some variant of the following lemma may be well known but for the reader's convenience we include a proof.

Lemma 7.6. *If $\Xi \subseteq \mathbb{R}^d$ is open and $\partial\Xi$ is a $(d-1)$-set, then for each $r_0, t_0 > 0$ there exists $C > 0$ such that*

$$|\{x \in \Xi : |x - x_0| < r,\, d(x, \mathbb{R}^d \setminus \Xi) \le tr\}| \le Ctr^d$$

for all $x_0 \in \overline{\Xi}$, $r \in (0, r_0]$, and $t \in (0, t_0]$.

Proof: For $x_0 \in \overline{\Xi}$, $r \in (0, r_0]$, and $t \in (0, t_0]$ put

$$E := \{x \in \Xi : |x - x_0| < r,\, d(x, \mathbb{R}^d \setminus \Xi) \le tr\}.$$

Then for each $x \in E$ there exists a boundary point $b_x \in \partial\Xi$ such that $x \in \overline{B(b_x, tr)}$. The Vitali covering lemma [**16**, Sec. 1.5] yields a countable subset $J \subseteq E$ such that the balls $\{B(b_x, tr)\}_{x \in J}$ are pairwise disjoint and such that $\{B(b_x, 6tr)\}_{x \in J}$ is a covering of E. Hence, $|E| \lesssim \#J(tr)^d$, where $\#J$ denotes the number of elements contained in J.

To get control on $\#J$ fix $z \in J$. If $y \in B(b_x, tr)$ for some $x \in J$ then by the triangle inequality $|y - b_z| \le 3tr + 2r < (3t_0 + 2)r$. The Ahlfors–David condition $\mathrm{m}_{d-1}(\partial\Xi \cap B(b_x, r)) \simeq r^{d-1}$ remains valid for all $b_x \in \partial\Xi$ and all $r \in (0, (3t_0 + 2)r_0]$ with implicit constants depending only on Ξ, r_0, and t_0. Hence,

$$((3t_0+2)r)^{d-1} \gtrsim \mathrm{m}_{d-1}\big(\partial\Xi \cap B(b_z, (3t_0+2)r)\big) \ge \sum_{x \in J} \mathrm{m}_{d-1}\big(\partial\Xi \cap B(b_x, tr)\big).$$

Again by the Ahlfors–David condition the right-hand side is comparable to $\#J(tr)^{d-1}$. Thus, $\#J \lesssim t^{1-d}$ and the conclusion follows. $\square$

As a corollary we record a connection between Ahlfors regular and plump sets that is of independent interest. Following [**32**] a bounded set $\Xi \subseteq \mathbb{R}^d$ is *κ-plump* if there exists $\kappa > 0$ such that for each $x_0 \in \overline{\Xi}$ and each $r \in (0, \operatorname{diam}(\Xi)]$ there exists $x \in \Xi$ such that $B(x, \kappa r) \subseteq \Xi \cap B(x_0, r)$.

Corollary 7.7. *If $\Xi \subseteq \mathbb{R}^d$ is a bounded open d-set and $\partial\Xi$ is a $(d-1)$-set, then Ξ is κ-plump.*

Proof: By the d-set property of Ξ fix $c > 0$ such that $|\Xi \cap B(x_0, r)| \geq cr^d$ for all $x_0 \in \overline{\Xi}$ and all $r \in (0, \operatorname{diam}(\Xi)]$. Choose $r_0 := \frac{1}{2}\operatorname{diam}(\Xi)$ and $t_0 = 1$ in Lemma 7.6 and apply the estimate with $t = \min\{\frac{c}{2C}, 1\}$ to conclude

$$\left|\left\{x \in \Xi : |x - x_0| < \frac{r}{2},\, d(x, \mathbb{R}^d \setminus \Xi) > \frac{tr}{2}\right\}\right| \geq \frac{cr^d}{2^{d+1}}$$

for all $x_0 \in \overline{\Xi}$ and all $r \in (0, \operatorname{diam}(\Xi)]$. In particular, these sets are non-empty so one can choose $\kappa = t$. $\square$

Corollary 7.8. *Under Assumptions 2.1(Ω) and 2.1($\partial\Omega$) there exist constants $\eta, C_2 > 0$ such that*

$$|\{x \in Q : d(x, \mathbb{R}^d \setminus Q) \leq t\delta^k\}| \leq C_2 t^\eta |Q|$$

for each $k \in \mathbb{N}_0$, $Q \in \Delta_{\delta^k}$, and $t > 0$.

Proof: Put $\eta := \min\{1, \widehat{\eta}\}$ where $\widehat{\eta}$ is given by Theorem 7.4. If $t \geq 1$ then the estimate in question holds with $C_2 = 1$. If $t < 1$ split

$$\begin{aligned} E &:= \big\{x \in Q : d(x, \mathbb{R}^d \setminus Q) \leq t\delta^k\big\} \\ &\subseteq \big\{x \in Q : d(x, \Omega \setminus Q) \leq t\delta^k\big\} \cup \big\{x \in Q : d(x, \mathbb{R}^d \setminus \Omega) \leq t\delta^k\big\}. \end{aligned}$$

Property (vi) of the dyadic decomposition and Lemma 7.6 applied with $r_0 := C_1$, $t_0 := \frac{1}{C_1}$, and r and t replaced by $C_1\delta^k$ and $\frac{t}{C_1}$ yield the estimate $|E| \lesssim \widehat{C}_2 t^{\widehat{\eta}} |Q| + t\delta^{kd}$. The conclusion follows from Remark 7.5 taking into account $t < 1$. $\square$

The boundedness assertions of Lemma 7.2 self-improve to off-diagonal estimates. These will be a crucial instrument in the following. Recall that given $z \in \mathbb{C}$ we write $\langle z \rangle = 1 + |z|$.

Proposition 7.9 (Off-diagonal estimates). *Let U_t be either of the operators R_t^B, P_t^B, Q_t^B, or Θ_t^B. Then for every $M \in \mathbb{N}_0$ there exists a constant $A_M > 0$ such that*

$$\left\|\mathbf{1}_E U_t(\mathbf{1}_F u)\right\| \lesssim A_M \left\langle \frac{d(E,F)}{t} \right\rangle^{-M} \|\mathbf{1}_F u\|$$

holds for all $u \in \mathcal{H}$, all $t \in \mathbb{R} \setminus \{0\}$, and all bounded Borel sets $E, F \subseteq \Omega$.

We skip the proof as it is literally the same as in [**7**, Prop. 5.2] with one minor modification: In the case $0 < |t| \leq d(E,F)$ one separates E and F by some $\eta \in \mathrm{C}_c^\infty(\widetilde{E})$ such that $\eta = 1$ on E and $\|\nabla \eta\|_\infty \leq c/d(E,F)$, where $\widetilde{E} := \{x \in \mathbb{R}^d : d(x,E) < \frac{1}{2} d(E,F)\}$ and c depending only on d, rather then the choices for η and $\widetilde{E}$ in [**7**]. This is due to the slight difference between our (H5) and (H6) in [**7**].

The next lemma helps to control the sums that naturally crop up when combining off-diagonal estimates with the dyadic decomposition.

Lemma 7.10. *The following hold true for each $M > d+1$.*

(i) *There exists $c_M > 0$ depending solely on M and Ω such that*

$$\sum_{R \in \Delta_t} \left\langle \frac{d(x,R)}{t} \right\rangle^{-M} \leq c_M \quad (x \in \mathbb{R}^d,\, t \in (0,1]).$$

(ii) *Let $l \in \mathbb{N}_0$, $t \in (0,1]$, $Q \in \Delta_t$, and $F \subseteq \mathbb{R}^d$ be such that $d(Q,F) \geq lt$. Then exist $c_{l,1}, c_{l,2} \geq 0$ depending solely on l, M, and Ω such that*

$$\sum_{R \in \Delta_t} \left\langle \frac{d(Q, R \cap F)}{s} \right\rangle^{-M} \leq c_{l,1} + c_{l,2} \left(\frac{s}{t}\right)^M \quad (s > 0).$$

If $l > 0$, then one can choose $c_{l,1} = 0$.

Proof: To show the first statement fix $x \in \mathbb{R}^d$ and $t \in (0,1]$. Fix $k \in \mathbb{N}_0$ such that $\delta^{k+1} < t \leq \delta^k$. Put $\Omega_n := B(x,(n+1)C_1\delta^k) \cap \Omega$ for $n \in \mathbb{N}_0$ and $\Omega_{-1} := \Omega_{-2} := \emptyset$. If $R \in \Delta_t$ intersects an annulus $\Omega_n \setminus \Omega_{n-1}$, $n \in \mathbb{N}$, then due to property (iv) of the dyadic decomposition

$$d(x,R) \geq d(x, \Omega_{n+1} \setminus \Omega_{n-2}) \geq (n-1)C_1\delta^k \geq (n-1)\delta^{-1}C_1 t. \tag{7.4}$$

It readily follows from Assumption 2.1 that there exists $c > 0$ such that $|\Omega \cap B(x,r)| \geq cr^d$ holds for all $x \in \Omega$ and all $r \in (0,a_0)$, where $a_0 > 0$ is given by Theorem 7.4. Properties (iv) and (v) of the dyadic decomposition yield

$$\#\big\{R \in \Delta_t : R \cap (\Omega_n \setminus \Omega_{n-1}) \neq \emptyset\big\} \leq \frac{|\Omega_{n+1}|}{c(a_0\delta^k)^d} \leq \frac{C_1^d(n+2)^d}{ca_0^d} \quad (n \in \mathbb{N}_0). \tag{7.5}$$

Now, rearrange the cubes in Δ_t according to the first annulus that they intersect to find

$$\sum_{R \in \Delta_t} \left\langle \frac{d(x,R)}{t} \right\rangle^{-M} \leq \sum_{n=0}^{\infty} \frac{C_1^d (n+2)^d}{c a_0^d} (1 + (n-1)\delta^{-1} C_1)^{-M} =: c_M < \infty$$

thanks to $M > d+1$.

The second claim is very similar. Choose an arbitrary $x \in Q$ and define Ω_n, $n \geq -2$, as before. By (7.5) there are at most $\frac{C_1^d (n+2)^d}{c a_0^d}$ cubes $R \in \Delta_t$ intersecting an annulus $\Omega_n \setminus \Omega_{n-1}$, $n \in \mathbb{N}_0$, and if this happens then by assumption on F, property (iv) of the dyadic decomposition, and (7.4),

$$d(Q, R \cap F) \geq \max\{d(Q,R), d(Q,F)\} \geq \max\{(n-2)\delta^{-1} C_1 t, lt\}.$$

Hence, the left-hand side of the estimate in question is bounded by

$$\frac{C_1^d}{c a_0^d} \sum_{n=0}^{l+2} (n+2)^d \left(1 + \frac{lt}{s}\right)^{-M} + \frac{C_1^d}{c a_0^d} \sum_{n=l+3}^{\infty} (n+2)^d \left(\frac{(n-2)\delta^{-1} C_1 t}{s}\right)^{-M}.$$

The second sum is controlled by a generic multiple of $s^M t^{-M}$ and so is the first one if $l > 0$. $\square$

A consequence of the preceding lemma is the following. Take $w \in \mathbb{C}^N$ and regard it as a constant function on Ω. Also fix $s \in (0,1]$. If $Q \in \Delta_t$ for some $t \in (0,1]$ then Proposition 7.9 and the second part of Lemma 7.10 assure

$$\sum_{R \in \Delta_t} \|\mathbf{1}_Q \Theta_s^B (\mathbf{1}_R w)\| \lesssim \sum_{R \in \Delta_t} \left\langle \frac{d(Q,R)}{s} \right\rangle^{-(d+2)} \|\mathbf{1}_R w\| < \infty.$$

As the measure of each cube $Q \in \Delta_t$ is comparable to t^d, cf. Remark 7.5, each bounded subset of Ω is covered up to a set of measure zero by finitely many cubes $Q \in \Delta_t$. Now, define $\Theta_s^B w \in \mathrm{L}^2_{\mathrm{loc}}(\Omega; \mathbb{C}^N)$ by setting it equal to $\sum_{R \in \Delta_t} \mathbf{1}_Q \Theta_s^B (\mathbf{1}_R w)$ on each $Q \in \Delta_t$. This definition is independent of the particular choice of t. Indeed, if $0 < t_1 < t_2 \leq 1$ and $Q_1 \in \Delta_{t_1}$ is a subcube of $Q_2 \in \Delta_{t_2}$ then

$$\begin{aligned} \mathbf{1}_{Q_1} \sum_{R_2 \in \Delta_{t_2}} \mathbf{1}_{Q_2} \Theta_s^B (\mathbf{1}_{R_2} w) &= \sum_{R_2 \in \Delta_{t_2}} \sum_{\substack{R_1 \in \Delta_{t_1} \\ R_1 \subseteq R_2}} \mathbf{1}_{Q_1} \Theta_s^B (\mathbf{1}_{R_1} w) \\ &= \sum_{R_1 \in \Delta_{t_1}} \mathbf{1}_{Q_1} \Theta_s^B (\mathbf{1}_{R_1} w) \end{aligned}$$

by properties (i), (ii), and (iii) of the dyadic decomposition. This gives rise to the following definition.

Definition 7.11. Let $0 < t \leq 1$. The *principal part* of Θ_t^B is defined as

$$\gamma_t \colon \Omega \to \mathcal{L}(\mathbb{C}^N), \quad \gamma_t(x) \colon w \mapsto (\Theta_t^B w)(x).$$

Remark 7.12. If Ω is bounded then $\mathcal{H}$ contains the constant $\mathbb{C}^N$ valued functions and the direct definition of $\Theta_t^B w$ for $t \in (0,1]$ and $w \in \mathbb{C}^N$ coincides with the one above.

Next, we introduce the dyadic averaging operator.

Proposition 7.13. *Let $t \in (0,1]$. The* dyadic averaging operator A_t, *defined for $u \in \mathcal{H}$ by*

$$A_t u(x) := \fint_{Q(x,t)} u(y) \,\mathrm{d}y \quad (x \in \Omega \setminus \mathfrak{N}),$$

where $Q(x,t)$ is uniquely characterized by $x \in Q(x,t) \in \Delta_t$, is a contraction on $\mathcal{H}$.

Proof: Split $\Omega \setminus \mathfrak{N}$ into the dyadic cubes Δ_t and apply Jensen's inequality to find

$$\begin{aligned} \|A_t u\|^2 &= \sum_{Q \in \Delta_t} \int_Q |A_t u|^2 \,\mathrm{d}y \\ &= \sum_{Q \in \Delta_t} |Q| \left| \fint_Q u \,\mathrm{d}y \right|^2 \leq \sum_{Q \in \Delta_t} |Q| \fint_Q |u|^2 \,\mathrm{d}y = \|u\|^2. \qquad \square \end{aligned}$$

Lemma 7.14. *Let $t \in (0,1]$. The operator $\gamma_t A_t \colon \mathcal{H} \to \mathcal{H}$ acting via $(\gamma_t A_t u)(x) = \gamma_t(x)(A_t u)(x)$ is bounded with operator norm uniformly bounded in t. Moreover,*

$$\fint_Q \|\gamma_t(x)\|_{\mathcal{L}(\mathbb{C}^N)}^2 \,\mathrm{d}x \lesssim 1 \quad (Q \in \Delta_t)$$

with an implicit constant independent of t.

Proof: The first claim follows straightforwardly from the second one, cf. also [**28**, Cor. 5.4]. To prove the second claim fix $Q \in \Delta_t$. With $\{e_j\}_{j=1}^N$ the standard unit vectors in $\mathbb{C}^N$,

$$\begin{aligned} \left(\int_Q \|\gamma_t(x)\|_{\mathcal{L}(\mathbb{C}^N)}^2 \,\mathrm{d}x \right)^{1/2} &\lesssim \sum_{j=1}^N \left(\int_Q |\gamma_t(x) e_j|^2 \,\mathrm{d}x \right)^{1/2} \\ &\leq \sum_{j=1}^N \sum_{R \in \Delta_t} \left(\int_Q |(\Theta_t^B(\mathbf{1}_R e_j))(x)|^2 \,\mathrm{d}x \right)^{1/2}. \end{aligned}$$

Proposition 7.9, item (i) of Remark 7.5, and Lemma 7.10 yield

$$\lesssim \sum_{j=1}^{N} \sum_{R\in\Delta_t} \left\langle \frac{d(R,Q)}{t} \right\rangle^{-(d+2)} |Q|^{1/2} \lesssim |Q|^{1/2}$$

uniformly in t. $\square$

For $u \in \mathcal{R}(\Gamma)$ integration over $t \in (0,1]$ on the left-hand side of (7.2) is now split as

$$\begin{aligned} \int_0^1 \|\Theta_t^B P_t u\|^2 \frac{\mathrm{d}t}{t} \lesssim \int_0^1 &\|(\Theta_t^B - \gamma_t A_t) P_t u\|^2 \frac{\mathrm{d}t}{t} \\ + \int_0^1 &\|\gamma_t A_t (P_t - 1)u\|^2 \frac{\mathrm{d}t}{t} \\ + \int_0^1 &\int_\Omega \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)} |A_t u(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}. \end{aligned} \tag{7.6}$$

The idea behind is to compensate the non-integrable singularity showing up at $t = 0$ as follows: In the first term $\Theta_t^B P_t u$ is compared with its averages over dyadic cubes. Letting $t \to 0$, the difference is expected to vanish since the diameter of the cubes used for the averaging shrinks to zero. In the second term P_t is compared with the identity operator, which is the strong limit of P_t as $t \to 0$. Finally, the third and most difficult term cries for a Carleson measure estimate. At the beginning of this section we have seen that it remains to bound each of the three terms on the right-hand side by a generic multiple of $\|u\|^2$. This will be done in the remaining sections.

8. The proof of Theorem 3.3: Principal part approximation

This section is concerned with estimating the first two terms on the right-hand side of (7.6). To start with, recall the classical Poincaré inequality as it can be deduced from Lemmas 7.12 and 7.16 in [**17**]. Throughout, $u_S := ⨍_S u \,\mathrm{d}x$ is the *mean value* of an integrable function $u\colon S \to \mathbb{C}^n$ over a set $S \subseteq \mathbb{R}^d$ with Lebesgue measure $|S| > 0$.

Lemma 8.1 (Poincaré inequality)**.** *Let $\Xi \subseteq \mathbb{R}^d$ be bounded and convex, and let S be a Borel subset of Ξ with $|S| > 0$. Then for all $u \in \mathrm{H}^1(\Xi;\mathbb{C})$,*

$$\|u - u_S\|_{\mathrm{L}^2(\Xi;\mathbb{C})} \leq \frac{(\operatorname{diam}\Xi)^d |B(0,1)|^{1-1/d} |\Xi|^{1/d}}{|S|} \|\nabla u\|_{\mathrm{L}^2(\Xi;\mathbb{C}^d)}.$$

The following weighted Poincaré inequality is the key to handle the first term in (7.6).

Proposition 8.2 (A weighted Poincaré inequality)**.** *For each $M > 2d+2$ there exists $C_M > 0$ such that*

$$\int_{\mathbb{R}^d} |u(x)-u_Q|^2 \Big\langle \frac{d(x,Q)}{t} \Big\rangle^{-M} \mathrm{d}x \leq C_M \int_{\mathbb{R}^d} |t\nabla u(x)|^2 \Big\langle \frac{d(x,Q)}{t} \Big\rangle^{2d+2-M} \mathrm{d}x$$

holds for all $t \in (0,1]$, all $Q \in \Delta_t$, and all $u \in \mathrm{H}^1(\mathbb{R}^d;\mathbb{C})$.

Proof: Let $t \in (0,1]$ and $Q \in \Delta_t$. Fix some arbitrary $x_0 \in Q$, let T be the affine transformation $x \mapsto x_0 - t^{-1}x$, and put $S := T(Q)$. Upon replacing u by $u \circ T^{-1}$ it suffices to prove

$$\int_{\mathbb{R}^d} |u(x)-u_S|^2 \langle d(x,S)\rangle^{-M} \, \mathrm{d}x \lesssim \int_{\mathbb{R}^d} |\nabla u(x)|^2 \langle d(x,S)\rangle^{2d+2-M} \, \mathrm{d}x \tag{8.1}$$

for arbitrary $u \in \mathrm{H}^1(\mathbb{R}^d;\mathbb{C})$ and an implicit constant independent of t, Q, and u.

Let C_1 and δ be given by Theorem 7.4. Due to property (iv) of the dyadic decomposition, $S \subseteq B(0, C_1\delta^{-1})$ and $|S| \simeq 1$. Hence, for $r \geq C_1\delta^{-1}$ Lemma 8.1 applies with $\Xi = B(0,r)$ and S as above yielding

$$\int_{\mathbb{R}^d} |u(x) - u_S|^2 \mathbf{1}_{B(0,r)}(x) \, \mathrm{d}x \lesssim r^{2d+2} \int_{\mathbb{R}^d} |\nabla u(x)|^2 \mathbf{1}_{B(0,r)}(x) \, \mathrm{d}x$$

with an implicit constant independent of u and r. Integration with respect to $r^{-M-1}\mathrm{d}r$ gives

$$\int_{\mathbb{R}^d} |u(x) - u_S|^2 \int_{C_1\delta^{-1}}^{\infty} \mathbf{1}_{B(0,r)}(x) r^{-M-1} \, \mathrm{d}r \, \mathrm{d}x$$
$$\lesssim \int_{\mathbb{R}^d} |\nabla u(x)|^2 \int_{C_1\delta^{-1}}^{\infty} r^{2d+1-M} \mathbf{1}_{B(0,r)}(x) \, \mathrm{d}r \, \mathrm{d}x.$$

For fixed $x \in \mathbb{R}^d$ the inner integrand becomes unequal to 0 precisely when r gets larger than $\max\{|x|, C_1\delta^{-1}\}$ and it is straightforward to verify (draw a sketch!) that

$$\frac{C_1\delta^{-1}}{1+C_1\delta^{-1}}(1 + d(x,S)) \leq \max\{|x|, C_1\delta^{-1}\} \leq (1 + C_1\delta^{-1})(1 + d(x,S)).$$

Thus, (8.1) follows from the previous estimate by a simple computation of the inner integrals. □

Now, we are in position to estimate the first term in (7.6).

Proposition 8.3 (First term estimate)**.** *It holds*

$$\int_0^1 \|(\Theta_t^B - \gamma_t A_t)P_t u\|^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \quad (u \in \mathcal{R}(\Gamma)).$$

Proof: We first inspect the integrand $\|(\Theta_t^B - \gamma_t A_t)v\|^2$ for arbitrary $t \in (0,1]$ and $v \in \mathcal{V}^k$. Split Ω into dyadic cubes $Q \in \Delta_t$ and decompose $v = \sum_{R \in \Delta_t} \mathbf{1}_R v$ to find by the definitions of the principal part and the dyadic averaging operator

$$\|(\Theta_t^B - \gamma_t A_t)v\|^2 = \sum_{Q \in \Delta_t} \bigg\| \sum_{R \in \Delta_t} \mathbf{1}_Q \Theta_t^B (\mathbf{1}_R v - \mathbf{1}_R v_Q) \bigg\|^2.$$

Off-diagonal estimates as in Proposition 7.9 yield

$$\lesssim \sum_{Q \in \Delta_t} \bigg\{ \sum_{R \in \Delta_t} \Big\langle \frac{d(R,Q)}{t} \Big\rangle^{-3d-4} \|\mathbf{1}_R (v - v_Q)\| \bigg\}^2$$

and by the Cauchy–Schwarz inequality and Lemma 7.10,

$$\lesssim \sum_{Q \in \Delta_t} \sum_{R \in \Delta_t} \Big\langle \frac{d(R,Q)}{t} \Big\rangle^{-3d-4} \|\mathbf{1}_R (v - v_Q)\|^2.$$

If $Q, R \in \Delta_t$ and $x \in R$ then $d(x,Q) \leq d(R,Q) + C_1 \delta^{-1} t$ as follows immediately from property (iv) of the dyadic decomposition. Consequently,

$$\lesssim \sum_{Q \in \Delta_t} \sum_{R \in \Delta_t} \int_R |v(x) - v_Q|^2 \Big\langle \frac{d(x,Q)}{t} \Big\rangle^{-3d-4} \mathrm{d}x$$

$$= \sum_{Q \in \Delta_t} \int_\Omega |v(x) - v_Q|^2 \Big\langle \frac{d(x,Q)}{t} \Big\rangle^{-3d-4} \mathrm{d}x.$$

Now, use $(\mathcal{V})$ of Assumption 2.1 coordinatewise to construct an extension $\mathfrak{E}v \in \mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^m)^k$ of v to which Proposition 8.2 applies coordinatewise. Switching sum and integral then leads to

$$\leq \int_{\mathbb{R}^d} |t \nabla (\mathfrak{E}v)(x)|^2 \sum_{Q \in \Delta_t} \Big\langle \frac{d(x,Q)}{t} \Big\rangle^{-d-2} \mathrm{d}x \lesssim t^2 \|v\|_{\mathcal{V}^k}^2,$$

the second step being due to Lemma 7.10 and the boundedness of $\mathfrak{E} \colon \mathcal{V}^k \to \mathrm{H}^1(\mathbb{R}^d; \mathbb{C}^m)^k$.

On the other hand, Lemmas 7.2 and 7.14 bound $\|\Theta_t^B - \gamma_t A_t\|_{\mathcal{L}(\mathcal{H})}$ uniformly in $t \in (0,1]$. Invoking (H7), complex interpolation with the previous estimate yields

$$\|(\Theta_t^B - \gamma_t A_t)v\|^2 \lesssim t^{2\beta_2} \|v\|^2_{[\mathcal{H},\mathcal{V}^k]_{\beta_2}} \lesssim \|(t^2\Pi^2)^{\beta_2/2} v\|^2$$

for all $v \in \mathcal{R}(\Gamma) \cap \mathcal{D}(\Pi^2)$ and all $t \in (0,1]$. In particular, if $u \in \mathcal{R}(\Gamma)$, then due to (7.3) the previous estimate applies to $v = P_t u$. Hence,

$$\int_0^1 \|(\Theta_t^B - \gamma_t A_t)P_t u\|^2 \frac{\mathrm{d}t}{t} \lesssim \int_0^1 \|(t^2\Pi^2)^{\beta_2/2} P_t u\|^2 \frac{\mathrm{d}t}{t} = \int_0^1 \|\Phi_t(\Pi)u\|^2 \frac{\mathrm{d}t}{t}$$

with regularly decaying holomorphic functions $\Phi_t := (t^2 z^2)^{\beta_2/2}(1+t^2z^2)^{-1}$. Now the conclusion follows by the Schur estimate presented in Remark 4.6: Indeed, as in the proof of Proposition 4.4 a direct estimate yields some $\zeta \in \mathrm{L}^1(0,\infty; \mathrm{d}r/r)$ such that $\|\Phi_t(\Pi)Q_s\|_{\mathcal{L}(\mathcal{H})} \leq \zeta(ts^{-1})$ for all $s, t > 0$ and moreover $\overline{\mathcal{R}(\Gamma)} \subseteq \overline{\mathcal{R}(\Pi)}$ holds by the unperturbed counterpart of (5.2). ☐

Remark 8.4. In contrast to [**6**] we do not require a weighted Poincaré inequality on Ω to handle the first term on the right-hand side of (7.6). This is a key observation in order to dispense with smooth local coordinate charts around $\partial\Omega$.

We head toward the second term in (7.6). The key ingredient is the following interpolation inequality for the unperturbed operators Γ, Γ^*, and Π. The proof follows the one of [**6**, Lem. 6] line by line except that one invokes Corollary 7.8 to estimate the measure of inner boundary strips of dyadic cubes. This results in an exponent η as in Corollary 7.8 instead of $\eta = 1$ in [**6**, Lem. 6].

Lemma 8.5. *If Υ is either of the operators Γ, Γ^*, or Π then with $\eta > 0$ given by Corollary 7.8,*

$$\left| \fint_Q \Upsilon u \,\mathrm{d}x \right|^2 \lesssim \frac{1}{t^\eta} \left(\fint_Q |u|^2 \,\mathrm{d}x \right)^{\eta/2} \left(\fint_Q |\Upsilon u|^2 \,\mathrm{d}x \right)^{1-\eta/2} + \fint_Q |u|^2 \,\mathrm{d}x$$

holds for all $t \in (0,1]$, all $Q \in \Delta_t$, and all $u \in \mathcal{D}(\Upsilon)$.

Proposition 8.6 (Second term estimate)**.** *It holds*

$$\int_0^1 \|\gamma_t A_t (P_t - 1)u\|^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \quad (u \in \mathcal{H}).$$

Proof: Since A_t is a dyadic averaging operator, $A_t^2 = A_t$. Lemma 7.14 bounds $\|\gamma_t A_t\|_{\mathcal{L}(\mathcal{H})}$ uniformly in $t \in (0,1]$ so that in fact it suffices to establish

$$\int_0^1 \|A_t(P_t - 1)u\|^2 \frac{\mathrm{d}t}{t} \lesssim \|u\|^2 \quad (u \in \mathcal{H}).$$

This is certainly true for $u \in \mathcal{N}(\Pi)$ since then $P_t u = u$ holds for all $t \in \mathbb{R}$. Since Π is bisectorial, $\mathcal{H} = \mathcal{N}(\Pi) \oplus \overline{\mathcal{R}(\Pi)}$. Whence, it remains to consider $u \in \overline{\mathcal{R}(\Pi)}$. In this case the conclusion follows by the Schur estimate presented in Remark 4.6 applied to $T_t := A_t(P_t - 1)$ if $t \leq 1$ and $T_t := 0$ if $t > 1$, provided that we can find some $\zeta \in \mathrm{L}^1(0,\infty; \mathrm{d}r/r)$ such that

$$\|A_t(P_t - 1)Q_s\|_{\mathcal{L}(\mathcal{H})} \lesssim \zeta(ts^{-1}) \quad (t \in (0,1],\, s > 0).$$

In fact one can choose $\zeta(r) := \min\{r, r^{-1} + r^{-\eta}\}$. We skip details, since the argument relying on Lemma 8.5, Lemma 7.10, and off-diagonal estimates for P_s and Q_s is the same as in [**6**, Prop. 5]. Note that $\eta = 1$ in [**6**] and that Proposition 7.9 holds for the unperturbed operators P_s and Q_s, since if $\{\Gamma, B_1, B_2\}$ satisfies (H1)–(H7), then so does $\{\Gamma, \mathrm{Id}, \mathrm{Id}\}$. □

9. The proof of Theorem 3.3: Principal part estimate

After all it remains to estimate the last term in (7.6) appropriately, that is to establish

$$\int_0^1 \int_\Omega \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)} |A_t u(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim \|u\|^2 \quad (u \in \mathcal{R}(\Gamma)). \tag{9.1}$$

The proof follows the usual strategy of reducing the problem to a Carleson measure estimate, which in turn is established by a $T(b)$ procedure, see e.g. [**2, 6, 7, 8, 28**]. However, since only the last two references deal with the case $\Omega \neq \mathbb{R}^d$ but under different underlying hypotheses, we include a more detailed argument for our setup.

Recall the notion of a (dyadic) Carleson measure.

Definition 9.1. The *Carleson box* R_Q of $Q \in \Delta$ is the Borel set given by $R_Q := Q \times (0, l(Q)]$. A positive Borel measure ν on $\Omega \times (0,1]$ satisfying *Carleson's condition*

$$\|\nu\|_{\mathcal{C}} := \sup_{Q \in \Delta} \frac{\nu(R_Q)}{|Q|} < \infty$$

is called *dyadic Carleson measure* on $\Omega \times (0,1]$.

The following dyadic version of Carleson's theorem can be found in [**28**, Thm. 4.3].

Theorem 9.2. *If ν is a dyadic Carleson measure on $\Omega \times (0,1]$, then*

$$\iint_{\Omega\times(0,1]} |A_t u(x)|^2 \,\mathrm{d}\nu(x,t) \lesssim \|\nu\|_{\mathcal{C}} \|u\|^2 \quad (u \in \mathcal{H}).$$

So, (9.1) follows if $\|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)} \frac{\mathrm{d}x\,\mathrm{d}t}{t}$ is a Carleson measure on $\Omega\times(0,1]$ and it is this property of the principal part γ_t we are going to establish in the following.

We begin by fixing $\sigma > 0$; its value to be chosen later. Also, by compactness, we fix a finite set $\mathcal{F}$ in the boundary of the unit ball of $\mathcal{L}(\mathbb{C}^N)$ such that the sets

$$K_\nu := \left\{ \nu' \in \mathcal{L}(\mathbb{C}^N) \setminus \{0\} : \left\| \frac{\nu'}{\|\nu'\|_{\mathcal{L}(\mathbb{C}^N)}} - \nu \right\|_{\mathcal{L}(\mathbb{C}^N)} \leq \sigma \right\} \quad (\nu \in \mathcal{F}) \tag{9.2}$$

cover $\mathcal{L}(\mathbb{C}^N) \setminus \{0\}$. By a standard argument using the John–Nirenberg Lemma, the following proposition implies Carleson's condition for the measure $\|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)} \frac{\mathrm{d}x\,\mathrm{d}t}{t}$, cf. e.g. [**28**, p. 906].

Proposition 9.3. *There exist $\beta, \beta' > 0$ such that for each $Q \in \Delta$ and for each $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $\|\nu\|_{\mathcal{L}(\mathbb{C}^N)} = 1$, there is a collection $\{Q_k\}_k \subseteq \Delta$ of pairwise disjoint subcubes of Q such that $|E_{Q,\nu}| > \beta|Q|$, where $E_{Q,\nu} := Q \setminus \bigcup\{Q_k\}_k$, and such that*

$$\iint_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma_t(x)\in K_\nu}} \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)} \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leq \beta'|Q|, \tag{9.3}$$

where $E^*_{Q,\nu} := R_Q \setminus \bigcup\{R_{Q_k}\}_k$.

Hence, our task is to prove Proposition 9.3. We closely follow [**7**, pp. 23–26]. For the proof keep $Q \in \Delta$ and $\nu \in \mathcal{L}(\mathbb{C}^N)$ with $\|\nu\|_{\mathcal{L}(\mathbb{C}^N)} = 1$ fixed and put $\tau := l(Q)$. Define $2Q := \{x \in \mathbb{R}^d : d(x,Q) \leq l(Q)\}$. Since the adjoint matrix $\nu^* \in \mathcal{L}(\mathbb{C}^N)$ has norm 1 there are $\omega, \hat{\omega} \in \mathbb{C}^N$ such that

$$|\omega| = |\hat{\omega}| = 1 \quad \text{and} \quad \omega = \nu^* \hat{\omega}. \tag{9.4}$$

We prepare for the usual $T(b)$ argument but similar to [**4**, Sec. 3.6] we use $\mathbf{1}_{2Q}\omega$ as a test function rather than some smoothened version of it. This leads to a simplification of the argument compared to [**6**, Sec. 4.4]. In the subsequent estimates a constant is called *admissible* if it neither depends on the quantities fixed above nor on σ its value still to be chosen. For $\varepsilon > 0$ we then put

$$\begin{aligned} f^\omega_{Q,\varepsilon} &:= (1 - \varepsilon\tau\mathrm{i}\Gamma R^B_{\varepsilon\tau})\mathbf{1}_{2Q}\omega \\ &= \mathbf{1}_{2Q}\omega - \varepsilon\tau\mathrm{i}\Gamma(1 + \varepsilon\tau\mathrm{i}\Pi_B)^{-1}\mathbf{1}_{2Q}\omega = (1 + \varepsilon\tau\mathrm{i}\Gamma^*_B)R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega \end{aligned} \tag{9.5}$$

and derive the following estimates.

Lemma 9.4. *There exist admissible constants $A_1, A_2, A_3 > 0$ such that for all $\varepsilon > 0$ it holds*

$$\|f^\omega_{Q,\varepsilon}\| \leq A_1 |Q|^{1/2},$$

$$\iint_{R_Q} |\Theta^B_t f^\omega_{Q,\varepsilon}(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} \leq \frac{A_2}{\varepsilon^2}|Q|,$$

$$\left| \fint_Q f^\omega_{Q,\varepsilon}(x)\,\mathrm{d}x - \omega \right|^2 \leq A_3(\varepsilon^\eta + \varepsilon^2).$$

Proof: Note $|2Q| \leq (1+C_1)^d l(Q)^d \lesssim |Q|$ by property (iv) of the dyadic decomposition. Hence, (5.2) and Lemma 7.2 yield

$$\|\Gamma R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega\| + \|\Gamma^*_B R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega\| = (\varepsilon\tau)^{-1}\|(1-R^B_{\varepsilon\tau})\mathbf{1}_{2Q}\omega\| \lesssim (\varepsilon\tau)^{-1}|Q|^{1/2} \tag{9.6}$$

with admissible implicit constants. From this, the first estimate follows. For the second estimate check by nilpotence of Γ and Γ^*_B that

$$\Theta^B_t f^\omega_{Q,\varepsilon} = t\Gamma^*_B P^B_t(1+\varepsilon\tau\mathrm{i}\Gamma^*_B)R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega = tP^B_t\Gamma^*_B R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega.$$

Recalling $l(Q) = \tau$, integration gives

$$\begin{aligned}\iint_{R_Q} |\Theta^B_t f^\omega_{Q,\varepsilon}(x)|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t} &\leq \int_0^\tau t\|P^B_t\Gamma^*_B R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega\|^2\,\mathrm{d}t \\ &\lesssim \int_0^\tau t\|\Gamma^*_B R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega\|^2\,\mathrm{d}t\end{aligned}$$

and (9.6) yields the claim. For the third estimate apply Lemma 8.5 with $\Upsilon = \Gamma$ to find

$$\begin{aligned}\left|\fint_Q f^\omega_{Q,\varepsilon}\,\mathrm{d}x - \omega\right|^2 &= \left|\fint_Q (f^\omega_{Q,\varepsilon} - \mathbf{1}_{2Q}\omega)\,\mathrm{d}x\right|^2 = (\varepsilon\tau)^2\left|\fint_Q \Gamma R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega\,\mathrm{d}x\right|^2 \\ &\lesssim \frac{(\varepsilon\tau)^2}{\tau^\eta}\left(\fint_Q |R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega|^2\,\mathrm{d}x\right)^{\eta/2}\left(\fint_Q |\Gamma R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega|^2\,\mathrm{d}x\right)^{1-\eta/2} \\ &\quad + (\varepsilon\tau)^2\fint_Q |R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega|^2\,\mathrm{d}x.\end{aligned}$$

By Lemma 7.2 and (9.6), keeping in mind $\tau \leq 1$, it follows

$$\lesssim \frac{(\varepsilon\tau)^2}{\tau^\eta}(\varepsilon\tau)^{\eta-2} + (\varepsilon\tau)^2 \leq \varepsilon^\eta + \varepsilon^2. \qquad \square$$

From now on keep $\varepsilon > 0$ fixed as the solution of $A_3(\varepsilon^\eta + \varepsilon^2) = \frac{1}{2}$ with A_3 as in the preceding lemma. We shall simply write f_Q^ω instead of $f_{Q,\varepsilon}^\omega$. Owing to Lemma 9.4 and $|\omega| = 1$ we find

$$2\operatorname{Re}\Big\langle \omega, \fint_Q f_Q^\omega \,\mathrm{d}x \Big\rangle = \Big| \fint_Q f_Q^\omega \,\mathrm{d}x \Big|^2 + |\omega|^2 - \Big| \fint_Q f_Q^\omega \,\mathrm{d}x - \omega \Big|^2 \geq \frac{1}{2}. \tag{9.7}$$

The following lemma now follows literally as in [**7**, Lem. 5.11].

Lemma 9.5. *There exist admissible constants $\beta, \rho > 0$ and a collection $\{Q_k\}_k \subseteq \Delta$ of dyadic subcubes of Q such that $|E_{Q,\nu}| > \beta|Q|$ where $E_{Q,\nu} := Q \setminus \bigcup\{Q_k\}_k$, and such that*

$$\operatorname{Re}\Big\langle \omega, \fint_{Q'} f_Q^\omega(x) \,\mathrm{d}x \Big\rangle \geq \rho \quad \textit{and} \quad \fint_{Q'} |f_Q^\omega(x)| \,\mathrm{d}x \leq \frac{1}{\rho} \tag{9.8}$$

for all dyadic subcubes $Q' \in \Delta$ of Q which satisfy $R_{Q'} \cap E_{Q,\nu}^* \neq \emptyset$, where $E_{Q,\nu}^* := R_Q \setminus \bigcup\{R_{Q_k}\}_k$.

Let ρ, $\{Q_k\}_k$, $E_{Q,\nu}$, and $E_{Q,\nu}^*$ be as provided by Lemma 9.5. We shall prove the estimates in Proposition 9.3 for these choices. Eventually, we fix the value of $\sigma > 0$ determining the size of the 'pizza slices' K_ν in (9.2) as $\sigma := \frac{\rho^2}{2}$. For the next lemma recall that $\mathfrak{N}$ is the exceptional set defined in Remark 7.5.

Lemma 9.6. *Suppose $(x,t) \in E_{Q,\nu}^*$ is such that $x \notin \mathfrak{N}$ and $\gamma_t(x) \in K_\nu$. Then*

$$\big|\gamma_t(x)(A_t f_Q^\omega(x))\big| \geq \frac{\rho}{2} \|\gamma_t(x)\|_{\mathcal{L}(\mathbb{C}^N)}.$$

Proof: Due to $x \notin \mathfrak{N}$ there exists a unique $Q' \in \Delta_t$ that contains x. Hence $R_{Q'} \cap E_{Q,\nu}^* \neq \emptyset$. Since by definition $A_t f_Q^\omega(x) = \fint_{Q'} f_Q^\omega(y)\,\mathrm{d}y$, the previous lemma and the relations between ν, ω, and $\hat{\omega}$, cf. (9.4), yield

$$\big|\nu(A_t f_Q^\omega(x))\big| \geq \operatorname{Re}\big\langle \hat{\omega}, \nu(A_t f_Q^\omega(x))\big\rangle = \operatorname{Re}\big\langle \omega, A_t f_Q^\omega(x)\big\rangle \geq \rho$$

and furthermore – due to $\gamma_t(x) \in K_\nu$ – also

$$\begin{aligned} \Big| \frac{\gamma_t(x)}{\|\gamma_t(x)\|_{\mathcal{L}(\mathbb{C}^N)}}(A_t f_Q^\omega(x)) \Big| &\geq |\nu(A_t f_Q^\omega(x))| \\ &\qquad - |A_t f_Q^\omega(x)| \Big\| \frac{\gamma_t(x)}{\|\gamma_t(x)\|_{\mathcal{L}(\mathbb{C}^N)}} - \nu \Big\|_{\mathcal{L}(\mathbb{C}^N)} \\ &\geq \frac{\rho}{2}. \end{aligned}$$

□

Finally we complete the proof of Proposition 9.3.

Proof of Proposition 9.3: It remains to establish (9.3). The crucial observation is that Lemma 9.6 allows to reintroduce the dyadic averaging operator:

$$\iint_{\substack{(x,t)\in E^*_{Q,\nu}\\ \gamma_t(x)\in K_\nu}} \|\gamma_t(x)\|^2_{\mathcal{L}(\mathbb{C}^N)} \frac{\mathrm{d}x\,\mathrm{d}t}{t} \lesssim \iint_{R_Q} |\gamma_t(x)(A_t f^\omega_Q(x))|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}$$
$$\leq 2\iint_{R_Q} |\Theta^B_t f^\omega_Q|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}$$
$$+ 2\iint_{R_Q} |(\Theta^B_t - \gamma_t A_t) f^\omega_Q|^2 \frac{\mathrm{d}x\,\mathrm{d}t}{t}.$$

Lemma 9.4 bounds the first term on the right-hand side by $2A_2\varepsilon^{-2}|Q|$. To handle the second one put $u := \varepsilon\tau\mathrm{i}\Gamma R^B_{\varepsilon\tau}\mathbf{1}_{2Q}\omega \in \mathcal{R}(\Gamma)$. Then due to $f^\omega_Q = \mathbf{1}_{2Q}\omega - u$, see (9.5), it remains to show

$$(9.9)\quad \int_0^\tau \|\mathbf{1}_Q(\Theta^B_t - \gamma_t A_t)\mathbf{1}_{2Q}\omega\|^2 \frac{\mathrm{d}t}{t} + \int_0^\tau \|\mathbf{1}_Q(\Theta^B_t - \gamma_t A_t)u\|^2 \frac{\mathrm{d}t}{t} \lesssim |Q|.$$

For the first term on the left-hand side note $A_t\mathbf{1}_{2Q}\omega(x) = \omega$ for all $x \in Q$ and $t \in (0,\tau)$ so that by definition of the principal part

$$\|\mathbf{1}_Q(\Theta^B_t\mathbf{1}_{2Q}\omega - \gamma_t A_t\mathbf{1}_{2Q}\omega)\| \leq \sum_{R\in\Delta_\tau} \|\mathbf{1}_Q\Theta^B_t(\mathbf{1}_{R\cap(\mathbb{R}^d\setminus 2Q)}\omega)\|.$$

Proposition 7.9 gives a bound by

$$\lesssim \sum_{R\in\Delta_\tau} \left\langle \frac{\mathrm{d}(Q, R\cap(\mathbb{R}^d\setminus 2Q))}{t} \right\rangle^{-(d+2)} \|\mathbf{1}_{R\cap(\mathbb{R}^d\setminus 2Q)}\omega\|.$$

Since dyadic cubes of the same step size are comparable in measure, we get for each $R \in \Delta_\tau$ that $\|\mathbf{1}_{(\mathbb{R}^d\setminus 2Q)\cap R}\omega\| \leq |R|^{1/2} \simeq |Q|^{1/2}$. Now, the latter sum is under control by the second part of Lemma 7.10 with $l = 1$. Altogether,

$$\|\mathbf{1}_Q(\Theta^B_t\mathbf{1}_{2Q}\omega - \gamma_t A_t\mathbf{1}_{2Q}\omega)\| \lesssim |Q|^{1/2}\frac{t^{d+2}}{\tau^{d+2}}.$$

Going back to (9.9), this gives the required bound for the first term. The second one is bounded by

$$\int_0^1 \|\Theta^B_t(1-P_t)u\|^2 + \|(\Theta^B_t - \gamma_t A_t)P_t u\|^2 + \|\gamma_t A_t(P_t - 1)u\|^2 \frac{\mathrm{d}t}{t}$$

and these three terms have already been taken care of in (7.1) and Propositions 8.3 and 8.6 bounding them by a multiple of $\|u\|^2$. However, in view of (9.6) we find $\|u\|^2 \lesssim |Q|$. This completes the proof of Proposition 9.3. □

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