A MONOTONICITY FORMULA FOR MINIMAL SETS WITH A SLIDING BOUNDARY CONDITION

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Abstract: We prove a monotonicity formula for minimal or almost minimal sets for the Hausdorff measure $\mathcal{H}^d$, subject to a sliding boundary constraint where competitors for E are obtained by deforming E by a one-parameter family of functions φ_t such that $\varphi_t(x) \in L$ when $x \in E$ lies on the boundary L. In the simple case when L is an affine subspace of dimension $d-1$, the monotone or almost monotone functional is given by $F(r) = r^{-d}\mathcal{H}^d(E \cap B(x,r)) + r^{-d}\mathcal{H}^d(S \cap B(x,r))$, where x is any point of E (not necessarily on L) and S is the shade of L with a light at x. We then use this, the description of the case when F is constant, and a limiting argument, to give a rough description of E near L in two simple cases.

Contents

1. Introduction

The central point of this paper is a monotonicity formula that is valid for minimal sets with a sliding boundary condition, as defined in [**D5**], for balls that are not necessarily centered on the boundary set and also when the boundary is not a cone with the same center as the balls.

In this introduction, we restrict to the special case of minimal sets of dimension d in some open set $U \subset \mathbb{R}^n$, and when the boundary condition is given by an affine plane L of dimension at most $d - 1$. The main monotonicity result of this paper is also valid in more general situations, but the statements are more complicated and the author is not sure that the extra generality will be used.

Let us say what we mean by sliding minimal sets in this simpler context. We only consider sets E that are closed in U, and have a locally finite Hausdorff measure, i.e., for which

$$(1.1) \qquad \mathcal{H}^d(E \cap \overline{B}) < +\infty \text{ for every closed ball } \overline{B} \subset U.$$

See for instance [**Fe**] or [**Ma**] for the definition of the Hausdorff measure; recall that for subsets of smooth d-dimensional surfaces, $\mathcal{H}^d$ is the same as the surface measure.

We compare E with images $\varphi_1(E)$, coming from one parameter families $\{\varphi_t\}$, $t \in [0,1]$, of continuous functions defined on E, such that

$$(1.2) \qquad (x,t) \to \varphi_t(x) \colon E \times [0,1] \to U \text{ is continuous,}$$

$$(1.3) \qquad \varphi_0(x) = x \text{ for } x \in E,$$

$$(1.4) \qquad \varphi_1 \text{ is Lipschitz on } E,$$

$$(1.5) \text{ if we set } W_t = \{x \in E;\ \varphi_t(x) \neq x\} \text{ and } \widehat{W} = \bigcup_{0 \leq t \leq 1} W_t \cup \varphi_t(W_t),$$

then $\widehat{W}$ is contained in a compact subset of U, and finally (the sliding boundary condition)

$$(1.6) \qquad \varphi_t(x) \in L \text{ for } 0 \leq t \leq 1 \text{ when } x \in E \cap L.$$

Such families $\{\varphi_t\}$ will be called *acceptable deformations*; note that the notion depends on E, U, and L.

Definition 1.1. We say that E (a closed set in U) is a *sliding minimal set in U, with boundary condition given by L*, or in short that $E \in \mathrm{SM}(U, L)$, when (1.1) holds and

$$(1.7) \qquad \mathcal{H}^d(E \cap \widehat{W}) \leq \mathcal{H}^d(\varphi_1(E) \cap \widehat{W})$$

for every acceptable deformation $\{\varphi_t\}$, and where $\widehat{W}$ is as in (1.5).

Thus, when we deform E to get $F = \varphi_1(E)$, we require points of L to stay on L. We do not require the φ_t to be injective. We added the Lipschitz constraint in (1.4) mostly by tradition, but our results would hold without it, because the resulting class of minimizers would be smaller. In (1.7) we took the intersections with $\widehat{W}$ because maybe our sets have infinite measure; otherwise we would just have said $\mathcal{H}^d(E) \leq \mathcal{H}^d(\varphi_1(E))$.

This definition and the more general ones that will be given in Section 2 are modifications of Almgren's initial definition of "restricted sets" (see [**A3**]), suited to fit a natural definition of Plateau problems. See [**D5**] for motivations. Maybe we should mention that some of the sets that arise from some other natural minimization problems are also sliding minimal (or almost minimal) sets. This is the case, for instance, of the sets that minimize $\mathcal{H}^d$ under the Reifenberg homology boundary conditions (see [**Re**], [**A2**], and more recently [**Fa**]), or of the support of size-minimizing currents. See for instance Section 7 of [**D4**] for a discussion and a proof of this fact.

We are interested in the regularity properties of sliding minimal sets near the boundary L, and a good monotonicity formula will clearly help. In [**D5**], it was checked that (among other things) if L is a cone (not necessarily an affine subspace) centered at the origin, and satisfies some mild regularity constraints, and E is a sliding minimizer as above, then the density

$$(1.8) \qquad \theta_0(r) = r^{-d} \mathcal{H}^d(E \cap B(0, r)) \text{ is nondecreasing on } (0, R_0)$$

when R_0 is such that $B(0, R_0) \subset U$. Here and below, $B(x, r)$ denotes the open Euclidean ball of radius r centered at x. This is a rather easy extension of a very standard result that applies to minimal sets (without boundary conditions), and even more general objects. As usual, the monotonicity of θ_0 is proved by comparing E with a cone, and the sliding boundary condition cooperates well with this when L is a cone.

Various consequences can be derived from this, but it will be good to have monotonicity properties for balls that are not necessarily centered on L, and this is the main point of this paper.

An obvious difficulty with the extension of (1.8) when L is not a cone is that we cannot use the same density. For instance, if $L = \big\{(x, y, z) \in \mathbb{R}^3;\ x = 1 \text{ and } z = 0\big\}$, the half plane $E = \big\{(x, y, z) \in \mathbb{R}^3;\ x \leq 1 \text{ and } z = 0\big\}$ is a sliding minimal set of dimension 2 in $\mathbb{R}^3$, with boundary condition given by L. The function θ_0 of (1.8) is constant on $[0, 1]$, and then decreasing. We shall add to E a missing piece, so that we get a nondecreasing function.

The case when E is a half space bounded by L suggests that we symmetrize E, but this is not what we are going to do. Instead, we shall add to $\mathcal{H}^d(E)$ the measure of the shade of L, seen from the origin. The shade in question is

$$(1.9) \qquad S = \big\{ y \in \mathbb{R}^n;\ \lambda y \in L \text{ for some } \lambda \in [0,1] \big\}$$

and the functional that we want to consider is

$$(1.10) \qquad F(r) = r^{-d}\big[\mathcal{H}^d(E \cap B(0,r)) + \mathcal{H}^d(S \cap B(0,r))\big].$$

Our basic theorem is the following.

Theorem 1.2. *Let $U \in \mathbb{R}^n$ be open, L be an affine space of dimension $m \leq d-1$, and E be a sliding minimal set of dimension d in U, with boundary condition given by L. Then the function F defined by (1.10) is nondecreasing on $[0, \mathrm{dist}(0, \mathbb{R}^n \setminus U))$.*

See Theorem 7.1 for a generalization, where more general sets L are allowed. For these more complicated sets L, we also change the formula for the added term in L; the shade only works well when the half lines starting from the origin do not meet L twice.

In the case mentioned above when L is a line, E is a half plane bounded by L, and $0 \in E \setminus L$, our formula is exact, in the sense that F is constant (the shade exactly compensates the missing half plane). The same thing is true when E is a truncated $\mathbb{Y}$-set centered at the origin, such that L is contained in one of the three branches of the $\mathbb{Y}$-set, and the truncation precisely consists in removing the interior of the shade S. See near (1.28) for the definition of the $\mathbb{Y}$-sets. Of course this is interesting because we believe that the truncated $\mathbb{Y}$-set is a sliding minimal set.

In contrast, the defect of the general monotonicity formula given in Theorem 7.1 below is that it is possibly not exact on minimal sets other than those above.

In the two applications that we give below, we shall see that knowing examples where the formula is exact helps a lot; otherwise, it still gives some information, but probably not precise enough. One can thus object that the only cases where we know that the formula is exact are the two examples given above (truncated planes and $\mathbb{Y}$-sets), plus their products by orthogonal $(d-2)$-planes. This observation is right, but should probably be tempered by the fact that we do not know so many minimal cones, and there are many places, in particular when $d = 2$, where a sliding minimal set looks like one of the examples above.

Notice also that although we do not exclude the case when $0 \in L$, Theorem 1.2 is not new in this case; we get that the shade S is reduced

to L, so F coincides with θ_0 and we rediscover (1.8). Similarly, when $r <$ dist$(0, L)$, $S \cap B(0, r)$ is empty, and we rediscover the more classical fact that θ_0 is nondecreasing for minimal sets (with no boundary condition), see for instance [**D2**], and which is true in much more general contexts (for instance [**All**]).

When L is less than $(d-1)$-dimensional, $H^d(S) = 0$ and we get again that θ_0 is monotone, but this is not an impressive result: the sliding condition (1.6), applied on a set L of dimension $< d - 1$, does not seem coercive enough to allow many sliding minimal sets that are not equal $\mathcal{H}^d$-almost everywhere to a minimal set.

A useful monotonicity formula should probably come with a toolbox, so we shall try to give a few connected tools. The description of the next results will be slightly simpler with the notion of coral sets. Let E be closed in U; we denote by E^* the closed support of the restriction of $\mathcal{H}^d$ to E. That is,

$$(1.11) \qquad E^* = \big\{ x \in E; \; \mathcal{H}^d(E \cap B(x, r)) > 0 \text{ for every } r > 0 \big\}.$$

Some times we call E^* the core of E, and we shall say that E is *coral* when $E = E^*$. It follows easily from the definition of E^* that $\mathcal{H}^d(E \setminus E^*) = 0$, and since $E^* \in \mathrm{SM}(U, L)$ when $E^* \in \mathrm{SM}(U, L)$ (see the discussion in Section 2, just below (2.8)), we may restrict our attention to coral minimal sets. The advantage is that coral sets are a little cleaner and easier to describe: from the description above, we see that a general minimal set is the union of a coral minimal set and a set of $\mathcal{H}^d$ measure 0, and this negligible set may be ugly. In fact, if we start from any (coral if we want) minimal set and add to it any $\mathcal{H}^d$-null set, it follows from the definitions that the resulting set is still minimal as long as it is closed.

Our first complement to Theorem 1.2 deals with the case when our functional F is constant on an interval.

Theorem 1.3. *Let $U \subset \mathbb{R}^n$, L, and E be as in Theorem 1.2, and suppose in addition that E is coral, and $0 < R_0 < R_1$ are such that $B(0, R_1) \subset U$ and F is constant on the interval (R_0, R_1). Set $A = B(0, R_0) \setminus B(0, R_1)$. Then*

$$(1.12) \qquad \qquad \mathcal{H}^d(A \cap E \cap S) = 0$$

and, if X denotes the cone over $A \cap E$, i.e.,

$$(1.13) \qquad \qquad X = \big\{ \lambda x; \; \lambda \geq 0 \text{ and } x \in A \cap E \big\},$$

then

$$(1.14) \qquad \qquad A \cap X \setminus S \subset E.$$

If in addition $R_0 < \text{dist}(0, L)$, then X is a coral minimal set in $\mathbb{R}^n$ (with no boundary condition), and

$$(1.15) \qquad\qquad \mathcal{H}^d(S \cap B(0, R_1) \setminus X) = 0.$$

Notice that $E \cap A \subset X$ by definition, and then (1.12) and (1.14) say that E and $X \setminus S$ coincide in A, modulo a set of vanishing $\mathcal{H}^d$-measure.

The last part is interesting, because with some luck it will allow us to identify X, and then $A \cap E$.

In the present case when L is an affine subspace, (1.15) says that if its dimension is $d-1$ and $B(0, R_1)$ meets L, then in fact X contains (the cone over) $L \cap B(0, R_1)$. But we stated the result like we did because it stays true for more general sets L. See Theorem 8.1.

The additional information (1.15) is not necessarily good news, because although it gives some extra information when the assumptions of the theorem are satisfied, it also says that the monotonicity formula is not exact when $R_0 < \text{dist}(0, L)$ and E coincides near $\partial B(0, R_0)$ with a minimal cone that does not satisfy (1.15). For instance, if $n = 3$, $d = 2$, L is a line that does not contain 0, and E is a plane through the origin that does not contain L, then $F(r)$ is strictly increasing for $r > \text{dist}(0, L)$.

We will also be interested in almost monotonicity results for almost minimal sets. We just give a simple statement here, and refer to Theorem 7.1 for a more general result, but also more complicated to state.

Even this way we need some definitions. Almost minimality will be defined in terms of some gauge function $h \colon (0, +\infty) \to [0, +\infty]$ (we allow $h(r) = 0$, which corresponds to minimal sets, and $h(r) = +\infty$, which is a brutal way of saying that we have no information at that scale). We always restrict to nondecreasing functions h, with

$$(1.16) \qquad\qquad \lim_{r \to 0} h(r) = 0,$$

and for our almost monotonicity result we shall assume that h satisfies the Dini condition

$$(1.17) \qquad\qquad \int_0^{r_0} h(r) \frac{dr}{r} < +\infty \text{ for some } r_0 > 0.$$

Definition 1.4. Let E be a closed set in U, such that (1.1) holds, L be a closed set in U, and let $h \colon (0, +\infty) \to [0, +\infty]$ be a nondecreasing function such that (1.16) holds. We say that E is a *sliding almost minimal set* with boundary condition defined by L and gauge function h, and in

short we write $E \in \mathrm{SAM}(U, L, h)$, when

$$(1.18) \qquad \mathcal{H}^d(E \cap \widehat{W}) \leq \mathcal{H}^d(\varphi_1(E) \cap \widehat{W}) + r^d h(r)$$

whenever $\{\varphi_t\}$, $0 \leq t \leq 1$, is an acceptable deformation such that the set $\widehat{W}$ of (1.5) is contained in a ball of radius r.

Other, slightly different, notions exist and will be treated the same way. See Section 2. Here is the generalization of Theorem 1.2 to the classes $\mathrm{SAM}(U, L, h)$.

Theorem 1.5. *There exist constants $a > 0$ and $\tau > 0$, which depend only on n and d, with the following property. Let U and L be as above, and let $E \in \mathrm{SAM}(U, L, h)$ be a sliding almost minimal set, for some gauge function h such that (1.17) holds. Suppose that*

$$(1.19) \qquad 0 \in E^*, \ B(0, R_1) \subset U, \ and \ h(R_1) < \tau,$$

and set

$$(1.20) \qquad A(r) = \int_0^r h(t) \frac{dt}{t} \ for \ 0 < r \leq R_1;$$

then

$$(1.21) \qquad F(r) \, e^{aA(r)} \ is \ a \ nondecreasing \ function \ of \ r \in (0, R_1).$$

See Theorem 7.1 for a more general version, where we allow more general boundary sets L, and also slightly different definitions of almost minimality.

Notice that because of (1.17), $e^{aA(r)}$ tends to 1 when r tends to 0, so (1.21) can really be interpreted as an almost monotonicity property. In particular, we get that $\lim_{r \to 0} F(r)$ exists and is finite; it is positive because E^* is locally Ahlfors regular (see Section 2).

See for instance Proposition 5.24 on p. 101 of [**D2**] for the analogue of Theorem 1.5 when $L = \emptyset$, and Theorem 28.15 in [**D5**] for the case of sliding almost minimal sets, but with the function θ_0 and balls centered on L.

The second important element of our toolbox says that when E is as above and $F(r)$ varies very little on an interval, E looks a lot like a sliding minimal set E_0 for which F is constant on a slightly smaller interval. Hence, in many cases, Theorem 1.3 says that E_0 and E look like truncated minimal cones. Here is the simplest result that corresponds to an interval $[0, r_1)$.

Theorem 1.6. *For each choice of L and $r_1 > 0$ and each small $\tau > 0$, we can find $\varepsilon > 0$, which depends only on τ, n, d, L, and r_1, with the following property. Let $E \in \mathrm{SAM}(U, L, h)$ be a coral sliding almost minimal set in the open set U, with boundary condition defined by L and some nondecreasing gauge function h. Suppose that*

$$(1.22) \qquad\qquad B(0, r_1) \subset U \ \text{and} \ h(r_1) < \varepsilon,$$

and

$$(1.23) \qquad\qquad F(r_1) \le \varepsilon + \inf_{0 < r < 10^{-3} r_1} F(r).$$

Then there is a coral set $E_0 \in \mathrm{SM}(B(0, r_1), L)$ (i.e., E_0 is sliding minimal in $B(0, r_1)$, with boundary condition defined by L), such that

$$(1.24) \qquad \text{the analogue of F for the set E_0 is constant on } (0, r_1),$$

$$(1.25) \qquad \mathrm{dist}(y, E_0) \le \tau r_1 \ \text{for } y \in E \cap B(0, (1 - \tau) r_1),$$

$$(1.26) \qquad \mathrm{dist}(y, E) \le \tau r_1 \ \text{for } y \in E_0 \cap B(0, (1 - \tau) r_1),$$

and

$$(1.27) \quad \left| \mathcal{H}^d(E \cap B(y, t)) - \mathcal{H}^d(E_0 \cap B(y, t)) \right| \le \tau r_1^d$$
$$\text{for all } y \in \mathbb{R}^n \text{ and } t > 0 \text{ such that } B(y, t) \subset B(0, (1 - \tau) r_1).$$

Notice that we do not need to assume (1.17) here. In many cases, in particular if (1.17) holds, we know that $\lim_{r \to 0} F(r)$ exists, and we could just require that $F(r_1) \le \varepsilon + \lim_{r \to 0} F(r)$ instead of (1.23). Finally observe that (1.27) does not say much when t is much smaller than r, because the error term in (1.27) is τr_1^d, not τt^d.

See Theorem 9.1 for a generalization of this to more general boundary sets L, and Theorem 9.7 for the analogue of Theorems 1.6 and 9.1 when F is only assumed to be nearly constant on some interval (r_0, r_1), $r_0 > 0$.

Once we have Theorem 9.1, we can use Theorem 1.3 or Theorem 8.1 to get more information on E_0; we do not do this in this introduction because the amount of information that we get depends on L, in particular through $\mathrm{dist}(0, L)$ and r_1.

Theorems 1.6 and 9.1 generalize Proposition 7.24 in [**D2**] (a version without boundary set L) and Proposition 30.19 in [**D5**] (a version with the density θ_0 and balls centered on L). All these results are fairly easy to obtain by compactness, because we have results that say that limits of almost minimal sets, with a fixed gauge function h, are almost minimal with the same gauge function.

It is unpleasant that, in both Theorems 1.6 and 9.1, ε depends on the specific choice of L and r_1. In the present case where we assume that L is an affine subspace, the dependence is in terms of $r_1^{-1} \operatorname{dist}(0, L)$, and we can try to eliminate this distance from the compactness argument that leads to Theorem 1.6. Still we get three different regimes. When $r_1 < \operatorname{dist}(0, L)$, the boundary condition does not play a role and we can use results from [**D2**]. When $\operatorname{dist}(0, L) \ll r_1$, we shall find it more convenient in practice to reduce to the case when $0 \in L$, and then use results from [**D5**]. So we concentrate on the intermediate case when $\operatorname{dist}(0, L) < r_1 < C \operatorname{dist}(0, L)$, with C large, and then we get an approximation of E by a set $E_0 = \overline{X_0 \setminus S}$, where X_0 is a minimal cone (without boundary condition) and S is the shade of an affine subspace. This is Corollary 9.3, where we also include the case when L is not exactly an affine space, but is very close to one (in a bilipschitz way), to allow subvarieties as well.

We shall give two rather elementary applications of the results above. Both concern the behavior of a sliding almost minimal set E, with a boundary condition defined by a $(d-1)$-dimensional space $L \subset \mathbb{R}^n$, and in a small ball where E looks simple enough. In both cases, we shall not try to give an optimal result, but instead explain how the results of the first part can be useful. A motivation for this type of results is that they provide first steps in the study of specific boundary behavior of minimal sets, subject to a Plateau condition. See Figure 13.9.3 on p. 134 of [**Mo**], or Figure 5.3 in [**LM**], for a list of conjectured behaviors for a 2-dimensional minimal set bounded by a curve.

We start with some notation which is common to the two results. We are given a $(d-1)$-dimensional vector space L in $\mathbb{R}^n$, and we shall use some classes of minimal cones. First we denote by $\mathbb{P}_0 = \mathbb{P}_0(n, d)$ the set of d-planes through the origin, and by $\mathbb{P} = \mathbb{P}(n, d)$ the set of all affine d-planes.

Next, $\mathbb{H}(L)$ denotes the class of closed half-d-planes bounded by L. That is, to get $H \in \mathbb{H}(L)$ we pick a d-plane P that contains L, select one of the two connected components of $P \setminus L$, and let H be the closure of this component.

For the sake of Corollary 1.8, we also denote by $\mathbb{V}(L)$ the set of *cones of type* $\mathbb{V}$ bounded by L. These are the unions $V = H_1 \cup H_2$, where H_1 and H_2 are elements of $\mathbb{H}(L)$ that make an angle at least $2\pi/3$ along L. This last means that if v_i denotes the unit vector which lies in H_i and is orthogonal to L, then $\langle v_1, v_2 \rangle \leq -1/2$. We add this angle constraint because it is easy to see that if it fails, then V is not a sliding minimal set with boundary condition given by L. We believe that the elements of $\mathbb{V}(L)$ are sliding minimal sets, but we shall not check this here.

Let us also define the set $\mathbb{Y}_0 = \mathbb{Y}_0(n,d)$ of *minimal cones of type* $\mathbb{Y}$ that are centered at 0. We first say that $Y \in \mathbb{Y}_0(n,1)$ when Y is the union of three half lines that start from 0, are contained in some plane $P = P(Y)$, and make $2\pi/3$ angles with each other at the origin. For $d > 1$, $\mathbb{Y}_0(n,d)$ is the set of products $Y \times W$, where $Y \in \mathbb{Y}_0(n,1)$ and W is a vector space of dimension $(d-1)$ that is orthogonal to the 2-plane $P(Y)$. Finally, we set

$$(1.28) \qquad \mathbb{Y}_x(n,d) = \big\{ x + Z;\ Z \in \mathbb{Y}_0(n,d) \big\}$$

(the cones of type $\mathbb{Y}$ centered at x). By $\mathbb{Y}$-set, we usually mean an element of any $\mathbb{Y}_x(n,d)$.

We are almost ready for our first application, which tries to say that if E is a coral sliding almost minimal set in $B(0,3)$, with a sufficiently small gauge function, and if it is close enough in $B(0,3)$ to a half plane $H \in \mathbb{H}(L)$, then E is Hölder-equivalent to H in $B(0,1)$. The initial distance from E to H will be expressed in terms of the following very useful, dimensionless local version of the Hausdorff distance: when E and F are two closed sets, we set

$$(1.29) \quad d_{x,r}(E,F) = \frac{1}{r} \sup_{y \in E \cap B(x,r)} \operatorname{dist}(y,F) + \frac{1}{r} \sup_{y \in F \cap B(x,r)} \operatorname{dist}(y,E),$$

where by convention we set $\sup_{y \in E \cap B(x,r)} \operatorname{dist}(y,F) = 0$ when $E \cap B(x,r) = \emptyset$, for instance. As we shall see, we express the conclusion in terms of distances $d_{x,r}$ and Reifenberg approximation condition; we shall explain why after the statement.

Corollary 1.7. *For each small $\tau > 0$ we can find $\varepsilon > 0$, which depend only on τ, n, and d, with the following property. Let L be a vector $(d-1)$-plane and let E be a coral sliding almost minimal set in $B(0,3)$, with boundary condition given by L and a gauge function h such that (1.17) holds. Suppose that*

$$(1.30) \qquad \int_0^3 \frac{h(t)\,dt}{t} < \varepsilon$$

and

$$(1.31) \qquad d_{0,3}(E,H) \leq \varepsilon \text{ for some } H \in \mathbb{H}(L).$$

Then

$$(1.32) \qquad L \cap B(0,2) \subset E,$$

and for each $x \in E \cap B(0,2)$ and $0 < r < 1/2$, we can find a set $Z = Z(x,r)$ with the following properties:

(1.33)   *if $r \leq \text{dist}(x,L)/2$, then $Z(x,r)$ is a plane through x;*

(1.34)   *if $\text{dist}(x,L)/2 < r \leq \tau^{-1}\text{dist}(x,L)$,*

then $Z(x,r)$ is the element of $\mathbb{H}(L)$ that contains x;

(1.35)   *if $r > \tau^{-1}\text{dist}(x,L)$, then $Z(x,r) \in \mathbb{H}(L)$;*

(1.36) $d_{x,r}(E,Z) \leq \tau;$

and

(1.37) $\left| \mathcal{H}^d(E \cap B(y,t)) - \mathcal{H}^d(Z \cap B(y,t)) \right| \leq \tau r^d$

for $y \in \mathbb{R}^n$ and $t > 0$ such that $B(y,t) \subset B(0,(1-\tau)r)$.

Recall that coral is defined near (1.11). We decided to work in $B(0,3)$ to simplify the statement, but by translation and dilation invariance we could easily get a statement for balls $B(x_0, r_0)$, with $x_0 \in L$ and $\int_0^{3r_0} h(t)\frac{dt}{t} < \varepsilon$ in (1.30).

Notice that we do not assume that $0 \in E$ (or that $E \cap L \neq \emptyset$); this comes as a conclusion.

We claim that the conclusion of Corollary 1.7 (even without (1.37)) probably implies that E is Hölder-equivalent to H in $B(0,1)$, as in the topological disk theorem of Reifenberg [**Re**]. More precisely, we claim that for each $\eta > 0$, we should find $\tau > 0$ such that if the conclusion of Corollary 1.7 holds (without (1.37)), then there is a Hölder homeomorphism f of $\mathbb{R}^n$ such that

(1.38) $f(x) = x$ for $x \in L$ and for $x \in \mathbb{R}^n \setminus B(0,2)$,

(1.39) $|f(x) - x| \leq \eta$ for $x \in \mathbb{R}^n$,

(1.40) $(1-\eta)|x{-}y|^{1+\eta} \leq |f(x){-}f(y)| \leq (1+\eta)|x{-}y|^{1-\eta}$ for $x,y \in B(0,3)$,

and

(1.41) $E \cap B(0,1) \subset f(H \cap B(0,1+\eta)) \subset E$.

We shall not prove this claim here, and unfortunately do not know of a proof in the literature. However, a small modification (mostly a simplification) of the argument in [**DDT**] should give this, and since this is not central to the present paper, we shall leave the claim with no proof for the moment, and content ourselves with the conclusions of Corollary 1.7.

The corollary probably stays true when L is a flat enough smooth submanifold of dimension $d-1$, but we shall not try to pursue this here; see Remark 11.4 though.

The local Hölder regularity that we claim here is probably far from optimal; we could expect slightly better than C^1 regularity, the proof given below obviously does not give this, but at least it is fairly simple. Here the important issue is near L; far from L, we can deduce additional regularity on E from its flatness (given by Corollary 1.7) and regularity results from [**A3**] or [**All**], but at this point we cannot exclude that E turns around L infinitely many times at some places.

Corollary 1.7 should be the simplest from a series of results that give a local description of E when we know that E looks like a given minimal cone in a small ball centered on L. The next case would be when E is close to a d plane through L, or a set of $\mathbb{V}(L)$. But new ingredients seem to be needed for this; the author will try to investigate the special case when $d = 2$, for which we have more control on the geometry and the list of minimal cones.

For our second application, we work on $B(0,3)$ again to simplify the statement, assume that the coral almost minimal set $E \in$ SAM$(B(0,3), L, h)$ is very close to a cone of type $\mathbb{V}$ in $B(0,3)$ and that the gauge function h is small enough, and get some information on the behavior of E near its singular points (if they exist). Since we don't know whether all the plain minimal cones (i.e., with no boundary condition) of dimension d in $\mathbb{R}^n$ that have a density at most $3\omega_d/2$ are necessarily cones of type $\mathbb{Y}$, we restrict to dimensions n and d for which we know that this is the case. Recall that $\omega_d = \mathcal{H}^d(\mathbb{R}^d \cap B(0,1))$ is the density of a d-plane.

Corollary 1.8. *Suppose that $d = 2$, or that $d = 3$ and $n = 4$. For each choice of $N > 1$ and $\tau > 0$ we can find $\varepsilon > 0$, which depends only on N, τ, and n, with the following property. Let L be a vector $(d-1)$-plane, and let E be a coral sliding almost minimal set in $B(0,3)$, with sliding condition defined by L and a gauge function h such that (1.17) holds. Suppose that*

$$(1.42) \qquad\qquad \int_0^3 \frac{h(t)\,dt}{t} < \varepsilon$$

and that we can find $V \in \mathbb{V}(L)$ such that

$$(1.43) \qquad\qquad d_{0,3r}(E, V) \leq \varepsilon.$$

Then for each $x \in E \cap B(0,1) \setminus L$,

$$(1.44) \qquad \theta_x(0) := \lim_{r \to 0} r^{-d} \mathcal{H}^d(E \cap B(x,r)) \leq \frac{3\omega_d}{2} + \tau.$$

In addition, let $x \in E \cap B(0,1) \setminus L$ be such that $\theta_x(0) > \omega_d$, and set $\delta(x) = \mathrm{dist}(x,L)$. Then $\delta(x) \leq N^{-1}$, and if Y denotes the cone of $\mathbb{Y}_x(n,d)$ (see the definition (1.28)) that contains L, and

(1.45) $W = \overline{Y \setminus S_x}$, *with* $S_x = \{y \in \mathbb{R}^n;\ x + \lambda(y-x) \in L \text{ for some } \lambda \in [0,1]\}$,

then

$$(1.46) \qquad\qquad d_{x,2N\delta(x)}(W,E) \leq \tau$$

and

$$(1.47) \quad \left| \mathcal{H}^d(E \cap B(y,t)) - \mathcal{H}^d(W \cap B(y,t)) \right| \leq \tau \delta(x)^d$$
$$\text{for } y \in \mathbb{R}^n \text{ and } t > 0 \text{ such that } B(y,t) \subset B(x, 2N\delta(x)).$$

The existence of the limit $\theta_x(0)$ in (1.44) is classical; for instance, it follows from (1.42) and Proposition 5.24 in [**D2**]. Our condition that $\theta_x(0) > \omega_d$ is another way to say that x is a singular point of E, and (because we may take τ arbitrarily small) (1.44) just forbids certain types of singularities, such as points of type $\mathbb{T}$ when $d = 2$ and $n = 3$.

Our assumption on the dimensions is a little strange and probably too conservative; we shall prove the result as soon as n and d satisfy assumption (10.9), which says that all the plain minimal cones with density at most $\frac{3\omega_d}{2}$ are d-planes or elements of $\mathbb{Y}_0(n,d)$; see Proposition 12.1. Assumption (10.9) is officially satisfied when $d = 2$ and when $d = 3$ and $n = 4$ (hence the statement above), but the author believes that the proof of the case when $d = 3$ and $n = 4$ that was given by Luu in [**Lu**] also works when $d = n - 1$ and $n \leq 6$, hence Corollary 1.8 should be valid in these dimensions as well, and maybe some other ones.

Corollary 1.8 gives a good description of E in $B(x, 2N\delta(x))$. The proof will also show that we have a similar description in balls $B(x,r)$, $\delta(x) < r < 2N\delta(x)$ (see Lemma 12.5), which we could even get form our main statement by taking τ even smaller, depending on N. In Lemma 12.4, we will also show that in the balls $B(x,r)$, $r < \delta(x)$, E is well approximated by a cone $Y(x,r) \in \mathbb{Y}_x(n,d)$. If $d = 2$ we can also use the description of E in $B(x, 2\delta(r))$, and the local regularity result of [**D3**], to show that E is also C^1-equivalent to a cone of type $\mathbb{Y}_x(n,d)$ in $B(x, \delta(x)/2)$, say. When $d = 3$ and $n = 4$ we can use the local regularity result of [**Lu**] instead, and we only get that E is Hölder-equivalent to a cone of type $\mathbb{Y}_x(n,d)$ in $B(x, \delta(x)/2)$.

We can also deduce a reasonable description of E in the part of $B(x, N\delta(x)/2)$ that lies closer to L than the singular set of Y, because (1.46) allows us to apply Corollary 1.7 there (notice that near L, W coincides with a half d-plane).

Our proof will also give some control on the shape of E in balls $B(x, r)$, $N\delta(x) \leq r \leq \frac{1}{2}$, where it will be shown that E looks a lot like a sliding minimal cone with boundary L and density close to ω_d; when $d = 2$, for instance, we believe that this should mean that E is close to a set of $\mathbb{V}(L)$ in these balls. See Proposition 12.7 and Remark 12.8.

But unfortunately, with only the methods of this paper, it seems very hard to get a description of E near the regular points of E, and in particular in balls that are far from the singular set of E (i.e., the points $x \in E$ such that $d(x) > \omega_d$). In such balls, we do not know that the functional $F(r)$ is nearly constant (it may increase from ω_d, its value for small r, to $3\omega_d/3$, its approximate value for $r = 1/2$), and we don't know what E looks like then.

An instance of the situation of Corollary 1.8 is when $n = 3$, $d = 2$, and E looks a lot, in $B(0, 3)$, like a plane that contains L. In this case, the following possibility seems to be expected by specialists, as a typical way for a soap film to leave a curve. The reader may first look at Figure 1.1, which we borrowed from J. Sullivan's site, and which explains how a soap film may leave a cylindrical boundary. Then Figure 1.2 tries to show how the picture may be deformed when the inner radius of the cylinder gets smaller. In both cases E has a singular point on the cylinder, near which E is composed of three walls that are perpendicular to the cylinder and make $120°$ angles with each other. The bottom curve that turns around the cylinder would become more and more horizontal, and the triangular wall would get thinner. At the limit, we would get a large piece of E that crosses L tangentially at the origin, plus a thin triangular piece that connects the upper part of L to the main piece, and meets it along a curve Γ where E has singularities of type $\mathbb{Y}$. See Figure 1.3 for a sketch of the limiting set E, that would be sliding minimal, and Figure 1.4 for its sections by some vertical planes. We also refer to [**Br**] for additional detail on the description. The initial goal of the author, when starting this paper, was not to control the curve Γ, as we can do (at least when $d = 2$ and when $d = 3$ and $n = 4$) as a consequence of Corollary 1.8, but to show that it does not exist, at least when V is a plane or the angle between its two branches is larger than $2\pi/3$. But apparently this attempt failed.
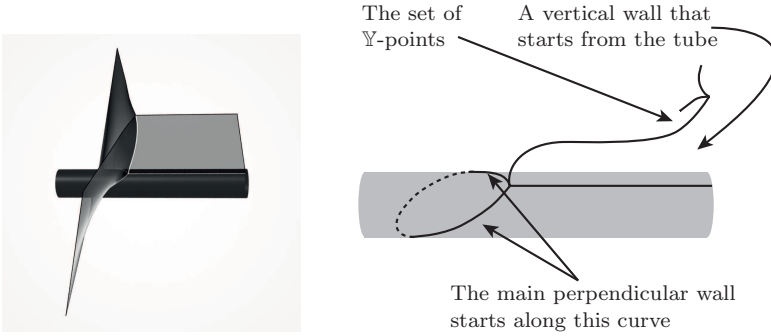
The set of
$\mathbb{Y}$-points

A vertical wall that
starts from the tube

The main perpendicular wall
starts along this curve

FIGURE 1.1. (Left) A soap film leaves a cylinder
(Picture by J. Sullivan).

FIGURE 1.2. (Right) The same picture with more tilt.

The set of $\mathbb{Y}$-points

A thin vertical triangular wall

L

Horizontal tangent plane here, where E crosses L

Here E is like a plane below L

FIGURE 1.3. The limiting set E.

A point of type $\mathbb{Y}$

The thin triangular wall

3 branches in this region

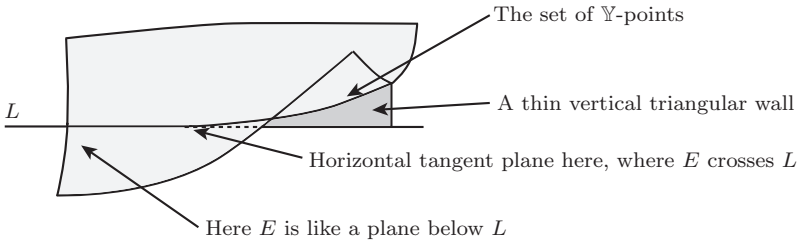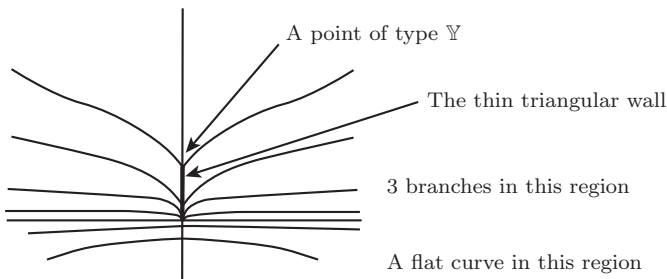A flat curve in this region

FIGURE 1.4. Sections of E by vertical planes.

Another possible behavior of a soap film that may perhaps occur is
the one depicted by Figure 1.5. On the left, E looks like a set of $\mathbb{V}(L)$;
at the origin, E has a blow-up limit $V_0 \in \mathbb{V}(L)$, possibly with a different
angle; on the left, E is composed of a thin triangular vertical wall, as
before, plus two surfaces that meet it along a singular set of points of
type $\mathbb{Y}$. When the two faces of V_0 make $120°$ angles along L, this picture
is not shocking at all; the question concerns the case when this angle is
larger, which looks more surprising (why this apparent discontinuity in
angles?), but not more than the example of Figure 1.3. Notice however
that in this case, E is still attached to L on the left of the origin, so
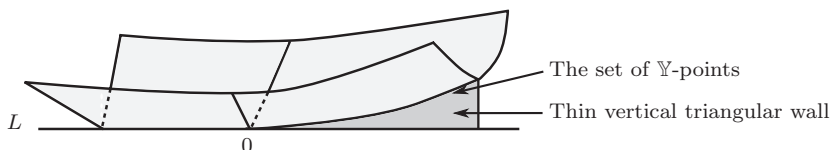maybe it does not pull itself down as much.

FIGURE 1.5. A minimal set E with a blow-up limit of
type $\mathbb{V}$ at 0.

Maybe we should also say that experiments with soap should not help
much here, to decide whether the pictures above are realistic, because
capillarity plays a strong role for thin wires.

The remaining part of this paper will be devoted to the proof of the
various results mentioned above, often in more generality than in the
statements above.

Section 2 mostly records notation and definitions for of sliding almost
minimal sets, and rapidly recalls some results from [**D5**].

In Section 3 we give general conditions on the boundary sets that
will allow us to prove a monotonicity property. These should never be a
restriction in practice; the main difficulty will be to find situations where
the monotonicity property is useful.

Then we prove the monotonicity and near monotonicity properties, in
Sections 4–7, with a comparison argument which looks technical (because
we think we need to be careful when taking some limits), but whose main
point is to compare E with a deformation of E in $B(0,r)$ which is as
close as possible to the cone over $E \cap \partial B(0,r)$. We need to add a piece
to that cone (for instance, the cone over $L \cap \overline{B}(0,r)$), because this way
we can deform on the new set; this is why we end up adding a term to
the density $\theta_0(r)$ in the definition (1.10) of $F(r)$.

In Section 8 we show that for sliding minimal sets E, and when the functional F is constant on an interval, E comes from a cone in the corresponding annulus (we may have to remove a piece of the shade of L, for instance if E is a cone of type $\mathbb{Y}$ that contains L, truncated by L, and this is why we refer to Theorem 8.1 for a more precise statement).

In Section 9 we deduce, from Theorem 8.1 and a compactness argument, that E is well approximated by truncated minimal cones when it is a sliding almost minimal set and the functional F is almost constant on an interval. See Theorem 9.1 for the statement with a fixed (but general) boundary and an approximation result in a ball, and Theorem 9.7 for a variant of Theorem 9.1 where F is nearly constant on an interval that does not start from the origin, and then the approximation only takes place in an annulus.

Corollary 9.3 is a more precise and uniform version of Theorem 9.1 but where L is assumed to be very close to an affine subspace, and which is neither too close nor to far from the origin.

In Section 10 we prepare the two applications and discuss a few simple properties (true or to be assumed) of plain minimal cones that will be used in Sections 11 and 12.

We prove Corollary 1.7 (the case when E looks like a half plane) in Section 11 and Corollary 1.8 (the case when E looks like a $\mathbb{V}$-set) in Section 12.

2. Three types of sliding almost minimal sets

We shall be working with more general sliding almost minimal sets than described in the introduction. In this section, we describe a set of assumptions that we import from [**D5**], and that should be (more than) general enough for us here. We do this because we do not necessarily want to restrict to the case of a unique boundary set L which is an affine subspace of dimension at most $d-1$, yet did not decide of an optimally nice set of assumptions, and need some results from [**D5**] anyway.

The (fairly weak) assumptions presented in this section will be satisfied if there is a unique boundary L, E is a coral sliding almost minimal set, as described in Definition 1.4, and there is a bilipschitz mapping

$\psi \colon U \to B(0,1) \subset \mathbb{R}^n$ such $\psi(L)$ is the intersection with $B(0,1)$ of a vector subspace of dimension at most $d-1$.

Also, we shall present two minor variants of the definition of sliding almost minimal sets, relative to the way we do the accounting in (1.18). The reader that would only be interested in the case presented in the introduction may skip the rest of this section, and will probably not be disturbed afterwards.

In [**D5**] (and from now on), we are in fact given a finite collection of boundary pieces L_j, $0 \le j \le j_{\max}$ (and not just one as above). We say that

(2.1)   the sets L_j satisfy the *Lipschitz assumption* in the domain U

when there is a constant λ (a scale normalization) and a bilipschitz mapping $\psi \colon \lambda U \to B(0,1)$, such that each of the sets $\widetilde{L}_j = \psi(\lambda L_j) \subset B(0,1)$ coincides in $B(0,1)$ with a finite union of (closed) dyadic cubes. We allow dyadic cubes of different dimensions in a single L_j. We refer to Section 2 of [**D5**], and in particular Definition 2.7 and above, for additional detail.

By convention, the first set L_0 in [**D5**] is a larger set than the others, which plays the role of a closed domain where things happen. See (1.1) in [**D5**]. This is a convenience, not a constraint, because we may take $\widetilde{L}_0 = \mathbb{R}^n$ and $L_0 = U$ if there is no need for this. Because of the constraint below, we shall not use this convenience here, and use $L_0 = U$. Indeed, for our monotonicity formula to make sense, we shall need to know that $\mathcal{H}^d(L_j) = 0$ for all the $j \ge 0$ that matter, so we shall always assume that

(2.2) $L_0 = U$ and all the dyadic cubes that compose the $\widetilde{L}_j$, $j \ge 1$,

are of dimensions $< d$.

For many results in [**D5**], an additional technical assumption (namely, (10.7) in [**D5**]) is needed; the reader does not need to worry, this condition, which concerns faces of dimensions larger than d, is automatically satisfied because of (2.2).

The author is aware that (2.1) is complicated; in particular, it is not true that if the L_j satisfy the Lipschitz assumption in U, then their restriction to a smaller domain $V \subset U$ also satisfy the Lipschitz assumption. For one thing, V may not be bilipschitz-equivalent to a ball. This is not a major issue, because we are interested in local properties, and we can always restrict first to a domain U where the L_j satisfy the Lipschitz assumption, and then apply the desired results. Even when U is a ball

and L is an affine subspace of dimension $< d$, we may have to restrict to a slightly smaller ball to make sure that (2.1) and (2.2) hold (think about the case when L is nearly tangent to ∂B), but in the rest of the paper we shall pretend that this has been done and use results where (2.1) and (2.2) hold without explaining again. Notice also that if L does not meet $\frac{1}{2}B$, we may deduce information on $E \cap \frac{1}{2}B$ from results about plain almost minimal sets (with no boundary) anyway.

The definition of an *acceptable deformation* is the same as above, except that we replace the unique condition (1.6) with the condition that for $1 \leq j \leq j_{\max}$,

$$(2.3) \qquad \varphi_t(x) \in L_j \text{ for } 0 \leq t \leq 1 \text{ when } w \in E \cap L_j.$$

We removed $j = 0$, because (2.3) is a tautology for $j = 0$, by (2.2).

In [**D5**] we have three slightly different notions of sliding almost minimal sets (with the given sets L_j and a given gauge function h), which we recall now.

Definition 2.1. Let E be closed in U and satisfy (1.1). For each acceptable deformation $\{\varphi_t\}$, $0 \leq t \leq 1$ (with (1.6) replaced by (2.3)), set

$$(2.4) \qquad W_1 = \big\{x \in E;\ \varphi_1(x) \neq x\big\}.$$

Also denote by r the radius of a ball that contains the set $\widehat{W}$ of (1.5). We say that E is a (sliding) A-almost minimal set (and write $E \in \mathrm{SAM}(U, L_j, h)$) when

$$(2.5) \qquad \mathcal{H}^d(W_1) \leq \mathcal{H}^d(\varphi_1(W_1)) + h(r)r^d$$

for all choice of acceptable deformations $\{\varphi_t\}$ and r as above.

We say that E is A$'$-almost minimal (and write $E \in \mathrm{SA'M}(U, L_j, h)$) when instead

$$(2.6) \qquad \mathcal{H}^d(E \setminus \varphi_1(E)) \leq \mathcal{H}^d(\varphi_1(E) \setminus E) + h(r)r^d$$

for $\{\varphi_t\}$ and r as above, and that E is A$_+$-almost minimal (and write $E \in \mathrm{SA_+M}(U, L_j, h)$) when instead

$$(2.7) \qquad \mathcal{H}^d(W_1) \leq (1 + h(r))\mathcal{H}^d(\varphi_1(W_1)).$$

Let us finally set $\mathrm{SA^*M}(U, L_j, h) = \mathrm{SAM}(U, L_j, h) \cup \mathrm{SA'M}(U, L_j, h) \cup \mathrm{SA_+M}(U, L_j, h)$ (where we don't care about which definition is taken).

See Definition 20.2 in [**D5**]. The definition (1.18) that we gave above is the definition of an A$'$-almost minimal set (notice that since E and $\varphi_1(E)$ coincide on $U \setminus \widehat{W}$, this set does not play any role in (2.6), or just refer to (20.7) in [**D5**]).

It turns out that A_+-almost minimal implies A-almost minimal and A′-almost minimal (but in a slightly smaller domain, and with a slightly larger gauge function); see the comments below (20.7) in [**D5**]. Moreover, the two notions of A-almost minimal and A′-almost minimal are equivalent (with the same gauge function); see Proposition 20.9 in [**D5**]. In the introduction we chose to give the A′-definition because it looks simpler, but finally this does not matter.

When $h = 0$, the three notions coincide, and yield the sliding minimal sets defined in the introduction.

The main reason why we give all this notation is that we shall be using some results from [**D5**], which we quote now. For our convenience, we shall always assume that

$$\text{(2.8)} \qquad E \text{ is coral and } E \in \text{SA}^*\text{M}(U, L_j, h),$$

which means that E is a coral (sliding) almost minimal set in U, with boundary conditions given by the L_j and the gauge function h, and with any of the three definitions. In some rare occasions, we shall need to specify. We like to ask E to be coral, because this way we don't need to worry about the fuzzy set $E \setminus E^*$, with E^* as in (1.11), because $E = E^*$. We know that we don't lose anything, because Proposition 3.3 in [**D5**] says that E^* is almost minimal when E is almost minimal, with the same gauge function. More precisely, that proposition is stated for more general quasiminimal sets, but its proof works for almost minimal sets, and we can also deduce the result for almost minimal sets by comparing the definitions of quasiminimal sets (Definition 2.3 in [**D5**]) and almost minimal sets (Definition 20.2 in [**D5**]).

The first main property of E that we shall use is its local Ahlfors regularity. There exist constants $\eta_0 > 0$ and $C \geq 1$ (that depend on U, and the L_j) such that

$$\text{(2.9)} \qquad C^{-1} r^d \leq \mathcal{H}^d(E \cap B(x, r)) \leq C r^d$$

when $x \in E$ and $r > 0$ are such that

$$\text{(2.10)} \qquad B(x, 2r) \subset U \text{ and } h(2r) \leq \eta_0.$$

This is Proposition 4.74 in [**D5**]; since the proposition is stated for (more general) quasiminimal sets, use the comment below (20.8) in [**D5**], or compare directly with Definition 2.3 in [**D5**].

We shall also use the fact that

$$\text{(2.11)} \qquad E \text{ is a rectifiable set of dimension } d;$$

see Theorem 5.16 in [**D5**], and observe that we can trivially reduce to small balls $B(x, r)$ such that $h(2r)$ is as small as we want, because the rectifiability of E is a local property.

We shall later quote from [**D5**] various limiting theorems, but we shall only mention them when we need them.

3. Specific assumptions on L; local retractions

We shall now describe the additional assumptions that we shall make on our sets L_j (or the unique L of the introduction), so that the proof below runs reasonably smoothly. We start with some notation.

We assume that $0 \in U$, and for $r > 0$ such that $\overline{B}(0, r) \subset U$, set

$$(3.1) \qquad L(r) = \overline{B}(0, r) \cap \left\{ \bigcup_{1 \leq j \leq j_{\max}} L_j \right\},$$

where the L_j are the boundary pieces of Section 2. If we have just one boundary set L, as the introduction, just take $L(r) = \overline{B}(0, r) \cap L$.

We shall often use the truncated cone

$$(3.2) \qquad L^*(r) = \overline{B}(0, r) \cap \left\{ \lambda z; \, z \in L(r) \text{ and } \lambda \geq 0 \right\}$$

over $L(r)$, and then its trace

$$(3.3) \quad L^\circ(r) = \partial B(0, r) \cap L^*(r) = \partial B(0, r) \cap \left\{ \lambda z; \, z \in L(r) \text{ and } \lambda > 0 \right\}$$

(think about the shadow of $L \setminus \{0\}$ on $\partial B(0, r)$, with a light at the origin).

Our first additional assumption is that

$$(3.4) \qquad \mathcal{H}^d(L^*(\rho)) < +\infty \text{ for some } \rho > 0.$$

Notice that as soon as we have this, then

$$(3.5) \qquad \mathcal{H}^d(L^*(r)) < +\infty \text{ for every } r > 0.$$

Indeed, $L^*(r) \subset L^*(\rho)$ for $r \leq \rho$, and if $r > \rho$, $L^*(r)$ is contained in the union of two cones. The first one is the cone over $L(\rho)$, which we control by (3.4). The second one is the cone C over $L(r) \setminus L(\rho)$. But by (2.2), $L(r) \setminus L(\rho)$ is contained in a finite union of bilipschitz images of cubes of dimensions at most $d - 1$; then $\mathcal{H}^{d-1}(L(r) \setminus L(\rho)) < +\infty$, and (for instance by the area theorem and because $L(r) \setminus L(\rho)$ stays away from the origin) $\mathcal{H}^d(C \cap \overline{B}(0, r)) < +\infty$; (3.5) follows.

If $L(r)$ does not contain the origin, (3.4) holds trivially. But we also want to allow the case when 0 lies in some L_j, $j \geq 1$, so we include (3.4) as an assumption.

We shall play with the sets $L^*(r)$, in particular to deform parts of $B(0,r)$ onto subsets of $L^*(r)$, and it will be good to have some local retractions.

Definition 3.1. Let $r > 0$ be such that $\overline{B}(0,r) \subset U$. We say that *r admits a local retraction* when we can find constants $\tau_0 > 0$ and $C_0 \geq 1$, and a C_0-Lipschitz function π_0, defined on the set

$$(3.6) \qquad R(r,\tau_0) = \big\{ x \in \partial B(0,r); \, \mathrm{dist}(x, L^\circ(r)) \leq \tau_0 \big\},$$

such that

$$(3.7) \qquad\qquad \pi_0(x) \in L^\circ(r) \text{ for } x \in R(r,\tau_0),$$

and

$$(3.8) \qquad\qquad \pi_0(x) = x \text{ for } x \in L^\circ(r).$$

Our typical assumption will be that almost every r in the interval of interest admits a local retraction. We won't need uniform bounds on τ_0 and C_0, because these two constants will disappear in a limiting argument.

Admittedly, this is not such a beautiful condition, but we only expect to use our results with fairly simple sets L, and then the local retractions will be easy to obtain. For instance, if L is an affine subspace, or is convex, π_0 is very easy to construct.

Probably a Lipschitz retraction on a neighborhood of $L^*(r)$ would have been easier to use, but we decided to use retractions on spheres, just for the hypothetical case when we would be interested in a set L with a loop, for which the cone $L^*(r)$ then has a small loop at 0, making retractions of $L^*(r)$ near 0 hard to get.

4. The main competitor for monotonicity

The typical way to obtain monotonicity results like (1.8) is to compare E with a cone which has the same trace on a sphere $\partial B(0,r)$. Here we cannot quite do that when the L_j are not cones centered at 0, so we shall need to add an extra piece to the cone (essentially, the set $L^\sharp$ below).

The assumptions for this section are the following. We work in a closed ball $\overline{B} = \overline{B}(0,r)$, and we assume that there is an open set U and boundary sets L_j, $0 \leq j \leq j_{\max}$, such that $\overline{B} \subset U$, (2.1), (2.2), and (3.4) hold, and E is a coral sliding almost minimal set in U, as in (2.8). Set

$$(4.1) \qquad\qquad L = L(r) = \overline{B} \cap \left\{ \bigcup_{1 \leq j \leq j_{\max}} L_j \right\},$$

(as in (3.1)),

(4.2) $$L^* = L^*(r) = \overline{B} \cap \{\lambda z; \; z \in L \text{ and } \lambda \geq 0\}$$

(as in (3.2)), and also

(4.3) $$L^\sharp = L^\sharp(r) = \{\lambda z; \; z \in L \text{ and } \lambda \in [0,1]\} \subset \overline{B}.$$

In this section we just construct some acceptable deformations in the ball $\overline{B}$; the idea is to compare $E \cap B$ with the union of $L^\sharp$ and the cone over $E \cap \partial B$, but we shall not do this exactly, and instead be very prudent and deform $E \cap B$ to sets that tend to this union. We shall only see later how to use the deformation of this section to prove our monotonicity results.

We assume that

(4.4) $$r \text{ admits a local retraction,}$$

as in Definition 3.1. Let us recall what this means with the present notation. Set

(4.5) $$R(\tau) = \{x \in \partial B; \; \text{dist}(x, L^* \cap \partial B) \leq \tau\}$$

for $\tau > 0$. If $L \cap \overline{B} = \emptyset$, we take $L^* = \emptyset$ and $R(\tau) = \emptyset$. If $L \cap \overline{B} = \{0\}$, we also take $R(\tau) = \emptyset$. Definition 3.1 gives us $\tau_0 > 0$ and a C_0-Lipschitz mapping $\pi_0 \colon R(\tau_0) \to L^* \cap \partial B$, such that

(4.6) $$\pi_0(x) = x \text{ for } x \in L^* \cap \partial B.$$

Our construction will have four parameters, a small τ that controls distances to L^*, and three radii r_j, $j = 0, 1, 2$, with $r \geq r_0 > r_1 > r_2$ (and $r - r_2$ very small). Later on, we shall take specific values for the r_j and take limits twice. We shall take $\tau < 1/4 \min(\tau_0, r)$, but rapidly it will tend to 0.

Our first task is to use π_0 to construct a new mapping π, which will be defined on the whole ∂B. Set

(4.7) $$\pi(x) = \pi_0(x) \text{ for } x \in R(\tau)$$

and

(4.8) $$\pi(x) = x \text{ for } x \in \partial B \setminus R(2\tau).$$

In the remaining region $R(2\tau) \setminus R(\tau)$, set

(4.9) $$\alpha(x) = \frac{\text{dist}(x, L^* \cap \partial B)}{\tau} - 1 \in [0, 1]$$

and then

(4.10) $$\pi(x) = \alpha(x)x + (1 - \alpha(x))\pi_0(x),$$

which is well defined because $x \in R(\tau_0/2)$. Let us check that

(4.11) π is $12C_0$-Lipschitz on ∂B.

First notice that (4.10) is still valid for $x \in R(2\tau_0)$, if we set $\alpha(x) = 0$ on $R(\tau)$ and $\alpha(x) = 1$ on $R(2\tau_0) \setminus R(2\tau)$. Next we show that π is $12C_0$-Lipschitz on $R(2\tau_0)$. Consider $x, y \in R(2\tau_0)$, and notice that

$$|\pi(x) - \pi(y)| = \big|[\pi_0(x) - \pi_0(y)] + \alpha(x)[x - \pi_0(x)] - \alpha(y)[y - \pi_0(y)]\big|$$

$$\leq C_0|x - y| + \alpha(x)|x - \pi_0(x) - y + \pi_0(y)|$$

(4.12)

$$+ |\alpha(x) - \alpha(y)||y - \pi_0(y)|$$

$$\leq C_0|x - y| + \alpha(x)(1 + C_0)|x - y| + \tau^{-1}|x - y||y - \pi_0(y)|,$$

where we used (4.9) to get a bound on $|\alpha(x) - \alpha(y)|$. If in addition $y \in R(4\tau)$, we can find $z \in L^* \cap \partial B$ such that $|z - y| \leq 4\tau$; then

(4.13) $|y - \pi_0(y)| \leq |y - z| + |\pi_0(z) - \pi_0(y)| \leq (1 + C_0)4\tau$

because $\pi_0(z) = z$ by (4.6); altogether

(4.14) $|\pi(x) - \pi(y)| \leq 6(1 + C_0)|x - y| \leq 12C_0|x - y|$

$$\text{for } x \in R(2\tau_0) \text{ and } y \in R(4\tau).$$

We get the same bound when $x \in R(4\tau)$ and $y \in R(2\tau_0)$ (just exchange x and y in the estimate). And when both x and y lie in $R(2\tau_0) \setminus R(4\tau)$, $|\pi(x) - \pi(y)| = |x - y|$ by (4.8); thus π is $12C_0$-Lipschitz on $R(2\tau_0)$.

To complete the proof of (4.11), we still need to estimate $|\pi(x) - \pi(y)|$ when $x \in \partial B \setminus R(2\tau_0)$. When $y \in R(2\tau)$,

(4.15) $|\pi(x) - \pi(y)| = |x - \pi(y)| \leq |x - y| + |y - \pi(y)| \leq |x - y| + |y - \pi_0(y)|$

$$\leq |x - y| + 4(1 + C_0)\tau \leq |x - y| + 2(1 + C_0)|x - y|$$

by (4.8), (4.10), and (4.13), and because $\text{dist}(x, L^* \cap \partial B) \geq 2\tau_0 \geq 4\tau$ and $\text{dist}(y, L^* \cap \partial B) \leq 2\tau$. When $y \in \partial B \setminus R(2\tau)$, $\pi(x) - \pi(y) = x - y$ by (4.8), and we are happy too. So (4.11) holds.

Next we extend π to B by homogeneity, i.e., set

(4.16) $\pi(\lambda x) = \lambda \pi(x)$ for $x \in \partial B$ and $0 \leq \lambda \leq 1$.

This completes our definition of π. Notice that

(4.17) π is $13C_0$-Lipschitz on $\overline{B}$;

this time, the simplest is to compute the radial and tangential derivatives, and notice that $|\pi(x)| \leq |x|$ (by (4.10) and because $|\pi_0(x)| = r$

on ∂B). Set

$$(4.18) \qquad B^* = \overline{B} \setminus \{0\} \text{ and } d(x) = \text{dist}\left(\frac{rx}{|x|}, L^* \cap \partial B\right) \text{ for } x \in B^*;$$

we use this to measure a radial distance to L^*. Notice that d is locally Lipschitz on B^*, with

$$(4.19) \qquad\qquad\qquad |\nabla d(x)| \leq \frac{r}{|x|}.$$

Then set

$$R_1 = \left\{x \in B^*; \, d(x) \leq \tau\right\} = \left\{x \in B^*; \, \frac{rx}{|x|} \in R(\tau)\right\},$$

$$(4.20) \qquad R_2 = \left\{x \in B^*; \, \tau < d(x) \leq 2\tau\right\}$$

$$= \left\{x \in B^*; \, \frac{rx}{|x|} \in R(2\tau) \setminus R(\tau)\right\}, \text{ and}$$

$$R_3 = B^* \setminus [R_1 \cup R_2] = \left\{x \in B^*; \, \frac{rx}{|x|} \in \partial B \setminus R(2\tau)\right\}.$$

In the easier special case when $L^* = \emptyset$, we just have $R_1 = R_2 = \emptyset$ and we can take $d(x) = +\infty$. Notice that

$$(4.21) \qquad\qquad\qquad \pi(x) = x \text{ for } x \in L^*$$

by (4.6), (4.7), and (4.16),

$$(4.22) \qquad\qquad\qquad \pi(x) \in L^* \text{ for } x \in R_1$$

by (4.4), (4.7), and (4.16), and

$$(4.23) \qquad\qquad\qquad \pi(x) = x \text{ for } x \in R_3,$$

by (4.8) and (4.16). Finally record that

$$(4.24) \qquad\qquad\qquad |\pi(x)| \leq |x| \text{ for } x \in \overline{B},$$

by (4.16) and the definition of π on ∂B.

We are ready to define our final mapping $\varphi = \varphi_1$. We don't need to restrict to E yet; φ will be defined on $\mathbb{R}^n$. Set

(4.25) $B_j = B(0, r_j)$, for $j = 0, 1, 2$, $A_1 = B_0 \setminus B_1$, and $A_2 = B_1 \setminus B_2$.

We start slowly and set

$$(4.26) \qquad\qquad\qquad \varphi(x) = x \text{ for } x \in \mathbb{R}^n \setminus B_0.$$

Next we set

$$(4.27) \qquad\qquad\qquad \varphi(x) = \pi(x) \text{ for } x \in \partial B(0, r_1),$$

and interpolate quietly in the middle. That is, we take

$$(4.28) \quad \alpha_1(x) = \frac{|x| - r_1}{r_0 - r_1} \text{ and } \varphi(x) = \alpha_1(x)x + (1 - \alpha_1(x))\pi(x)$$

$$\text{for } x \in A_1 = B_0 \setminus B_1.$$

In the remaining ball $B_1 = B(0, r_1)$, we shall use a function $\hbar$ to contract along radii whenever this is possible. Set

$$(4.29) \qquad \hbar(x) = \left[1 - \tau^{-1}\operatorname{dist}(x, L) - \tau^{-1}d(x)\right]_+ \in [0, 1] \text{ for } x \in \overline{B}^*$$

(where d and B^* come from (4.18) and $[a]_+$ is a notation for $\max(a, 0)$). Thus $\hbar \equiv 0$ when $L^* = \emptyset$. Then define φ on B_2 by

$$(4.30) \qquad\qquad \varphi(x) = \hbar(x)\pi(x) \text{ for } x \in B_2 \setminus \{0\}$$

and $\varphi(0) = 0$; this makes a continuous function, by (4.24) and because $0 \leq \hbar \leq 1$ everywhere. On the remaining annulus A_2, we interpolate. That is, we set

$$(4.31) \qquad\qquad \alpha_2(x) = \frac{|x| - r_2}{r_1 - r_2} \text{ for } x \in A_2 = B_1 \setminus B_2,$$

and take

$$(4.32) \qquad \varphi(x) = \alpha_2(x)\pi(x) + (1 - \alpha_2(x))\hbar(x)\pi(x) \text{ for } x \in A_2.$$

Since d has a singularity at the origin, we feel obligated to check that

$$(4.33) \qquad\qquad\qquad \varphi \text{ is Lipschitz on } \overline{B}.$$

Since it is continuous along the boundaries of our different pieces, it is enough to check that φ is Lipschitz on each piece separately. The various cut-off functions are Lipschitz, so it will be enough to check that

$$(4.34) \qquad\qquad \hbar(x)\pi(x) \text{ is } \frac{15C_0 r}{\tau}\text{-Lipschitz on } \overline{B}.$$

We know that $\hbar\pi$ is locally Lipschitz on B^* (by (4.19) in particular), so we just need to bound the derivative $D(\hbar\pi)(x)$; but

$$|D(\hbar\pi)(x)| \leq |D\pi(x)|\hbar(x) + |\pi(x)||D\hbar(x)|$$

$$\leq 13C_0\hbar(x) + |\pi(x)|\left[\tau^{-1} + \tau^{-1}|\nabla d(x)|\right]$$

$$(4.35)$$

$$\leq 13C_0 + \tau^{-1}|x| + \tau^{-1}|x||\nabla d(x)|$$

$$\leq 13C_0 + 2\tau^{-1}r \leq \frac{15C_0 r}{\tau}$$

by (4.17), (4.29), (4.24), and (4.19), and because $|x| \leq r$ and $\tau \leq r/4$; (4.34) follows because $\hbar\pi$ is also continuous across 0.

We now complete the family by taking

(4.36) $\varphi_t(x) = tx + (1 - t)\varphi(x)$ for $x \in \mathbb{R}^n$ and $t \in [0, 1]$,

and check that

(4.37)   the restriction to E of the φ_t, $0 \le t \le 1$,

$$\text{forms an acceptable deformation, with } \widehat{W} \subset B_0.$$

By (4.33) in particular, the mapping $(x, t) \to \varphi_t(x)$ is Lipschitz on $\mathbb{R}^n \times [0, 1]$, which takes care of (1.2) and (1.4); (1.3) is trivial. All the sets W_t, $0 \le t \le 1$, of (1.5) are contained in B_0, by (4.26), and

(4.38) $\varphi_t(W_t) \subset \varphi_t(B_0) \subset B_0$

by (4.24) and because $0 \le \hbar(x) \le 1$. Thus $\widehat{W} \subset B_0$ and (1.5) holds.

We are left with (2.3) (the current replacement for (1.6)) to check. So let $1 \le j \le j_{\max}$ and $x \in E \cap L_j$ be given, and let us check that $\varphi_t(x) \in L_j$ for $0 \le t \le 1$. If $x \in \mathbb{R}^n \setminus B_0$, (4.26) and (4.36) say that $\varphi_t(x) = x$, and we are happy because $x \in L_j$. Otherwise, notice that $x \in L \subset L^*$, by (4.1) and (4.2). Then $\pi(x) = x$, by (4.21), and $\hbar(x) = 1$ by (4.29). This yields $\varphi_t(x) = x \in L_j$, which proves (2.3); (4.37) follows.

Thus we get one of the formulas (2.5)–(2.7), with $r = r_0$. Our next task is to estimate the measure of $\varphi_1(E \cap B_0) = \varphi(E \cap B_0)$, which we will cut into many small pieces.

We start with B_2. Notice that by (4.29) and the definitions (4.18) and (4.20),

(4.39) $\hbar(x) = 0$ for $x \in R_2 \cup R_3$;

thus (4.30) yields $\varphi(x) = 0$ for $x \in B_2 \cap [R_2 \cup R_3]$, hence

(4.40) $\mathcal{H}^d(\varphi(B_2 \cap [R_2 \cup R_3])) = 0$.

We are left with $B_2 \cap R_1$. Let us show that

(4.41) $\varphi(B_2 \cap R_1) \subset \{z \in L^*; \operatorname{dist}(z, L^\sharp) \le (C_0 + 2)\tau\}$.

Let $x \in B_2 \cap R_1$ be given. If $\operatorname{dist}(x, L^\sharp) \ge \tau$, then $\operatorname{dist}(x, L) \ge \tau$ (because $L \subset L^\sharp$), $\hbar(x) = 0$ by (4.29), $\varphi(x) = 0$ by (4.30), and we are happy because R_1 is not empty, L^* and L are not empty either, and (4.3) says that $0 \in L^\sharp$. So we may assume that

(4.42) $\operatorname{dist}(x, L^\sharp) \le \tau$.

Set $x^\circ = \frac{rx}{|x|}$. By (4.16), $\pi(x) = \frac{|x|\pi(x^\circ)}{r}$. Since $x \in R_1$, (4.20) says that $x^\circ \in R(\tau)$, then $\pi(x^\circ) = \pi_0(x^\circ) \in L^*$ by (4.7) and (4.4). Since

$x° ∈ R(τ)$, we can find $y ∈ L^* ∩ ∂B$ such that $|y − x°| ≤ τ$ (see (4.5)), and then

(4.43) $\quad |\pi_0(x°)−x°|≤|\pi_0(x°)−\pi_0(y)|+|y−x°|≤(C_0+1)|y−x°|≤(C_0+1)τ$

because $\pi_0(y) = y$ by (4.6). Next

(4.44) $\quad |\pi(x) − x| = \dfrac{|x|}{r}|\pi(x°) − x°| = \dfrac{|x|}{r}|\pi_0(x°) − x°| ≤ (C_0 + 1)τ$

and so $\mathrm{dist}(\pi(x), L^♯) ≤ (C_0 + 2)τ$ by (4.42). But $λy ∈ L^♯$ when $y ∈ L^♯$ and $λ ∈ [0, 1]$ (by (4.2)), so

(4.45) $\quad \mathrm{dist}(\varphi(x), L^♯) = \mathrm{dist}(\hbar(x)\pi(x), L^♯) ≤ \mathrm{dist}(\pi(x), L^♯) ≤ (C_0 + 2)τ$

by (4.30) and because $\hbar(x) ∈ [0, 1]$. By (4.22), $\pi(x) ∈ L^*$ and hence also $\varphi(x) = \hbar(x)\pi(x) ∈ L^*$; this completes our proof of (4.41).

Next we consider the (more interesting) interior annulus A_2. The largest piece is $A_2 ∩ R_3$. Recall from (4.23) that $\pi(x) = x$ for $x ∈ R_3$. In addition, $\hbar(x) = 0$ by (4.39), so (4.32) simplifies and becomes

(4.46) $\qquad \varphi(x) = α_2(x)x = \dfrac{|x| − r_2}{r_1 − r_2}x \text{ for } x ∈ A_2 ∩ R_3,$

by (4.31); this is a rather simple dilation that maps $A_2∩R_3$ to $B_1∩R_3$. In this region, we have no other option but to compute $\mathcal{H}^d(\varphi(E∩A_2∩R_3))$ with the area formula, and later take a limit.

Recall from (2.11) that E is rectifiable. Then it has an approximate tangent d-plane $P(x)$ at $\mathcal{H}^d$-almost every point x. In fact, thanks to the local Ahlfors regularity (2.9), this approximate tangent plane is even a true tangent plane; this is reassuring, but we shall not really need this remark. The mapping φ, given by (4.46), is smooth, and we can compute its Jacobian $J(x)$ relative to the plane $P(x)$. Again, an approximate differential would be enough, but we are happy that we can compute $J(x)$ in terms of $P(x)$. Denote by $P'(x)$ the vector space of dimension d parallel to $P(x)$. The main quantity here is the smallest angle $θ(x) ∈ [0, π/2]$ between the line $(0, x)$ and a vector of $P'(x)$. Said in other words,

(4.47) $\qquad \cos θ(x) = \sup\left\{\left⟨v, \dfrac{x}{|x|}\right⟩; v ∈ P'(x) \text{ and } |v| = 1\right\}.$

If E were a cone centered at 0, we would get $\cos θ(x) = 1$ almost everywhere, for instance.

Denote by $D = D\varphi(x)$ the differential of φ at x. Set $e = \dfrac{x}{|x|}$ and define $α: [r_2, r_1] → [0, 1]$ by $α(ρ) = \dfrac{ρ−r_2}{r_1−r_2}$. From (4.46) we deduce that

for $v \in \mathbb{R}^n$,

$$D(v) = \alpha_2(x)v + \langle \nabla \alpha_2(x), v \rangle x$$

(4.48)
$$= \alpha_2(x)v + \alpha'(|x|)\langle e, v \rangle x = \alpha_2(x)v + \frac{\langle e, v \rangle}{r_1 - r_2}x.$$

Then we compute the Jacobian $J(x)$. Use (4.47) to choose a first unit vector $v_1 \in P'(x)$, such that $\cos\theta(x) = \langle v, e \rangle$, and then choose unit vectors $v_2, \dots, v_d$ such that $(v_1, \dots, v_d)$ is an orthonormal basis of $P'(x)$. By (4.47) again,

(4.49)
$$\langle v_1 + tv_j, e \rangle \leq \cos\theta(x)|v_1 + tv_j|$$

for $j \geq 2$ and $t \in \mathbb{R}$. When we take the derivative at $t = 0$, we get that $\langle v_j, e \rangle = 0$.

Now return to (4.48). Set $\rho = |x|$; for $j \geq 2$, we get that

(4.50)
$$D(v_j) = \alpha_2(x)v_j = \alpha(\rho)v_j.$$

For v_1, we further decompose v_1 as $v_1 = \cos\theta(x)e + \sin\theta(x)w$, where w is a unit normal vector orthogonal to e, and get that

$$D(v_1) = \alpha_2(x)v_1 + \frac{x\cos\theta(x)}{r_1 - r_2}$$

(4.51)
$$= \alpha(\rho)[\cos\theta(x)e + \sin\theta(x)w] + \frac{\rho\cos\theta(x)e}{r_1 - r_2}.$$

Both e and $\sin\theta(x)w = v_1 - \cos\theta(x)e$ are orthogonal to the v_j, $j \geq 2$, so all the vectors $D(v_j)$, $j \geq 1$, are orthogonal, and

(4.52)
$$J(x) = \prod_{j \geq 1} |D(v_j)| = \alpha(\rho)^{d-1}|D(v_1)| = \alpha(\rho)^{d-1}\beta(x),$$

with

$$\beta(x) = \left\{ \cos^2\theta(x)\left(\alpha(\rho) + \frac{\rho}{r_1 - r_2}\right)^2 + \sin^2\theta(x)\alpha^2(\rho) \right\}^{1/2}$$

(4.53)
$$= \left\{ \cos^2\theta(x)\left(\frac{2\rho - r_2}{r_1 - r_2}\right)^2 + \sin^2\theta(x)\alpha^2(\rho) \right\}^{1/2}.$$

We now write the area formula for the injective mapping φ (see for instance [**Fe**]):

$$\mathcal{H}^d(\varphi(E \cap A_2 \cap R_3)) = \int_{E \cap A_2 \cap R_3} J(x) \, d\mathcal{H}^d(x)$$

(4.54)
$$= \int_{E \cap A_2 \cap R_3} \alpha(\rho)^{d-1}\beta(x) \, d\mathcal{H}^d(x),$$

where we set

$$(4.55) \qquad \rho = |x| \text{ and } \alpha(\rho) = \frac{\rho - r_2}{r_1 - r_2}.$$

We leave this as it is for the moment, to be evaluated later, and return to the other pieces of A_2, starting with $A_2 \cap R_2$. In this region, we still have that $\hbar(x) = 0$, by (4.39), but we need to keep $\pi(x)$ as it is, and (4.32) only yields

$$(4.56) \qquad \varphi(x) = \alpha_2(x)\pi(x) = \frac{|x| - r_2}{r_1 - r_2}\pi(x) \text{ for } x \in A_2 \cap R_2.$$

Let us again apply the area formula. Let $x \in E$ be such that E has a tangent plane $P(x)$ at x (as before), but also φ has an approximate differential $D\varphi(x)$ at x in the direction of $P(x)$; since φ is Lipschitz, this happens for $\mathcal{H}^d$-almost every $x \in E \cap B$; see for instance [**Fe**], to which we shall systematically refer concerning the area formula on rectifiable sets. At such a point x, π also has an approximate differential $D\pi(x)$ in the direction of $P(x)$; we compute as in (4.48) and get that for $v \in P'(x)$,

$$(4.57) \qquad \begin{aligned} D\varphi(x)(v) &= \alpha_2(x)D\pi(x)(v) + \langle \nabla\alpha_2(x), v\rangle\pi(x) \\ &= \alpha_2(x)D\pi(x)(v) + \frac{\langle e, v\rangle\pi(x)}{r_1 - r_2}. \end{aligned}$$

Let us use the same orthonormal basis $(v_1, \ldots, v_d)$ of $P'(x)$ as before. For $j \geq 2$, we now get that $D\varphi(x)(v_j) = \alpha_2(x)D\pi(x)(v_j)$ (because $v_j \perp e$), and we shall remember that $|D\varphi(x)(v_j)| \leq C$ for $\mathcal{H}^d$-almost every x, where C depends on C_0, just because (4.17) says that π is $13C_0$-Lipschitz. For v_1 we have an extra term, and we can only say that

$$(4.58) \qquad |D\varphi(x)(v_1)| \leq C + \frac{|\pi(x)|}{r_1 - r_2} \leq C + \frac{r}{r_1 - r_2}.$$

Then we estimate $J(x)$, the Jacobian of φ on E at x, brutally, and get that $J(x) \leq \frac{Cr}{r_1-r_2}$. The area formula now yields

$$(4.59) \qquad \begin{aligned} \mathcal{H}^d(\varphi(E \cap A_2 \cap R_2)) &\leq \int_{E \cap A_2 \cap R_3} J(x)\, d\mathcal{H}^d(x) \\ &\leq \frac{Cr}{r_1 - r_2}\mathcal{H}^d(E \cap A_2 \cap R_2), \end{aligned}$$

where we only get an inequality in the first part because do not know whether φ is injective. This looks large because of $\frac{r}{r_1-r_2}$, but we hope that the fact that A_2 is quite thin, plus the small angle of R_2, will compensate.

Our next piece is $A_2 \cap R_1$, which we decompose again. Set

$$(4.60) \qquad V = \left\{ x \in B; \; \mathrm{dist}(x, L^\sharp) \leq \tau \right\}.$$

On $A_2 \cap R_1 \setminus V$, we again have that $\hbar(x) = 0$ (directly by (4.29) and because $\mathrm{dist}(x, L) \geq \mathrm{dist}(x, L^\sharp) \geq \tau$), so the formula (4.56) still holds, and we can compute as for $A_2 \cap R_2$. We avoid L^* for the moment, and we get that

$$
\mathcal{H}^d(\varphi(E \cap A_2 \cap R_1 \setminus (V \cup L^*))) \leq \frac{Cr}{r_1 - r_2} \mathcal{H}^d(E \cap A_2 \cap R_1 \setminus (V \cup L^*))
$$
$$(4.61)$$
$$
\leq \frac{Cr}{r_1 - r_2} \mathcal{H}^d(E \cap A_2 \cap R_1 \setminus L^*).
$$

Next we claim that

$$(4.62) \qquad \varphi(A_2 \cap R_1 \cap V) \subset \left\{ z \in L^*; \; \mathrm{dist}(z, L^\sharp) \leq (C_0 + 2)\tau \right\}.$$

We can repeat the proof that we gave below (4.42), up to (4.45) which is the first time where we used the fact that $x \in B_2$ (to compute $\varphi(x)$). Here (4.31) and (4.32) say that $\varphi(x) \in [\hbar(x)\pi(x), \pi(x)]$, so we still can write $\varphi(x) = \lambda \pi(x)$ for some $\lambda \in [0, 1]$, and we can replace (4.45) by

$$(4.63) \quad \mathrm{dist}(\varphi(x), L^\sharp) = \mathrm{dist}(\lambda \pi(x), L^\sharp) \leq \mathrm{dist}(\pi(x), L^\sharp) \leq (C_0 + 2)\tau.$$

Then our claim follows as before.

The last part of A_2 is $A_2 \cap L^* \setminus V$. By (4.21), $\pi(x) = x$ on this set, so $\varphi(x) \in [0, x]$, and we just record that

$$(4.64) \quad \varphi(E \cap A_2 \cap L^* \setminus V) \subset \bigcup_{x \in E \cap A_2 \cap L^*} [0, x]$$
$$= \left\{ \lambda x; \; x \in E \cap A_2 \cap L^* \text{ and } \lambda \in [0, 1] \right\}.$$

We are left with the contribution of the exterior annulus A_1, where φ interpolates between x and $\pi(x)$ (see (4.28)). When $x \in R_3$, (4.23) says that $\pi(x) = x$, so let us record that

$$(4.65) \qquad \varphi(x) = x \text{ for } x \in A_1 \cap R_3.$$

Let us check that

$$(4.66) \qquad \varphi \text{ is } 28C_0 \left(1 + \frac{\tau}{r_0 - r_1} \right)\text{-Lipschitz on } A_1.$$

As usual, for $x, y \in A_1$, we write

$$
\begin{aligned}
\varphi(x) - \varphi(y) &= \alpha_1(x)x + (1 - \alpha_1(x))\pi(x) - \alpha_1(y)y - (1 - \alpha_1(y))\pi(y) \\
&= [\pi(y) - \pi(x)] + \alpha_1(x)[x - \pi(x)] - \alpha_1(y)[y - \pi(y)] \\
&= [\pi(y) - \pi(x)] + \alpha_1(x)[x - \pi(x) - y + \pi(y)] \\
&\quad + [\alpha_1(x) - \alpha_1(y)][y - \pi(y)].
\end{aligned}
$$

(4.67)

Then we observe that $\operatorname{dist}(y, R_3) \leq 2\tau$ by (4.20) and (4.5), so we can choose $z \in R_3$ such that $|z - y| \leq 2\tau$, and

$$
(4.68) \quad |y - \pi(y)| \leq |y - z| + |\pi(z) - \pi(y)| \leq 14C_0|z - y| \leq 28C_0\tau
$$

by (4.23) and (4.17). Thus (4.67) yields

$$
(4.69) \quad |\varphi(x) - \varphi(y)| \leq 13C_0|x - y| + 14C_0|x - y| + \frac{28C_0\tau|x - y|}{r_0 - r_1}
$$

by (4.28) and (4.68); the Lipschitz bound (4.66) follows. We deduce from this that

$$
(4.70) \quad \mathcal{H}^d(\varphi(E \cap A_1 \cap (R_1 \cup R_2)))
$$

$$
\leq \left[28C_0\left(1 + \frac{\tau}{r_0 - r_1}\right) \right]^d \mathcal{H}^d(E \cap A_1 \cap (R_1 \cup R_2)).
$$

Let us summarize our estimates so far. Let us first assume that E is A'-almost minimal (see Definition 2.1). As a consequence of (4.37), we can apply (2.6) (see above (4.39)). We get that

$$
(4.71) \quad \mathcal{H}^d(E \setminus \varphi(E)) \leq \mathcal{H}^d(\varphi(E) \setminus E) + h(r_0)r_0^d \leq \mathcal{H}^d(\varphi(E) \setminus E) + h(r)r^d
$$

because $\varphi_1 = \varphi$ and h is nondecreasing. Next observe that E and $\varphi(E)$ coincide on $\mathbb{R}^n \setminus B_0$, because $\varphi(x) = x$ on $\mathbb{R}^n \setminus B_0$ and $\varphi(B_0) \subset B_0$. Then add $\mathcal{H}^d(B_0 \cap E \cap \varphi(E))$ to both sides of (4.71). We get that

$$
(4.72) \quad \mathcal{H}^d(E \cap B_0) \leq \mathcal{H}^d(\varphi(E) \cap B_0) + h(r)r^d = \mathcal{H}^d(\varphi(E \cap B_0)) + h(r)r^d,
$$

where the last part holds because $\varphi(x) = x$ on $\mathbb{R}^n \setminus B_0$. Then we add the various pieces from above and get that

$$
(4.73) \quad \mathcal{H}^d(E \cap B_0) \leq \mathcal{H}^d(L^\sharp) + I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + h(r)r^d,
$$

where we get no contribution from (4.40),

$$
(4.74) \quad I_1 = \mathcal{H}^d(Z_1), \text{ with } Z_1 = \left\{ z \in L^* \setminus L^\sharp; \operatorname{dist}(z, L^\sharp) \leq (C_0 + 2)\tau \right\}
$$

comes from (4.41),

$$
(4.75) \quad I_2 = \mathcal{H}^d(\varphi(E \cap A_2 \cap R_3)) = \int_{E \cap A_2 \cap R_3} \alpha(\rho)^{d-1}\beta(x)\, d\mathcal{H}^d(x)
$$

comes from (4.54),

$$(4.76) \qquad I_3 = \mathcal{H}^d(\varphi(E \cap A_2 \cap R_2)) \leq \frac{Cr}{r_1 - r_2} \mathcal{H}^d(E \cap A_2 \cap R_2)$$

comes from (4.59),

$$(4.77) \qquad I_4 = \frac{Cr}{r_1 - r_2} \mathcal{H}^d(E \cap A_2 \cap R_1 \setminus L^*)$$

comes from (4.61), the contribution of (4.62) was already accounted for in I_1,

(4.78) $I_5 = \mathcal{H}^d(Z_5)$, with $Z_5 = \{\lambda x; \ x \in E \cap A_2 \cap L^* \text{ and } \lambda \in [0,1]\} \setminus L^\sharp$

comes from (4.64) and we removed $L^\sharp$ because it was accounted for in (4.73), and

$$I_6 = \mathcal{H}^d(\varphi(E \cap A_1))$$

$$\leq \mathcal{H}^d(E \cap A_1 \cap R_3)$$

$$(4.79) \qquad\qquad + \left[28 C_0 \left(1 + \frac{\tau}{r - r_1}\right)\right]^d \mathcal{H}^d(E \cap A_1 \cap (R_1 \cup R_2))$$

$$\leq \left[28 C_0 \left(1 + \frac{\tau}{r - r_1}\right)\right]^d \mathcal{H}^d(E \cap A_1)$$

comes from (4.65) and (4.70). We shall estimate all these terms in the next section.

If E is A-almost minimal, we can of course use Proposition 20.9 in [**D5**] to say that E is A′-almost minimal, with the same gauge function, and use the estimate above. Since this is a little heavy, we can instead use (2.5) and work a little bit more.

Since (2.5) is written in terms of $W_1 = \{x \in E; \ \varphi(x) \neq x\}$, we shall need to control $E \setminus W_1$. We claim that

$$(4.80) \qquad E \cap B_0 \setminus W_1 \subset (E \cap A_1) \cup (E \cap L) \cup \{0\}.$$

Indeed, let $x \in E \cap B_0 \setminus W_1$ be given; by definition, $\varphi(x) = x$. We may assume that $x \notin L$, and then (4.29) says that $\hbar(x) < 1$. Also recall from (4.24) that $|\pi(x)| \leq |x|$.

If $x \in A_1$, we are happy. If $x \in A_2$, (4.31) and (4.32) say that $\varphi(x)$ lies between $\hbar(x)\pi(x)$ and $\pi(x)$, and even $\varphi(x) \neq \pi(x)$ because $\hbar(x) < 1$; hence $|\varphi(x)| < |\pi(x)| \leq |x|$, and $\varphi(x) \neq x$ (a contradiction). If $x \in B_2 \setminus \{0\}$, then again $|\varphi(x)| = |\hbar(x)||\pi(x)| < |x|$. Finally, we are also happy if $x = 0$. So (4.80) holds.

Now we want to estimate $\mathcal{H}^d(E \cap B_0)$. We say that

$$
\begin{aligned}
\mathcal{H}^d(E \cap B_0) &= \mathcal{H}^d(W_1) + \mathcal{H}^d(E \cap B_0 \setminus W_1) \\
&\leq \mathcal{H}^d(\varphi(W_1)) + h(r)r^d + \mathcal{H}^d(E \cap B_0 \setminus W_1) \\
&\leq \mathcal{H}^d(\varphi(E \cap B_0)) + h(r)r^d + \mathcal{H}^d(E \cap B_0 \setminus W_1) \\
&\leq \mathcal{H}^d(\varphi(E \cap B_0)) + h(r)r^d + \mathcal{H}^d(E \cap A_1)
\end{aligned}
$$

(4.81)

because $W_1 \subset E \cap B_0$ (by (2.4)), by (2.5), and then by (4.80) and because $\mathcal{H}^d(L) = 0$ (recall that L is at most $(d-1)$-dimensional). This almost the same thing as (4.72), and then we get the same thing as (4.73), except that we need to add the extra term

$$(4.82) \qquad\qquad I_7 = \mathcal{H}^d(E \cap A_1).$$

This term will not bother, as it is dominated by I_6.

Finally assume that E is A_+-almost minimal. This time we can only use (2.7), whose error term is $h(r)\mathcal{H}^d(\varphi(W_1))$ instead of $h(r)r^d$. Notice that if $\mathcal{H}^d(\varphi(W_1)) \geq \mathcal{H}^d(W_1)$, we have (2.5) with no error term, and otherwise we can replace the error term $h(r)\mathcal{H}^d(\varphi(W_1))$ with the larger $\mathcal{H}^d(W_1)$). Thus we get that

$$
\begin{aligned}
\mathcal{H}^d(E \cap B_0) &\leq \mathcal{H}^d(L^\sharp) + \sum_{j=1}^{7} I_j + h(r) \min(\mathcal{H}^d(\varphi(W_1)), \mathcal{H}^d(W_1)) \\
&\leq \mathcal{H}^d(L^\sharp) + \sum_{j=1}^{7} I_j + h(r)\mathcal{H}^d(E \cap B)
\end{aligned}
$$

(4.83)

when E is sliding A_+-almost minimal.

This may be better than (4.73), if the origin lies outside of E. It is not much worse, because of the local Ahlfors regularity of E. That is, if we assume (as in (2.10)) that

$$(4.84) \qquad\qquad B(0, 2r) \subset U \text{ and } h(2r) \text{ is small enough,}$$

then we get that

$$(4.85) \qquad\qquad \mathcal{H}^d(E \cap B) \leq Cr^d,$$

with a constant C that depend only on n, d, and the L_j (through the constants in (2.1)), regardless of whether $x \in E$ or not, and (4.83) is nearly as good as (4.73). If we do not want to assume that $B(0, 2r) \subset U$, we still get (4.85), but with a constant C that depends also on $r^{-1} \operatorname{dist}(B, \mathbb{R}^n \setminus U)$ (apply (2.8) to balls of size $\operatorname{dist}(B, \mathbb{R}^n \setminus U)$, and then count how many you need to get an upper bound for $\mathcal{H}^d(E \cap B)$).

5. We take a first limit and get an integral estimate

In this section we integrate the estimate obtained in the previous section, and take a first limit. We get a bound on $\mathcal{H}^d(E \cap B(0,a))$, for $a < r$, in terms of the restriction of E to an annulus $B(0,b) \setminus B(0,a)$; see Lemma 5.1 below. Later on, we will let a tend to b.

We continue with the fixed radius r, and keep the same assumptions as in Section 4. Let $a, b \in [r/2, r]$ be given, with $a < b \leq r$. We want to do the following computations. For each small $\tau > 0$ and $t \in [a, b]$, we shall write down the main estimate (4.73) (with the added term I_7 if E is A-almost minimal, or even (4.83) if E is A_+-almost minimal), with

$$(5.1) \qquad r_0 = t, \; r_1 = t - \tau, \text{ and } r_2 = t - 2\tau,$$

average in t, and then take the limit when τ tends to 0.

The main reason why we do this slow and cautious limiting process is that the author was not able to handle the more natural process when one would take $r_0 = r$, $r_1 = r - \tau$, and $r_2 = r - 2\tau$, and go to the limit directly. It seems harder to control the contribution of the regions $R_2 \cap A_2$ when we do that.

The average of (4.73) (adapted to A-almost minimal sets) yields

$$(5.2) \qquad \mathcal{H}^d(E \cap B(0,a)) \leq \mathcal{H}^d(L^\sharp \cap B(0,b)) + \sum_{j=1}^{7} J_j(\tau) + h(r)r^d,$$

with

$$(5.3) \qquad J_j(\tau) = \frac{1}{b-a} \int_{t \in [a,b]} I_j(t, \tau) \, dt$$

for $1 \leq j \leq 7$, and where $I_j(t, \tau)$ is the value of I_j for the choice of r_0, r_1, r_2 of (5.1). For A_+-almost minimal sets, we replace $h(r)r^d$ with $h(r)\mathcal{H}^d(E \cap B)$, or $Ch(r)r^d$, as in (4.83) or (4.85).

Our next task is to take the $J_j(\tau)$ one after the other, and estimate them. Notice that the sets L, L^*, $L^\sharp$ depend on r, but not on t or τ, and the sectors R_1, R_2, and R_3 depend on r and τ, but not on t. (See (4.20) and (4.18).)

The advantage of I_1 (in (4.74)) is that it does not depend on t. Recall that $\mathcal{H}^d(L^*) < +\infty$ (because we assumed (3.4) and (3.5) follows), and since the set $Z_1 = Z_1(\tau)$ is contained in L^* and decreases to the empty set when τ tends to 0, we get that

$$(5.4) \qquad \lim_{\tau \to 0} J_1(\tau) = \lim_{\tau \to 0} H^d(Z_1(\tau)) = 0.$$

Next consider the integral

$$(5.5) \qquad J_2(\tau) = \frac{1}{b-a} \int_{t \in [a,b]} \int_{E \cap A_2(t,\tau) \cap R_3(\tau)} \alpha_t(\rho)^{d-1} \beta_t(x) \, d\mathcal{H}^d(x)$$

(coming from (4.75)), where α_t and β_t are as in (4.53) and (4.55), and we still work with $\rho = |x|$. Here

$$(5.6) \qquad A_2(t,\tau) = B_1(t,\tau) \setminus B_2(t,\tau) = \{t - 2\tau \le |x| < t - \tau\}$$

by (4.25) and (5.1), the set $R_3(\tau)$ depends on τ, but not on t (see (4.20)),

$$(5.7) \qquad\qquad\qquad \alpha_t(\rho) = \frac{\rho - r_2}{r_1 - r_2} = \frac{\rho - t + 2\tau}{\tau}$$

by (4.55), and

$$
\begin{aligned}
\beta_t(x) &= \left\{ \cos^2 \theta(x) \left(\frac{2\rho - r_2}{r_1 - r_2} \right)^2 + \sin^2 \theta(x) \alpha^2(\rho) \right\}^{1/2} \\
&= \left\{ \cos^2 \theta(x) \left(\frac{2\rho - t + 2\tau}{\tau} \right)^2 + \sin^2 \theta(x) \alpha_t^2(\rho) \right\}^{1/2}
\end{aligned}
$$

(5.8)

by (4.53). We apply Fubini and get

$$(5.9) \qquad\qquad\qquad J_2(\tau) = \frac{1}{b-a} \int_E f(x) \, d\mathcal{H}^d(x),$$

where in fact we only integrate on

$$(5.10) \qquad A(\tau) = \bigcup_{t \in [a,b]} A_2(t,\tau) = \{a - 2\tau \le |x| < b - \tau\}$$

and

$$(5.11) \qquad f(x) = \int_{t \in [a,b]} \mathbb{1}_{t - 2\tau \le |x| < t - \tau}(x) \, \mathbb{1}_{R_3(\tau)}(x) \, \alpha_t(\rho)^{d-1} \beta_t(x) \, dt.$$

We want to estimate β_t. Notice that $0 \le \alpha_t(\rho) \le 1$ when $x \in A_2(t,\tau)$, so

$$
\begin{aligned}
\beta_t(x)^2 &\le \cos^2 \theta(x) \left(\frac{2\rho - t + 2\tau}{\tau} \right)^2 + 1 \\
&\le \cos^2 \theta(x) \left(\frac{t}{\tau} \right)^2 + 1 \le \frac{b^2 \cos^2 \theta(x)}{\tau^2} + 1
\end{aligned}
$$

(5.12)

because $\rho \le t - \tau$ and then $t \le b$. Since $\sqrt{1 + u^2} \le 1 + u$ for $u \ge 0$, we get that

$$(5.13) \qquad\qquad \beta_t(x) \le 1 + \tau^{-1} b \cos \theta(x) \text{ for } x \in A_2(t,\tau).$$

Thus

$$(5.14) \qquad f(x) \leq \mathbb{1}_{R_3(\tau)}(x) \left[1 + \tau^{-1} b \cos \theta(x)\right] g(x),$$

with

$$
\begin{aligned}
g(x) &= \int \mathbb{1}_{t-2\tau \leq |x| < t-\tau}(x)\, \alpha_t(\rho)^{d-1}\, dt = \int_{[\rho+\tau, \rho+2\tau]} \left(\frac{\rho - t + 2\tau}{\tau}\right)^{d-1} dt \\
(5.15) \\
&= \int_{u \in [0,\tau]} \left(\frac{\tau - u}{\tau}\right)^{d-1} du = \tau^{1-d} \int_{v \in [0,\tau]} v^{d-1}\, dv = \frac{\tau}{d},
\end{aligned}
$$

where we set $t = u + \rho + \tau$ and then $v = \tau - u$. We return to (5.9) and get that

$$
\begin{aligned}
J_2(\tau) &\leq \frac{1}{d(b-a)} \int_{E \cap A(\tau) \cap R_3(\tau)} \left[b \cos \theta(x) + \tau\right] d\mathcal{H}^d(x) \\
(5.16) \\
&\leq \frac{1}{d(b-a)} \int_{E \cap A(\tau) \setminus L^*} \left[b \cos \theta(x) + \tau\right] d\mathcal{H}^d(x)
\end{aligned}
$$

by (5.14) and because the set $R_3(\tau)$ never meets L^* (see (4.20)). Set $A(a,b) = B(0,b) \setminus B(0,a)$. We claim that

$$(5.17) \qquad \limsup_{\tau \to 0} J_2(\tau) \leq \frac{b}{d(b-a)} \int_{E \cap A(a,b) \setminus L^*} \cos \theta(x)\, d\mathcal{H}^d(x).$$

Indeed, for every $c < a$, $A(\tau) \subset A(c,b)$ for τ small, so (5.16) yields (5.17), but where we integrate on $A(c,b)$ instead of $A(a,b)$. We easily deduce (5.17) from this, because $\mathcal{H}^d(E \cap B(0,b)) < +\infty$.

Recall from (4.76) that

$$
\begin{aligned}
I_3(t) &\leq \frac{Cr}{r_1 - r_2} \mathcal{H}^d(E \cap A_2(t,\tau) \cap R_2(\tau)) \\
(5.18) \\
&\leq \frac{2Cb}{\tau} \mathcal{H}^d(E \cap A_2(t,\tau) \cap R_2(\tau))
\end{aligned}
$$

with the same sort of notation as above. We average over $[a,b]$ and get that

$$
\begin{aligned}
J_3(\tau) &\leq \frac{Cb}{\tau(b-a)} \int_{t \in [a,b]} \mathcal{H}^d(E \cap A_2(t,\tau) \cap R_2(\tau))\, dt \\
(5.19) \\
&= \frac{Cb}{\tau(b-a)} \int_{x \in E \cap A(\tau) \cap R_2(\tau)} \int_{t \in [a,b]} \mathbb{1}_{x \in A_2(t,\tau)}(t)\, dt\, d\mathcal{H}^d(x).
\end{aligned}
$$

But $\int_{t\in[a,b]} \mathbb{1}_{x\in A_2(t,\tau)}(t)\,dt \leq \tau$ by (5.6), so we are left with

$$(5.20) \qquad J_3(\tau) \leq \frac{Cb}{(b-a)}\mathcal{H}^d(E \cap A(\tau) \cap R_2(\tau)).$$

Now $E \cap A(\tau) \cap R_2(\tau) \subset H(\tau)$, where $H(\tau) = \{x \in E \cap B(0,b);\ 0 < d(x) \leq 2\tau\}$ (see (4.20)). Since $\mathcal{H}^d(E \cap B(0,b)) < +\infty$ and the $H(\tau)$ decrease to the empty set, we get that

$$(5.21) \qquad \lim_{\tau \to 0} J_3(\tau) = 0.$$

We turn to

$$(5.22) \qquad \begin{aligned} I_4(t) &= \frac{Cr}{r_1 - r_2}\mathcal{H}^d(E \cap A_2(t,\tau) \cap R_1(\tau) \setminus L^*) \\ &\leq \frac{2Cb}{\tau}\mathcal{H}^d(E \cap A_2(t,\tau) \cap R_1(\tau) \setminus L^*) \end{aligned}$$

(see (4.77)). This term can be treated exactly like $I_3(t)$, and we get that

$$(5.23) \qquad \lim_{\tau \to 0} J_4(\tau) = 0.$$

Next we study $I_5(t,\tau) = \mathcal{H}^d(Z_5(t,\tau))$, where $Z_5(t,\tau)$ is as in (4.78). For each $c \in (0,a]$, set

$$(5.24) \quad Z(c,b) = \{\lambda x;\ x \in E \cap L^* \cap B(0,b) \setminus B(0,c) \text{ and } \lambda \in [0,1]\} \setminus L^\sharp.$$

If $c < a$, then for τ small, $Z_5(t,\tau) \subset Z(c,b)$ for every $t \in [a,b]$. We take the average and get that $J_5(\tau) \leq \mathcal{H}^d(Z(c,b))$. Then we let c tend to a, use the fact that $\mathcal{H}^d(L^*) < +\infty$, and get that

$$(5.25) \qquad \limsup_{t \to 0} J_5(\tau) \leq \mathcal{H}^d(Z(a,b)).$$

Recall from (5.1) that $r_0 - r_1 = \tau$; then by (4.79)

$$(5.26) \qquad I_6(t) \leq [56C_0]^d\mathcal{H}^d(E \cap A_1(t,\tau)),$$

where by (5.1)

$$(5.27) \qquad A_1(t,\tau) = B(0,r_0) \setminus B(0,r_1) = B(0,t) \setminus B(0,t-\tau).$$

Set $A_1(\tau) = \bigcup_{t\in[a,b]} A_1(t,\tau) = B(0,b) \setminus B(0,a-\tau)$. We proceed as for $J_3(\tau)$ and use Fubini to estimate

$$(5.28) \qquad \begin{aligned} J_6(\tau) &= \frac{1}{b-a}\int_{t\in[a,b]} I_6(t)\,dt \\ &\leq [56C_0]^d\frac{1}{b-a}\int_{x\in E\cap A_1(\tau)} \mathbb{1}_{x\in A_1(t,\tau)}(t)\,dt\,d\mathcal{H}^d(x). \end{aligned}$$

As before, $\int_{t\in[a,b]} \mathbb{1}_{x\in A_1(t,\tau)}(t)\,dt \leq \tau$, so

$$J_6(\tau) \leq [56C_0]^d \frac{\tau}{b-a}\mathcal{H}^d(E\cap A_1(\tau))$$

(5.29)

$$\leq [56C_0]^d \frac{\tau}{b-a}\mathcal{H}^d(E\cap B(0,b)).$$

Recall from (4.82) that $I_7(t) = H^d(E\cap A_1(t))$; this term is smaller than the right-hand side of (5.26), so (5.29) also holds for $J_7(\tau)$. Then of course

(5.30)
$$\lim_{\tau\to 0}(J_6(\tau)+J_7(\tau)) = 0.$$

Let us summarize the estimates from this section as a lemma.

Lemma 5.1. *Let U, E, and r satisfy the assumptions of Section 4. If E is of type A or A' (as in (2.5) or (2.6)), then for all choices of $r/2 \leq a < b \leq r$*

(5.31) $\mathcal{H}^d(E\cap B(0,a)) \leq \mathcal{H}^d(L^\sharp \cap B(0,b))$

$$+ \frac{1}{d}\frac{b}{(b-a)}\int_{E\cap A(a,b)\setminus L^*} \cos\theta(x)\,d\mathcal{H}^d(x) + \mathcal{H}^d(Z(a,b)) + h(r)r^d,$$

with $Z(a,b)$ as in (5.24). If E is of type A_+ (as in (2.7)), replace $h(r)r^d$ by $h(r)\mathcal{H}^d(E\cap B)$, or $Ch(r)r^d$, as in (4.83) or (4.85).

This follows from (5.2), (5.4), (5.17), (5.21), (5.23), (5.25), and (5.30). $\square$

6. The second limit and a differential inequality

In this section we still work with U, E, and a fixed $r > 0$, with the same assumptions as in the previous sections, and we try to see what happens to the estimates of Lemma 5.1 when we fix b (in a suitable Lebesgue set) and let a tend to b.

We start with $\mathcal{H}^d(Z(a,b))$, for which no special caution needs to be taken. Set

(6.1) $\qquad Z(b) = \{\lambda x;\ x\in E\cap L^*\cap \partial B(0,b) \text{ and } \lambda\in[0,1]\}\setminus L^\sharp.$

We claim that

(6.2)
$$\limsup_{a\to b^-}\mathcal{H}^d(Z(a,b)) \leq \mathcal{H}^d(Z(b)).$$

Indeed, the sets $Z(a,b)\cup Z(b)$ all trivially contain $Z(b)$, are contained in a set L^* such that $\mathcal{H}^d(Z^*) < +\infty$, and their monotone intersection is the set $Z(b)$ (see (5.24)); (6.2) follows.

We want to evaluate $\mathcal{H}^d(Z(b))$, and for this we shall use the coarea formula. First we check that

(6.3) L^* is a rectifiable set of dimension d.

For $k \geq 0$, set $L^k = L \setminus B(0, 2^{-k})$. By the definition (4.1) and our assumptions (2.1) and (2.2), L^k is a rectifiable set of dimension $d - 1$, with finite measure. Then the set $L^*_k = \overline{B} \cap \{\lambda x;\ \lambda \geq 0 \text{ and } x \in L^k\}$ is a rectifiable set of dimension d, with finite measure. Since L^* is the union of these sets, it is rectifiable as well. We already know from (3.4) and (3.5) that $\mathcal{H}^d(L^*) < +\infty$, so (6.3) holds.

Since $Z(b) \subset L^*$, it is rectifiable as well, with finite measure, and we can write the coarea formula, applied to $Z(b)$ and the radial projection π defined by $\pi(x) = |x|$. By [**Fe**, Theorem 3.2.22], we have the following identity between measures:

$$(6.4) \qquad J d\mathcal{H}^d_{|Z(b)} = \int_0^b d\mathcal{H}^{d-1}_{|\pi^{-1}(t) \cap Z(b)}\, dt = \int_0^b d\mathcal{H}^{d-1}_{|\partial B(0,t) \cap Z(b)}\, dt,$$

where J is a Jacobian function that we shall discuss soon, and (6.4) means that we can take any positive Borel function, integrate both sides of (6.4) against this function, and get the same result. Let us say that, if needed, we normalized the Hausdorff measures so that they coincide with the Lebesgue measures of the same dimensions; then we do not need a normalization constant in (6.4) (i.e., we can take $J = 1$ when $Z(b)$ is a d-plane through the origin).

Now the Jacobian J is the same as if we computed it for L^* (either go to the definitions, or observe that (6.4) is the restriction of the coarea formula on L^*). But in $B(0, r)$, L^* coincides with a cone, so its approximate tangent planes, wherever they exist, contain the radial direction. In these direction, the derivative of π is ± 1, so $J(x) \geq 1$ almost everywhere on L^* and $Z(b)$. Also, π is 1-Lipschitz, so $J \leq 1$. Altogether, $J = 1$.

We apply (6.4) to the function 1 and get that

$$(6.5) \qquad \mathcal{H}^d(Z(b)) = \int_0^b \mathcal{H}^{d-1}(\partial B(0,t) \cap Z(b))\, dt.$$

Set $X = E \cap L^* \cap \partial B(0, b)$. For our main estimate, we can forget about removing $L^\sharp$ in (6.1), say that $Z(b)$ is contained in the cone over X, and get that

$$(6.6) \quad \mathcal{H}^{d-1}(\partial B(0,t) \cap Z(b)) \leq \mathcal{H}^{d-1}((t/b)X) = (t/b)^{d-1} \mathcal{H}^{d-1}(X),$$

and hence

$$(6.7) \qquad \mathcal{H}^d(Z(b)) \leq \int_0^b (t/b)^{d-1} \mathcal{H}^{d-1}(X) \, dt = \frac{b}{d} \mathcal{H}^{d-1}(X).$$

But we want to prepare the case of equality in the monotonicity formula, so we also evaluate the intersection with $L^\sharp$. Define a function g on X by

$$(6.8) \qquad g(x) = \sup\{t \in [0,1]; \, tx \in L\}.$$

Since L is closed, this a Borel (even semicontinuous) function. Also (again because L is closed), $g(x)x \in L$, which implies that $\lambda x \in L^\sharp$ for $0 \leq \lambda \leq g(x)$, by (4.3).

Let us check that for $0 < t \leq b$,

$$(6.9) \qquad \partial B(0,t) \cap Z(b) = b^{-1}t\{x \in X; \, g(x) < b^{-1}t\}.$$

If $z \in \partial B(0,t) \cap Z(b)$, then by (6.1) we can find $x \in X$ and $\lambda \in [0,1]$ such that $z = \lambda x$. Clearly, $\lambda = b^{-1}t$, and also $\lambda > g(x)$ because otherwise $z = \lambda x \in L^\sharp$. That is, $g(x) < b^{-1}t$. Conversely, if $x \in X$ and $g(x) < b^{-1}t$, then $z = b^{-1}tx \in Z(b)$ because otherwise $z = \lambda y$ for some $y \in L$ and $\lambda \in [0,1]$ (by (4.3)), and then $g(x) \geq b^{-1}t$.

Now (6.5) yields

$$\mathcal{H}^d(Z(b)) = \int_0^b (b^{-1}t)^{d-1} \int_X \mathbb{1}_{g(x)<b^{-1}t} \, d\mathcal{H}^{d-1}(x) \, dt$$

$$(6.10) \qquad = b^{1-d} \int_X \left\{ \int_{bg(x)}^b t^{d-1} \, dt \right\} d\mathcal{H}^{d-1}(x)$$

$$= b^{1-d} \int_X \frac{b^d}{d}(1 - g(x)^d) \, d\mathcal{H}^{d-1}(x)$$

$$= \frac{b}{d} \mathcal{H}^{d-1}(X) - \frac{b}{d} \int_X g(x)^d \, d\mathcal{H}^{d-1}(x).$$

For the moment, just set

$$(6.11) \qquad \Delta = \frac{b}{d} \int_X g(x)^d \, d\mathcal{H}^{d-1}(x),$$

and remember that $\Delta \geq 0$ and

$$(6.12) \qquad \mathcal{H}^d(Z(b)) = \frac{b}{d} \mathcal{H}^{d-1}(X) - \Delta.$$

Next we evaluate integrals on the annulus $A(a,b)$. We start with integrals on $E \cap L^*$. Denote by μ the restriction of $\mathcal{H}^d$ to $E \cap L^*$, and

by ν its pushforward by the radial projection π. Thus

(6.13) $$\nu(K) = \mu(\pi^{-1}(K)) = \mathcal{H}^d(E \cap L^* \cap \pi^{-1}(K))$$

for Borel subsets K of $[0,r]$. Let us use again the coarea formula (6.4), but now on the set $E \cap L^*$. The Jacobian is still $J = 1$, for the same reason. We apply the formula to the function $F = \mathbb{1}_{\pi^{-1}(K)}$, and we get that

(6.14)
$$\nu(K) = \mathcal{H}^d(E \cap L^* \cap \pi^{-1}(K))$$
$$= \int_{E \cap L^*} F \, d\mathcal{H}^d = \int_{t=0}^{r} \int_{E \cap L^* \cap \partial B(0,t)} F \, d\mathcal{H}^{d-1} \, dt$$
$$= \int_{t=0}^{r} \mathbb{1}_K(t) H^{d-1}(E \cap L^* \cap \partial B(0,t)) \, dt.$$

This proves that ν is absolutely continuous with respect to the Lebesgue measure on $[0,r]$, with the density $f(t) = H^{d-1}(E \cap L^* \cap \partial B(0,t))$ (the measurability of f is included in the formula). We shall restrict to points $b \in [r/2, r]$ that are Lebesgue points for f, because for such points b,

(6.15) $$\lim_{a \to b, \, a < b} \frac{1}{b-a} \int_{[a,b)} f(t) \, dt = f(b).$$

Since

(6.16)
$$\int_{[a,b)} f(t) \, dt = \nu([a,b)) = H^d(E \cap L^* \cap \pi^{-1}([a,b)))$$
$$= H^d(E \cap L^* \cap A(a,b))$$

by (6.13) and with the notation introduced above (5.17), we get that

(6.17)
$$\lim_{a \to b, \, a < b} \frac{1}{b-a} H^d(E \cap L^* \cap A(a,b)) = f(b)$$
$$= H^{d-1}(E \cap L^* \cap \partial B(0,b))$$
$$= H^{d-1}(X)$$
$$= \frac{d}{b} \mathcal{H}^d(Z(b)) + \frac{d}{b} \Delta$$
$$\geq \frac{d}{b} \mathcal{H}^d(Z(b))$$

by various definitions and (6.12).

Let us now consider the measures

$$(6.18) \quad \mu_0 = \mathcal{H}^d_{|E \cap \overline{B}(0,r)}, \ \mu_1 = \mathcal{H}^d_{|E \cap \overline{B}(0,r) \setminus L^*} = \mu_0 - \mu, \ \text{and} \ \mu_2 = \cos \theta(x) \mu_1,$$

where $\theta(x)$ is the same angle as in (5.31), for instance. For $j = 0, 1, 2$, define the pushforward measure ν_j of μ_j by π, as we did for μ in (6.13), and then decompose ν_j into its absolutely continuous part $\nu_{j,a}$ and its singular part $\nu_{j,s}$. Finally, let f_j denote the density of $\nu_{j,a}$ with respect to the Lebesgue measure $d\lambda$ on $[0, r]$.

Observe that f_j can be computed from density ratios, i.e.,

$$(6.19) \qquad\qquad f_j(t) = \lim_{\tau \to 0} \tau^{-1} \mu_j([t - \tau, t))$$

for (Lebesgue)-almost every $t \in [0, r]$. See for instance Theorem 2.12 in [**Ma**], in a much more general context.

Again we shall assume that b is such a point. Then the definitions yield

$$
\lim_{a \to b,\, a < b} \frac{1}{b-a} \int_{E \cap A(a,b) \setminus L^*} \cos \theta(x)\, d\mathcal{H}^d(x) = \lim_{a \to b,\, a < b} \frac{1}{b-a} \mu_2(A(a,b))
$$

$$(6.20)$$

$$
= \lim_{a \to b,\, a < b} \frac{1}{b-a} \nu_2([a,b))
$$

$$
= f_2(b).
$$

We now let a tend to b in (5.31), and get that

$$
\mathcal{H}^d(E \cap B(0,b)) \leq \mathcal{H}^d(L^\sharp \cap B(0,b)) + \frac{b}{d} f_2(b) + \mathcal{H}^d(Z(b)) + h(r)r^d
$$

$$
= \mathcal{H}^d(L^\sharp \cap B(0,b)) + \frac{b}{d} f_2(b) + \frac{b}{d} f(b) - \Delta + h(r)r^d
$$

$$(6.21) \qquad = \mathcal{H}^d(L^\sharp \cap B(0,b)) + \frac{b}{d} f_2(b)
$$

$$
+ \frac{b}{d}(f_0(b) - f_1(b)) - \Delta + h(r)r^d
$$

$$
\leq \mathcal{H}^d(L^\sharp \cap B(0,b)) + \frac{b}{d} f_0(b) + h(r)r^d
$$

by (6.20) and (6.2), then the second part of (6.17), then (6.18), and (6.19) and its analogue (6.15) for f. This holds for Lebesgue-almost every $b \in [r/2, r]$ and when E is of type A or A'; when E is of type A_+, we modify the last term $h(r)r^d$ as usual. This comment about A_+ will remain valid, but we shall not always repeat it.

We are now ready to take a third limit, and let b tend to r, but since additional constraints on r will arise, let us review some of the notation. Set

$$(6.22) \qquad R = \text{dist}(0, \mathbb{R}^n \setminus U) = \sup\{r > 0; \ B(0,r) \subset U\},$$

define the measure $\overline{\mu}_0 = \mathcal{H}^d_{|E \cap B(0,R)}$, and let $\overline{\nu}_0$ denote the pushforward measure of $\overline{\mu}_0$ by π, as usual. Thus our measure ν_0 is the restriction of $\overline{\nu}_0$ to $[0, r]$.

Write $\overline{\nu}_0 = \overline{\nu}_{0,a} + \overline{\nu}_{0,s}$, and denote by f_0 the density of the absolutely continuous part $\overline{\nu}_{0,a}$. There is no confusion here, the previous f_0 was just the restriction of the new one to $[0, r]$. In addition to the assumptions from the beginning of Section 4, let us assume that r is a Lebesgue point of f_0, so that in particular

$$(6.23) \qquad f_0(r) = \lim_{\tau \to 0} \frac{1}{\tau} \int_{r-\tau}^{r} f_0(t)\, dt.$$

Then we can take a limit in (6.21) (which is valid for almost every $b \in [r/2, r]$), and get that

$$(6.24) \qquad \mathcal{H}^d(E \cap B(0,r)) \le \mathcal{H}^d(L^\sharp \cap B(0,r)) + \frac{r}{d} f_0(r) + h(r) r^d.$$

7. The almost monotonicity formula

We start with the assumptions for the next theorem. We are given a domain U, which contains 0, and a finite collection of boundary sets L_j, $0 \le j \le j_{\max}$. We assume that

$$(7.1) \qquad\qquad \text{the } L_j \text{ satisfy (2.1), (2.2), and (3.4).}$$

We are also given an interval (R_0, R_1), such that

$$(7.2) \qquad\qquad\qquad B(0, R_1) \subset U$$

and

$$(7.3) \quad \text{almost every } r \in (R_0, R_1) \text{ admits a local retraction}$$

$$\text{(as in Definition 3.1).}$$

Then we consider a coral sliding almost minimal set E in U, with boundary conditions defined by the L_j, and with a gauge function h, as in (2.8).

Finally we assume that h satisfies the Dini condition (1.17), and we set, as in (1.20),

$$(7.4) \qquad\qquad A(r) = \int_0^r h(t) \frac{dt}{t} \text{ for } 0 < r < R_1.$$

Next we introduce more notation and define our functional F. First set

$$(7.5) \qquad L' = \bigcup_{1 \leq j \leq j_{\max}} L_j.$$

If, as in the introduction, there is only one boundary set L, then $L' = L$. For $0 < r < R_1$, set

$$(7.6) \qquad L^\sharp(r) = \{\lambda z;\, z \in L' \cap \overline{B}(0, r) \text{ and } \lambda \in [0, 1]\}$$

(this is the same thing as $L^\sharp$ in the previous sections), then

$$(7.7) \qquad m(r) = \mathcal{H}^d(L^\sharp(r)) \text{ and } H(r) = dr^d \int_0^r \frac{m(t)\,dt}{t^{d+1}}.$$

With our current assumptions we are not sure that $H(r) < \infty$ for r small. For instance, when $d = 2$, L' could be a logarithmic spiral in the plane; then $m(r) = ar^2$ for r small, and the integral in (7.7) diverges. But if L' is a $C^{1+\varepsilon}$ curve through the origin, $m(r) \leq Cr^{2+\varepsilon}$ (only a small sector is seen), and the integral converge. Of course $H(r) = 0$ for r small when L' lies at positive distance from the origin.

If $H \equiv +\infty$, the theorem below is true, but useless, so we may as well assume that the integral in (7.7) converges.

The reason why we choose this function H is that it is a solution of a differential equation. Namely, (7.7) yields

$$(7.8) \qquad H'(r) = \frac{d}{r}H(r) + dr^d \frac{m(r)}{r^{d+1}} = \frac{d}{r}H(r) + \frac{d}{r}m(r)$$

for $0 < r < R_1$.

We shall use the functions G and F defined by

$$(7.9) \qquad G(r) = \mathcal{H}^d(E \cap B(0, r)) + H(r) \text{ and } F(r) = r^{-d}G(r)$$

for $r \in (0, R_1)$. In our mind, H is a correcting term which we add to $\mathcal{H}^d(E \cap B(0, r))$ so that F becomes nondecreasing for minimal sets, and almost nondecreasing for almost minimal sets. Notice that although it seems complicated, it depends only on the geometry of L'. We will check later that in the special case of the introduction where L' is an affine subspace of dimension $d - 1$, $H(r) = \mathcal{H}^d(S \cap B(0, r))$, where the shade S is as in (1.9). See Remark 7.3.

Theorem 7.1. *Let U, the L_j, E, and h satisfy the assumptions (7.1)–(7.3) and (1.17). Then there exist constants $a > 0$ and $\tau > 0$, that depend only on n, d, and the constants that show up in (2.1), with the following property. Suppose in addition that*

$$(7.10) \qquad 0 \in E \text{ and } h(R_1) \leq \tau.$$

Then

(7.11) $F(r)e^{aA(r)}$ *is nondecreasing on the interval* (R_0, R_1).

We start with a few remarks, and then we shall prove the theorem.

When E is a minimal set, i.e., $h = 0$, then $A = 0$ and (7.11) says that F is nondecreasing.

In the special case when all the L_j are composed of cones of dimensions at most $d-1$ centered at the origin, $m \equiv 0$, F is the same as θ_0 in (1.8), and we recover a special case of Theorem 28.7 in [**D5**].

When the L_j are almost cones (again, of dimensions smaller than d), $r^{-d}m(r)$ is rather small near 0, and we get some form of near mono-tonicity for θ_0 as well. The author did not try to compare this to The-orem 18.15 in [**D5**], but bets that Theorem 18.15 in [**D5**] is at least as good because the competitor used in the proof looks more efficient.

When E is an A_+-almost minimal set, we do not even need (7.10), and and we can take $a = d$. See Remark 7.2 below.

In the special case when L' is an affine subspace, or more generally when L' has at most one point on each ray starting from the origin, $H(r) = H^d(S \cap B(0, r))$, where S is the shade

(7.12) $S = \big\{ x \in \mathbb{R}^n; \, \lambda x \in L' \text{ for some } \lambda \in (0, 1] \big\}$.

See Remark 7.3. We thus recover Theorems 1.2 and 1.5 as special cases of Theorem 7.1.

Even though Theorem 7.1 looks quite general, it is not clear to the author that all this generality will be useful (in fact, the author did not know exactly where to stop and ended up not taking tough decisions). In the proof, we spend some time making sure that it works in many complicated situations, but this does not mean that it is efficient there. For instance, if $n = 3$, $d = 2$, and L consists in two parallel lines at a small distance from each other, our main competitor uses two half planes bounded by the two lines, and it would be more efficient to use one of these half planes, plus a small thin stripe that connects the two lines.

Many of our assumptions (for instance (2.2), (3.4), or (7.3)) are used in the limiting process, but do not show up in the final estimate. This explains why we do not need uniform bounds for C_0 in (7.3).

We shall now complete the proof of the theorem. The main point will be a differential inequality, which we shall derive from (6.24). Recall from Sections 4–6 that (6.24) holds for almost every $r \in (0, R_1)$ that admits a local retraction, as in (4.4). Because of our assumption (7.3), this means almost every $r \in (R_0, R_1)$.

The set $L^\sharp$ in (6.24) is the same as our $L^\sharp(r)$; see (7.6), (4.3), and (4.1). Then we get that for almost every $r \in (R_0, R_1)$,

$$
\begin{aligned}
G(r) &= \mathcal{H}^d(E \cap B(0, r)) + H(r) \\
&\leq \mathcal{H}^d(L^\sharp(r) \cap B(0, r)) + \frac{r}{d} f_0(r) + h(r)r^d + H(r) \\
&\leq m(r) + \frac{r}{d} f_0(r) + h(r)r^d + H(r) \\
&= \frac{r}{d}(f_0(r) + H'(r)) + h(r)r^d
\end{aligned}
$$

(7.13)

by (7.9), (6.24), (7.7), and (7.8).

This is our main differential inequality. We now need to integrate it to obtain the desired conclusion (7.11). Let a and A be as in (7.11), and set

(7.14)
$$
g(r) = r^{-d} e^{aA(r)} \text{ for } R_0 < r < R_1.
$$

Since $F(r) = r^{-d}G(r)$ by (7.9), (7.11) amounts to checking that

(7.15)
$$
gG \text{ is nondecreasing on } (R_0, R_1).
$$

Recall from the definitions below (6.22) that for $0 < r < s \leq R_1$,

(7.16)
$$
\begin{aligned}
\mathcal{H}^d(E \cap B(0, s)) - \mathcal{H}^d(E \cap B(0, r)) &= \bar\nu_0([0, s)) - \bar\nu_0([0, r)) \\
&= \bar\nu_0([r, s)) \geq \bar\nu_{0,a}([r, s)) \\
&= \int_r^s f_0(u)\, du.
\end{aligned}
$$

By (7.7),

(7.17)
$$
H(s) - H(r) = \int_r^s H'(t)\, dt.
$$

Set $d\nu = d\bar\nu_0 + H'(t)\, dt$ for the moment. We sum the first part of (7.16) and (7.17) and get that

(7.18)
$$
G(s) - G(r) = \nu([r, s)),
$$

by (7.9). Next we check that for $0 < r < t < R_1$,

(7.19)
$$
g(t)G(t) - g(r)G(r) = \int_r^t g'(s)G(s)\, ds + \int_{[r,t)} g(s)\, d\nu(s).
$$

To see this without integrating by parts, we use Fubini's theorem to compute the integral $I = \iint_{r \leq s \leq u < t} g'(s)\, d\nu(u)$ in two different ways. When we integrate in s first, we get that

(7.20)
$$
I = \int_{r \leq u < t} (g(u) - g(r))\, d\nu(u) = \int_{r \leq u < t} g(u)\, d\nu(u) - g(r)[G(t) - G(r)].
$$

When we integrate in u first, we get

$$I = \int_{r \leq s < t} g'(s)[G(t) - G(s)]\, ds$$

(7.21)

$$= G(t)(g(t) - g(r)) - \int_{r \leq s < t} g'(s) G(s)\, ds.$$

We compare the two and get (7.19).

The positive measure ν is at least as large as its absolutely continuous part ν_a, whose density is $f_0 + H'$ (see (7.16)). Thus (7.19) yields

$$(7.22) \quad g(t)G(t) - g(r)G(r) \geq \int_r^t g'(s)G(s)\, ds + \int_r^t g(s)(f_0 + H')(s)\, ds.$$

But $g'(s) = g(s)\left[\frac{-d}{s} + aA'(s)\right] = g(s)\left[\frac{-d}{s} + \frac{ah(s)}{s}\right]$ by (7.14) and (7.4). So (7.15) and (7.11) will follow if we prove that

$$(7.23) \qquad\qquad (f_0 + H')(s) \geq \left[\frac{d}{s} - \frac{ah(s)}{s}\right]G(s)$$

for almost every $s \in (R_0, R_1)$. But (7.13) says that

$$(7.24) \qquad\qquad \frac{d}{s}G(s) \leq (f_0 + H')(s) + \frac{d}{s}h(s)s^d,$$

so it is enough to prove that

$$(7.25) \qquad\qquad \frac{d}{s}h(s)s^d \leq \frac{ah(s)}{s}G(s).$$

This is the place where we use our assumption (7.10): the (lower) local Ahlfors regularity (2.9) yields

$$(7.26) \qquad G(s) \geq \mathcal{H}^d(E \cap B(0, s/2)) \geq C^{-1}2^{-d}s^d$$

(we used $s/2$ to make sure that (2.10) holds). We now choose a sufficiently large, depending on C, and (7.25) follows from (7.26). This completes the proof of Theorem 7.1. $\qquad\qquad\qquad\qquad\qquad\square$

Remark 7.2. When E is a sliding A_+-almost minimal set (as in (2.7)), we may drop (7.10) from the assumptions of Theorem 7.1. Indeed, our initial error term in (4.83) could be taken to be $h(r)\mathcal{H}^d(E \cap B)$. When we follow the computations, we see that we can replace s^d with $\mathcal{H}^d(E \cap B(0, s))$ in (7.24) and (7.25). Then we just need to observe that $G(s) \geq \mathcal{H}^d(E \cap B(0, s))$, and we get (7.25) if $a \geq d$, without using (7.10) or the local Ahlfors regularity. Some constraint on h, namely the fact that it tends to 0, is needed to obtain the qualitative properties of E (mainly the rectifiability; the local Ahlfors regularity was only comfort), but not the

specific bound in (7.10). On the other hand, if the Dini condition (1.17) fails, (7.11) should be written differently, and is not useful near 0.

Remark 7.3. Suppose that for some $R \in (0, R_1)$

(7.27) the set $L' \cap B(0, R)$ never meets any radius $(0, x]$ more than once.

Then

$$(7.28) \qquad H(r) = \mathcal{H}^d(S \cap B(0, r)) \text{ for } 0 < r < R,$$

where S is the shade set of (7.12).

Since we also want to prepare the next remark, we shall make a slightly more general computation than needed for (7.28). First we define a function N.

Set $N(0) = 0$ and, for $x \neq 0$, denote by $N(x)$ the number of points of L' that lie on the line segment $(0, x]$. In short,

$$(7.29) \qquad N(x) = \sharp(L' \cap (0, x]) \in [0, +\infty].$$

It is not hard to check that N is a Borel function. Notice that

$$(7.30) \qquad S \setminus \{0\} = \big\{ x \in \mathbb{R}^n; \ N(x) > 0 \big\}.$$

Then set

$$(7.31) \qquad H_1(r) = \int_{S \cap B(0, r)} N(x) \, dH^d(x);$$

we will later show that H_1 could have been used instead of H, but let us not discuss this yet. When we assume (7.27), we have that

$$(7.32) \qquad N(x) \leq 1 \text{ for } x \in B(0, R)$$

and we shall see that $H_1 = H$, but let us not use this assumption for the moment.

Since we prefer to work with finite measures, let us first replace L' with $L'_\rho = L' \setminus B(0, \rho)$, where the small $\rho > 0$ will soon tend to 0. Define N_ρ, S_ρ, and $H_{1,\rho}$ as above, but with the smaller set L'_ρ. As for (7.30) and because $0 \notin S_\rho$,

$$(7.33) \qquad S_\rho = \big\{ x \in \mathbb{R}^n; \ N_\rho(x) > 0 \big\}.$$

For $0 < r \leq R_1$, set

$$(7.34) \qquad L_\rho(r) = L'_\rho \cap \overline{B}(0, r) = L' \cap \overline{B}(0, r) \setminus B(0, \rho)$$

and

$$(7.35) \qquad L^*_\rho(r) = \overline{B}(0, r) \cap \big\{ \lambda z; \ z \in L_\rho(r) \text{ and } \lambda \geq 0 \big\}.$$

Recall that by the description of L' by (2.1) and (2.2), L'_ρ is contained in a finite union of bilipschitz images of dyadic cubes of dimensions

at most $d - 1$. Since it also stays far from the origin, we get that $\mathcal{H}^d(L_\rho^*(R_1)) < +\infty$.

Set $\mu_\rho = N_\rho \mathcal{H}^d_{|B(0,R_1)} = N_\rho \mathcal{H}^d_{|S_\rho \cap B(0,R_1)}$ (by (7.33)). Our nice description of L_ρ' also implies, with just a little more work, that μ_ρ is a finite measure. Finally denote by ν_ρ the pushforward measure of μ_ρ by π; we want to show that ν_ρ is absolutely continuous and control its density, and then we shall use the coarea formula again.

By (7.12), $S_\rho \cap B(0, R_1) \subset L_\rho^*(R_1)$, which is a rectifiable truncated cone of finite $\mathcal{H}^d$ measure. This allows us to apply the coarea formula (6.4), with $Z(b)$ replaced by S_ρ and $J = 1$, as we did near (6.13)–(6.14) with the set $E \cap L^*$. That is, for every nonnegative Borel function h on $B(0, R_1)$,

$$(7.36) \qquad \int_{S_\rho \cap B(0,R_1)} h(x) \, d\mathcal{H}^d(x) = \int_{t=0}^{R_1} \int_{S \cap \partial B(0,t)} h(x) \, d\mathcal{H}^{d-1}(x) \, dt.$$

We apply this with $h = \mathbb{1}_{\pi^{-1}(K)} N_\rho$, where K is a Borel subset of $[0, R_1]$, and get that

$$
\begin{aligned}
\nu_\rho(K) &= \int_{S_\rho \cap B(0,R_1)} \mathbb{1}_{\pi^{-1}(K)}(x) \, d\mu_\rho(x) \\
&= \int_{S_\rho \cap B(0,R_1)} \mathbb{1}_{\pi^{-1}(K)}(x) N_\rho(x) \, d\mathcal{H}^d(x) \\
(7.37) \qquad &= \int_{t=0}^{R_1} \int_{S \cap \partial B(0,t)} \mathbb{1}_{\pi^{-1}(K)}(x) N_\rho(x) \, d\mathcal{H}^{d-1}(x) \, dt \\
&= \int_{t=0}^{R_1} \mathbb{1}_K(t) \int_{S \cap \partial B(0,t)} N_\rho(x) \, d\mathcal{H}^{d-1}(x) \, dt;
\end{aligned}
$$

this proves that ν_ρ is absolutely continuous with respect to the Lebesgue measure, and its density is

$$(7.38) \qquad f_\rho(t) = \int_{S_\rho \cap \partial B(0,t)} N_\rho(x) \, d\mathcal{H}^{d-1}(x).$$

The function f_ρ is locally integrable, but we can say a bit more: since N_ρ is radially nondecreasing, f_ρ is also nondecreasing, so its restriction to any $[0, T]$, $T < R_1$, is bounded.

Let us record that by the analogue of (7.31) for $H_{1,\rho}$,

$$(7.39) \quad H_{1,\rho}(r) = \int_{S_\rho \cap B(0,r)} N_\rho(x) \, dH^d(x) = \nu_\rho([0, r)) = \int_0^r f_\rho(t) \, dt.$$

For $r > 0$, define a function g_r on $B(0, r)$ by

(7.40) $g_r(0) = 0$ and $g_r(x) = N_\rho(xr/|x|)$ for $x \neq 0$.

Notice that

(7.41) $g_r(x) \geq N_\rho(x)$ for $x \in B(0, r)$,

just because N_ρ is radially nondecreasing (see the definition (7.29)). In addition, we claim that

(7.42) $g_r(x) \geq N_\rho(x) + 1$ for $x \in B(0, r) \cap L_\rho^\sharp(r) \setminus L_\rho(r)$,

where

(7.43) $L_\rho^\sharp(r) = \left\{ \lambda z; \, z \in L_\rho(r) \text{ and } \lambda \in [0, 1] \right\}$

is the analogue of $L^\sharp(r)$ for L_ρ' (see (7.6) and (7.34)). Indeed, if $x \in B(0, r) \cap L_\rho^\sharp(r) \setminus L_\rho(r)$, (7.43) says that there exists $z \in L_\rho(r)$ and $\lambda \in [0, 1]$ such that $x = \lambda z$. In addition, $\lambda \neq 1$ because $x \notin L_\rho(r)$. Thus $z \in (x, rx/|x|]$, and by (7.40) and (7.29), $g_r(x) = N_\rho(x, rx/|x|) > N_\rho(x)$, as claimed. This yields

$$
\begin{aligned}
(7.44) \qquad H_{1,\rho}(r) + \mathcal{H}^d(L_\rho^\sharp(r) \cap B(0, r)) &= \int_{S_\rho \cap B(0,r)} N_\rho(x) \, dH^d(x) \\
&\quad + \int_{L_\rho^\sharp(r) \cap B(0,r)} dH^d(x) \\
&\leq \int_{B(0,r)} g_r(x) \, dH^d(x)
\end{aligned}
$$

by (7.39), (7.41), and (7.42), and because $\mathcal{H}^d(L_\rho(r)) = 0$.

In the special case when (7.27) holds and for $r < R$, we claim that in fact

(7.45) $H_{1,\rho}(r) + \mathcal{H}^d(L_\rho^\sharp(r) \cap B(0, r)) = \int_{B(0,r)} g_r(x) \, dH^d(x)$.

By the first line of (7.44), we just need to check that $N_\rho(x) + \mathbb{1}_{L_\rho^\sharp(r)}(x) = g_r(x)$ for H^d-almost every $x \in B(0, r)$. So let $x \in B(0, r)$ be given. If $g_r(x) = 0$, then $N_\rho(x) = 0$ by (7.41), and x cannot lie in $L_\rho^\sharp(r) \setminus L_\rho(r)$, by (7.42). Since $\mathcal{H}^d(L_\rho(r)) = 0$, we get that $N_\rho(x) + \mathbb{1}_{L_\rho^\sharp(r)}(x) = 0$ almost surely. If instead $g_r(x) \neq 0$, then $(0, xr/|x|]$ meets $L_\rho(r)$ by (7.40) and (7.29). By (7.27), the intersection is unique. Call z the only point of $(0, xr/|x|] \cap L_\rho(r)$; if $|z| > |x|$, then $N_\rho(x) = 0$, but $x \in L^\sharp(r) \setminus L_\rho(r)$. If $|z| \leq |x|$, then $N_\rho(x) = 1$, but $x \notin L^\sharp(r) \setminus L_\rho(r)$. In both cases $N_\rho(x) + \mathbb{1}_{L_\rho^\sharp(r)}(x) = 1$. The claim follows.

We return to the general case. The coarea formula (7.36), applied with $h = g_r \mathbb{1}_{B(0,r)}$, yields

$$
\int_{S_\rho \cap B(0,r)} g_r(x)\, dH^d(x) = \int_{t=0}^{r} \int_{S_\rho \cap \partial B(0,t)} g_r(x)\, d\mathcal{H}^{d-1}(x)\, dt
$$

$$
= \int_{t=0}^{r} \int_{S_\rho \cap \partial B(0,t)} N_\rho(xr/t)\, d\mathcal{H}^{d-1}(x)\, dt
$$

(7.46)

$$
= \int_{t=0}^{r} (t/r)^{d-1} \int_{S_\rho \cap \partial B(0,r)} N_\rho(y)\, d\mathcal{H}^{d-1}(y)\, dt
$$

$$
= \frac{r}{d} f_\rho(r)
$$

by (7.40) and (7.38). Thus by (7.44) and (7.45)

$$
(7.47) \qquad H_{1,\rho}(r) + \mathcal{H}^d(L_\rho^\sharp(r) \cap B(0,r)) \le \frac{r}{d} f_\rho(r)
$$

for almost every $r \in [0, R_1)$, with equality when (7.27) holds and $0 < r < R$.

Let us now complete the proof of Remark 7.3. Set $\varphi = H_{1,\rho}$; by (7.39) and the paragraph above it, φ is differentiable almost everywhere on $[0, R)$, with a derivative $\varphi' = f_\rho$ which is bounded on compact subintervals, and in addition $\varphi(r) = \int_0^r \varphi'(t)\, dt$ for $0 < r < R$. Finally, set $m_\rho(r) = \mathcal{H}^d(L_\rho^\sharp(r)) \cap B(0,r))$ and recall that

$$
(7.48) \qquad \frac{r}{d} \varphi'(r) = \frac{r}{d} f_\rho(r) = \varphi(r) + m_\rho(r)
$$

almost everywhere on $[0, R)$, by (7.47) and the line that follows it.

Notice that thanks to the fact that we removed $B(0, \rho)$ from L', $m_\rho(r) = 0$ for $r < \rho$ (see (7.43) and (7.34)). Similarly, $\varphi(r) = H_{1,\rho}(r) = 0$ for $r < \rho$, by the analogue for $L_\rho' = L' \setminus B(0, \rho)$ of (7.31) and (7.29). On any compact subinterval of $(0, R)$ our equation (7.48) is very nice; let us say why it implies that

$$
(7.49) \qquad H_{1,\rho}(r) = \varphi(r) = dr^d \int_0^r \frac{m_\rho(t)\, dt}{t^{d+1}} \quad \text{for } 0 < r < R.
$$

For instance we may call ψ the right-hand side of (7.49), observe that $\eta = \varphi - \psi$ vanishes near 0, is the integral of its derivative, and satisfies $\eta'(r) = \frac{d}{r} \eta(r)$ almost-everywhere, and then apply Grönwall's inequality to show that it vanishes on $(0, R)$.

We may now let ρ tend to 0 in (7.49), notice that both $H_{1,\rho}$ and m_ρ are nondecreasing functions of ρ (see (7.43) and (7.7) for m_ρ, and (7.29), (7.31), (7.34), and (7.39) for $H_{1,\rho}$).

Thus by Beppo-Levi we get that $H_1(r) = H(r)$ for $0 < r < R$. Since $N(x) = \mathbb{1}_S(x)$ by (7.32) and (7.33), (7.31) yields $H_1(r) = \mathcal{H}^d(S \cap B(0, r))$, as needed for Remark 7.3. $\qquad\square$

Return to the general case. Notice that $H_{1,\rho}$, $L_\rho^\sharp(r)$, and f_ρ, are nondecreasing functions of ρ; then by (7.47) and Beppo-Levi,

$$(7.50) \qquad H_1(r) + \mathcal{H}^d(L^\sharp(r) \cap B(0, r)) \leq \frac{r}{d} f(r)$$

for almost every $r \in (0, R_1)$, with

$$(7.51) \qquad f(r) = \lim_{\rho \to 0} f_\rho(r) = \int_{S \cap \partial B(0,t)} N(x) \, d\mathcal{H}^{d-1}(x).$$

Let us assume that

$$(7.52) \qquad H_1(r) < +\infty \text{ for some } r > 0.$$

Since the formulas (7.29) and (7.31) are additive in terms of L', and we checked earlier that the function $\mathcal{H}_{1,\rho}$ associated to $L' \setminus B(0, \rho)$ is bounded on $[0, R_1]$ (see (7.39) and recall that ν_ρ is a finite measure), we get that $H_1(r) \leq H_1(R_1) < +\infty$ for $0 < r \leq R_1$ (because H_1 is clearly nondecreasing). We are ready for the following.

Remark 7.4. In the statement of Theorem 7.1, we may replace the function H of (7.7) with the function H_1 of (7.31).

Of course the statement is only useful when the functional F that we build with H_1 is finite somewhere, which forces (7.52). When this happens, H_1 is bounded by $H_1(R_1)$, H_1 is the integral of its derivative f (start from (7.39) and apply Beppo-Levi), and we have the differential inequality (7.50). We can then reproduce the proof of Theorem 7.1, starting at (7.17), and conclude as before. $\qquad\square$

8. E is contained in a cone when F is constant

The main result of this section is the following theorem, which gives some information on the case of equality in Theorem 7.1.

Theorem 8.1. *Let U and the L_j satisfy assumptions (7.1)–(7.3) of Theorem 7.1, and let E be a coral sliding minimal set in U, with boundary conditions given by the L_j. Suppose in addition that the functional F defined by (7.6), (7.7), and (7.9) is finite and constant on (R_0, R_1) (the same interval as in (7.3)). Set $A = B(R_1) \setminus \overline{B}(0, R_0)$. Then*

$$(8.1) \qquad \mathcal{H}^d(E \cap S \cap A) = 0,$$

where S is the shade set defined by (7.12), and, if

$$(8.2) \qquad\qquad X = \big\{\lambda x;\ \lambda \geq 0\ \text{and}\ x \in A \cap E\big\}$$

denotes the cone over $A \cap E$, then

$$(8.3) \qquad\qquad\qquad A \cap X \setminus S \subset E.$$

If in addition $R_0 < \mathrm{dist}(0, L')$, then X is also a coral minimal set of dimension d in $\mathbb{R}^n$ (with no boundary condition), and

$$(8.4) \qquad\qquad \mathcal{H}^d(S \cap B(0, R_1) \setminus X) = 0.$$

By sliding minimal, we mean sliding almost minimal with the gauge function $h = 0$, as in Definition 2.1; then the three ways to measure minimality (that is, A, A$'$, and A$_+$) are equivalent. Recall also that "coral" is defined near (1.11). When $R_0 = 0$, we will prove the result with $A = B(0, R_1) \setminus \{0\}$, but it immediately implies the result with $A = B(0, R_1)$.

In the special case of the introduction, we recover Theorem 1.3. Theorem 8.1 is also a generalization of Theorem 6.2 in [**D2**] and a partial generalization of Theorem 29.1 in [**D5**].

Of course we can feel very good when we can apply Theorem 8.1, because this means that the choice of functional F was locally optimal. We are lucky that there is at least one example (the case when L is an affine space of dimension $d - 1$) where this happens.

Most of the proof of Theorem 8.1, which we shall start now, consists in checking the proof of Theorem 7.1 for places where we could have had strict inequalities, and our main target is the string of inequalities that lead to the differential inequality (6.24).

Let E be as in the statement. This means that on (R_0, R_1), F is finite and constant. In the present situation, (7.14) just says that $g(r) = r^{-d}$, and then $gG = F$ (see (7.9)). By (7.19),

$$(8.5) \qquad \int_{[r,t)} g(s)\, d\nu(s) = -\int_r^t g'(s) G(s)\, ds = d\int_r^t s^{-d-1} G(s)\, ds$$

for $R_0 < r < t < R_1$. This forces ν, and then $\overline{\nu}_0$ to be absolutely continuous on (R_0, R_1) (recall from below (7.17) that $d\nu = d\overline{\nu}_0 + H'(t)\, dt$), and since the density of $\overline{\nu}_0$ is f_0 (see below (6.22)), we get that

$$(8.6) \quad s^{-d}(f_0 + H')(s) = g(s)(f_0 + H')(s) = d s^{-d-1} G(s)$$

$$\text{for almost every } s \in (R_0, R_1).$$

That is, (7.13) is an identity almost everywhere on (R_0, R_1) (recall that $h \equiv 0$), which means (because (7.13) was derived from (6.24)) that (6.24) is an identity almost everywhere on (R_0, R_1). Let us record this:

$$(8.7) \quad \mathcal{H}^d(E \cap B(0, r)) = \mathcal{H}^d(L^\sharp(r) \cap B(0, r)) + \frac{r}{d} f_0(r)$$
$$\text{for almost every } r \in (R_0, R_1);$$

we noted above (7.13) that $L^\sharp$ in (6.24) is the same as $L^\sharp(r)$ in (7.6).

We first prove (8.1). We proceed by contradiction, and suppose that $\mathcal{H}^d(E \cap S \cap A) > 0$. First define a function $\overline{g}$ by

$$(8.8) \qquad \overline{g}(x) = \sup\{t \in [0, 1]; \, tx \in L'\} \text{ for } x \in E \cap S.$$

Notice that $\overline{g}(x) > 0$ by the definition (7.12) of S. It is easy to check that g is a Borel function, and we can use our contradiction assumption that $\mathcal{H}^d(E \cap S \cap A) > 0$ to find $x_0 \in E \cap S \cap A$ such that

$$(8.9) \qquad \liminf_{\rho \to 0} \rho^{-d} \mathcal{H}^d(E \cap S \cap B(x_0, \rho)) = \omega_d$$

(this density requirement holds $\mathcal{H}^d$-almost everywhere on $E \cap S \cap A$, because this is a subset of the rectifiable set $L^*(R_1)$, which also has a finite Hausdorff measure (see the proof of (6.3)); in fact bounds on the lower and upper densities would also be enough for the present proof),

$$(8.10) \qquad \liminf_{\rho \to 0} \rho^{-d} \int_{E \cap S \cap B(x_0, \rho)} |\overline{g}(x)^d - \overline{g}(x_0)^d| \, d\mathcal{H}^d(x) = 0$$

(i.e., x_0 is a Lebesgue point for $\overline{g}^d$ on $E \cap S$), and finally $x_0 \notin L'$, which we may add because L' is at most $(d - 1)$-dimensional. We shall restrict our attention to radii ρ so small that

$$(8.11) \qquad B(x_0, 4\rho) \subset A \setminus L'.$$

For $r \in (R_0, R_1)$, set

$$(8.12) \qquad L^*(r) = \overline{B}(0, r) \cap \{\lambda z; \, z \in \overline{B}(0, r) \cap L' \text{ and } \lambda \geq 0\};$$

this is the same set that was called L^* in (4.2) (also see (7.5) for L') but since we shall let r vary, we include it in the notation. Set $B = B(x_0, \rho)$ and notice that

$$(8.13) \qquad S \cap B \subset L^*(|x_0| + \rho)$$

because if $x \in S \cap B$, (7.12) says that we can find $\lambda \in (0, 1]$ such that $\lambda x \in L'$; by (8.11), $|\lambda x - x_0| \geq 4\rho$, which implies that $|\lambda x - x| \geq 3\rho$

and forces $\lambda x \in B(0, |x| - 3\rho) \subset B(0, |x_0| - 2\rho)$. Hence $x \in L^*(|x_0| + \rho)$ by (8.12). Let us also check that

(8.14) $B \cap L^*(r) = B \cap \overline{B}(0, r) \cap L^*(|x_0| + \rho)$ for $|x_0| - \rho \leq r \leq |x_0| + \rho$.

The direct inclusion is clear from the definitions; conversely, if $x \in B \cap \overline{B}(0, r) \cap L^*(|x_0| + \rho)$, then $x = \lambda z$ for some $z \in L' \cap \overline{B}(0, |x_0| + \rho)$ and $\lambda \geq 0$. By definition of B and then (8.11), $|z - x| \geq |z - x_0| - \rho \geq 3\rho$, and since z is collinear with x, this forces $|z| \leq |x| - 3\rho$ or $|z| \geq |x| + 3\rho$. The second one is impossible because $z \in \overline{B}(0, |x_0| + \rho)$, and so $|z| \leq |x| - 3\rho < |x_0| - \rho \leq r$. Thus $z \in L' \cap \overline{B}(0, r)$ and $x \in L^*(r)$, which proves (8.14).

Now suppose that

(8.15) $$|x_0| - \rho \leq b < r \leq |x_0| + \rho,$$

and recall from the intermediate estimate in (6.21) that

(8.16)
$$\mathcal{H}^d(E \cap B(0, b)) \leq \mathcal{H}^d(L^\sharp(r) \cap B(0, b))$$
$$+ \frac{b}{d} f_0(b) - \frac{b}{d}(f_1(b) - f_2(b)) - \Delta(b, r)$$
$$\leq \mathcal{H}^d(L^\sharp(r) \cap B(0, b)) + \frac{b}{d} f_0(b) - \Delta(b, r),$$

where we give a more explicit notation for Δ because we will let b and r vary, and where the last line follows because $f_1(b) \geq f_2(b)$ by (6.18)). Also recall that

(8.17) $$\Delta(b, r) = \frac{b}{d} \int_X g(x)^d \mathcal{H}^{d-1}(x)$$

(by (6.11)), with $X = E \cap L^*(r) \cap \partial B(0, b)$ (see below (6.5)) and

(8.18) $$g(x) = \sup\{t \in [0, 1]; tx \in L' \cap \overline{B}(0, r)\}$$

(see (6.8), and recall the definitions (4.1) of L and (7.5) of L'). We claim that

(8.19) $$g(x) = \overline{g}(x) \text{ for } x \in X \cap B$$

when r and b as in (8.15). Indeed, if $t \in [0, 1]$ is such that $tx \in L'$, then $|tx - x| \geq |tx - x_0| - \rho \geq 3\rho$ by (8.11), hence $tx \in \overline{B}(0, |x| - 3\rho) \subset B(0, r)$; hence the supremums in (8.8) and (8.18) are the same, and (8.19) follows. Thus

(8.20)
$$\Delta(b, r) \geq \frac{b}{d} \int_{X \cap B} \overline{g}(x)^d \mathcal{H}^{d-1}(x)$$
$$\geq \frac{b}{d} \int_{B \cap E \cap S \cap \partial B(0, b)} \overline{g}(x)^d \mathcal{H}^{d-1}(x)$$

because $B \cap E \cap S \cap \partial B(0,b) \subset B \cap E \cap L^*(|x_0| + \rho) \cap \partial B(0,b) = B \cap X$ by (8.13) and (8.14). Thus (8.16) implies that

$$
(8.21) \quad
\begin{aligned}
\mathcal{H}^d(E \cap B(0,b)) \leq &\; \mathcal{H}^d(L^{\sharp}(r) \cap B(0,b)) \\
&+ \frac{b}{d} f_0(b) - \frac{b}{d} \int_{B \cap E \cap S \cap \partial B(0,b)} \overline{g}(x)^d \mathcal{H}^{d-1}(x).
\end{aligned}
$$

Fix b and let $r > b$ tend to b. Notice that by (7.6), $L^{\sharp}(r)$ is a nondecreasing set function that tends to $L^{\sharp}(b)$; since all these sets have finite $\mathcal{H}^d$-measure by (3.5), we can take a limit in (8.21) and get that

$$
(8.22) \quad
\begin{aligned}
\mathcal{H}^d(E \cap B(0,b)) \leq &\; \mathcal{H}^d(L^{\sharp}(b) \cap B(0,b)) \\
&+ \frac{b}{d} f_0(b) - \frac{b}{d} \int_{B \cap E \cap S \cap \partial B(0,b)} \overline{g}(x)^d \mathcal{H}^{d-1}(x).
\end{aligned}
$$

We compare this to (8.7) and get that

$$
(8.23) \quad \int_{B \cap E \cap S \cap \partial B(0,b)} \overline{g}(x)^d \mathcal{H}^{d-1}(x) = 0
$$

$$
\text{for almost every } b \in (|x_0| - \rho, |x_0| + \rho).
$$

We are going to use the coarea formula again. By (8.13), $B \cap E \cap S$ is contained in the truncated rectifiable cone $L^*(|x_0| + \rho)$, which has a finite $\mathcal{H}^d$-measure; as in (6.4) above, the coarea formula yields

$$
(8.24) \quad
\begin{aligned}
Jd\mathcal{H}^d_{|B \cap E \cap S} &= \int_{|x_0|-\rho}^{|x_0|+\rho} d\mathcal{H}^{d-1}_{|\pi^{-1}(t) \cap B \cap E \cap S} \, dt \\
&= \int_{|x_0|-\rho}^{|x_0|+\rho} d\mathcal{H}^{d-1}_{|\partial B(0,t) \cap B \cap E \cap S} \, dt,
\end{aligned}
$$

where in addition $J = 1$ because $L^*(|x_0| + \rho)$ is a truncated cone. We apply this to the function $\overline{g}$ and get that

$$
(8.25) \quad \int_{B \cap E \cap S} \overline{g}(x) \, d\mathcal{H}^d(x) = \int_{|x_0|-\rho}^{|x_0|+\rho} \int_{\partial B(0,t) \cap B \cap E \cap S} \overline{g}(x) \, d\mathcal{H}^{d-1}(x) \, dt = 0
$$

by (8.23). On the other hand, recall that $B = B(x_0, \rho)$, where we still may choose ρ as small as we want. Then by (8.9) and (8.10),

$$\int_{B \cap E \cap S} \overline{g}(x) \, d\mathcal{H}^d(x) \geq g(x_0) \, \mathcal{H}^d(B \cap E \cap S)$$

$$(8.26) \qquad\qquad\qquad - \int_{B \cap E \cap S} |\overline{g}(x) - \overline{g}(x_0)| \, d\mathcal{H}^d(x)$$

$$\geq g(x_0) \frac{\omega_d \rho^d}{2} - \int_{B \cap E \cap S} |\overline{g}(x) - \overline{g}(x_0)| \, d\mathcal{H}^d(x) > 0$$

if ρ is small enough. This contradiction with (8.25) completes our proof of (8.1).

Next we worry about the angle $\theta(x)$. Recall that $\theta(x) \in [0, \pi/2]$ was defined, for $\mathcal{H}^d$-almost every point $x \in E$, to be the smallest angle between the radial direction $[0, x)$ and the (approximate) tangent plane $P(x)$ to E at x; see near (4.47). We want to check that

$$(8.27) \qquad \theta(x) = 0 \text{ for } \mathcal{H}^d\text{-almost every point } x \in E \cap A.$$

Suppose that (8.27) fails. Then we can find $\eta < 1$ and a set $E_0 \subset E \cap A$ such that $\mathcal{H}^d(E_0) > 0$ and $\cos \theta(x) \leq \eta$ for $\mathcal{H}^d$-almost every $y \in E_0$.

Choose $x_0 \in E_0$ such that (as in (8.9))

$$(8.28) \qquad\qquad \liminf_{\rho \to 0} \rho^{-d} \mathcal{H}^d(E_0 \cap B(x_0, \rho)) > 0.$$

As before, we can take $x_0 \in A \setminus L'$, because $\mathcal{H}^d(L') = 0$. Then pick $\rho > 0$ so small that (8.11) holds and set $B = B(x_0, \rho)$. Notice that (8.14) holds for the same reason as before. In fact, we just need the (trivial) first inclusion. Observe that since $L^*(|x_0| + \rho)$ is a truncated rectifiable cone with finite $\mathcal{H}^d$-measure, the tangent measure to any of its points, when it exists, passes through the origin. Since this is almost never the case on the set E_0 (and by the uniqueness of the tangent plane almost everywhere on $E_0 \cap L^*(|x_0| + \rho)$), we get that

$$(8.29) \qquad \mathcal{H}^d(E_0 \cap L^*(r)) \leq \mathcal{H}^d(E_0 \cap L^*(|x_0| + \rho)) = 0$$

for $r \leq |x_0| + \rho$.

Denote by μ_B the restriction of $\mathcal{H}^d$ to $E_0 \cap B$, and let ν_B be direct image of μ_B by π. Observe that since $\mu_B \leq \overline{\mu}_0$ (the restriction of $\mathcal{H}^d$ to $E \cap B(0, R) \supset E \cap A$; see (6.22) and below), we get that $\nu_B \leq \overline{\nu}_0$ is absolutely continuous as well (see below (8.5)). Denote by f_B the density of ν_B.

Let $r \in (|x_0| - \rho, |x_0| + \rho)$ be given; for $b < r$, we defined in (6.18) the two measures $\mu_1 = \mathcal{H}^d_{E \cap \overline{B}(0,r) \setminus L^*(r)}$ and $\mu_2 = \cos \theta(x)\mu_1$, and then defined the pushed measures ν_1 and ν_2 as usual. Notice that $\mathbb{1}_{B(0,r)}\mu_B \leq \mathbb{1}_{E_0 \cap B}\mu_1$ because of (8.29).

Next consider b such that $|x_0| - \rho < b < r$. For $\tau > 0$ (small),

$$\nu_1([b-\tau, b)) - \nu_2([b-\tau, b))$$

$$= \int_{B(0,b) \setminus B(0,b-\tau)} (1 - \cos \theta(x)) \, d\mu_1(x)$$

(8.30)

$$\geq \int_{B(0,b) \setminus B(0,b-\tau)} \mathbb{1}_{E_0 \cap B}(x)(1 - \cos \theta(x)) \, d\mu_1(x)$$

$$\geq (1-\eta) \int_{B(0,b) \setminus B(0,b-\tau)} d\mu_B(x) = (1-\eta)\nu_B([b-\tau, b))$$

because $\cos \theta(x) \leq \eta$ almost everywhere on E_0, then by definition of μ_B and ν_B.

If in addition b is a Lebesgue point for the absolutely continuous parts of ν_1, ν_2, and of ν_B, we can divide (8.30) by τ, take a limit, and get that

(8.31) $$f_1(b) - f_2(b) \geq (1-\eta)f_B(b).$$

Then (6.21) says that

$$\mathcal{H}^d(E \cap B(0,b)) \leq \mathcal{H}^d(L^\sharp(r) \cap B(0,b)) + \frac{b}{d}f_2(b) + \frac{b}{d}(f_0(b) - f_1(b))$$

(8.32)

$$\leq \mathcal{H}^d(L^\sharp(r) \cap B(0,b)) + \frac{b}{d}f_0(b) - \frac{b}{d}(1-\eta)f_B(b).$$

With b fixed, we may let $r > b$ tend to b, observe that $\mathcal{H}^d(L^\sharp(r) \cap B(0,b))$ tends to $\mathcal{H}^d(L^\sharp(b) \cap B(0,b))$ (see above (8.22)), and get as in (8.22) that

(8.33) $$\mathcal{H}^d(E \cap B(0,b)) \leq \mathcal{H}^d(L^\sharp(b) \cap B(0,b)) + \frac{b}{d}f_0(b) - \frac{b}{d}(1-\eta)f_B(b).$$

Then we compare with (8.7) and get that $f_B(b) = 0$ for almost every $b \in (|x_0| - \rho, |x_0| + \rho)$. Since ν_B is absolutely continuous and f_B is its density, we get that $\nu_B = 0$, and hence $\mathcal{H}^d(E_0 \cap B) = \mu_B(E_0 \cap B) = 0$. If ρ was chosen small enough, this contradicts (8.28), and proves (8.27).

Now we shall worry about cones. For $x \in A$, set

(8.34) $$\ell(x) = \{\lambda x; \lambda > 0\}.$$

The next stage is to prove the following: for $\mathcal{H}^d$-almost every $x \in E \cap A \setminus S$ (in fact, every $x \in E \cap A \setminus S$ such that the tangent plane to E at x goes

through the origin),

(8.35) the connected component of x in $A \cap \ell(x) \setminus L'$ is contained in E.

Notice that $x \in \ell(x) \setminus L'$ because $L' \subset S$ by (7.12), so (8.35) makes sense. The proof of this is long and painful (especially since this should morally be easy once we know (8.27)), but fortunately it was already done in [**D2**, Proposition 6.11]. Of course the statement in [**D2**] does not mention sliding boundary conditions, but the argument applies for the following reason. The proof goes by contradiction: if (8.35) fails, we first find a point $y \in \ell'(x)$, the component of x in $A \cap \ell(x) \setminus L'$, which does not lie in E. Then we deform E into a set E' with smaller measure and get a contradiction. The deformation takes place in a neighborhood of the segment $[x, y]$ which is as small as we want, and in particular can be taken not to meet L' (because L' is closed and does not meet $[x, y]$). Then the boundary constraints (2.3) are automatically satisfied, the competitor constructed in [**D2**] is also valid here, and we can use the estimates from [**D2**] to get the conclusion. So (8.35) holds for $\mathcal{H}^d$-almost every $x \in E \cap A \setminus S$.

We are ready to prove (8.3). Let $z \in A \cap X$ be given, and let $x \in E \cap A$ and $\lambda \geq 0$ be such that $z = \lambda x$.

Since E is coral, $\mathcal{H}^d(E \cap B(x, r)) > 0$ for every $r > 0$, so we can find a sequence $\{x_k\}$ in E, that converges to x, such that for each $k \geq 0$, $x_k \in A \setminus S$ (possible by (8.1)) and (8.35) holds.

For each k, the segment $(0, x_k]$ lies in $\mathbb{R}^n \setminus L'$ (see the definition (7.12)), so $A \cap (0, x_k] \subset E$ by (8.35), and now $A \cap (0, x] \subset E$ because E is closed. Thus $z \in E$ if $\lambda \leq 1$.

Suppose that $\lambda > 1$ instead, and assume in addition that $z \in A \cap X \setminus S$. Then the segment $[x, z] = [x, \lambda x]$ does not meet L' (because $z \notin S$ and by the definition (7.12)), and is contained in A (because $x \in A$). For k large, $[x_k, \lambda x_k]$ is also contained in A, and does not meet L' either (because L' is closed). By (8.35), $[x_k, \lambda x_k] \subset E$. Hence $[x, \lambda x] \subset E$, and in particular $z \in E$. This completes our proof of (8.3).

Let us now assume that $R_0 < \text{dist}(0, L')$ and get additional information on X. First we want to show that

(8.36) X is a minimal set in $\mathbb{R}^n$,

with no boundary condition.

Set $\delta = \text{dist}(0, L')$ and observe that for $0 < r < \delta$, $H(r) = 0$ and so $F(r) = r^{-d} \mathcal{H}^d(E \cap B(0, r))$. See (7.6), (7.7), and (7.9). Similarly, $S \cap B(0, \delta) = \emptyset$ (see (7.12)).

By (8.2) and (8.3), E and X coincide on $B(0,\delta) \setminus \overline{B}(0,R_0)$. In particular, X is closed. If D denotes the constant value of F on (R_0, R_1); then for $R_0 < r < \delta$,

$$
\mathcal{H}^d(X \cap B(0,r)) = \frac{r^d}{\delta^d - R_0^d} \mathcal{H}^d(X \cap B(0,\delta) \setminus B(0,R_0))
$$

(8.37)
$$
= \frac{r^d}{\delta^d - R_0^d} \mathcal{H}^d(E \cap B(0,\delta) \setminus B(0,R_0))
$$

$$
= r^d D = \mathcal{H}^d(E \cap B(0,r))
$$

because X is a cone and by the discussion above. Notice also that E is a minimal set in $B(0,\delta)$, with no boundary condition (because $L' \cap B(0,\delta) = \emptyset$). Now if X was not a minimal in $\mathbb{R}^n$, it would be possible to find a strictly better competitor $\widetilde{X}$ of X, with a deformation inside $B(0,\delta)$ (we may reduce to this because X is a cone). We could then deform E inside $B(0,\delta)$, first to X and then to $\widetilde{X}$, and contradict the minimality of X. See the argument on p. 125–126 of [**D2**] (near (6.98)) for detail. This proves (8.36).

Since E is coral and $X = E$ on $B(0,\delta) \setminus \overline{B}(0,R_0)$, we easily get that X is coral too. Then we prove (8.4). Observe that

(8.38)
$$
R_1^d D = H^d(X \cap B(0,R_1))
$$
$$
= \mathcal{H}^d(E \cap (B(0,R_1)) + H^d(X \cap S \cap B(0,R_1))
$$

by (8.37) and because X is a cone, and because (8.2) and (8.3) say that $A \cap E = A \cap X \setminus S$ modulo a $\mathcal{H}^d$-negligible set. Also,

(8.39) $R_1^d D = R_1^d F(R_1) = G(R_1) = \mathcal{H}^d(E \cap (B(0,R_1)) + H(R_1)$

by (7.9). Thus $H^d(X \cap S \cap B(0,R_1)) = H(R_1)$. We shall now prove that

(8.40) $H(r) \geq \mathcal{H}^d(S \cap B(0,r))$ for $0 < r \leq R_1$,

and (8.4) will follow at once. Let us proceed as in Remark 7.3, and compare $H_2(r) = \mathcal{H}^d(S \cap B(0,r))$ with $H(r)$ by means of the differential equalities that they satisfy. We claim that for $0 < r < R_1$,

(8.41) $H_2(r) = \displaystyle\int_0^r f_2(t)\, dt$, with $f_2(t) = \mathcal{H}^{d-1}(S \cap \partial B(0,t))$.

Indeed, $S \cap B(0,R_1) \subset L^*(R_1)$, where $L^*(R_1)$ is the same as in (7.35). We observed above (7.36) that $L^*(R_1)$ is a rectifiable truncated cone with $\mathcal{H}^d(L^*(R_1)) < +\infty$; then we can use the coarea formula on $S \cap B(0,R_1) \subset L^*(R_1)$, and we get (8.41), as in (7.37)–(7.38), where we use

$\mathbb{1}_S$ instead of N_ρ. If we prove that for almost all $r < R_1$, we have the differential inequality

$$(8.42) \qquad H_2'(r) = f_2(r) \leq \frac{d}{r} H_2(r) + \frac{d}{r} m(r),$$

where $m(r) = \mathcal{H}^d(L^\sharp(r))$ is as in (7.7), then when we compare with H, which satisfied the corresponding identity (by (7.8)) and has the same vanishing values on $(0, \delta)$, we will deduce (8.40) from Grönwall's inequality, as we did near (7.49). But by the coarea formula again, $\frac{r}{d} f_2(r) = \frac{r}{d} \mathcal{H}^{d-1}(S \cap \partial B(0, r))$ is the $\mathcal{H}^d$-measure of the cone $\Gamma = \{\lambda z; z \in S \cap \partial B(0, r) \text{ and } \lambda \in [0, 1)\}$. Let us check that

$$(8.43) \qquad \Gamma \subset (S \cap B(0, r)) \cup L^\sharp(r).$$

Let $x \in \Gamma$ be given, and write $x = \lambda z$ as in the definition of Γ. Since $z \in S$, the segment $(0, z]$ meets L' at some point w (see (7.12)). If $|w| \leq |x|$, then $(0, x]$ contains $w \in L'$, and $x \in S$. Otherwise, $x \in [0, w]$ and $x \in L^\sharp(r)$ (see (7.6)). This proves (8.43), the differential inequality (8.42), and finally (8.40) and (8.4). This also completes the proof of Theorem 8.1. $\qquad \square$

9. Nearly constant F and approximation by a cone

The main result of this section is a simple consequence of the theorems on limits of sliding almost minimal sets that are at the center of [**D5**]. Roughly speaking, we start from a sliding almost minimal set E with sufficiently small gauge function h, assume that its function F is almost constant on an interval, and get that in a slightly smaller annulus, E is close to a sliding minimal set E_0 for which F is constant.

Then we should be able to apply Theorem 8.1 to the set E_0, prove that it is close to a truncated cone, and get the same thing for E, but we shall only do this here in very special cases; see Sections 11 and 12.

We start with a statement where the interval is of the form $(0, r_1)$.

Theorem 9.1. *Let U and the L_j be as in Section 7. In particular, assume (2.1), (2.2), and (3.4). Let $r_1 > 0$ be such that $B(0, r_1) \subset U$ and almost every $r \in (0, r_1)$ admits a local retraction (as in Definition 3.1). For each small $\tau > 0$ we can find $\varepsilon > 0$, which depends only on τ, n, d, U, the L_j, and r_1, with the following property. Let $E \in \mathrm{SA}^*\mathrm{M}(U, L_j, h)$ be a coral sliding almost minimal set in U, with sliding condition defined by L and some nondecreasing gauge function h (see Definition 2.1). Suppose that*

$$(9.1) \qquad B(0, r_1) \subset U \text{ and } h(r_1) < \varepsilon,$$

and

$$(9.2) \qquad F(r_1) \leq \varepsilon + \inf_{0 < r < 10^{-3} r_1} F(r) < +\infty.$$

Then there is a coral minimal set E_0 in $B(0, r_1)$, with sliding condition defined by the L_j, such that

$(9.3) \qquad$ *the analogue of F for the set E_0 is constant on $(0, r_1)$,*

$(9.4) \quad$ *E_0 satisfies the conclusion of Theorem 8.1,*

$$\textit{with the radii } R_0 = 0 \textit{ and } R_1 = r_1,$$

$(9.5) \qquad \mathrm{dist}(y, E_0) \leq \tau r_1 \textit{ for } y \in E \cap B(0, (1 - \tau) r_1),$

$(9.6) \qquad \mathrm{dist}(y, E) \leq \tau r_1 \textit{ for } y \in E_0 \cap B(0, (1 - \tau) r_1),$

and

$(9.7) \quad \left| \mathcal{H}^d(E \cap B(y, t)) - \mathcal{H}^d(E_0 \cap B(y, t)) \right| \leq \tau r_1^d$

$\qquad \textit{for all } y \in \mathbb{R}^n \textit{ and } t > 0 \textit{ such that } B(y, t) \subset B(0, (1 - \tau) r_1).$

Clearly this is not a perfect statement, because we would prefer ε not to depend on r_1 or the L_j. The difficulty is that we should then understand how the part H of the functional depends on L', and in particular its values near the origin.

Instead we decided to make a simple statement, show how the proof goes, and maybe revise it later if a specific need arises.

See Corollary 9.3 for a statement that concerns a single boundary condition that comes from a set L which is very close to an affine subspace.

We wrote the strange condition (9.2) with an infimum because we did not want to assume the Dini condition (1.17), hence we are not sure that $F(r)$ has a limit when r tends to 0; otherwise we would just require that $F(r_1) \leq \varepsilon + \lim_{r \to 0} F(r) < +\infty$.

We now prove the theorem by contradiction and compactness, following the arguments for Proposition 7.24 in [**D2**] and Proposition 30.19 in [**D5**]. Let U, the L_j, r_1, and τ be given, and suppose we cannot find ε as in the statement. In particular, $\varepsilon = 2^{-k}$ does not work, which means that for $k \geq 1$ we can find E_k and h_k as in the statement (and in particular that satisfy (9.1) and (9.2) with $\varepsilon = 2^{-k}$), and for which no minimal set E_0 satisfies the conclusion.

First we want to replace $\{E_k\}$ with a converging subsequence. Let us say what we mean by that. For each ball $B(x, r)$, let $d_{x,r}$ be the normalized local variant of the Hausdorff distance between closed sets that was defined by (1.29). Standard compactness results on the Hausdorff

distance imply that we can replace $\{E_k\}$ with a subsequence for which there is a closed set E_0 such that

$$(9.8) \qquad \lim_{k \to +\infty} d_{x,r}(E_k, E_0) = 0$$

for every ball $B(x, r)$ such that $\overline{B}(x, r) \subset U$. In short, we will just say that the E_k converge to E_0 locally in U. We want to show that

$$(9.9) \qquad E_0 \text{ is a sliding A-almost minimal set in } U,$$

where it is understood that the boundary conditions are still given by the L_j, for the gauge function h_0 defined by

$$(9.10) \qquad h_0(r) = 0 \text{ for } 0 < r < r_1 \text{ and } h_0(r) = +\infty \text{ for } r \geq r_1.$$

For this we shall apply Theorem 10.8 in [**D5**].

We may replace $\{E_k\}$ with a subsequence so that all the E_k are almost minimal of the same type (i.e., A, A′, or A$_+$; see Definition 2.1). Let us first suppose that the E_k are almost minimal of type A, i.e., satisfy (2.5). Then let us check the assumptions with $M = 1$, $\delta = r_1$, and any small number $h > 0$.

The Lipschitz assumption ((10.1) in [**D5**]) is satisfied, by (2.1). For k large enough, $h_k(r_1) \leq 2^{-k} < h$, and then the main assumption that E_k lies in the class GSAQ(U, M, δ, h) follows from (2.5) (compare with Definition 2.3 in [**D5**]). This takes care of the following hypotheses of [**D5**]: (10.3), (10.4) that holds because the E_k are coral, (10.5) that holds (with the limit E_0) precisely by (9.8), and finally the technical assumption (10.7) that holds because we said in (2.2) that the faces of the L_j are at most $(d-1)$-dimensional. So Theorem 10.8 in [**D5**] applies, and we get that E_0 lies in GSAQ$(U, 1, r_1, h)$ for any $h > 0$. Looking again at Definition 2.3 in [**D5**], we see that $E_0 \in$ GSAQ$(U, 1, r_1, 0)$, and this means that (9.9) holds with the function h_0.

If the E_k are almost minimal of type A′, i.e., satisfy (2.6), the easy part of Proposition 20.9 in [**D5**] says that they are also of type A, with the same function h_k, and we can conclude as before. Finally, if the E_k are of type A$_+$, as in (2.7), we apply Theorem 10.8 in [**D5**] with $M = 1 + a$, where $a > 0$ is any small number, $\delta = r_1$, and $h = 0$. The main assumption that $E_k \in$ GSAQ$(U, 1 + a, r_1, 0)$ is satisfied as soon as $h_k(r_1) < a$ (compare (2.7) with Definition 2.3 in [**D5**]). This time we get that $E_0 \in$ GSAQ$(U, 1 + a, r_1, 0)$ for any $a > 0$, hence $E_0 \in$ GSAQ$(U, 1, r_1, 0)$, and E is A-almost minimal with the gauge h_0, as before. So we get (9.9).

Next we worry about Hausdorff measure. By Theorem 10.97 in [**D5**], which has the same hypotheses as Theorem 10.8 there, we get that for

every open set $V \subset U$,

(9.11) $$\mathcal{H}^d(E_0 \cap V) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap V).$$

In addition, Lemma 22.3 in [**D5**], which has also the same assumptions as Theorem 10.8 there, can be applied with M as close to 1 and h as small as we want; thus we get that for every compact set $K \subset U$,

(9.12) $$\mathcal{H}^d(E_0 \cap K) \geq \limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap K).$$

Because of (2.9), for each $\rho < r_1$ there is a constant $C(\rho)$ such that $\mathcal{H}^d(E_k \cap B(0, \rho)) \leq C(\rho)$ for all $k \geq 0$. Then by (9.11), $\mathcal{H}^d(E_0 \cap B(0, \rho)) \leq C(\rho) < +\infty$ for each $\rho < r_1$, and $\mathcal{H}^d(E_0 \cap \partial B(0, r)) = 0$ for almost every $r \in (0, r_1)$, because the $\partial B(0, r)$ are disjoint. For such r,

(9.13)
$$\limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap B(0, r)) \leq \mathcal{H}^d(E_0 \cap \overline{B}(0, r)) = \mathcal{H}^d(E_0 \cap B(0, r))$$
$$\leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap B(0, r)),$$

by (9.11) and (9.12), so

(9.14) $$\lim_{k \to +\infty} \mathcal{H}^d(E_k \cap B(0, r)) = \mathcal{H}^d(E_0 \cap B(0, r)).$$

Denote by F_k the analogue of F for E_k; that is, set

(9.15) $$F_k(r) = r^{-d}\mathcal{H}^d(E_k \cap B(0, r)) + r^{-d}H(r) \text{ for } 0 < r \leq r_1,$$

where $H(r)$ is as in (7.7) (see (7.9) for the definition of F). Notice that $H(r)$ stays the same (it depends on the geometry of L', not on E_k). Similarly, set

(9.16) $$F_\infty(r) = r^{-d}\mathcal{H}^d(E_0 \cap B(0, r)) + r^{-d}H(r) \text{ for } 0 < r \leq r_1.$$

We want to show that

(9.17) $$F_\infty \text{ is finite and constant on } (0, r_1).$$

We start with the finiteness. We checked below (9.12) that for $0 < r < r_1$, there is a constant $C(r)$ such that $\mathcal{H}^d(E_0 \cap B(0, r)) \leq C(r) < +\infty$, so we just need to show that $H(r) < +\infty$. This last follows from our assumption (9.2) (for any k).

Let us apply Theorem 7.1 to E_0, with the function h_0. First observe that E_0 is also A_+-almost minimal, with the same gauge function h_0 (just compare (2.5) and (2.7) when $h(r) = 0$). By Remark 7.2, we do not need to check (7.10). Also observe that the function A defined by (7.4) with the gauge function h_0 vanishes on $(0, r_1)$. Then Theorem 7.1 says that F_∞ is nondecreasing on $(0, r_1)$.

Next let r and s be such that $0 < r < 10^{-3}r_1 < s < r_1$ and (9.14) holds for r and s. Then

$$
\begin{aligned}
F_\infty(s) = \lim_{k \to +\infty} F_k(s) &\leq (r_1/s)^d \limsup_{k \to +\infty} F_k(r_1) \\
&\leq (r_1/s)^d \lim_{k \to +\infty} (2^{-k} + F_k(r)) \\
&= (r_1/s)^d \lim_{k \to +\infty} F_k(r) = (r_1/s)^d F_\infty(r)
\end{aligned}
$$

(9.18)

by (9.14), the definition of F_k in (9.15) (twice), because E_k satisfies (9.2) with $\varepsilon = 2^{-k}$, and by (9.14) again. We know that F_∞ is nondecreasing; then $F_\infty(s)$ has a limit when s tends to r_1, and by (9.18) this limit is at most $F_\infty(r)$; thus F_∞ is constant on (r, r_1) and since this holds for all $r \in (0, 10^{-3}r_1)$ we get (9.17).

Now we claim that (9.4) holds. This is not an immediate consequence of the statement, because (9.9) does not exactly say that E_0 is minimal in U (the gauge function h_0 is not identically null), but the proof of Theorem 8.1 only uses deformations in compact subsets of $B(0, r_1)$, for which the values of $h_0(r)$, $r \geq r_1$, do not matter. So (9.4) holds.

The set E_0 also satisfies the constraints (9.5) and (9.6) for k large, by (9.8). We want to check that it satisfies (9.7) as well, and as soon as we do this, we will get the desired contradiction with the definition of E_k. Since we shall use this sort of argument a few times, let us state a lemma.

Lemma 9.2. *Let the open set $U \subset \mathbb{R}^n$ and the closed sets L_j satisfy the Lipschitz assumption (as in (2.1)), and let $B_0 = B(0, r_0) \subset U$ be given. Then let $\{E_k\}$ be a sequence of coral sliding almost minimal sets, such that*

$$
(9.19) \qquad\qquad E_k \in \mathrm{SA}^*\mathrm{M}(U, L_j, h_k) \ \text{for } k \geq 0,
$$

for a sequence $\{h_k\}$ of nondecreasing gauge functions h_k such that

$$
(9.20) \qquad\qquad \lim_{k \to +\infty} h_k(r_0) = 0.
$$

Also suppose that there is a closed set E_0 such that

$$
(9.21) \quad \lim_{k \to +\infty} d_{x,r}(E_k, E_0) \ \text{for every ball } B(x, r) \text{ such that } \overline{B}(x, r) \subset U.
$$

Finally assume that $E_0 \cap B(x_0, r_0)$ is contained in a cone X, which is a countable union of closed d-rectifiable cones X_m such that $\mathcal{H}^d(X_m \cap B(0, R)) < +\infty$ for every $R > 0$. Then for each $\tau > 0$, the following

measure estimate holds for k large:

$$(9.22) \quad \left| \mathcal{H}^d(E \cap B(y,t)) - \mathcal{H}^d(E_0 \cap B(y,t)) \right| \le \tau r_0^d$$

for all $y \in \mathbb{R}^n$ and $t > 0$ such that $B(y,t) \subset B(x_0, (1-\tau)r_0)$.

Proof: Recall that the distance d_{x_0,r_0} is defined in (1.29). We start as in the argument above. We first check that (9.9) holds with the gauge function h_0 of (9.10); the proof relies on Theorem 10.8 in [**D5**] and is the same as above. We also have the two semicontinuity estimates (9.11) and (9.12), with the same proof, and we deduce from this that

$$(9.23) \qquad \mathcal{H}^d(E_0 \cap B(y,t)) = \lim_{k \to +\infty} \mathcal{H}^d(E_k \cap B(y,t))$$

for every ball $B(y,t)$ such that $\overline{B}(y,t) \subset U$ and

$$(9.24) \qquad\qquad \mathcal{H}^d(E_0 \cap \partial B(y,t)) = 0.$$

The proof is the same as for (9.14).

Now this almost the same thing as (9.22); the difference is that maybe some balls do not satisfy (9.24), but also that in the lemma, we announced some uniformity in $B(y,t)$. This is where we use our extra assumption about X. Again we follow the proof of Proposition 7.1 in [**D2**]. First we observe that if X_m is a rectifiable cone of locally finite $\mathcal{H}^d$-measure (as in the statement), not necessarily minimal, Lemma 7.34 in [**D2**] says that $H^d(X \cap \partial B) = 0$ for every ball B. This stays true for a countable union X of such cones, and then also for $E_0 \cap B(x_0, r_0)$.

So (9.24), and then (9.23) hold for all balls $B(y,t)$ such that $\overline{B}(y,t) \subset B(x_0, r_0)$.

Finally we need to deduce from this the conclusion of the lemma. This can be done with a small uniformity argument. We do not repeat it here, and instead send the reader either to [**D2**], starting from (7.15) on p. 128 and ending after (7.19) on the next page, where the proof is done in almost the same context, or to Lemma 9.4 below and the argument that follows it, where we shall do it in a slightly more complicated situation. Lemma 9.2 follows. $\square$

We may now return to the proof of Theorem 9.1. We observed just before Lemma 9.2 that all we have to do is prove that the sets E_k that we constructed satisfy the condition (9.7) for k large.

For this, it is enough to show that we can apply Lemma 9.2 to the sets E_k, in the ball $B(0, r_1)$. We already checked all the assumptions except one, the existence of the closed rectifiable cone X.

Recall from (9.4) that E_0 satisfies the conclusions of Theorem 8.1. Then let X be, as in (8.2), the cone over $A \cap E_0$, where here $A = B(0, r_1)$

(see the remark below the statement of Theorem 8.1). By definition, X contains $E_0 \cap A = E_0 \cap B(0, r_1)$. By (8.3), and for any $r < r_1$,

$$
(9.25) \quad
\begin{aligned}
\mathcal{H}^d(X \cap B(0,r)) &\leq \mathcal{H}^d(E_0 \cap B(0,r)) + \mathcal{H}^d(S \cap B(0,r)) \\
&\leq \mathcal{H}^d(E_0 \cap B(0,r)) + H(r) \leq r^d F_\infty(r) < +\infty,
\end{aligned}
$$

where the bound on $\mathcal{H}^d(S \cap B(0,r))$ follows from (8.40). So $\mathcal{H}^d(X \cap B) < +\infty$ for every ball B. Next, X is (locally) rectifiable by (8.3), because E_0 is locally rectifiable (see (2.11)), and S is also locally rectifiable.

Maybe X is not closed because of bad things that may happen near the origin, but it is a countable union of closed cones $X_m \subset X$; for instance, we may take for X_m the cone over $E_0 \cap \overline{B}(0, r_1 - 2^{-m}) \setminus B(0, 2^{-m})$. So Lemma 9.2 applies, and we get (9.7) and the desired contradiction.

This completes our proof of Theorem 9.1 by contradiction. $\square$

In the special case of the introduction when L' is composed of a unique boundary piece L which is an affine subspace of $\mathbb{R}^n$, we can try to choose an ε in Theorem 9.1 that does not depend on the position of L. We shall only do this when 0 is not too close to the origin, see the condition (9.26) below, but we shall include the possibility that L be very close (in a bilipschitz way) to an affine subspace, without necessarily being one. We leave open the case when $\mathrm{dist}(0, L)$ is very small, but in this case we claim that it is probably just as convenient to center the balls at some $x_0 \in L$, use the simple density $r^{-d}\mathcal{H}^d(E \cap B(x_0, r))$, and apply Proposition 30.3 in [**D5**]. See Remark 9.6.

Corollary 9.3. *For each small $\tau > 0$ we can find $\varepsilon > 0$, which depends only on τ, n, and d, with the following property. Let $E \in \mathrm{SA}^*\mathrm{M}(U, L_j, h)$ be a coral sliding almost minimal set (as in Definition 2.1), associated to a single boundary set L, a gauge function h, and an open set U that contains $B(0, 1)$. Suppose that*

$$
(9.26) \qquad\qquad \tau \leq \mathrm{dist}(0, L) \leq \frac{9}{10},
$$

and let $y_0 \in L$ be such that $\mathrm{dist}(0, L) = |y_0|$. Also suppose that there exists a vector subspace L_0, whose dimension m is at most $d - 1$, and a bilipschitz mapping $\xi \colon \mathbb{R}^n \to \mathbb{R}^n$, such that $\xi(L_0) = L$ and

$$
(9.27) \quad (1 - \varepsilon)|y - z| \leq |\xi(y) - \xi(z)| \leq (1 + \varepsilon)|y - z| \text{ for } y, z \in \mathbb{R}^n.
$$

Finally suppose that

$$
(9.28) \qquad\qquad h(1) < \varepsilon \text{ and } F(1) \leq \varepsilon + \inf_{0 < r < 10^{-3}} F(r).
$$

Then we can find a coral minimal cone X_0 in $\mathbb{R}^n$ (with no sliding boundary condition), and an affine subspace L' of dimension m through y_0, such that

$$(9.29) \qquad L' \cap B(0, 99/100) \subset X_0 \text{ if } m = d - 1,$$

and, if

$$(9.30) \qquad S = \left\{ z \in \mathbb{R}^n;\ \lambda z \in L' \text{ for some } \lambda \in (0, 1] \right\}$$

denotes the shade of L',

$$(9.31) \quad E_0 = \overline{X_0 \setminus S} \text{ is a coral minimal set in } B(0, 1 - \tau),$$

$$\text{with sliding boundary condition defined by } L',$$

$$(9.32) \qquad \operatorname{dist}(y, E_0) \le \tau \text{ for } y \in E \cap B(0, 1 - \tau),$$

$$(9.33) \qquad \operatorname{dist}(y, E) \le \tau \text{ for } y \in E_0 \cap B(0, 1 - \tau),$$

and

$$(9.34) \quad \left| \mathcal{H}^d(E \cap B(y, t)) - \mathcal{H}^d(E_0 \cap B(y, t)) \right| \le \tau$$

$$\text{for all } y \in \mathbb{R}^n \text{ and } t > 0 \text{ such that } B(y, t) \subset B(0, 1 - \tau).$$

We are mostly interested in the case when L is an affine subspace, in which case Definition 2.1 could be replaced by the simpler Definition 1.4 (or its A$'$ and A$_+$ variants). In this case we take L_0 to be the vector space parallel to L, and the proof will show that the conclusion holds with $L' = L$.

For our main application, X_0 will turn out to be a plane through 0 or a $\mathbb{Y}$-set centered at 0, (9.29) will say that X_0 contains (a piece of) L, and this will determine X_0 and E_0.

Of course we don't really need to have a bilipschitz mapping defined on the whole $\mathbb{R}^n$; a ball of radius 3 centered at $\xi^{-1}(y_0)$ would be more than enough to cover $B(0, 1)$ and prove the theorem.

When L lies very close to the origin, we can still get an approximation result, but only by a sliding minimal cone, and we claim that it should be as simple to use results concerning ball that are centered on the boundary. See Remark 9.6. When $\operatorname{dist}(x, L) \ge 9/10$, the simplest is to restrict to $B(0, 9/10)$ and apply Proposition 7.24 in [**D2**] to the plain almost minimal set E, with no boundary condition. We would still get something like the conclusion above, maybe in a smaller ball (notice that then $E_0 = X_0$ in $B(0, 9/10)$, and (9.29) and (9.31) are void anyway).

We start the proof of the corollary as for Theorem 9.1 above. We may assume that the dimension of L is a given integer $m \leq d - 1$, and that L_0 is a fixed m-dimensional vector space.

We assume that we can find $\tau > 0$ such that the corollary fails for all ε, and we let E_k provide a counterexample for the statement, with $\varepsilon = 2^{-k}$. This time, E_k is a sliding almost minimal set associated to a (changing) boundary set L_k, of fixed dimension m, and which is bilipschitz equivalent, through some mapping ξ_k whose bilipschitz constants that tend to 1, to the set L_0.

By precomposing each ξ_0 with a translation in L_0 if needed, we may assume that $\xi_k(0) = y_{0,k}$, where $y_{0,k}$ denotes a point of L_k such that $|y_{0,k}| = \mathrm{dist}(0, L_k)$. Then we can use the uniform Lipschitz bounds on the ξ_k to choose our subsequence so that the ξ_k converge, uniformly in $\overline{B}(0,1)$, to some Lipschitz mapping ξ_∞. Since the ξ_k satisfy (9.27) with bilipschitz constants that tend to 1, ξ_∞ is an isometry from $\overline{B}(0,1)$ to its image, and it is known that there is an affine isometry of $\mathbb{R}^n$ that coincides with ξ_∞ on $\overline{B}(0,1)$. We subtract $\xi_\infty(0)$ and we get a linear isometry ξ of $\mathbb{R}^n$ such that

$$
(9.35) \quad
\begin{aligned}
\xi(y) = \xi_\infty(y) - \xi_\infty(0) &= \lim_{k \to +\infty} [\xi_k(y) - \xi_k(0)] \\
&= \lim_{k \to +\infty} [\xi_k(y) - y_{0,k}] \text{ for } y \in \overline{B}(0,1).
\end{aligned}
$$

By rotation invariance of our problem, the sets $\xi^{-1}(E_k)$ still provide a counterexample, so we may replace the E_k by $\xi^{-1}(E_k)$; thus we may assume that ξ is the identity, i.e., that

$$
(9.36) \qquad \lim_{k \to +\infty} [\xi_k(y) - y_{0,k}] = y \text{ for } y \in \overline{B}(0,1).
$$

We want to use the m-planes $L_0 + y_{0,k}$ and their limit

$$
(9.37) \qquad L_\infty = L_0 + y_{0,\infty}, \text{ where } y_{0,\infty} = \xi_\infty(0) = \lim_{k \to +\infty} y_{0,k},
$$

as boundaries. Maybe we should mention that in the simpler case of Corollary 9.3 when L is an affine space, we may decide in advance that L_0 was the vector space parallel to L, and then ξ_∞ is a translation. When we follow the argument above, we see that we do not need to modify the E_k, and that we get $L_0 + y_{0,k} = L_k$.

We replace $\{E_k\}$ with a subsequence for which $\{E_k\}$ converges, locally in $B(0,1)$, to a closed set E_∞. This just means that $d_{0,\rho}(E, E_\infty)$ tends to 0 for every $\rho \in (0,1)$, and the existence of a subsequence like this is classical.

We claim that

(9.38) E_∞ is a coral almost minimal set in $B(0, 1 - \tau/2)$,

with boundary condition coming from L_∞,

and with the special gauge function h_0 given by (9.10) with $r_1 = 1$.

Compared to what we did for Theorem 9.1, the situation is slightly different, because we have variable boundary conditions, so we shall have to use a limiting result of Section 23 in [**D5**] rather than its Section 10. We want to apply Theorem 23.8 in [**D5**], so let us check the assumptions.

We use the domain $U = B(0, 1 - \tau/2)$ and the single boundary piece L_∞, and then the Lipschitz assumption (23.1) in [**D5**] is satisfied. We use the bilipschitz mappings $\widetilde{\xi}_k$ defined by

(9.39) $\widetilde{\xi}_k(y) = \xi_k(y - y_{0,\infty})$.

Notice that if we set $U_k = \widetilde{\xi}_k(U)$, then

$$U_k = \widetilde{\xi}_k\left(B\left(0, 1 - \frac{\tau}{2}\right)\right)$$

(9.40)

$$= \xi_k\left(B\left(-y_{0,\infty}, 1 - \frac{\tau}{2}\right)\right) \subset y_{0,k}$$

$$+ B\left(-y_{0,\infty}, 1 - \frac{\tau}{3}\right) \subset B\left(0, 1 - \frac{\tau}{4}\right)$$

by (9.36) and for k large; in addition,

(9.41) $\widetilde{\xi}_k(L_\infty) = \xi_k(L_0) = L_k$,

so the condition (23.2) of [**D5**] is satisfied, the asymptotically optimal Lipschitz bound (23.3) comes from (9.27), and (23.4) (the pointwise convergence of the $\widetilde{\xi}_k$ to the identity) follows from (9.36), (9.39), and (9.37).

With the current notation, E_k is sliding minimal in a domain that contains U_k, with a boundary condition given by $\widetilde{\xi}_k(L_\infty)$, and the gauge function h_k. This implies that (23.5) in [**D5**] holds with constants M that are arbitrarily close to 1, δ arbitrarily close to 1, and h as small as we want because of (9.28) (see the discussion below (9.9)).

Assumption (23.6) comes from the convergence of the E_k to E_∞, we don't need to check the technical assumptions (10.7) or (19.36) there, because the L_k are m-dimensional, and so Theorem 23.8 in [**D5**] applies. We get that E_∞ is sliding quasiminimal, relative to L_∞, and with constants M arbitrarily close to 1, δ arbitrarily close to 1, and h arbitrarily small; (9.38) follows.

Next we estimate Hausdorff measures. We start with the analogue of (9.11). By Remark 23.23 in [**D5**], we can apply Theorem 10.97 in [**D5**] as soon as the assumptions of Theorem 23.8 there are satisfied. We checked this to prove (9.38), so we get that if V is an open set such that $V \subset B(0, \rho)$ for some $\rho < 1$,

$$(9.42) \qquad \mathcal{H}^d(E_\infty \cap V) \leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap V),$$

as in (23.23) in [**D5**]. This stays true for any open set $V \subset B(0, 1)$ (just apply (9.42) to $V \cap B(0, \rho)$ and take a limit). Similarly, Lemma 23.31 in [**D5**] (applied with constants $M > 1$ arbitrarily close to 1 and $h > 0$ arbitrarily small) says that if K is a compact subset of $B(0, 1)$,

$$(9.43) \qquad \limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap K) \leq \mathcal{H}^d(E_\infty \cap K).$$

Alternatively we could use Lemma 23.36 in [**D5**], or copy its proof. As in (9.14), we deduce from (9.42) and (9.43) that

$$(9.44) \qquad \lim_{k \to +\infty} \mathcal{H}^d(E_k \cap B(0, r)) = \mathcal{H}^d(E_\infty \cap B(0, r))$$

for almost every $r \in (0, 1)$.

Denote by F_k the functional associated to E_k and the boundary L_k. By (7.9),

$$(9.45) \qquad F_k(r) = r^{-d}\mathcal{H}^d(E_k \cap B(0, r)) + r^{-d}H_k(r),$$

where $H_k(r)$ is given by (7.6) and (7.7) in terms of $L' = L_k$. That is,

$$(9.46) \qquad H_k(r) = dr^d \int_0^r \frac{m_k(t)\, dt}{t^{d+1}},$$

with $m_k(t) = \mathcal{H}^d(L_k^\sharp(t) \cap B(x, t))$ and $L_k^\sharp(t) = \{\lambda z;\ \lambda \in [0, 1]$ and $z \in L_k \cap \overline{B}(0, t)\}$.

We also define F_∞, H_∞, m_∞, and the $L_\infty^\sharp(t)$ similarly, but in terms of E_∞ and $L_\infty = L_0 + y_{0,\infty}$ (see (9.37)). We want to check that

$$(9.47) \qquad F_\infty(r) = \lim_{k \to +\infty} F_k(r)$$

for almost every $r \in (0, 1)$ and, because of (9.44), it will follow from the next lemma.

Lemma 9.4. *We have that*

$$(9.48) \qquad H_\infty(r) = \lim_{k \to +\infty} H_k(r) \text{ for } 0 < r < 1.$$

Proof: First notice that the lemma is particularly simple when the sets L_k are affine subspaces of dimension m (and in this case it is also slightly simpler to use Remark 7.3 and the more direct formula (7.28) to compute H_∞ and the H_k in terms of shades), and also when $m < d - 1$ (because then $H_k = H_\infty = 0$). When the L_k are bilipschitz images of $(d-1)$-planes, we need to be a little careful about how the H_k tend to a limit, but hopefully the reader will not be surprised by the result.

So let us assume that $m = d - 1$. The main step will be to check that for $0 < t < 1$,

$$(9.49) \qquad m_\infty(t) = \lim_{k \to +\infty} m_k(t).$$

Recall from (9.41) that $L_k = \widetilde{\xi}_k(L_\infty)$, so we get a parameterization of $L_k^\sharp(t)$ as

$$(9.50) \qquad L_k^\sharp(t) = \big\{ \lambda \widetilde{\xi}_k(y); \; (\lambda, y) \in [0,1] \times Z_k(t) \big\},$$

where $Z_k(t) = \big\{ y \in L_\infty; \; \widetilde{\xi}_k(y) \in \overline{B}(0,t) \big\}$. We use the area formula and get that

$$
(9.51) \begin{aligned}
\mathcal{H}^d(L_k^\sharp(t)) &\leq \int_{y \in Z_k(t)} \int_{t \in [0,1]} \lambda^{m-1} \left| \frac{\partial \widetilde{\xi}_k(y)}{\partial x_1} \wedge \cdots \wedge \frac{\partial \widetilde{\xi}_k(y)}{\partial x_m} \wedge \widetilde{\xi}_k(y) \right| dy \, d\lambda, \\
&= \frac{1}{d} \int_{y \in Z_k(t)} \left| \frac{\partial \widetilde{\xi}_k(y)}{\partial x_1} \wedge \cdots \wedge \frac{\partial \widetilde{\xi}_k(y)}{\partial x_m} \wedge \widetilde{\xi}_k(y) \right| dy,
\end{aligned}
$$

where we took coordinates $(x_1, \ldots, x_m)$ in L_∞ to compute the Jacobian of the parameterization. Unfortunately for the converse computation, we only have an inequality because the parameterization may fail to be injective.

Recall from (9.39), (9.36), and (9.37) that the $\widetilde{\xi}_k$ converge uniformly to the identity; then for each $\rho \in (t, 1)$, $Z_k(t) \subset Z_\rho =: L_\infty \cap \overline{B}(0, \rho)$ for k large, so

$$(9.52) \quad \limsup_{k \to +\infty} \mathcal{H}^d(L_k^\sharp(t)) \leq \frac{1}{d} \limsup_{k \to +\infty} \int_{y \in Z_\rho} \left| \frac{\partial \widetilde{\xi}_k(y)}{\partial x_1} \wedge \cdots \wedge \frac{\partial \widetilde{\xi}_k(y)}{\partial x_m} \wedge \widetilde{\xi}_k(y) \right| dy.$$

Next we claim that the wedge product $\frac{\partial \widetilde{\xi}_k(y)}{\partial x_1} \wedge \cdots \wedge \frac{\partial \widetilde{\xi}_k(y)}{\partial x_m}$ converges weakly to the constant m-vector $e_1 \wedge \cdots \wedge e_m$, where the e_j are the corresponding basis vectors of L_0 (the vector space parallel to L_∞), in the sense that for each small product Q of intervals (with faces parallel

to the axes) in L_∞,

$$(9.53) \qquad \lim_{k\to+\infty} \int_Q \frac{\partial\widetilde\xi_k(y)}{\partial x_1} \wedge \cdots \wedge \frac{\partial\widetilde\xi_k(y)}{\partial x_m}\, dy = |Q|e_1 \wedge \cdots \wedge e_m.$$

This last follows from integrating coordinate by coordinate, and using the fact that the $\widetilde\xi_k(y)$ converge uniformly to y. Also, we have uniform bounds on $\nabla\widetilde\xi_k$, by (9.27), which allows one to pass from the dense class of linear combinations of characteristic functions $\mathbb{1}_Q$ to the vector-valued function $y\,\mathbb{1}_{Z_\rho}$, so

$$(9.54)\quad \lim_{k\to+\infty} \frac{1}{d}\int_{y\in Z_\rho}\left|\frac{\partial\widetilde\xi_k(y)}{\partial x_1}\wedge\cdots\wedge\frac{\partial\widetilde\xi_k(y)}{\partial x_m}\wedge y\right| = \frac{1}{d}\int_{y\in Z_\rho}\left|e_1\wedge\cdots\wedge e_m\wedge y\right|.$$

We can replace y by $\widetilde\xi_k(y)$ in the right-hand side of (9.54) because the $\widetilde\xi_k$ are uniformly Lipschitz, and converge uniformly to the identity; we get that

$$(9.55)\quad \limsup_{k\to+\infty}\mathcal{H}^d(L_k^\sharp(t)) \le \frac{1}{d}\int_{y\in Z_\rho}\left|e_1\wedge\cdots\wedge e_m\wedge y\right| = \mathcal{H}^d(L_\infty^\sharp(\rho)),$$

where the last part comes from the same computations as above, i.e., applying the area formula to

$$(9.56)\qquad L_\infty^\sharp(\rho) = \left\{\lambda z;\ \lambda\in[0,1]\text{ and } y\in L_\infty\cap B(0,\rho)\right\}$$

(this time, our parameterization is injective). We let ρ tend to t from above and get that

$$(9.57)\qquad \limsup_{k\to+\infty} m_k(t) = \limsup_{k\to+\infty}\mathcal{H}^d(L_k^\sharp(t)) \le \mathcal{H}^d(L_\infty^\sharp(t)) = m_\infty(t)$$

by (7.7). We may now concentrate on the other inequality in (9.49), which we shall obtain by topology.

For $\rho < 1$, still set $Z_\rho = L_\infty\cap\overline{B}(0,\rho)$, and set $T_\rho = \{\lambda y;\ \lambda\in [0,1]\text{ and } y\in Z_\rho\}$. The T_ρ are homothetic cones in a fixed space V of dimension d; let π_V denote the orthogonal projection on that space.

Let $t\in(0,1)$ be given, and pick $\rho < t$. For k large, $Z_\rho\mathbb{1}Z_k(t)$, so

$$(9.58)\qquad L_k^\sharp(t) \supset \left\{\lambda\widetilde\xi_k(y);\ (\lambda, y)\in[0,1]\times Z_\rho\right\},$$

by (9.50). We can even define a continuous mapping $h_k\colon T_\rho \to L_k^\sharp(t)$, by setting $h_k(z) = \lambda\widetilde\xi_k(y)$ when $z = \lambda y$ for some $\lambda\in[0,1]$ and $y\in Z_\rho$ (notice that y can be computed from z, since it is its radial projection on L_∞).

Next let $\rho_1 < \rho$ and $\xi \in T_{\rho_1}$ be given. Notice that for k large and $z \in \partial T_\rho$ (the boundary of T_ρ in the space V),

$$(9.59) \quad \operatorname{dist}(\pi_V(h_k(z)), \xi) \geq \operatorname{dist}(\pi_V(h_k(z)), T_{\rho_1}) \geq \frac{1}{2} \operatorname{dist}(\partial T_\rho, T_{\rho_1}) > 0$$

because $\pi_V \circ h_k(z)$ tends to z uniformly on T_ρ (since $\widetilde{h}_k(y)$ tends to y uniformly on Z_ρ). The same argument also shows that

$$(9.60) \qquad\qquad \operatorname{dist}(w, \xi) \geq \frac{1}{2} \operatorname{dist}(\partial T_\rho, T_{\rho_1}) > 0$$

for $w \in [z, \pi_V(h_k(z))]$ (and z as above). This means that there is a homotopy, from the identity on ∂T_ρ to the mapping $\pi_V \circ h_k$ on ∂T_ρ, among continuous mappings with values in V and that do not take the value ξ.

Denote by π_ξ the radial projection, centered at ξ, that maps any point $v \in V \setminus \{\xi\}$ to the point $\pi_\xi \in \partial T_\rho$ that lies on the half line from ξ through v. The mapping $\pi_\xi \circ \pi_V \circ h_k$ is continuous, from ∂T_ρ to itself, and had degree 1 (when we identify ∂T_ρ with a $(d-1)$-sphere) because it is homotopic to the identity. This implies that it cannot be extended to a continuous mapping from T_ρ to ∂T_ρ, and in turns this implies that $\xi \in \pi_V \circ h_k(T_\rho)$ (otherwise, $\pi_\xi \circ \pi_V \circ h_k$ would be a continuous extension).

We proved that for k large, $\pi_V \circ h_k(T_\rho)$ contains T_{ρ_1}, hence by (9.58) $\pi_V(L_k^\sharp(t))$ contains T_{ρ_1}. Then

$$(9.61) \qquad m_k(t) = \mathcal{H}^d(L_k^\sharp(t)) \geq \mathcal{H}^d(\pi_V(L_k^\sharp(t))) \geq \mathcal{H}^d(T_{\rho_1})$$

by (7.7) and because π_V is 1-Lipschitz. This is true for all choices of $\rho_1 < \rho < t$; when ρ_1 tends to t, the right-hand side of (9.61) tends to $\mathcal{H}^d(T_t) = m_\infty(t)$ (by (7.7) again). Thus

$$(9.62) \qquad\qquad \liminf_{k \to +\infty} m_k(t) \geq m_\infty(t),$$

which gives the second half of (9.49).

Notice that (by the proof of (9.57)) the m_k are uniformly bounded on $[0, r]$, $r < 1$. They also vanish on $[0, \tau)$, because since the L_k satisfy (9.26), $B(0, \tau)$ does not meet L_k and $L_k^\sharp(t) = \emptyset$ for $t < \tau$; see below (9.46). Our conclusion (9.48) now follows from the dominated convergence theorem, the definition (9.46), (9.49), and (7.7) (for H_∞).   □

By Lemma 9.4, (9.44), and the definitions (7.9) and its analogue for F_∞, we get (9.47). We then proceed as in the proof of (9.17). We first apply Theorem 7.1 to E_∞ with the gauge function h_0 (of course L_∞ satisfies the conditions of Section 7, and Remark 7.2 still allows us

not to check (7.10)); then Theorem 7.1 says that F_∞ is nondecreasing on $(0,1)$ and the argument of (9.18) yields that F_∞ is constant on $(0,1)$.

Now let us apply Theorem 1.3, with $R_1 = 1$ and R_0 arbitrarily small. The main assumption is satisfied because F_∞ is constant on $(0,1)$. Notice that

$$(9.63) \qquad\qquad \mathrm{dist}(0, L_\infty) \geq \tau$$

because $\mathrm{dist}(0, L_k) \geq \tau$ by (9.26), $\widetilde{\xi}_k(L_\infty) = L_k$ by (9.41), and the $\widetilde{\xi}_k$ converge pointwise to the identity (see below (9.41)).

Set $A = B(0,1) \setminus B(0, R_0)$, and denote by X_∞ the cone over $A \cap E_\infty$ (as in (1.13)); we take $R_0 < \tau$, and then we know that X_∞ is a coral minimal cone (with no boundary condition). Set $A' = B(0,1) \setminus \overline{B}(0, R_0)$, and let us check that

$$(9.64) \qquad\qquad A' \cap E_\infty = A' \cap \overline{X_\infty \setminus S_\infty},$$

where S_∞ is the shade of L_∞ seen from the origin (as in (7.12)). By (1.12), $H^d(A' \cap E_\infty \cap S_\infty) = 0$; then each $x \in E_\infty \cap A'$ is the limit of a sequence $\{x_j\}$ in $E_\infty \setminus S_\infty$ (recall that E_∞ is coral). Obviously, $x_j \in A'$ for j large, hence, by definition of X_∞, $x_j \in X_\infty$; it follows that $x \in A' \cap \overline{X_\infty \setminus S_\infty}$. Conversely, if $x \in A' \cap \overline{X_\infty \setminus S_\infty}$, (1.14) says that $x \in E_\infty$; (9.64) follows.

Since (9.64) and (9.63) say that $E_\infty = X_\infty$ on $B(0, \tau) \setminus \overline{B}(0, R_0)$, and X_∞ is a cone, we see that it does not depend on R_0 (provided that we take $R_0 < \tau$), and then (letting R_0 tend to 0),

$$(9.65) \qquad\qquad B(0,1) \cap E_\infty = B(0,1) \cap \overline{X_\infty \setminus S_\infty}.$$

Here we took the intersection with $B(0,1)$, but we do not really need to: all our sets are initially defined as subsets of $B(0,1)$. We still need to modify the sets X_∞ and $E_\infty = \overline{X_\infty \setminus S_\infty}$ a little to get the desired X_0 and E_0, because E_∞ is minimal with a sliding boundary condition defined by L_∞, while we promised in (9.31) that $E_0 = E_{0,k}$ would be sliding minimal with respect to an affine d-plane $L' = L'_k$ through $y_{0,k}$.

So we replace X_∞ and E_∞ with slightly different sets. Recall from (9.37) that $L_\infty = L_0 + y_{0,\infty}$ and that $y_{0,\infty}$ is the limit of the $y_{0,k}$; also, $|y_{0,\infty}| = \mathrm{dist}(0, L_\infty) \geq \tau$ because $|y_{0,k}| = \mathrm{dist}(0, L_k)$, by (9.41), and by (9.63). Then we can find numbers ρ_k, that tend to 1, and linear isometries I_k, that converge to the identity, so that $y_{0,k} = \rho_k I_k(y_{0,\infty})$. We set

$$(9.66) \quad X_0 = X_{0,k} = I_k(X_\infty), \; E_0 = E_{0,k} = \rho_k I_k(E_\infty),$$

$$\text{and } L' = L'_k = \rho_k I_k(L_\infty).$$

Notice that L'_k is a m-dimensional affine subspace that contains $y_{0,k}$, because L_∞ contains $y_{0,\infty}$ (by (9.37)). We want to show that for k large, these set satisfy the properties announced in Corollary 9.3.

For the special case of Corollary 9.3 where L is an affine space, we can assume that the L_k are affine subspaces, all of the same dimension and parallel to the same vector space L_0. For this case, we announced that we would take $L'_k = L_k$, so let us check that we can choose ρ_k and I_k so that this is the case. Recall that $y_{0,k}$ is the orthogonal projection of 0 on L_k, and hence, since the L_k converge to L_∞, $y_{0,\infty}$ is the projection of 0 on L_∞. We need to set $\rho_k = |y_{0,k}|/|y_{0,\infty}|$. If $y_{0,k}$ is collinear to $y_{0,\infty}$, we just take I_k to be the identity. Otherwise, let us first define I_k on the 2-plane P_k that contains $y_{0,k}$ and $y_{0,\infty}$, so that it preserves P_k and maps $y_{0,\infty}$ to $y_{0,k}|y_{0,\infty}|/|y_{0,k}|$. Then set $I_k(z) = z$ on $P_k^\perp$. It is easy to see that I_k is an isometry; since it preserves $L_0 \subset P_k^\perp$, the mapping $\rho_k I_k$ sends L_∞ to L_k, as needed.

The fact that X_∞ and X_0 are coral minimal cones (no boundary condition) comes from our application of Theorem 1.3 (see below (9.63)).

For (9.29), we need to check that if $m = d-1$, $L'_k \cap B(0, 99/100) \subset X_0$, or equivalently (by (9.66)),

$$(9.67) \qquad L_\infty \cap B(0, 99\rho_k^{-1}/100) \subset I_k^{-1}(X_0) = X_\infty.$$

Recall from above (9.63) that we were able to apply Theorem 1.3 to E_∞, with $R_1 = 1$ and R_0 arbitrarily small; then (1.15) holds, which says that $\mathcal{H}^d(S_\infty \cap B(0,1) \setminus X_\infty) = 0$; since S_∞ is a d-dimensional set (because L_∞ does not contain the origin), we deduce from this that X_0 contains $L_\infty \cap B(0,1)$ (recall that X_∞ is closed too); (9.67) (for k large) follows. So (9.29) holds.

The next condition (9.31) is equivalent to the fact that E_∞ is a coral minimal set in $B(0, (1-\tau)\rho_k^{-1})$, with boundary condition given by $L_\infty = \rho_k^{-1} I_k^{-1}(L')$, and this follows from (9.38) as soon as k is so large that $(1-\tau)\rho_k^{-1} < 1 - \tau/2$.

The next conditions (9.32) and (9.33) come from the fact that E_∞ is the limit, locally in $B(0,1)$, of the E_k, and that $\rho_k I_k$ tends to the identity. We are thus left with (9.34) to check. That is, we need to show that for k large,

$$(9.68) \qquad |\mathcal{H}^d(B \cap E_k) - \mathcal{H}^d(B \cap E_{0,k})| \leq \tau$$

for every ball $B = B(y, t)$ such that $B \subset B(0, (1 - \tau))$. We intend to proceed as in Lemma 9.2, and the main step is the following small lemma.

Lemma 9.5. *Denote by H the set of pairs (y,t), with $y \in \overline{B}(0,(1-\tau/2))$ and $t \geq 0$, such that $\overline{B}(y,t) \subset \overline{B}(0,(1-\tau/2))$. We include $t = 0$ (and set $B(x,0) = \emptyset$) to make sure that H is compact. Set*

$$(9.69) \qquad h(y,t) = \mathcal{H}^d(E_\infty \cap B(y,t)) \text{ for } (y,t) \in H.$$

Then h is a continuous function on H.

Proof: We prove the continuity of h at the point $(y,t) \in H$. Suppose $\{(y_j, t_j)\}$ is a sequence in H that tends to (y,t), and denote by Δ_j the symmetric difference

$$(9.70) \qquad \Delta_j = [B(y_j, t_j) \setminus B(y,t)] \cup [B(y,t) \setminus B(y_j, t_j)];$$

then

$$(9.71) \qquad \begin{aligned} \limsup_{j \to \infty} |h(y,t) - h(y_j,t_j)| &\leq \limsup_{j \to \infty} \mathcal{H}^d(E_\infty \cap \Delta_j) \\ &\leq \mathcal{H}^d(E_\infty \cap \partial B(y,t)) \end{aligned}$$

because if V is any open neighborhood of $\partial B(y,t)$, Δ_j is contained in V for j large (when $t = 0$, we replace $\partial B(y,t)$ by $\{y\}$ to make this work).

But Lemma 7.34 in [**D2**] implies, as for (9.24) above, that

$$(9.72) \qquad \mathcal{H}^d(E_\infty \cap \partial B(y,t)) = 0;$$

thus $h(y,t) - h(y_j,t_j)$ tends to 0 and the lemma follows. $\square$

Let us return to the proof of (9.34), fix a point $(y,t) \in H$, and prove that

$$(9.73) \qquad \begin{aligned} \lim_{k \to +\infty} \mathcal{H}^d(B(y,t) \cap E_k) &= \mathcal{H}^d(B(y,t) \cap E_\infty) \\ &= \lim_{k \to +\infty} \mathcal{H}^d(B(y,t) \cap E_{0,k}). \end{aligned}$$

The first part follows from (9.42), (9.43), and the fact that $\mathcal{H}^d(E_\infty \cap \partial B(y,t)) = 0$ exactly as for (9.44) or (9.14). For the second part, we use the definition (9.66) and get that

$$(9.74) \quad B(y,t) \cap E_{0,k} = B(y,t) \cap \rho_k I_k(E_\infty) = \rho_k I_k(B(y_k, t_k) \cap E_\infty),$$

with $y_k = \rho_k^{-1} I_k^{-1}(y)$ and $t_k = \rho_k^{-1} t$. Then

$$(9.75) \qquad \mathcal{H}^d(B(y,t) \cap E_{0,k}) = \rho_k^d \mathcal{H}^d(B(y_k, t_k) \cap E_\infty),$$

which tends to $\mathcal{H}^d(B(y,t) \cap E_\infty)$ by Lemma 9.5. So (9.73) holds.

Next let $\tau > 0$ be given (as in the statement). Since H is compact and h is continuous on H, there is a constant $\eta > 0$ such that

(9.76) $\quad |h(y,t) - h(y',t')| \leq \tau/10$ for $(y,t), (y',t') \in H$

$$\text{such that } |y - y'| + |t - t'| \leq 5\eta.$$

We may assume that $10\eta < \tau$. Then let Y be a finite subset of $B(0,1)$ which is η-dense in $B(0,1)$, and T a finite subset of $[0,1]$ which is η dense. Let us include $t = 0$ in T. Then let H_0 denote the set of pairs $(y,t) \in H$ such that $y \in Y$ and $t \in T$. By (9.73), we get that for k large

(9.77) $\quad |\mathcal{H}^d(B(y,t) \cap E_k) - \mathcal{H}^d(B(y,t) \cap E_\infty)|$

$$+ |\mathcal{H}^d(B(y,t) \cap E_\infty) - \mathcal{H}^d(B(y,t) \cap E_{0,k})| \leq \tau/10$$

for $(y,t) \in H_0$. For (9.68), we want a similar estimate for every pair (y,t) such that $B(y,t) \subset B(0, 1 - \tau)$. Let us fix such a pair and first try to choose pairs $(y',t_1), (y',t_2) \in H_0$, so that if we set $B = B(y,t)$, $B_1 = B(y',t_1)$, and $B_2 = B(y',t_2)$, then

(9.78) $$B_1 \subset B \subset B_2 \text{ and } t_2 - t_1 \leq 4\eta.$$

Let us take $y' \in Y$ such that $|y' - y| \leq \eta$. If $t > \eta$, take $t_1 \in T \cap [t - 2\eta, t - \eta]$; otherwise, take $t_1 = 0$. In both cases, take $t_2 \in T \cap [t + \eta, t + 2\eta]$. With these choices, we get (9.78), and also both (y', t_j) lie in H_0. Then for instance

(9.79) $$\mathcal{H}^d(B_1 \cap E_k) \leq \mathcal{H}^d(B \cap E_k) \leq \mathcal{H}^d(B_2 \cap E_k),$$

which by (9.77) yields

(9.80) $\quad \mathcal{H}^d(B_1 \cap E_\infty) - \tau/10 \leq \mathcal{H}^d(B \cap E_k) \leq \mathcal{H}^d(B_2 \cap E_\infty) + \tau/10$

for k large, and since

(9.81) $\mathcal{H}^d(B_1 \cap E_\infty) \leq \mathcal{H}^d(B \cap E_\infty) \leq \mathcal{H}^d(B_2 \cap E_\infty) \leq \mathcal{H}^d(B_1 \cap E_\infty) + \tau/10$

by (9.78) and (9.76), we get that

(9.82) $$|\mathcal{H}^d(B \cap E_k) - \mathcal{H}^d(B \cap E_\infty)| \leq \tau/5$$

for k large. The same proof also yields

(9.83) $$|\mathcal{H}^d(B \cap E_{0,k}) - \mathcal{H}^d(B \cap E_\infty)| \leq \tau/5,$$

and (9.68) follows. This completes our proof that for k large, the sets $X_{0,k}$ and $E_{0,k}$ satisfy all the properties (9.29)–(9.34) (relative to E_k and L_k) that were required in the statement of Corollary 9.3. This contradicts the initial definitions, and completes our proof of Corollary 9.3. The

additional statement below the corollary (concerning the case of affine subspaces) was checked below (9.66). □

Remark 9.6. We could try to prove an analogue of Corollary 9.3 when the origin lies very close to the boundary set L, but if L is $(d-1)$-dimensional and without more precise assumptions on L (for instance, uniform C^1 estimates, rather than bilipschitz, that say that L is close to an affine subspace), we will not get a good convergence of the functions H_k to their analogue for L_∞, as in Lemma 9.4 above, and then we shall not be able to show that F_∞ is constant and apply Theorem 1.3 as above.

Even if we do (for instance, if L is assumed to be an affine subspace), we do not get the same conclusion as before, because we may get that $y_{0,\infty} = 0$ and $L_\infty = L_0$, and then we cannot take $R_0 < \mathrm{dist}(0, L_\infty)$ to prove that X_0 is a minimal cone. Instead, we only get the approximation of E_k by a sliding minimal cone, with boundary condition given by L_0.

We shall not try to pursue this here, because it seems as convenient, when 0 is very close to L, to consider balls centered at a point $x_0 \in L$, try to deduce some interesting information on the density $\theta_{x_0}(r)$ from assumption (9.28), and then apply Proposition 30.3 in [**D5**] to show that E is well approximated by a sliding minimal cone, with boundary condition given by an affine subspace of the same dimension as the L_0 from (9.27). See the argument below (11.42) and the proof of Proposition 12.7 for illustrations of this scheme.

Generally speaking, if we we have sets L_j that are not necessarily a single m-plane, and we still want an analogue of Theorem 9.1 where ε depends only on n and d, and not on the specific choices of L_j or r_1, we can always try to mimic the proof of Theorem 9.1 or Corollary 9.3, but we will need to understand how the function H depends on the specific L_j, and this seems easier to do on a case by case basis. Notice that when each L_k in the proof above is composed of two planes that both tend to the same plane L_∞, Lemma 9.4 fails in general.

We relax a little and end this section with the variant of Theorem 9.1 that corresponds to an annulus.

Theorem 9.7. *Let U and the L_j be as in Section 7, and in particular, assume (2.1), (2.2), and (3.4). Let r_0, r_1 be such that $0 < r_0 < r_1$ and $B(0, r_1) \subset U$, and assume that almost every $r \in (r_0, r_1)$ admits a local retraction (as in Definition 3.1). For each small $\tau > 0$ we can find $\varepsilon > 0$, which depends only on τ, n, d, U, the L_j, r_0, and r_1, with the following property. Let E be a coral sliding almost minimal set in U, with sliding condition defined by L and some nondecreasing gauge function h.*

Suppose that

$$(9.84) \qquad B(0, r_1) \subset U \text{ and } h(r_1) < \varepsilon,$$

and

$$(9.85) \qquad F(r_1) \leq F(r_0) + \varepsilon < +\infty.$$

Then there is a coral minimal set E_0 in $B(0, r_1)$, with sliding condition defined by L, such that

$$(9.86) \qquad \text{the analogue of } F \text{ for the set } E_0 \text{ is constant on } (r_0, r_1),$$

$$(9.87) \quad \text{the conclusions of Theorem 8.1 hold for } E_0,$$
$$\text{with the radii } R_0 = r_0 \text{ and } R_1 = r_1,$$

$$(9.88) \qquad \text{dist}(y, E_0) \leq \tau r_1 \text{ for } y \in E \cap B(0, r_1),$$

$$(9.89) \qquad \text{dist}(y, E) \leq \tau r_1 \text{ for } y \in E_0 \cap B(0, r_1),$$

and

$$(9.90) \quad \left| \mathcal{H}^d(E \cap B(y, t)) - \mathcal{H}^d(E_0 \cap B(y, t)) \right| \leq \tau r_1^d \text{ for all } y \in \mathbb{R}^n \text{ and } t > 0$$
$$\text{such that } B(y, t) \subset B(0, (1-\tau)r_1) \setminus B(0, (1+\tau)r_0).$$

There would probably be a way to state Theorem 9.7 so that ε does not depend on r_0, but we shall not do it. Also, we shall not try to generalize Corollary 9.3 to the case of an annulus.

The proof is almost the same as for Theorem 9.1. We change nothing up to (9.18), which we replace with

$$
\begin{aligned}
(9.91) \quad F_\infty(r_1) &= r_1^{-d} \mathcal{H}^d(E_0 \cap B(0, r_1)) + r_1^{-d} H(r_1) \\
&\leq \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap B(0, r_1)) + r_1^{-d} H(r_1) \\
&= \liminf_{k \to +\infty} F_k(r_1) \leq \liminf_{k \to +\infty} (F_k(r_0) + 2^{-k}) = \liminf_{k \to +\infty} F_k(r_0) \\
&= r_0^{-d} H(r_0) + r_0^{-d} \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap B(0, r_0))
\end{aligned}
$$

by (9.11) and (9.85). But for $r_0 < r$, $H(r_0) \leq H(r)$ and

$$
\begin{aligned}
(9.92) \quad \liminf_{k \to +\infty} \mathcal{H}^d(E_k \cap B(0, r_0)) &\leq \limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap \overline{B}(0, r_0)) \\
&\leq \mathcal{H}^d(E_0 \cap \overline{B}(0, r_0)) \\
&\leq \mathcal{H}^d(E_0 \cap B(0, r)),
\end{aligned}
$$

by (9.12), so $F_\infty(r_1) \leq F_\infty(r)$ for $r_0 < r < r_1$. The fact that F_∞ is non-decreasing on (r_0, r_1) is proved as before, so we get that F_∞ is constant

on (r_0, r_1), as in (9.17). The rest of the argument is as before; we only prove (9.90) for balls that do not meet $B(0, (1 + \tau)r_0)$, because we use the fact that in the annulus A, E is contained in the cone X.   □

10. Simple properties of minimal cones

Before we start proving the applications mentioned in the introduction, we shall introduce some properties of minimal cones (even, without sliding boundary condition), that will be used in the proofs.

Let us denote by $\mathrm{MC} = \mathrm{MC}(n, d)$ the set of coral minimal sets of dimension d in $\mathbb{R}^n$ which are also cones centered at the origin.

The set $\mathrm{MC}(n, d)$ is only known explicitly when $d = 1$ (and then MC is composed of lines and sets $Y \in \mathbb{Y}_0(n, 1)$; see the definition above (1.28)), and when $d = 2$ and $n = 3$, where $\mathrm{MC}(3, 2)$ is composed of 2-planes, sets $Y \in \mathbb{Y}_0(3, 2)$, and cones of type $\mathbb{T}$ (our name for a cone over the union of the edges of a regular tetrahedron centered at the origin); see for instance [**Mo**]. Even $\mathrm{MC}(4, 2)$ is not known explicitly, but at least we have a rough description of the minimal cones of $\mathrm{MC}(n, 2)$ for all $n > 3$.

For $X \in \mathrm{MC}(n, d)$, the density of X is

$$(10.1) \qquad\qquad d(X) = \mathcal{H}^d(X \cap B(0, 1)).$$

Notice that by the monotonicity of density (see near (1.8)), we get that for $X \in \mathrm{MC}(n, d)$, $x \in \mathbb{R}^n$, and $r > 0$,

$$
\begin{aligned}
(10.2) \qquad r^{-d}\mathcal{H}^d(X \cap B(x, r)) &\leq \lim_{\rho \to +\infty} \rho^{-d}\mathcal{H}^d(X \cap B(x, \rho)) \\
&\leq \lim_{\rho \to +\infty} \rho^{-d}\mathcal{H}^d(X \cap B(0, \rho + |x|)) \\
&= \lim_{\rho \to +\infty} \rho^{-d}(\rho + |x|)^d d(X) = d(X).
\end{aligned}
$$

Denote by

$$(10.3) \qquad\qquad \omega_d = \mathcal{H}^d(\mathbb{R}^d \cap B(0, 1))$$

the $\mathcal{H}^d$-measure of the unit ball in $\mathbb{R}^d$. Since we know that the minimal sets are rectifiable, and also that

$$(10.4) \qquad\qquad \lim_{r \to 0} r^{-d}\mathcal{H}^d(X \cap B(x, r)) = \omega_d$$

for $\mathcal{H}^d$-almost every point x of a rectifiable set X (see for instance the easy part of Theorem 16.2 in [**Ma**]), we deduce from (10.2) that $d(X) \geq \omega_d$ for $X \in \mathrm{MC}(n, d)$. We can then try to classify the minimal cones by their density. The beginning is easy. To prove that

$$(10.5) \qquad \text{if } X \in \mathrm{MC}(n, d) \text{ and } d(X) \leq \omega_d, \text{ then } X \text{ is a } d\text{-plane,}$$

one observes that if $d(X) = \omega_d$, then (10.2) is an identity for almost-every point $x \in X$. A close look at the proof of the monotonicity of density shows that for almost every $y \in X$, the tangent plane to X at y goes through x. A complicated (but not surprising) argument (see Section 6 in [**D2**]) shows that then X is also a cone centered at x, and (10.5) follows easily. But in the present case we could also use the fact the tangent plane at almost all $y \in X$ contains almost every $x \in X$ to get (10.5) directly. Here is a quantitative (but not explicit) version of this.

Lemma 10.1. *For each choice of integers $0 < d < n$, there is a constant $d(n, d) > \omega_d$ such that $d(X) \geq d(n, d)$ for $X \in \mathrm{MC}(n, d) \setminus \mathbb{P}_0(n, d)$.*

Proof: We prove this by contradiction and compactness. Suppose that for each integer $k \geq 0$ we can find $X_k \in \mathrm{MC}(n, d) \setminus \mathbb{P}_0(n, d)$ such that $d(X) \leq \omega_d + 2^{-k}$. We may replace $\{X_k\}$ with a subsequence for which the X_k converge to a limit X_∞, i.e., that $d_{0,N}(X_k, X_\infty)$ tends to 0 for each integer N (see the definition (1.29)). Since every X_k is a cone, X_∞ is a cone too. By Theorem 4.1 in [**D1**] (with $\Omega = \mathbb{R}^n$, $M = 1$, and $\delta = +\infty$), X_∞ is a coral minimal set in $\mathbb{R}^n$. By the low-ersemicontinuity of $\mathcal{H}^d$ along that sequence (for instance Theorem 3.4 in [**D1**]), $d(X_\infty) \leq \liminf_{k \to +\infty} d(X_k) = \omega_d$. Since $d(X_\infty) \geq \omega_d$ anyway, (10.5) says that $X_\infty \in \mathbb{P}_0(n, d)$. Since $d_{0,2}(X_k, X_\infty)$ tends to 0, X_k is arbitrarily close to a plane in $B(0, 1)$. In addition, by assumption $\mathcal{H}^d(X_k \cap B(0, 1)) \leq \omega_d + 2^{-k}$. We may now apply either Almgren's regularity result for almost minimal sets [**A3**], or Allard's regularity result for stationary varifolds [**All**] (whichever the reader finds easiest), and get that for k large, X_k is C^1 near the origin. But X_k is a cone, and this means that X_k is a plane. This proves the lemma. $\square$

The following consequence of Lemma 10.1 will be used in the next section.

Corollary 10.2. *Let L be a vector space of dimension $m < d$ in $\mathbb{R}^n$, and let E be a coral sliding minimal cone, with boundary condition given by L. Suppose that $\mathcal{H}^d(E \cap B(0, 1)) \leq \frac{\omega_d}{2}$. Then $m = d - 1$ and E is a half d-plane bounded by L.*

Proof: Of course we assume that $E \neq \emptyset$. Then, since E is coral and rectifiable (by (2.11)), we can find $x \in E \setminus L$ such that (10.4) holds. Let us apply Theorems 1.2 and 1.3 to the set $E_x = E - x$, which is sliding minimal with a boundary condition coming from $L_x = L - x$. We take $U = \mathbb{R}^n$, $R_0 = 0$, and $R_1 = +\infty$ (or arbitrarily large). Theorem 1.2 says

that the function F defined by (1.10) is nondecreasing. But

(10.6) $\lim\limits_{r \to 0} F(r) = \lim\limits_{r \to 0} r^{-d} \mathcal{H}^d(E_x \cap B(0,r)) = \lim\limits_{r \to 0} r^{-d} \mathcal{H}^d(E \cap B(x,r)) = \omega_d$

by (1.10), because $\mathrm{dist}(0, L_x) = \mathrm{dist}(x, L) > 0$, and by (10.4), while

$$\lim\limits_{r \to +\infty} F(r) = \lim\limits_{r \to +\infty} r^{-d}[\mathcal{H}^d(S \cap B(0,r)) + \mathcal{H}^d(E_x \cap B(0,r))]$$

(10.7)
$$= \frac{\omega_d}{2} + \lim\limits_{r \to 0} r^{-d} \mathcal{H}^d(E_x \cap B(0,r))$$

$$\leq \frac{\omega_d}{2} + \lim\limits_{r \to 0} r^{-d} \mathcal{H}^d(E \cap B(0, r + |x|)) \leq \omega_d$$

by (1.10), where S is the shade of L_x as in (1.9), because $B(x,r) \subset B(0, r + |x|)$, and by assumption.

So F is constant on $(0, +\infty)$, and Theorem 1.3 applies to any interval. Notice that $\mathrm{dist}(0, L_x) > 0$; so we get a coral minimal cone X (no boundary condition) such that $X \setminus S \subset E_x$ (as in (1.14)). Then

$$d(X) = \lim\limits_{r \to +\infty} r^{-d} \mathcal{H}^d(X \cap B(0,r))$$

(10.8)
$$\leq \liminf\limits_{r \to +\infty} r^{-d}[\mathcal{H}^d(E_x \cap B(0,r)) + \mathcal{H}^d(S \cap B(0,r))]$$

$$= \lim\limits_{r \to +\infty} F(r) = \omega_d,$$

and by (10.5) X is a d-plane. Notice that if $m < d - 1$ (the dimension of L), the proof of (10.8) even gives that $d(X) \leq \frac{\omega_d}{2}$, which is impossible.

In addition, (1.15) tells us that $\mathcal{H}^d(S \setminus X) = 0$, which implies that $S \subset X$ (because X is closed and S is d-dimensional). So X is the d-plane that contains S. We know from (1.14) that $X \setminus S \subset E_x$, and the definition (1.13) says that $E_x \subset X$. Since in addition $\mathcal{H}^d(E_x \cap S) = 0$ by (1.12), we get that $E_x = (X \setminus S) \cup L_x$, and E is a half plane bounded by L, as announced. $\square$

For Section 12 we will need to restrict to dimensions n and d such that

(10.9) $d(X) > \dfrac{3\omega_d}{2}$ for $X \in \mathrm{MC}(n,d) \setminus [\mathbb{P}_0(n,d) \cup \mathbb{Y}_0(n,d)]$,

i.e., when $X \in \mathrm{MC}(n,d)$ is neither a vector d-plane nor a cone of type $\mathbb{Y}$ (see the definitions above (1.28)).

The author does not know for which values of n and d this assumption is satisfied. When $d = 1$, (10.9) holds trivially because $\mathrm{MC}(n,1) = \mathbb{P}_0 \cup \mathbb{Y}_0$. When $d = 2$ and $n = 3$, it follows from the explicit description of $\mathrm{MC}(3,2)$ as the union of $\mathbb{P}_0$, $\mathbb{Y}_0$, and the cones of type $\mathbb{T}$. When $d = 2$ and $n > 3$ we also get in Proposition 14.1 of [**D2**] a description of the

minimal cones that implies (10.9). In all these cases, there is even a constant $d_{n,d} > \frac{3\omega_d}{2}$ such that

(10.10) $\qquad\qquad d(X) \geq d_{n,d}$ when $X \in \mathrm{MC}(n,d) \setminus (\mathbb{P}_0 \cup \mathbb{Y}_0)$.

See Lemma 14.2 in [**D2**] for the last case when $n > 3$.

Finally, we claim that (10.9) probably holds when $d = n - 1$ and $n \leq 6$. The proof is written in Lemmas 2.2 and 2.3 of [**Lu**], in the special case of $d = 3$, $n = 4$, and Luu uses a result of Almgren [**A1**] that says that the only minimal cones of dimension 3 in $\mathbb{R}^4$ whose restriction to the unit sphere are smooth hypersurfaces are the 3-planes. The same proof should work in codimension 1 when $n \leq 6$, with a dimension reduction argument, and starting from the generalization of Almgren's result by Simons [**Si**]. But even when $d = 3$ and $n = 4$, it does not seem to be known whether the analogue of (10.10) holds for some constant $d_{n,d} > \frac{3\omega_d}{2}$.

Let us check that assumption (10.9) implies an apparently slightly stronger one.

Lemma 10.3. *If n and d are such that* (10.9) *holds, then for each small $\tau > 0$ we can find $\eta > 0$ such that if $X \in \mathrm{MC}(n,d)$ is such that $d(X) \leq \frac{3\omega_d}{2} + \eta$, then either $X \in \mathbb{P}_0$ or else there is a cone $Y \in \mathbb{Y}_0$ such that $d_{0,1}(X,Y) \leq \tau$ (where $d_{0,1}$ is as in (1.29)).*

Proof: The standard proof is the same as for the first part of Lemma 10.1. We suppose that this fails for some $\tau > 0$, and take a sequence $\{X_k\}$ in $\mathrm{MC}(n,d)$ such that $d(X_k) \leq \frac{3\omega_d}{2} + 2^{-k}$, but $X_k \notin \mathbb{P}_0$ and there is no cone $Y \in \mathbb{Y}_0$ as in the statement. By (10.9), $d(X_k) \geq \frac{3\omega_d}{2}$.

Then extract a subsequence that converges to a limit X_∞. Observe that $X_\infty \in \mathrm{MC}(n,d)$ by Theorem 4.1 in [**D1**], that X_∞ is a coral cone (as a limit of coral cones), and that $d(X_\infty) = \lim_{k \to +\infty} d(X_k) = \frac{3\omega_d}{2}$ by Theorem 3.4 in [**D1**] (for the lowersemicontinuity inequality) and Lemma 3.12 in [**D2**] (for the uppersemicontinuity, which unfortunately was not already included in [**D1**]). By (10.9), $X_\infty \in \mathbb{Y}_0$, this contradicts the definition of the X_k or the fact that they tend to X_∞, and this proves the lemma. $\qquad\square$

11. Sliding almost minimal sets that look like a half plane

In this section we use the main results of the previous sections to prove Corollary 1.7.

Let L and E be as in the statement. In particular, E is a coral sliding almost minimal set in $B(0,3)$, associated to a unique boundary piece L,

which is a $(d-1)$-plane through the origin. We may choose the type of almost minimality as we wish (that is, A, A$'$, or A$_+$ in Definition 2.1).

We assume that h is sufficiently small, as in (1.30), and that E is sufficiently close in $B(0,3)$ to a half-plane $H \in \mathbb{H}(L)$, as in (1.31), and we want to approximate E by planes and half planes, in the Reifenberg way.

Recall that $\mathbb{H}(L)$ is the set of d-dimensional half planes bounded by L, and let $\mathbb{P}$ be the set of (affine) d-planes. The proof below will also follow known tracks. See for instance Section 16 of [**D2**]. We first check that E does not have too much mass in a slightly smaller ball.

Lemma 11.1. *Set $r_1 = \frac{28}{10}$ and $B_1 = B(0, r_1)$. There is a constant $C \geq 0$, that depends only on n and d, such that*

$$\text{(11.1)} \qquad \mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(H \cap B) + C\sqrt{\varepsilon}$$

for each ball B centered on E and such that $\overline{B} \subset B_1$.

Proof: We are shall use the local Ahlfors regularity of E (see (2.9)). First observe that

$$\text{(11.2)} \qquad h(r_1) \leq C \int_{r_1}^3 \frac{h(t)\,dt}{t} \leq C\varepsilon$$

because h is nondecreasing and by (1.30). Then for $x \in E \cap B_1$ and $\rho \leq 10^{-2}$, the pair (x, ρ) satisfies (2.10) if ε is small enough, hence by (2.9)

$$\text{(11.3)} \qquad \mathcal{H}^d(E \cap B(y, \rho)) \leq C\rho^d.$$

Here and in the next lines, C is a constant that depends only on n and d.

Next let $B = B(x, r)$ be as in the statement. If $r^d \leq \sqrt{\varepsilon}$, then $r \leq 10^{-2}$ too (if ε is small enough), and (11.1) follows brutally from (11.3). So let us assume that $r^d \geq \sqrt{\varepsilon}$. Define a cut-off function α by

$$\text{(11.4)} \qquad \begin{aligned} &\alpha(y) = 0 \text{ when } |y - x| \geq r, \\ &\alpha(y) = 1 \text{ when } |y - x| \leq r - 6\varepsilon, \\ &\alpha(y) = (6\varepsilon)^{-1}(r - |y - x|) \text{ otherwise.} \end{aligned}$$

Denote by π the smallest distance projection on the convex set H; notice that π is 1-Lipschitz. We set

$$\text{(11.5)} \qquad \varphi(y) = \alpha(y)\pi(y) + (1 - \alpha(y))y$$

for $y \in \mathbb{R}^n$, and then $\varphi_t(y) = t\varphi(y) + (1-t)y$ for $0 \leq t \leq 1$. Notice that the φ_t define an acceptable deformation (see near (1.2)), in particular because $\pi(y) = y$ on $L \subset H$, and that $\widehat{W}$ is compactly contained in B_1,

by (11.4) and because $\overline{B} \subset B_1$, $\pi(y) \in B_1$ when $y \in B_1$, and B_1 is convex. Let us assume for the moment that E is an almost minimal set of type A$'$. We deduce from (2.6) that

(11.6) $\mathcal{H}^d(E \setminus \varphi(E)) \leq \mathcal{H}^d(\varphi(E) \setminus E) + r_1^d h(r_1) \leq \mathcal{H}^d(\varphi(E) \setminus E) + C\varepsilon.$

Notice that if $x \in E \cap B$, then either $x \in E \setminus \varphi(E)$, or else $x \in E \cap \varphi(E)$. In the last case, $x \in \varphi(E \cap B)$, because $\varphi(y) = y$ on $\mathbb{R}^n \setminus B$. Thus

(11.7) $\qquad \mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(E \setminus \varphi(E)) + \mathcal{H}^d(E \cap \varphi(E \cap B)).$

By (11.6) this yields

(11.8) $\begin{aligned} \mathcal{H}^d(E \cap B) &\leq \mathcal{H}^d(\varphi(E) \setminus E) + \mathcal{H}^d(E \cap \varphi(E \cap B)) + C\varepsilon \\ &= \mathcal{H}^d(\varphi(E \cap B)) + C\varepsilon, \end{aligned}$

where the last part holds because $\varphi(E) \setminus E = \varphi(E \cap B) \setminus E$ (since $\varphi(y) = y$ when $y \in \mathbb{R}^n \setminus B$). Set $B' = B(x, r - 6\varepsilon)$, and let us check that

(11.9) $\qquad\qquad \varphi(y) \in H \cap B \text{ for } y \in E \cap B'.$

First recall from (1.31) and the definition (1.29) that

(11.10) $\qquad\qquad \operatorname{dist}(y, H) \leq 3\varepsilon \text{ for } y \in E \cap B(0, 3),$

hence

(11.11) $\qquad\qquad |\pi(y) - y| \leq 6\varepsilon \text{ for } y \in E \cap B(0, 3)$

(pick $z \in H$ such that $|z - y| \leq 3\varepsilon$, then use the fact that π is 1-Lipschitz and $\pi(z) = z$). If furthermore $y \in E \cap B'$, then $\pi(y) \in H \cap B$ and, since $\varphi(x) = \pi(x)$ by (11.5), we get (11.9). This takes care of the major part of $\varphi(E \cap B)$.

We are left with the contribution of $A = B \setminus B'$. Let P_0 denote the d-plane that contains H, and let P denote the d-plane through x parallel to P_0. By (11.10), $\operatorname{dist}(P, P_0) \leq 3\varepsilon$, and every point of $E \cap A$ lies within 3ε of P_0, hence within 6ε of P and within 12ε of $P \cap \partial B$. If we cover $P \cap \partial B$ by balls B_j of radius 20ε, the double balls $2B_j$ will then cover $E \cap A$. We can do this with less than $C(r/\varepsilon)^{d-1}$ balls B_j, and $\mathcal{H}^d(E \cap 2B_j) \leq C\varepsilon^d$ by (2.9) (maybe applied to a twice larger ball centered on E, if $2B_j$ meets E but is not centered on E). Thus

(11.12) $\qquad\qquad \mathcal{H}^d(E \cap A) \leq C(r/\varepsilon)^{d-1}\varepsilon^d \leq C\varepsilon r^{d-1}.$

Also, we claim that φ is C-Lipschitz on $E \cap A$. This is because α is $(6\varepsilon)^{-1}$-Lipschitz, but $|\pi(y) - y| \leq 6\varepsilon$ on $E \cap A$; the verification is the same as for (4.11), so we skip it. Thus

(11.13) $\qquad\qquad \mathcal{H}^d(\varphi(E \cap A)) \leq C\mathcal{H}^d(E \cap A) \leq C\varepsilon r^{d-1}.$

We put things together and get that

$$\mathcal{H}^d(E \cap B) \leq \mathcal{H}^d(\varphi(E \cap B)) + C\varepsilon$$

(11.14)
$$\leq \mathcal{H}^d(\varphi(E \cap B')) + \mathcal{H}^d(\varphi(E \cap A)) + C\varepsilon$$

$$\leq \mathcal{H}^d(H \cap B) + C\varepsilon r^{d-1} + C\varepsilon$$

by (11.8), (11.9), and (11.13). But we are in the case when $r^d \geq \sqrt{\varepsilon}$, so $\sqrt{\varepsilon} r^{d-1} \leq r^{2d-1} \leq C r^d$ and (11.1) follows from (11.14).

We still need to say how we proceed when E is of type A or A_+. Of course we could say that in both cases, E is also an A'-almost minimal set, but the long proof can be avoided with a small trick. Choose a possibly different d-dimensional half plane H_1, with the same boundary L as H, so that

(11.15) $\mathrm{dist}(y, H_1) \leq 4\varepsilon$ for $y \in E \cap B(0,3)$.

We have uncountably many choices of H_1, all disjoint except for the set L of vanishing H^d-measure, so we can choose H_1 so that $\mathcal{H}^d(E \cap B_1 \cap H_1) = 0$. Then we repeat the construction above, with 3ε replaced by 4ε. Since the points of B' are now sent to H_1, we get that $\mathcal{H}^d$-almost every point of $E \cap B'$ lies in $W_1 = \{x \in E;\ \varphi(x) \neq x\}$. Suppose that E is of type A; then

(11.16) $\mathcal{H}^d(E \cap B') \leq \mathcal{H}^d(W_1) \leq \mathcal{H}^d(\varphi(W_1)) + h(r_1) r_1^d \leq \mathcal{H}^d(\varphi(W_1)) + C\varepsilon$

by the discussion above and (2.5), and in turn

(11.17)
$$\mathcal{H}^d(\varphi(W_1)) \leq \mathcal{H}^d(\varphi(E \cap B)) \leq \mathcal{H}^d(\varphi(E \cap B')) + \mathcal{H}^d(\varphi(E \cap A))$$
$$\leq \mathcal{H}^d(H \cap B) + C\varepsilon r^{d-1}$$

because $\varphi(y) = y$ on $\mathbb{R}^n \setminus B$, and by (11.9) and (11.13). Therefore $\mathcal{H}^d(E \cap B') \leq \mathcal{H}^d(H \cap B) + C\varepsilon r^{d-1} + C\varepsilon$, and we can conclude as before.

The case when E is of type A_+ is as simple; just observe that the error term in (11.16) is replaced by $h(r_1)\mathcal{H}^d(W_1) \leq h(r_1)\mathcal{H}^d(E \cap B) \leq C\varepsilon$.   □

For each $x \in E \cap B(0,2)$, we define a shade S_x and a functional F_x as we did in the introduction, but with the center x. That is,

(11.18) $S_x = \{y \in \mathbb{R}^n;\ x + \lambda(y - x) \in L$ for some $\lambda \in [0,1]\}$

(see (1.9) and (7.12)), then

(11.19) $\theta_x(r) = r^{-d}\mathcal{H}^d(E \cap B(x,r))$

and

(11.20) $F_x(r) = \theta_x(r) + r^{-d}\mathcal{H}^d(S_x \cap B(x,r))$

for $r > 0$ (compare with (1.10)). Here and below, densities will often be compared to the $\mathcal{H}^d$-measure of the unit d-disk, i.e.,

$$(11.21) \qquad \omega_d = \mathcal{H}^d(\mathbb{R}^d \cap B(0,1)).$$

Lemma 11.2. *Set*

$$(11.22) \qquad r_2 = \frac{7}{10} \ and \ B_2 = B(0, 21/10).$$

There is a constant $C \geq 0$, that depends only on n and d, such that

$$(11.23) \qquad F_x(r_2) \leq \omega_d + C\sqrt{\varepsilon} \ for \ x \in E \cap B_2.$$

Proof: First assume that $\text{dist}(x, L) \leq \sqrt{\varepsilon}$ and choose $z \in L$ such that $|z - x| \leq \sqrt{\varepsilon}$. Then

$$(11.24) \qquad \begin{aligned} \mathcal{H}^d(E \cap B(x, r_2)) &\leq \mathcal{H}^d(H \cap B(x, r_2)) + C\sqrt{\varepsilon} \\ &\leq \mathcal{H}^d(H \cap B(z, r_2 + \sqrt{\varepsilon})) + C\sqrt{\varepsilon} \\ &\leq \frac{\omega_d(r_2 + \sqrt{\varepsilon})^d}{2} + C\sqrt{\varepsilon} \leq \frac{\omega_d r_2^d}{2} + C\sqrt{\varepsilon} \end{aligned}$$

by Lemma 11.1 and because $B(x, r_2) \subset B(z, r_2 + \sqrt{\varepsilon})$. Since $\mathcal{H}^d(S_x \cap B(x, r_2)) \leq \frac{\omega_d r_2^d}{2}$ because S_x is contained in a half plane centered at x, we add and get (11.23).

If instead $\text{dist}(x, L) \geq \sqrt{\varepsilon}$, (11.10) says that $\text{dist}(x, H) \leq 3\varepsilon$, hence the two half planes H and S_x make an angle $\alpha \geq \pi - C\sqrt{\varepsilon}$ along L. Let P_x denote the d-plane that contains x, L, and hence S_x, and let π be the orthogonal projection on P_x; since $\pi(B(x, r_2)) \subset B(x, r_2)$, we get that $\pi(H \cap B(x, r_2)) \subset P_x \cap B(x, r_2)$. Also, $\pi(H \cap B(x, r_2)) \cap S_x \subset L$ (because of the small angle), and hence

$$(11.25) \qquad \begin{aligned} r_2^d F_x(r_2) &= \mathcal{H}^d(S_x \cap B(x, r_2)) + \mathcal{H}^d(E \cap B(x, r_2)) \\ &\leq \mathcal{H}^d(S_x \cap B(x, r_2)) + \mathcal{H}^d(H \cap B(x, r_2)) + C\sqrt{\varepsilon} \\ &\leq \mathcal{H}^d(S_x \cap B(x, r_2)) \\ &\quad + \frac{1}{\cos(\pi - \alpha)}\mathcal{H}^d(\pi(H \cap B(x, r_2))) + C\sqrt{\varepsilon} \\ &\leq \mathcal{H}^d(S_x \cap B(x, r_2)) + \mathcal{H}^d(\pi(H \cap B(x, r_2))) + C\sqrt{\varepsilon} \\ &= \mathcal{H}^d(P_x \cap B(x, r_2)) + C\sqrt{\varepsilon} \leq \omega_d r_2^d + C\sqrt{\varepsilon} \end{aligned}$$

by Lemma 11.1 and because $\cos(\pi - \alpha) \geq 1 - C\varepsilon$; Lemma 11.2 follows. $\quad\square$

Let $a > 0$ denote the constant in Theorem 1.5, and choose ε so small that (11.2) implies that $h(r_2) \leq h(r_1) \leq \tau$, where τ is the small constant in Theorem 1.5. We deduce from Theorem 1.5 (applied to the translation of E by $-x$) that for $x \in E \cap B_2$ and $0 < r < s \leq r_2$,

$$(11.26) \quad F_x(r) \leq e^{a(A(s)-A(r))} F_x(s) \leq e^{a\varepsilon} F_x(s) \leq e^{2a\varepsilon} F_x(r_2) \leq \omega_d + C\sqrt{\varepsilon}$$

by (1.21), (1.20), (11.2), or (1.30), a second application of the same inequalities, and (11.23). Because of the first inequality (or directly (1.21)), there exists a limit

$$(11.27) \qquad\qquad F_x(0) = \lim_{r \to 0} F_x(r),$$

and (11.26) implies that

$$(11.28) \qquad F_x(0)e^{-a\varepsilon} \leq F_x(s) \leq \omega_d + C\sqrt{\varepsilon} \text{ for } 0 < s \leq r_2.$$

Let us restrict our attention to $x \in E \setminus L$. Then

$$(11.29) \qquad F_x(r) = \theta_x(r) \text{ for } 0 < r \leq \operatorname{dist}(x, L)$$

by (11.20), and there exists

$$(11.30) \qquad \theta_x(0) = \lim_{r \to 0} \theta_x(r) = F_x(0).$$

In fact, we already knew this, just from the almost monotonicity of θ_x for almost minimal sets with no sliding condition (see for instance Proposition 5.24 in [**D2**]).

We claim that for H^d-almost every $x \in E \cap B_2$,

$$(11.31) \qquad\qquad F_x(0) = \theta_x(0) = \omega_d.$$

Since $H^d(L) = 0$, we may restrict to $x \in E \setminus L$. But we know that $E \setminus L$ is rectifiable; since we are far from the boundary set L, we can even use the result of Almgren [**A3**] instead of (2.11)). Now (11.31) follows directly from this and known density properties of rectifiable sets (see for instance Theorem 16.2 in [**Ma**]).

$$(11.32) \qquad \omega_d e^{-a\varepsilon} \leq F_x(s) \leq \omega_d + C\sqrt{\varepsilon} \text{ for } 0 < s \leq r_2;$$

that is, F_x is nearly constant on $(0, r_2]$, and this will allow us to show that E look like a minimal cone at the corresponding scales.

Lemma 11.3. *For each $\tau > 0$, there is a constant $\varepsilon_0 > 0$, that depends only on n, d, and τ, such that the following property holds as soon as L and E are as above and $\varepsilon < \varepsilon_0$. Let $x \in E \cap B_2 \setminus L$ be such that (11.31) holds, and let $r \in (0, r_2)$ be given.*

If $0 < r < \operatorname{dist}(x, L)$, there a plane $X = X(x, r)$ through x such that

$$(11.33) \qquad\qquad d_{x,3r/4}(E, X) \leq 4\tau/3$$

and

(11.34) $\left|\mathcal{H}^d(E \cap B(y,t)) - \mathcal{H}^d(X \cap B(y,t))\right| \leq \tau r^d$

$\qquad$ *for $y \in \mathbb{R}^n$ and $t > 0$ such that $B(y,t) \subset B(x, (1-\tau)r)$.*

If $\frac{1}{2}\operatorname{dist}(x,L) < r < \tau^{-1}\operatorname{dist}(x,L)$, let X denote the affine d-plane that contains x and L, and set $X' = \overline{X \setminus S_x}$. Then

(11.35) $\qquad\qquad\qquad d_{x,3r/4}(E, X') \leq \tau$

and

(11.36) $\left|\mathcal{H}^d(E \cap B(y,t)) - \mathcal{H}^d(X' \cap B(y,t))\right| \leq \tau r^d$

$\qquad$ *for $y \in \mathbb{R}^n$ and $t > 0$ such that $B(y,t) \subset B(x, (1-\tau)r)$.*

Proof: See (1.29) for the definition of $d_{x,r}$. Let τ, x, and r be as in the statement, and set $B = B(x,r)$. We start with the case when $r < \operatorname{dist}(x,L)$; then E is a (plain) almost minimal set in B, and we want to apply Proposition 7.24 in [**D2**] to the set E in the ball B. The fact that E is almost minimal in B follows at once from our assumptions (since $L \cap B = \emptyset$), we can use the same gauge function h as here, and the assumption that $h(2r) \leq \varepsilon$ (for the small ε of [**D2**]) follows from (1.30) if ε_0 is small enough. Then there is the assumption that

(11.37) $\qquad\qquad\qquad \theta_x(r) \leq \inf_{0 < t < r/100} \theta_x(t) + \varepsilon,$

where again ε comes from Proposition 7.24 in [**D2**], with the same τ as here. This follows from (11.32), because $F_x(t) = \theta_x(t)$ for $t \leq r < \operatorname{dist}(x,L)$, and if ε_0 is small enough. Then Proposition 7.24 in [**D2**] says that there is a minimal cone X, centered at x, that satisfies our two conditions (11.33) and (11.34). In addition, (11.34) with the ball $B(x, r/2)$ yields

$$\left|\mathcal{H}^d(X \cap B(x,1)) - \omega_d\right|$$

(11.38)
$$= \left(\frac{r}{2}\right)^{-d}\left|\mathcal{H}^d\left(X \cap B\left(x, \frac{r}{2}\right)\right) - \left(\frac{r}{2}\right)^d \omega_d\right|$$

$$\leq \left(\frac{r}{2}\right)^{-d}\left|\mathcal{H}^d\left(X \cap B\left(x, \frac{r}{2}\right)\right) - \mathcal{H}^d\left(E \cap B\left(x, \frac{r}{2}\right)\right)\right| + \left|\theta_x\left(\frac{r}{2}\right) - \omega_d\right|$$

$$\leq 2^d \tau + C\sqrt{\varepsilon}$$

by (11.34) and (11.32). We may assume that τ was chosen smaller than $\frac{1}{2}(d(n,d) - \omega_d)$, where $d(n,d)$ is as in Lemma 10.1; then, if ε_0 is small

enough, Lemma 10.1 and (11.38) imply that X is a plane, and this completes our proof when $r < \operatorname{dist}(x, L)$.

Next we assume that $\frac{10}{9} \operatorname{dist}(x, L) < r < \tau^{-1} \operatorname{dist}(x, L)$. In this case we want to apply Corollary 9.3 to $E_x = r^{-1}(E - x)$, $L_x = r^{-1}(E - x)$, and some small constant $\tau_1 < \tau$ that will be chosen soon. Set $t = \operatorname{dist}(0, L_x) = r^{-1} \operatorname{dist}(x, L)$; our assumption says that

$$(11.39) \qquad\qquad \tau \leq t \leq \frac{9}{10},$$

and so the first assumption (9.26) is satisfied as soon as $\tau_1 \leq \tau$. Here L_x is a $(d-1)$-plane, which takes care of (9.27), and (9.28) holds (if ε_0 is small enough), by (1.30) and (11.32). Thus we get a minimal cone X_0, which satisfies (9.29)–(9.34) with the constant τ_1 (and with respect to E_x), and with $L' = L_x$ (by the comment after the statement of Corollary 9.3).

By (11.39) we can apply (9.34) to $B(0, t)$; since $E_0 = X_0$ on $B(0, t)$ we get that

$$
\begin{aligned}
\mathcal{H}^d(X_0 \cap B(0,t)) &= \mathcal{H}^d(E_0 \cap B(0,t)) \leq \mathcal{H}^d(E_x \cap B(0,t)) + \tau_1 \\
&= r^{-d}\mathcal{H}^d(E \cap B(x, tr)) + \tau_1 = t^d \theta_x(tr) + \tau_1 \\
&= t^d F_x(tr) + \tau_1 \leq t^d \omega_d + Ct^d \sqrt{\varepsilon_0} + \tau_1
\end{aligned}
$$
(11.40)

because $E_x = r^{-1}(E - x)$ and $B(x, tr)$ does not meet L, then by (11.32). With the notation (10.1),

$$(11.41) \qquad d(X_0) = \mathcal{H}^d(X_0 \cap B(0,1)) \leq \omega_d + \sqrt{\varepsilon_0} + t^{-d}\tau_1;$$

since $t \geq \tau$ by (11.39), we see that if τ_1 is chosen small enough, and ε_0 is small enough, (11.41) and Lemma 10.1 imply that X_0 is a plane.

By (9.29) and because $L' = L_x$ really meets $B(0, 99/100)$ (by (11.39) again), X_0 contains L_x. That is, X_0 is the only vector d-plane that contains L_x, and $X = x + rX_0 = x + X_0$ is the affine d-plane that contains x and L. We are now ready co conclude; the set $X' = \overline{X \setminus S_x}$ is the same as $x + r\overline{X_0 \setminus S} = x + rE_0$, where E_0 is as in Corollary 9.3, so (11.35) follows from (9.32) and (9.33), and (11.36) follows from (9.34) (if τ_1 is small enough, as before).

We are left with the intermediate case when $\frac{1}{2} \operatorname{dist}(x, L) \leq r \leq \frac{10}{9} \operatorname{dist}(x, L)$. If $3r \leq r_2$, we simply observe that we can apply the proof above to the radius $3r$, and with smaller choices of τ and τ_1, and get the desired result for r. When $r \geq r_2/3$ but $r \leq \frac{10}{9} \operatorname{dist}(x, L)$, we deduce (11.35) directly from (1.31), because since x itself lies close to H and far from L, H is quite close to X' at the unit scale. As for (11.36), we can

easily deduce it from the same result with X' replaced by H, and this last is itself easy to deduce from (1.31) and the same compactness argument as for Lemma 9.2 (also see near Lemma 9.5). We skip the details; anyway, the case when $r \geq r_2/3$ is far from being the most interesting. This completes our proof of Lemma 11.3. $\qquad\square$

Let us now check (1.32). Let $z \in L \cap B(0, 2)$ be given, suppose that $z \notin E$, and set $d = \operatorname{dist}(z, E) > 0$. By (1.31) and the definition (1.29), $d = \operatorname{dist}(z, E) \leq 3\varepsilon$. Pick $x_0 \in E$ be such that $|x_0 - z| = d \leq 3\varepsilon$. By the local Ahlfors regularity property (2.9) (which is valid near $B(x_0, d)$ because x_0 lies well inside B_2), and because L is a $(d-1)$-plane, we can find $x \in E \cap B(x_0, d)$ such that $\operatorname{dist}(x, L) \geq C^{-1}d$. We can also choose x so that $\theta_x(0) = \omega_d$ (as in (11.31)), since this is true for almost every x.

Notice that $x \in B_2$, and apply Lemma 11.3 with $r = 4d$ and τ smaller than $(4C)^{-2}$, C as above. The extra assumption that $\frac{1}{2}\operatorname{dist}(x, L) \leq r \leq \tau^{-1}\operatorname{dist}(x, L)$ is satisfied (by our choice of r and τ), so we have the conclusions (11.35) and (11.36). That is, if X denotes the plane through x that contains L, then by (11.35) $X' = \overline{X \setminus S_x}$ is very close to E in $B(x, 3r/4) = B(x, 3d)$. But X' meets $B(z, d/20)$, and all the points of $X' \cap B(z, d/20)$ lie in $B(x, 3d)$, hence very close to E; thus E meets $B(z, d/10)$; this contradiction with the definition of d proves (1.32).

Next we check that Lemma 11.3 gives the existence of the desired sets $Z = Z(x, r)$, $x \in E \cap B(0, 2)$, as long as $r \leq \tau^{-1}\operatorname{dist}(x, L)$ and x satisfies (11.31). When $r \leq \operatorname{dist}(x, L)/2$, we take $Z = X(x, 4r/3)$, where the d-plane $X(x, 4r/3)$ is obtained by applying Lemma 11.3, with the slightly smaller constant $3\tau/4$; observe that $4r/3 \leq r_2 = 7/10$ when $r \leq 1/2$ (see (11.22)). Then (1.33) holds by definition and (1.36) and (1.37) follow from (11.33) and (11.34).

When $\operatorname{dist}(x, L)/2 \leq r < \tau^{-1}\operatorname{dist}(x, L)$, we apply the second case of Lemma 11.3 to the pair $(x, 4r/3)$ and with the constant $\tau/4$. We get that E is well approximated in $B(x, r)$ by $X' = \overline{X \setminus S_x}$, where X is the d-plane that contains x and L; thus X' is the same half plane as $Z(x, r)$ in (1.34), and (1.36) and (1.37) follow from (11.35) and (11.36).

In fact, by applying Lemma 11.3 with a much smaller τ_1, we even get the desired set $Z(x, r) \in \mathbb{H}(L)$ for $\tau^{-1}\operatorname{dist}(x, L) \leq r \leq (2\tau_1)^{-1}\operatorname{dist}(x, L)$; we even get a better approximation and the extra information that $Z(x, r)$ goes through x (which was not required in (1.35)).

Also, our constraint that x satisfies (11.31) can be lifted, because if $x \in E \cap B(0, 2) \setminus L$ does not satisfy (11.31), we can write x as a limit of points $x_k \in E \cap B(0, 2)$ that satisfy (11.31) (because (11.31) holds almost everywhere on E), apply the result to these x_k and a slightly

larger radius r', and get the desired $Z(x, r)$ as a minor modification of some $Z(x_k, r')$.

We are thus left with the case when

$$(11.42) \qquad\qquad (2\tau_1)^{-1} \operatorname{dist}(x, L) \le r \le 1/2,$$

where τ_1 is as small as we want (we shall choose it soon, depending on τ). As before, it is enough to find $Z(x, r)$ when $x \in E \setminus L$ and x satisfies (11.31), which is useful because then (11.32) holds.

Since we declined to prove an analogue for Corollary 9.3 for points that lie too close to the boundary L, we'll have to apply Proposition 30.3 in [**D5**], which provides a similar result for balls centered on the boundary.

Let z denote the point of L that lies closest to x; thus

$$(11.43) \qquad\qquad |z - x| = \operatorname{dist}(x, L) \le 2\tau_1 r.$$

We want to use (11.32) to find good bounds on the density θ_z. We start with the radius $r_3 = \sqrt{\tau_1} r$; set $r_3' = r_3 - |z - x|$ and notice that

$$
\begin{aligned}
(11.44) \qquad \theta_z(r_3) &= r_3^{-d} \mathcal{H}^d(E \cap B(z, r_3)) \ge r_3^{-d} \mathcal{H}^d(E \cap B(x, r_3')) \\
&= (r_3'/r_3)^d \, \theta_x(r_3') \\
&= (r_3'/r_3)^d F_z(r_3') - r_3^{-d} \mathcal{H}^d(S_x \cap B(z, r_3')) \\
&\ge (r_3'/r_3)^d F_z(r_3) - \frac{\omega_d}{2} - C|z - x| r_3^{-1} \\
&\ge (1 - |z-x| r_3^{-1})^d e^{-a\varepsilon} \omega_d - \frac{\omega_d}{2} - C|z-x| r_3^{-1} \ge \frac{\omega_d}{2} - C\sqrt{\tau_1}
\end{aligned}
$$

by (11.20), (11.32), a brutal estimate on $\mathcal{H}^d(S_x \cap B(z, r_3'))$ that uses the fact that $|z - x| = \operatorname{dist}(x, L) \le 2\tau_1 r = 2\sqrt{\tau_1} r_3$, and if ε is small enough.

We also want an upper bound on $\theta_z(4r/3)$. Set $r_4 = 4r/3$ and $r_4' = r_4 + |z - x| < r_2$; then

$$
\begin{aligned}
(11.45) \qquad \theta_z(r_4) &= r_4^{-d} \mathcal{H}^d(E \cap B(z, r_4)) \le r_4^{-d} \mathcal{H}^d(E \cap B(x, r_4')) \\
&= (r_4'/r_4)^d \, \theta_x(r') \\
&= (r_4'/r_4)^d \left[F_z(r_1') - (r_4')^{-d} \mathcal{H}^d(S_x \cap B(z, r_4')) \right] \\
&\le (r_4'/r_4)^d \left[F_z(r_1') - \frac{\omega_d}{2} + \frac{C|z - x|}{r_4'} \right] \\
&\le \left(1 + \frac{|z - x|}{r} \right)^d \left([\omega_d + C\sqrt{\varepsilon}] - \frac{\omega_d}{2} + \frac{C|z - x|}{r} \right) \le \frac{\omega_d}{2} + C\tau_1
\end{aligned}
$$

by the same sort of estimates as above, including (11.32) and (11.43).

We want to apply Proposition 30.3 in [**D5**], with τ replaced by a small constant $\eta > 0$ that will be chosen soon (depending on our τ), $x_0 = z$, the same r_1, and $r_0 = r_2 = r_4$. The main assumption (30.6) in [**D5**] (the fact that $\theta_z(4r/3)$ is at most barely larger than $\theta_z(r_1)$) follows from (11.44) and (11.45), if τ_1 is small enough (depending on η). The other assumptions are that L be sufficiently close to a d-plane through z (in the bilipschitz sense), and that E is a sliding quasiminimal set with small enough constants, and these are satisfied if ε is small enough. We get a coral sliding minimal cone T centered at z, and which is sufficiently close to E. In particular,

$$(11.46) \quad \mathrm{dist}(y, T) \leq \eta r_4 \text{ for } y \in E \cap B(z, (1-\eta)r_4) \setminus B(z, r_3 + \eta r_4)$$

and

$$(11.47) \quad \mathrm{dist}(y, E) \leq \eta r_4 \text{ for } y \in T \cap B(z, (1-\eta)r_4) \setminus B(z, r_3 + \eta r_4).$$

Notice that this is not exactly the same as in (30.7) and (30.8) from [**D5**], where ηr_4 is replaced by η. So in fact we apply Proposition 30.3 there to a dilation of E by a factor r_4^{-1}, and then we get (11.46) and (11.47). The dilation does not matter here; we did not do it in [**D5**] because we were also authorizing boundaries L_j^0 that were not planes, and the Lipschitz assumptions on those have less dilation invariance.

We also get, from (30.10) in that proposition, that

$$(11.48) \qquad \left| \mathcal{H}^d(E \cap B(z, r)) - \mathcal{H}^d(T \cap B(z, r)) \right| \leq \eta r_4^d.$$

Since we also have that $|\theta_z(r) - \frac{\omega_d}{2}| \leq C\sqrt{\tau_1}$ by the proof of (11.44) and (11.45), we get that

$$(11.49) \quad \begin{aligned} \left| \mathcal{H}^d(T \cap B(z, 1)) - \frac{\omega_d}{2} \right| &= r^{-d} \left| \mathcal{H}^d(T \cap B(z, r)) - \frac{\omega_d \, r^d}{2} \right| \\ &\leq C\sqrt{\tau_1} + 2^d \eta. \end{aligned}$$

Let $\tau_2 > 0$ be small, to be chosen soon (depending on τ); we claim that if τ_1 and η are small enough, depending on τ_2, there is a half plane $Z \in \mathbb{H}(L)$ such that

$$(11.50) \qquad\qquad\qquad d_{z,1}(T, Z) \leq \tau_2.$$

In fact, we even claim the following apparently stronger result: there is a constant $\eta_1 > 0$ such that, if H is a (nonempty) coral sliding minimal cone centered at the origin, relative to a boundary which is a vector space L_0 of dimension $d - 1$, and if $\mathcal{H}^d(H \cap B(0, 1)) \leq \frac{\omega_d}{2} + \eta_1$, then there is a half plane $Z \in \mathbb{H}(L_0)$ such that $d_{0,1}(H, Z) \leq \tau_2$.

Let us prove this (stronger) claim by compactness. The proof is essentially the same as for Lemma 10.1, to which we refer for details. By rotation invariance, we may assume that L_0 is a given vector $(d-1)$-plane. If the claim fails, then for each $k \geq 0$ we can find a coral sliding minimal cone H_k centered at the origin, with $\mathcal{H}^d(H_k \cap B(0,1)) \leq \frac{\omega_d}{2} + 2^{-k}$, and which is τ_2-far from all $Z \in \mathbb{H}(L_0)$. We extract a subsequence for which H_k tends to a limit H_∞, we observe that H_∞ is also a coral sliding minimal cone (by Theorem 4.1 in [**D5**]), and $\mathcal{H}^d(H_\infty \cap B(0,1)) \leq \frac{\omega_d}{2}$ by Theorem 3.4 in [**D5**]. Then Corollary 10.2 says that $H_\infty \in \mathbb{H}(L_0)$, which contradicts the definition of the H_k or the fact that they tend to H_∞. This proves our two claims.

Return to E. If η and τ_1 are small enough, depending on τ_2, it follows from (11.46), (11.47), and (11.50) that

$$(11.51) \qquad\qquad d_{z,9r_4/10}(E,Z) \leq 2\tau_2;$$

we do not need to worry about what happens in the hole created by $B(z, r_3 + \eta r_4)$, because $r_3 + \eta r_4 = \sqrt{\tau_1} r + \eta r_4$ is much smaller than $\tau_2 r_4$, and z lies on both sets E (by (1.32)) and Z.

Clearly Z satisfies (1.35) and (11.51) implies (1.36) if $\tau_2 < \tau/2$. We still need to check that (1.37) holds, and again it follows from (11.51) if τ_2 is chosen small enough, depending on τ. Otherwise, we could find a sequence that contradicts Lemma 9.2.

Let us summarize. Given $\tau > 0$ and $\tau_1 \ll \tau$, we used the almost constant density property (11.32) and Corollary 9.3 or its simpler variant with no boundary to prove (1.32) and establish the existence of $Z(x,r)$ for $r \leq \tau_1^{-1} \operatorname{dist}(x,L)$ (provided $\varepsilon \leq \varepsilon_0$ is small enough). Now we just checked that we can find constants τ_2, then η and τ_1, so that if ε is small enough (depending on these constants too), we can find $Z(x,r)$ also when $r \geq \tau_1^{-1} \operatorname{dist}(x,L)$. This completes the proof of Corollary 1.7. □

Remark 11.4. The author claims that Corollary 1.7 can be extended to the case when L is a smooth embedded variety of dimension $(d-1)$ through the origin, which is flat enough in $B(0,1)$.

The main ingredients, namely Corollary 9.3 and Proposition 30.3 in [**D5**], both work in this context (and even for bilipschitz images with enough control); then it should only be a matter of checking that the other estimates, such as Lemmas 11.1 and 11.2, only bring small additional error terms.

But the author was a little too lazy to write down the argument. An excuse is that probably the right result is stronger, and says that E is a $C^{1+\alpha}$ version of a half plane locally, with a proof that looks more like

those of [**D3**] when $d = 2$. The main advantage of Corollary 1.7 is that it is relatively simple.

12. Sliding almost minimal sets that look like a V

In this section we prove Corollary 1.8, in a slightly more general setting than stated in the introduction. That is, we replace the precise assumption that $d = 2$, or else $d = 3$ and $n = 4$ with the assumption that (10.9) holds.

We thus consider sliding almost minimal sets with respect to a single boundary set L, which we assume to be a vector space of dimension $d-1$. As in the introduction, denote by $\mathbb{V}(L)$ the set of unions $V = H_1 \cup H_2$ of two half d-planes H_i, both bounded by L, and which make an angle at least equal to $2\pi/3$ along L; see above (1.28). We may call such a V a *cone of type* $\mathbb{V}$. Notice that V is allowed to be a d-plane that contain L.

In this section we prove the following slight extension of Corollary 1.8.

Proposition 12.1. *The statement of Corollary 1.8 is valid for all the integers d and n such that* (10.9) *holds.*

Let us start the proof. Let L, E, and $x \in E \cap B(0,1) \setminus L$ be as in the statement. We want to apply Theorem 1.5 to the set $E_x = E - x$ and the boundary $L_x = L - x$. Let S_x denote the shade of L seen from x, as in (1.45), and notice that $S = S_x - x$ is the usual shade of L_x (seen from the origin). We are interested in the functional F_x defined by

$$(12.1) \qquad F_x(r) = r^{-d}\mathcal{H}^d(E \cap B(x,r)) + r^{-d}\mathcal{H}^d(S_x \cap B(x,r)).$$

Note that $F_x(r) = r^{-d}\mathcal{H}^d(E_x \cap B(0,r)) + r^{-d}\mathcal{H}^d(S \cap B(0,r))$, by (1.9) and (1.10); this is the same thing as $F(r)$ relative to E_x and L_x. If ε is small enough, the assumptions of Theorem 1.5 are satisfied (with $U = B(0,2)$ and $R_1 = 2$), and (1.21) says that

$$(12.2) \qquad\qquad F_x(r)e^{aA(r)} \text{ is nondecreasing on } (0,2),$$

where $a > 0$ is a constant that depends on n and d, and A is the same function as in (1.20).

Our next task is to evaluate $F_x(r)$ for some large r. Let $\tau_1 > 0$ be small, to be chosen later.

Lemma 12.2. *If ε is small enough (depending on τ_1), then*

$$(12.3) \qquad F_x(r) \leq \frac{3\omega_d}{2} + 2\tau_1 \text{ for } x \in E \cap B(0,1) \setminus L \text{ and } 0 < r \leq \frac{19}{10}.$$

Proof: Because of (12.2) and the fact that $A(3)$ is as small as we want (by (1.42)), it is enough to show that

$$(12.4) \qquad F_x(19/10) \leq \frac{3\omega_d}{2} + \tau_1.$$

Suppose the lemma fails for some $\tau_1 > 0$; this means that if we take $\varepsilon = 2^{-k}$, we can find L_k, E_k, and h_k as above, and then $x_k \in E_k \cap B(0,1) \backslash L_k$ such that (12.4) fails (for E_k and x_k). By rotation invariance, we may even assume that $L_k = L$ for some fixed vector space L. Let us also replace $\{E_k\}$ by a subsequence which converges locally in $B(0,3)$ to a limit E_∞, and for which x_k has a limit $x_\infty \in E_\infty \cap \overline{B}(0,1)$.

Since E_k satisfies (1.43) with the constant $\varepsilon = 2^{-k}$, we see that $E_\infty \in \mathbb{V}(L)$, i.e., is the union of two half d-planes H_1 and H_2, that are bounded by L and make an angle at least $2\pi/3$ along L. The following sublemma will help us with measure estimates.

Lemma 12.3. *If $E_\infty \in \mathbb{V}(L)$, then*

$$(12.5) \qquad r^{-d}\mathcal{H}^d(E_\infty \cap B(x_\infty, r)) + r^{-d}\mathcal{H}^d(S_{x_\infty} \cap B(x_\infty, r)) \leq \frac{3\omega_d}{2}$$

for all $x_\infty \in E_\infty \setminus L$ and $r > 0$.

We shall prove this lemma soon, but let us first see why it implies Lemma 12.2. First assume that $x_\infty \notin L$. Then

$$(12.6) \qquad \lim_{k \to +\infty} \mathcal{H}^d(S_{x_k} \cap B(x_k, 19/10)) = \mathcal{H}^d(S_{x_\infty} \cap B(x_\infty, 19/10))$$

(because x_k too stays far from L).

Denote by F_{x_k} the functional associated to E_k and x_k as in (12.1), and pick any $r_1 \in (19/10, 2)$. Then set $\ell = \limsup_{k \to +\infty} F_{x_k}(19/10)$, and observe that

$$\ell = (19/10)^{-d} \limsup_{k \to +\infty} \Big[\mathcal{H}^d(S_{x_k} \cap B(x_k, 19/10))$$

$$+ \mathcal{H}^d(E_k \cap B(x_k, 19/10))\Big]$$

$$\leq (19/10)^{-d}\mathcal{H}^d(S_{x_\infty} \cap B(x_\infty, 19/10))$$

$$(12.7) \qquad\qquad + (19/10)^{-d} \limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap \overline{B}(x_\infty, r_1))$$

$$\leq (19/10)^{-d}\mathcal{H}^d(S_{x_\infty} \cap B(x_\infty, 19/10))$$

$$+ (19/10)^{-d}\mathcal{H}^d(E_\infty \cap \overline{B}(x_\infty, r_1))$$

$$\leq (19/10)^{-d}\frac{3\omega_2 r_1^d}{2}$$

by (12.6) and Lemma 22.3 in [**D5**], applied to the compact set $\overline{B}(x_\infty, r_1)$ as we did for (9.12), and then (12.5). If we choose r_1 close enough to 19/10, we deduce (12.4) for E_k and x_k (and if k is large enough) from (12.7), and this contradiction with the definition of E_k proves the lemma.

If $x_\infty \in L$, we simply notice that $\mathcal{H}^d(S_{x_k} \cap B(x_k, 19/10)) \leq (19/10)^d \frac{\omega_d}{2}$, and $\mathcal{H}^d(E_\infty \cap B(x_k, r_1)) = r_1^d \omega_d$, so

$$\ell = (19/10)^{-d} \limsup_{k \to +\infty} \Big[\mathcal{H}^d(S_{x_k} \cap B(x_k, 19/10))$$

$$+ \mathcal{H}^d(E_k \cap B(x_k, 19/10)) \Big]$$

$$(12.8) \qquad \leq \frac{\omega_d}{2} + (19/10)^{-d} \limsup_{k \to +\infty} \mathcal{H}^d(E_k \cap \overline{B}(x_\infty, r_1))$$

$$\leq \frac{\omega_d}{2} + (19/10)^{-d} \mathcal{H}^d(E_\infty \cap \overline{B}(x_\infty, r_1))$$

$$\leq \frac{\omega_d}{2} + (19/10)^{-d} \frac{\omega_2 r_1^d}{2},$$

again by Lemma 22.3 in [**D5**]. We can let r_1 tend to 19/10 and get (12.4) as above, so Lemma 12.2 follows from Lemma 12.3.

Proof of Lemma 12.3: We start with the case of a set V of dimension 1 in a 2-plane P. That is, let the points $y, \ell \in P$ be given, with $y \neq \ell$, and let v denote the union of two half lines h_1, h_2 in P, that both start at ℓ and make an angle at least $\frac{2\pi}{3}$; also denote by s the shade of ℓ seen from y, i.e., the set of points $w \in P$ such that $y + \lambda(w - y) = \ell$ for some $\lambda \in [0, 1]$. See Figure 12.1 for an illustration. We want to check that for $\rho > 0$,

$$(12.9) \qquad \mathcal{H}^1(v \cap B(y, \rho)) + \mathcal{H}^1(s \cap B(y, \rho)) \leq 3\rho.$$

By rotation and dilation invariance, it is enough to prove this when $P = \mathbb{R}^2$, $y = 0$, $\rho = 1$, and ℓ lies on the positive real axis.

When $\ell \geq 1$, $\mathcal{H}^1(s \cap B(y, 1)) = 0$; the result is clear when $\mathcal{H}^1(h_1 \cap B(y, \rho)) \leq 1$ because $\mathcal{H}^1(h_2 \cap B(y, \rho)) \leq 2$; otherwise, h_1 makes an angle at least $2\pi/3$ with the positive real axis (see Figures 12.1 and 12.2); that, is h_1 lies in the shaded region of Figure 12.3, and our angle condition forces h_2 to make an angle at most $2\pi/3$ with the positive real axis, hence to lie on the right of the dotted half line of Figure 12.3, so that $\mathcal{H}^1(h_2 \cap B(y, \rho)) \leq 1$, and the desired inequality follows.
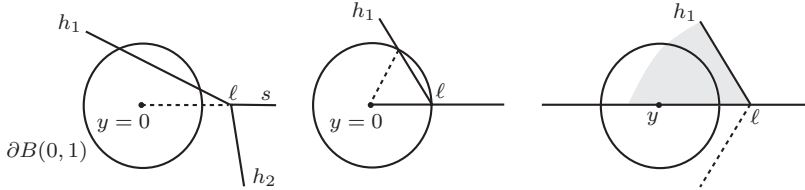
FIGURE 12.1. (Left) General position with $\ell > 1$.

FIGURE 12.2. (Center) The limiting case for
$\mathcal{H}^1(h_1 \cap B(y, \rho)) \geq 1$.

FIGURE 12.3. (Right) h_1 lies on the shaded region;
then h_2 lies on the right of the dotted line.

So we may assume that $0 < \ell \leq 1$. For $i = 1, 2$, let $\theta_i \in [0, \pi]$ denote
the angle between h_i and the positive real axis; we may assume that
$\theta_1 \leq \theta_2$. Also denote by θ the angle between h_1 and h_2. If h_1 and h_2 lie
on the same side of the real axis, our constraint that $\theta \geq 2\pi/3$ implies
that $\theta_1 \leq \pi/3$ (see Figure 12.4). Then $\mathcal{H}^1(h_1 \cap B(y, \rho)) < 1$, while
$\mathcal{H}^1(h_2 \cap B(y, \rho)) + \mathcal{H}^1(s \cap B(y, \rho)) \leq 2$ (corresponding to $\theta_2 = \pi$, look
at Figure 12.4 again, and recall that $\theta_2 \geq 2\pi/3$); then (12.9) holds.

So we may assume that h_1 and h_2 lie on different sides of the real axis;
then $\theta_1 + \theta_2 + \theta = 2\pi$, our constraint that $\theta \geq 2\pi/3$ yields $\theta_1 + \theta_2 \leq 4\pi/3$,
and since $\mathcal{H}^1(h_i \cap B(y, \rho))$ is easily seen to be an increasing function
of θ_i, we may assume that $\theta_1 + \theta_2 = 4\pi/3$ and $\theta = 2\pi$. Let h_3 denote
the half line in P that makes an angle of $2\pi/3$ with h_1 and h_2, and set
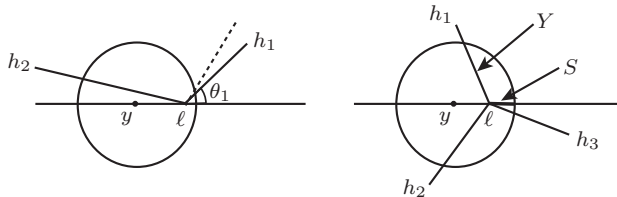$Y = h_1 \cup h_2 \cup h_3$ (see Figure 12.5).

FIGURE 12.4. (Left) General position when $\ell \leq 1$ and
h_1 and h_2 lie above the axis.

FIGURE 12.5. (Right) The case when h_1 and h_2 lie on
different sides, and the set $Y = h_1 \cup h_2 \cup h_3$.

Notice that $s \cap \overline{B}(0, 1)$ is the shortest line segment from ℓ to $\partial B(0, 1)$, hence $\mathcal{H}^1(s \cap B(y, 1)) \leq \mathcal{H}^1(h_3 \cap B(y, 1))$, and (12.9) will follow as soon as we prove that

$$(12.10) \qquad\qquad \mathcal{H}^1(Y \cap B(0, 1)) \leq 3.$$

This would probably not be so hard to compute, but it is simpler to notice that, for ℓ fixed, $r^{-1}\mathcal{H}^1(Y \cap B(0, r))$ is a nondecreasing function of r, either by Proposition 5.16 in [**D2**] (the lazy way; notice that 0 does not need to lie on the minimal set Y), or (repeating the proof of monotonicity) because

$$(12.11) \quad \frac{\partial}{\partial r}(\mathcal{H}^1(Y \cap B(0, r)) \geq \sharp(Y \cap \partial B(0, r)) = 3 \geq r^{-1}\mathcal{H}^1(Y \cap B(0, r))$$

for $r > \ell$, and where the last inequality comes from the minimality of Y (compare Y with the cone over $Y \cap \partial B(0, r)$).

This proves (12.9), and now we deduce (12.5) from (12.9) and a slicing argument. Let L, E_∞, x_∞, and r be as in Lemma 12.3. Recall that since $E_\infty \in \mathbb{V}(L)$, it is the union of two half d-planes H_1 and $H_2 \in \mathbb{H}(L)$ bounded by L. For $i = 1, 2$, let e_i be the unit vector such that $e_i \in H_i$ and $e_i \perp L$, and let h_i denote the half line $h_i = \{te_i; \, t \geq 0\}$; notice that $H_i = h_i \times L$ (an orthogonal product), and that h_1 makes an angle at least $2\pi/3$ with h_2, by definition of $\mathbb{V}(L)$. See Figure 12.6.

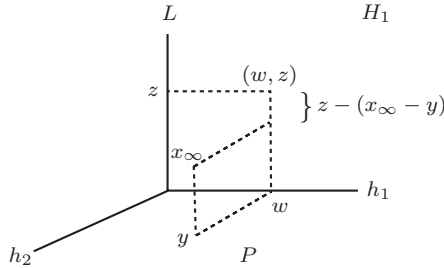FIGURE 12.6. Notation for the slicing argument (h_2 is not orthogonal to h_1).

Denote by P the 2-plane that contains e_1 and e_2 and by y the orthogonal projection of x_∞ on P. If $e_2 = -e_1$, pick any plane that contains e_1, or notice that (12.5) is easy to prove directly (since E_∞ is a d-plane through L). Then fix $i \in \{1, 2\}$, and write the current point of H_i

as (w, z), with $w \in h_i$ and $z \in L$. For $z \in L$, the slice of $H_i \cap B(x_\infty, r)$ at height z is the possibly empty set

(12.12) $W_i(z) = \{w \in h_i;\ (w, z) \in B(x_\infty, r)\}.$

Write $(w, z) - x_\infty$ as the sum of $w - y \in P$ and $z - x_\infty + y \in P^\perp$ (recall that $z \in L \subset P^\perp$); then $|(w, z) - x_\infty|^2 = |w - y|^2 + |z - (x_\infty - y)|^2$, the condition $|(w, z) - x_\infty|^2 < r^2$ becomes $|w - y|^2 < r_z^2$, with

(12.13) $r_z^2 = \max(0, r^2 - |z - (x_\infty - y)|^2),$

and then

(12.14) $W_i(z) = h_i \cap B(y, r_z).$

Then we apply Fubini's theorem on the d-plane that contains H_i and get that

(12.15)
$$\mathcal{H}^d(H_i \cap B(x_\infty, r)) = \int_{z \in L} \mathcal{H}^1(W_i(z))\, dz$$
$$= \int_{z \in L} \mathcal{H}^1(h_i \cap B(y, r_z))\, dz,$$

where for this computation we have assumed that the various Hausdorff measures were normalized so that they coincide with the corresponding Lebesgue measures on vector spaces. (After this, even if we chose different normalizations, (12.5) will follow because the result of the computation is exact when E_∞ is a plane through x_∞.)

Denote by s the shade of y seen from 0; the same computation, with H_i replaced by the half d-space S_{x_∞}, shows that

(12.16) $\mathcal{H}^d(S_{x_\infty} \cap B(x_\infty, r)) = \int_{z \in L} \mathcal{H}^1(s \cap B(y, r_z))\, dz.$

Thus, setting $I = \mathcal{H}^d(E_\infty \cap B(x_\infty, r)) + \mathcal{H}^d(S_{x_\infty} \cap B(x, r))$, (12.15) and (12.16) yield

(12.17)
$$I = \int_{z \in L} \left[\mathcal{H}^1(h_1 \cap B(y, r_z)) + \mathcal{H}^1(h_2 \cap B(y, r_z)) + \mathcal{H}^1(s \cap B(y, r_z)) \right] dz$$

$$= \int_{z \in L} \left[\mathcal{H}^1(v \cap B(y, r_z)) + \mathcal{H}^1(s \cap B(y, r_z)) \right] dz \leq \int_{z \in L} 3 r_z\, dz$$

by (12.9) and because the angle of h_1 and h_2 is the same as the (smallest) angle of H_1 and H_2. Denote by z_0 the orthogonal projection of x_∞ on L; then $|z - (x_\infty - y)|^2 \geq |z - z_0|^2$ (because z_0 is also the orthogonal

projection of $x_\infty - y$), and hence $r_z^2 \leq \max(0, r^2 - |z - z_0|^2)$. Thus (12.17) yields

$$
\begin{aligned}
(12.18) \qquad I &\leq 3 \int_{z \in L \cap B(z_0, r)} \left(r^2 - |z - z_0|^2 \right)^{1/2} dz \\
&= 3 \int_{w \in \mathbb{R}^{d-1} \cap B(0, r)} \left(r^2 - |w|^2 \right)^{1/2} dw = \frac{3\omega_d r^d}{2},
\end{aligned}
$$

because we recognize the way to compute the measure of a (half) ball of $\mathbb{R}^d$ by vertical slicing and induction. This completes our proof of (12.5); Lemmas 12.3 and 12.2 follow. $\qquad\square$

Let $x \in E \cap B(0, 1) \setminus L$ be given. Since $F_x(r) = r^{-d}\mathcal{H}^d(E \cap B(x, r))$ for r small, Lemma 12.2 implies that

$$
(12.19) \qquad \theta_x(0) = \lim_{r \to 0} r^{-d}\mathcal{H}^d(E \cap B(x, r)) \leq \frac{3\omega_d}{2} + 2\tau_1 \leq \frac{3\omega_d}{2} + \tau
$$

if we choose $2\tau_1 \leq \tau$, and so (1.44) holds.

From now on, we assume in addition that $\theta_x(0) > \omega_d$, and set $\delta(x) = \mathrm{dist}(x, L) > 0$, as in the statement of Corollary 1.8. Let X be any blow-up limit of E at x; standard arguments show that such things exist. By Proposition 7.31 in [**D2**] (for instance), X is a minimal cone with constant density $d(X) = \theta_x(0)$ (see the notation (10.1)). We assumed that $\theta_x(0) > \omega_d$, so $d(X) > \omega_d$ and X is not a plane. By (10.9), $d(X) \geq \frac{3\omega_d}{2}$ (because $d(X) = \frac{3\omega_d}{2}$ when $X \in \mathbb{Y}_0(n, d)$). Hence

$$
(12.20) \qquad \lim_{r \to 0} F_x(r) = \theta_x(0) \geq \frac{3\omega_d}{2},
$$

by the definitions (12.1) and (1.44), and because $x \in E \setminus L$. We now deduce from (12.2) that for $0 < r \leq 19/10$,

$$
(12.21) \qquad F_x(r) \geq e^{-aA(r)}\theta_x(0) \geq e^{-a\varepsilon}\theta_x(0) \geq \frac{3\omega_d}{2} - \tau_1
$$

by (1.42) and if ε is small enough (depending on τ_1). Because of (12.3), we thus get that

$$
(12.22) \qquad \frac{3\omega_d}{2} - \tau_1 \leq F_x(r) \leq \frac{3\omega_d}{2} + 2\tau_1 \text{ for } 0 < r \leq \frac{19}{10}.
$$

Our next task is to use (12.22), in conjunction with Corollary 9.3, to get a local control of E in balls $B(x, r)$, where $x \in E \setminus L$ is as above. We start with the small radii, for which the boundary L does not really interfere.

Lemma 12.4. *Let n, d, E, and L satisfy the assumptions of Corollary 1.8 and let $x \in E \cap B(0,1) \setminus L$ be such that $\theta_x(0) > \omega_d$. In particular, suppose that ε in (1.42) and (1.43) is small enough, depending on n, d, and $\tau > 0$. Then for $0 < r < \frac{1}{2}\,\mathrm{dist}(y,L)$ there is a minimal cone $Y \in \mathbb{Y}_x(n,d)$ (i.e., of type $\mathbb{Y}$ and centered at x) such that*

$$(12.23) \qquad\qquad d_{x,r}(E,Y) \leq \tau$$

and

$$(12.24) \quad \big|\mathcal{H}^d(E \cap B(y,t)) - \mathcal{H}^d(Y \cap B(y,t))\big| \leq \tau r^d$$
$$\text{for } y \in \mathbb{R}^n \text{ and } t > 0 \text{ such that } B(y,t) \subset B(x,r).$$

Proof: First observe that $r < \frac{1}{2}\,\mathrm{dist}(y,L) \leq \frac{1}{2}$ because $x \in B(0,1)$, so E is almost minimal in $B(x,2r)$, with no boundary condition, and we may try to apply Proposition 7.24 in [**D2**] in $B(x,2r)$, and with some constant τ_2 that will be chosen soon, to find an approximating cone. The condition that $h(4r)$ be small enough comes from (1.42), and the almost constant density condition (7.25) of this prroposition, which requires that $\theta_x(2r) \leq \int_{0<t<r/50} \theta_x(t) + \varepsilon'$, for some small ε' that depends on τ_2, is a consequence of (12.22) (choose τ_1 small, depending on τ_2, and then ε even smaller), because $\theta_x(t) = F_x(t)$ for $t < \frac{1}{2}\,\mathrm{dist}(y,L)$. So Proposition 7.24 in [**D2**] yields the existence of a coral minimal cone X, centered at x, such that

$$(12.25) \qquad \mathrm{dist}(y,X) \leq 2\tau_2 r \text{ for } y \in E \cap B(x,2(1-\tau_2)r),$$

$$(12.26) \qquad \mathrm{dist}(y,E) \leq 2\tau_2 r \text{ for } y \in X \cap B(x,2(1-\tau_2)r),$$

and

$$(12.27) \quad \big|\mathcal{H}^d(E \cap B(y,t)) - \mathcal{H}^d(X \cap B(y,t))\big| \leq 2^d \tau_2 r^d$$
$$\text{for } y \in \mathbb{R}^n \text{ and } t > 0 \text{ such that } B(y,t) \subset B(x,2(1-\tau_2)r).$$

Let us first apply (12.27) with $B(y,t) = B(x,r)$; we get that

$$(12.28) \quad |d(X) - \theta_x(r)| = r^{-d}\big|\mathcal{H}^d(X \cap B(x,r)) - \mathcal{H}^d(E \cap B(x,r))\big| \leq 2^d \tau_2.$$

Since $\theta_x(r) = F_x(r)$ because $B(x,r)$ does not meet L, we deduce from (12.22) that

$$(12.29) \qquad\qquad \left| d(X) - \frac{3\omega_d}{2} \right| \leq 2^d \tau_2 + 2\tau_1.$$

Now we use our assumption (10.9), and its consequence in Lemma 10.3, which we apply with some small constant τ_3 that will be chosen soon.

We get that if τ_1 and τ_2 are small enough, depending on τ_3, we can find a minimal cone $Y \in \mathbb{Y}_x(n, d)$ such that

$$(12.30) \qquad\qquad d_{x,1}(X, Y) \leq \tau_3$$

(the case when X is a plane is excluded by (12.29)). We just need to check that if τ_3 is chosen small enough, Y satisfies our conditions (12.23) and (12.24).

For (12.23), this is a simple consequence of (12.25), (12.26), (12.30), and the triangle inequality, so we skip the details. For (12.24) we proceed, as usual, by contradiction and compactness: we suppose that (12.25), (12.26), and (12.30) do not imply (12.23) and (12.24) (for a same Y), take a sequence $\{E_k\}$ for which they are satisfied with $\varepsilon + \tau_1 + \tau_3 < 2^{-k}$ (some center x_k, and some radius r_k), but (12.24) fails whenever (12.23) holds. By (12.25) and (12.26), the sets $E'_k = r_k^{-1}(E_k - x_x)$ converge, modulo extraction of a subsequence, to a cone $Y_0 \in \mathbb{Y}_0(n, d)$, and then we apply Lemma 9.2 (with $2^{-d-1}\tau$, say) in $B(0, 3/2)$, and get that (12.24) holds for k large, because of (9.22). Notice that we can take the cone $x + Y_0$ to check this, and then (12.23) still holds with the cone $x + Y_0$; then we get the desired contradiction. This completes the proof of Lemma 12.4. $\qquad\square$

Let N be the large number that shows up in the statement of Corollary 1.8. A simple consequence of Lemma 12.4 (applied with a small enough τ) is that, if x is as in Corollary 1.8 or Lemma 12.4, then

$$(12.31) \qquad\qquad \mathrm{dist}(x, L) \leq (10N)^{-1},$$

because otherwise the good approximation of E by a cone of type $\mathbb{Y}$ in $B(x, \frac{1}{3}\mathrm{dist}(L))$ would contradict its good approximation in $B(0, 3)$ by a set of type $\mathbb{V}$, given by (1.43). Notice that (12.31) is slightly better than what we announced in Corollary 1.8.

We continue our local description of E in balls $B(x, r)$ with the case of intermediate radii r, for which we shall combine (12.22) with Corollary 9.3.

Lemma 12.5. *Let n, d, E, and L satisfy the assumptions of Corollary 1.8 and let $x \in E \cap B(0, 1) \backslash L$ be such that $\theta_x(0) > \omega_d$. In particular, suppose that ε in (1.42) and (1.43) is small enough, depending on n, d, and $\tau > 0$. Let $r > 0$ be such that*

$$(12.32) \qquad\qquad \delta(x) \leq r \leq 10N\delta(x),$$

where $\delta(x) = \operatorname{dist}(x, L)$ as before. Let Y be the (unique) minimal cone $Y \in \mathbb{Y}_x(n, d)$ such that

(12.33) $L \subset Y,$

denote by S_x the shade of L seen from x, and set $W = \overline{Y \setminus S_x}$ (as in (1.45)). Then

(12.34) $d_{x,r}(E, W) \leq \tau$

and

(12.35) $\left| \mathcal{H}^d(E \cap B(y, t)) - \mathcal{H}^d(W \cap B(y, t)) \right| \leq \tau r^d$

$\qquad\qquad$ *for all $y \in \mathbb{R}^n$ and $t > 0$ such that $B(y, t) \subset B(x, r)$.*

Proof: We want to apply Corollary 9.3, with a small constant τ_4 that will be chosen later, to control E in $B(x, r)$. Set $r_1 = \frac{10r}{9}$ to have some room to play. Because of the normalization in the corollary, we apply it to the set $E_x = r_1^{-1}(E - x)$, which is sliding almost minimal in an open set that contains $B(0, 1)$ (because $B(x, r_1) \subset B(0, 3)$ since $x \in B(0, 1)$ and $r \leq 10N\delta(x) \leq 1$ by (12.32) and (12.31)), and with a boundary condition that comes from the set $L_x = r_1^{-1}(L - x)$.

The distance requirement (9.26) is that $\tau_4 \leq \operatorname{dist}(0, L_x) \leq \frac{9}{10}$, and since

(12.36) $\operatorname{dist}(0, L_x) = r_1^{-1}\delta(x) = \dfrac{9\delta(x)}{10r}$

we see that (9.26) follows from (12.32) as soon as we take $\tau_4 < \frac{9}{100N}$.

The bilipschitz condition on L_x is trivially satisfied, because L_x is an affine subspace; the first half of (9.28) (i.e., the fact that $h(r)$ is very small) follows from (1.42) if ε is small enough; and the more important second half follows from (12.22).

Then Corollary 9.3 yields the existence of a coral minimal cone X_0, centered at the origin and with no boundary condition, with the following properties. First,

(12.37) $L_x \cap B(x, 99/100) \subset X_0,$

by (9.29) and because the comment below Corollary 9.3 says that we can take $L' = L_x$. Next denote by S the shade of L_x seen from the origin, and set $E_0 = \overline{X_0 \setminus S}$; then by (9.31) E_0 is a coral minimal set in $B(0, 1 - \tau_4)$, with sliding boundary condition defined by L_x, and is τ_4-close to E_x in $B(0, 1 - \tau_4)$, in the sense of (9.32)–(9.34). By (9.32) and (9.33),

(12.38) $d_{0, 1-\tau_4}(E_x, E_0) \leq \tau_4.$

Let us also apply (9.34) to the ball $B = B(0, t)$, with $t = (20N)^{-1}$. Notice that $t \leq \frac{\delta(x)}{2r} < \text{dist}(0, L_x)$ by (12.32) and (12.36), so B does not meet L_x; then

$$
\begin{aligned}
(12.39) \quad d(X_0) &= \mathcal{H}^d(X_0 \cap B(0,1)) = t^{-d}\mathcal{H}^d(X_0 \cap B) = t^{-d}\mathcal{H}^d(\overline{X_0 \setminus S} \cap B) \\
&= t^{-d}\mathcal{H}^d(E_0 \cap B) \leq t^{-d}\mathcal{H}^d(E_x \cap B) + t^{-d}\tau_4 \\
&= t^{-d}r_1^{-d}\mathcal{H}^d(E \cap B(x, tr_1)) + (20N)^d\tau_4 \\
&= F(tr_1) + (20N)^d\tau_4 \leq \frac{3\omega_d}{2} + 2\tau_1 + (20N)^d\tau_4
\end{aligned}
$$

by the definition (10.1) of $d(X_0)$, where S still denotes the shade of L_x, because $S \cap B = \emptyset$, by definition of E_0, by (9.34) and because $t = (20N)^{-1}$, then by (12.1) and because $B(x, tr_1) \cap S_x = \emptyset$, and finally by (12.22). Thus $d(X_0)$ is as close to $\frac{3\omega_d}{2}$ as we want.

When $d = 2$, and more generally when there is a constant $d_{n,d} > \frac{3\omega_d}{2}$ such that (10.10) holds, we can deduce from this that $X_0 \in \mathbb{Y}_0(n, d)$, and this will simplify our life. In the general case when we only have (10.9), we can still use Lemma 10.3, as for (12.30) above, to show that if τ_1 and τ_4 are small enough, we can find a minimal cone $Y_0 \in \mathbb{Y}_0(n, d)$ such that

$$
(12.40) \qquad\qquad d_{0,1}(X_0, Y_0) \leq \tau_5,
$$

where $\tau_5 > 0$ is a new small constant, that will be chosen soon, depending on τ.

Pick a point $z_0 \in L_x \cap \overline{B}(0, 9/10)$; such a point exists because $\text{dist}(0, L_x) = \frac{9\delta(x)}{10r} \leq \frac{9}{10}$. Set $D = z_0 + [L \cap B(0, 1/20)]$; notice that $D \subset L_x \cap B(99/100) \subset X_0$ because L is the vector space parallel to L_x, and by (12.37). By (12.40),

$$
(12.41) \qquad\qquad \text{dist}(z, Y_0) \leq \tau_5 \text{ for } z \in D.
$$

Also set $D' = z_0 + [L \cap B(0, 1/40)]$; if τ_5 is small enough, it follows from (12.41) and the elementary geometry of $Y_0 \in \mathbb{Y}_0(n, d)$ that there is a single face F of Y_0 such that

$$
(12.42) \qquad\qquad \text{dist}(z, F) \leq \tau_5 \text{ for } z \in D'.
$$

Finally denote by Y_1 the cone of $\mathbb{Y}_0(n, d)$ that contains L_x (or equivalently D'). The face of Y_1 that contains L_x is quite close to F, by (12.42), and hence

$$
(12.43) \qquad\qquad d_{0,1}(Y_1, Y_0) \leq C\tau_5,
$$

where C depends only on n and d. For the record, let us mention that we can take $Y_1 = Y_0 = X_0$ when (10.10) holds for some $d_{n,d} > \frac{3\omega_d}{2}$, rather than the weaker (10.9).

Now set $Y = x + Y_1 \in \mathbb{Y}_x(n, d)$. Notice that $L \subset Y$ because $L_x \subset Y_1$; we just need to check that Y satisfies the conditions (12.34) and (12.35). For the general case we will need the following lemma.

Lemma 12.6. *Recall from below* (12.37) *that S is the shade of L_x seen from the origin, and that $E_0 = \overline{X_0 \setminus S}$. Also set $E_1 = \overline{Y_1 \setminus S}$. For each small $\tau_6 > 0$, we can choose τ_5 so small that in the present situation,*

$$(12.44) \qquad\qquad d_{0,95/100}(E_0, E_1) \le \tau_6$$

and

$$(12.45) \quad \left| \mathcal{H}^d(E_0 \cap B(y,t)) - \mathcal{H}^d(E_1 \cap B(y,t)) \right| \le \tau_6$$
$$\text{for } y \in \mathbb{R}^n \text{ and } t > 0 \text{ such that } B(y,t) \subset B(0, 96/100).$$

Before we prove this, let us mention that in the simpler situation when we have (10.10), the lemma holds trivially because $Y_1 = X_0$. Also, let us first see how Lemma 12.5 follows from Lemma 12.6, and prove Lemma 12.6 afterwards.

Recall that we just need to check that $Y = x + Y_1$ satisfies (12.34) and (12.35). We deduce from (12.38) and (12.44) that $d_{0,94/100}(E_x, E_1) \le 2\tau_4 + 2\tau_6$. Since $E = x + r_1 E_x$ and $W = \overline{Y \setminus S_x} = x + r_1 E_1$ (see above (12.34)), we also get that

$$(12.46) \qquad\qquad d_{x, 94r_1/100}(E, W) \le 2\tau_4 + 2\tau_6;$$

(12.34) follows, because $94r_1/100 = 94r/90 > r$, and if τ_4 and τ_6 are small enough. As for (12.34), let $B = B(y,t) \subset B(x,r)$ be given, set $B' = r_1^{-1}(B - x)$; then $B' \subset B(0, 95/100)$ and

$$
\begin{aligned}
(12.47) \qquad \left| \mathcal{H}^d(E \cap B) - \mathcal{H}^d(W \cap B) \right| &= r_1^d \left| \mathcal{H}^d(E_x \cap B') - \mathcal{H}^d(E_1 \cap B') \right| \\
&= r_1^d \left| \mathcal{H}^d(E_x \cap B') - \mathcal{H}^d(E_0 \cap B') \right| \\
&\quad + r_1^d \left| \mathcal{H}^d(E_0 \cap B') - \mathcal{H}^d(E_1 \cap B') \right| \\
&\le \tau_4 r_1^d + \tau_6 r_1^d = (\tau_4 + \tau_6)(10r/9)^d
\end{aligned}
$$

by (9.34) (for E_x and with the constant τ_4; see below (12.37)) and (12.45) (recall that $E_0 = \overline{X_0 \setminus S}$ and $E_1 = \overline{Y_1 \setminus S}$). This proves (12.35); thus Lemma 12.5 will follow as soon as we prove Lemma 12.6.

Proof of Lemma 12.6: The reason why we need to prove something is that although

$$(12.48) \qquad d_{0,1}(Y_1, X_0) \leq C\tau_5$$

by (12.40) and (12.43), removing S could change the situation. At least, both cones contain $L_x \cap B(0, 99/100)$ by (12.37) and the definition of Y_1, hence they both contain $S \cap B(0, 99/100)$. Even that way, we still need to check that X_0 does not contain a piece that lies close to $S \cap B(0, 99/100)$ (hence, to Y_1), but not close to E_1, and similarly with X_0 and Y_1 exchanged.

Set $\rho = 98/100$ and $B = B(0, \rho)$; we want to evaluate the size of $E_0 \cap S \cap B$. In fact, because of the way E_0 was obtained in the proof of Corollary 9.3, it is such that the functional F is constant on $(0, 1)$ (see below (9.64)), and we could deduce from (1.12) that $\mathcal{H}^d(E_0 \cap B(0, 1) \cap S) = 0$. But let us pretend we did not notice and check that $\mathcal{H}^d(E_0 \cap B \cap S)$ is small in a way which is more complicated, but easier to track. As we just said,

$$(12.49) \qquad S \cap B \subset X_0,$$

because X_0 is a cone that contains $L_x \cap B$ (by (12.37)). Let us check that

$$(12.50) \qquad S \cap B \setminus E_0 = X_0 \cap B \setminus E_0.$$

The direct inclusion follows from (12.49), and the converse from the fact that $E_0 = \overline{X_0 \setminus S} \supset X_0 \setminus S$. Then

$$
\begin{aligned}
(12.51) \qquad \mathcal{H}^d(X_0 \cap B) &= \mathcal{H}^d(E_0 \cap B) + \mathcal{H}^d(X_0 \cap B \setminus E_0) \\
&= \mathcal{H}^d(E_0 \cap B) + \mathcal{H}^d(S \cap B \setminus E_0) \\
&= \mathcal{H}^d(E_0 \cap B) + \mathcal{H}^d(S \cap B) - \mathcal{H}^d(S \cap B \cap E_0).
\end{aligned}
$$

But $\mathcal{H}^d(E_0 \cap B) \leq \mathcal{H}^d(E_x \cap B) + \tau_4$ by (9.34) and $\mathcal{H}^d(X_0 \cap B) = \rho^d d(X_0) \geq \frac{3\omega_d \rho^d}{2}$ by (10.9) and because X_0 is not a plane, so

$$
\begin{aligned}
(12.52) \qquad \mathcal{H}^d(S \cap B \cap E_0) &\leq \mathcal{H}^d(E_x \cap B) + \tau_4 + \mathcal{H}^d(S \cap B) - \frac{3\omega_d \rho^d}{2} \\
&= r_1^{-d} \mathcal{H}^d(E \cap B(x, \rho r_1)) \\
&\quad + \tau_4 + r_1^{-d} \mathcal{H}^d(S_x \cap B(x, \rho r_1)) - \frac{3\omega_d \rho^d}{2} \\
&= \rho^d F_x(\rho r_1) + \tau_4 - \frac{3\omega_d \rho^d}{2} \leq \tau_4 + 2\rho^d \tau_1 \leq \tau_4 + 2\tau_1
\end{aligned}
$$

by (12.1) (and because $E_x = r_1^{-1}(E - x)$ and $L_x = r_1^{-1}(L - x)$), then by (12.22).

For $B' = B(y, t) \subset B$, the proof of (12.51) also yields

$$(12.53) \quad \mathcal{H}^d(X_0 \cap B') = \mathcal{H}^d(E_0 \cap B') + \mathcal{H}^d(S \cap B') - \mathcal{H}^d(S \cap B' \cap E_0),$$

which implies that

$$(12.54) \quad |\mathcal{H}^d(X_0 \cap B') - \mathcal{H}^d(E_0 \cap B') - \mathcal{H}^d(S \cap B')|$$
$$\leq \mathcal{H}^d(S \cap B \cap E_0) \leq \tau_4 + 2\tau_1.$$

For the other cone Y_1, we have the simpler formula

$$(12.55) \quad \begin{aligned} \mathcal{H}^d(Y_1 \cap B') &= \mathcal{H}^d(\overline{Y_1 \setminus S} \cap B') + \mathcal{H}^d(S \cap B') \\ &= \mathcal{H}^d(E_1 \cap B') + \mathcal{H}^d(S \cap B') \end{aligned}$$

because Y_1 is a cone in $\mathbb{Y}_0(n, d)$ that contains L_x, and hence S.

By (12.48) and the same compactess argument using Lemma 9.2 as below (11.51) or (12.30),

$$(12.56) \quad |\mathcal{H}^d(X_0 \cap B') - \mathcal{H}^d(Y_1 \cap B')| \leq \frac{\tau_6}{2}$$

for every B' as above, where τ_6 is the small constant that is given for Lemma 12.6, provided that we take τ_5 accordingly small. Thus

$$(12.57) \quad \begin{aligned} |\mathcal{H}^d(E_0 \cap B') - \mathcal{H}^d(E_1 \cap B')| &\leq \big|[\mathcal{H}^d(X_0 \cap B') - \mathcal{H}^d(S \cap B')] \\ &\quad - \mathcal{H}^d(E_1 \cap B')\big| + \tau_4 + 2\tau_1 \\ &= \big|\mathcal{H}^d(X_0 \cap B') - \mathcal{H}^d(Y_1 \cap B')\big| \\ &\quad + \tau_4 + 2\tau_1 \leq \frac{\tau_6}{2} + \tau_4 + 2\tau_1, \end{aligned}$$

by (12.54), (12.55), and (12.56); this implies (12.45) if τ_1 and τ_4 are small enough.

Now we check (12.44). First we show that

$$(12.58) \quad \operatorname{dist}(y, E_0) \leq C\tau_5 \text{ for } y \in B(0, 96/100) \cap E_1.$$

Let $y \in B(0, 96/100) \cap E_1$ be given; since $y \in Y_1$, (12.48) says that we can find $z \in X_0$ such that $|z - y| \leq C\tau_5$. If $z \in X_0 \setminus S$, then $z \in E_0$ and $\operatorname{dist}(y, E_0) \leq |z - y| \leq C\tau_5$, as needed. So we may assume that $z \in S$.

Notice that $\operatorname{dist}(y, S) = \operatorname{dist}(y, L_x)$ by elementary geometry (because $y \in E_1 = \overline{Y_1 \setminus S}$ and Y_1 is the cone of $\mathbb{Y}_0(n, d)$ that contains L_x). Then

$$(12.59) \quad \operatorname{dist}(y, L_x) = \operatorname{dist}(y, S) \leq |y - z| \leq C\tau_5,$$

and then $\mathrm{dist}(y, E_0) \leq C\tau_5$ too, because X_0 is a cone that contains $B \cap L_x$, and then the points $\lambda\xi$, $\xi \in B \cap L_x$, and $\lambda \in (0,1)$, lie in E_0. This completes our proof of (12.58).

Now we want to show that

$$(12.60) \qquad \mathrm{dist}(y, E_1) \leq \tau_7 \text{ for } y \in E_0 \cap B(0, 95/100),$$

with a constant τ_7 that is as small as we want; (12.44) will follow at once from this and (12.58), except that formally we would need to replace τ_6 with τ_7 in the statement. Suppose this fails; then of course

$$(12.61) \qquad \mathcal{H}^d(E_1 \cap B(z, \tau_7)) = 0.$$

Recall from the lines below (12.37) that E_0 is a coral minimal set in $B(0, 1 - \tau_4)$; hence, by the local Ahlfors-regularity property (2.9),

$$(12.62) \qquad \mathcal{H}^d(E_0 \cap B(z, \tau_7)) \geq C^{-1}\tau_7^d,$$

where C depends only on n and d. But $B(z, \tau_7) \subset B(0, 96/100)$, and (12.45) says that $|\mathcal{H}^d(E_0 \cap B(z, \tau_7)) - \mathcal{H}^d(E_1 \cap B(z, \tau_7))| \leq \tau_6$. We can prove this with any $\tau_6 > 0$, and if we choose τ_6 small enough, we get a contradiction with (12.62) or (12.61) which proves (12.60). This concludes our proof of (12.44); Lemma 12.6 follows, and as was checked earlier, so does Lemma 12.5. $\qquad \square$

We easily deduce Proposition 12.1 and Corollary 1.8 from Lemma 12.5: the assumptions (for the more general Proposition 12.1) are the same, the fact that $\delta(x) \leq N^{-1}$ follows from (12.31), and the description of E and W in (1.45)–(1.47) follows by applying Lemma 12.5 with $r = 3\delta(x)$ and the constant $3^{-d}\tau$.

We complete this section with a rapid description of E at the large scales that are not covered by Corollary 1.8.

Proposition 12.7. *Let the dimensions n and d satisfy (10.9) (thus $d = 2$, or $d = n - 1$ and $n = 4$ (but probably $n \leq 6$) work). For each choice of constants $\tau > 0$ and $A \geq 10$, we can find $\varepsilon > 0$ and $N \geq 100 + \tau^{-1}$, with the following properties. Let L be a vector $(d-1)$-plane and let E be a coral sliding almost minimal set in $B(0, 3)$, with boundary condition coming from L and a gauge function h that satisfies (1.42), such that $d_{0,3}(E, V) \leq \varepsilon$ (as in (1.43)) for some $V \in \mathbb{V}(L)$. Then let $x \in E \cap B(0, 1) \setminus L$ be a singular point, in the sense that $\theta_x(0) > \omega_d$ (see the definition in (1.44)), and denote by y the orthogonal projection of x on L. Then*

$$(12.63) \qquad \qquad \mathrm{dist}(x, L) \leq N^{-1}$$

and, for every radius r such that

(12.64) $N \operatorname{dist}(x, L) \leq r \leq (2A)^{-1}$

there is a cone X centered at y, which is a coral sliding minimal set in $\mathbb{R}^n$, with the boundary condition coming from L, such that

(12.65) $\operatorname{dist}(z, X) \leq \tau r$ *for $z \in E \cap B(y, Ar) \setminus B(y, r)$,*

(12.66) $\operatorname{dist}(z, E) \leq \tau r$ *for $z \in X \cap B(y, Ar) \setminus B(y, r)$,*

(12.67) $\left| \mathcal{H}^d(E \cap B(z, t)) - \mathcal{H}^d(X \cap B(z, t)) \right| \leq \tau r^d$
$\qquad$ *for $z \in \mathbb{R}^n$ and $t > 0$ such that $B(z, t) \subset B(y, Ar) \setminus B(y, r)$,*

and

(12.68) $\left| \mathcal{H}^d(E \cap B(y, t)) - \mathcal{H}^d(X \cap B(y, t)) \right| \leq \tau r^d$ *for $r \leq t \leq Ar$.*

Moreover,

(12.69) $\left| \mathcal{H}^d(X \cap B(y, 1)) - \omega_d \right| \leq 2\tau$.

Notice that since N is larger than τ^{-1} (maybe much larger), (12.65)–(12.67) do not give information at the scale of $\operatorname{dist}(x, L)$; this is fair, because E does not look like a cone at that scale.

Our measure estimate (12.67) only works outside of $B(y, r)$, which is why we needed to add (12.68).

In the good cases, (12.69) should help us determine the type of X, but we shall not try to do this here. See Remark 12.8.

Proof: Let E and x be as in the statement. Notice that the assumptions of Proposition 12.1 are satisfied (if ε is small enough, depending on N, which itself will be chosen later), so we may use the previous results of this section.

Set $\delta(x) = \operatorname{dist}(x, L)$; the fact that $\delta(x) \leq N^{-1}$ (i.e., that (12.63) holds) as soon as ε is small enough follows from Proposition 12.1, or directly (12.31). The main point is thus the existence of X.

Again we shall use (12.22), but since we want to apply a near monotonicity result for balls centered at y, we will translate it in terms of the functional F_y defined as F_x in (12.1), but with x replaced by y. Notice that for $r > 2|y - x| = 2\delta(x)$,

(12.70)
$$\theta_y(r) = r^{-d} \mathcal{H}^d(E \cap B(y, r))$$
$$\geq r^{-d} \mathcal{H}^d(E \cap B(y, r - \delta(x))) = \left(\frac{r - \delta(x)}{r} \right)^d \theta_x(r - \delta(x)).$$

Similarly,

$$(12.71) \qquad \theta_y(r) \leq \left(\frac{r + \delta(x)}{r}\right)^d \theta_x(r + \delta(x)).$$

We also deduce from the definition of S_x (see (1.45)) that

$$(12.72) \qquad \frac{\omega_d}{2} - C\frac{\delta(x)}{\rho} \leq \rho^{-d}\mathcal{H}^d(S_x \cap B(x, \rho)) \leq \frac{\omega_d}{2}$$

for $\rho \geq \delta(x)$. Let us use this to show that

$$(12.73) \qquad |\theta_y(r) - \omega_d| \leq C\frac{\delta(x)}{r} + 3\tau_1 \text{ for } 2\delta(x) < r < \frac{18}{10}.$$

We first apply (12.70), set $\rho = r - \delta(x)$, and then apply (12.1), (12.72), and (12.22) to get that

$$\theta_y(r) \geq \rho^d r^{-d}\theta_x(\rho) = \rho^d r^{-d}\big(F_x(\rho) - \rho^{-d}\mathcal{H}^d(S_x \cap B(x, \rho))\big)$$

$$(12.74) \qquad \geq \rho^d r^{-d}\left(F_x(\rho) - \frac{\omega_d}{2}\right) \geq \rho^d r^{-d}\big(\omega_d - \tau_1\big)$$

$$\geq \left(1 - C\frac{\delta(x)}{r}\right)\big(\omega_d - \tau_1\big) \geq \omega_d - C\omega_d\frac{\delta(x)}{r} - \tau_1,$$

which gives the lower bound in (12.73). Similarly, we apply (12.71), set $\rho = r + \delta(x)$, and continue as above to get that

$$\theta_y(r) \leq \rho^d r^{-d}\theta_x(\rho) = \rho^d r^{-d}\big(F_x(\rho) - \rho^{-d}\mathcal{H}^d(S_x \cap B(x, \rho))\big)$$

$$(12.75) \qquad \leq \rho^d r^{-d}\left(F_x(\rho) - \frac{\omega_d}{2} + C\frac{\delta(x)}{r}\right) \geq \rho^d r^{-d}\left(\omega_d + C\frac{\delta(x)}{r} + 2\tau_1\right)$$

$$\leq \left(1 + C\frac{\delta(x)}{r}\right)\left(\omega_d + C\frac{\delta(x)}{r} + 2\tau_1\right) \leq \omega_d + C\frac{\delta(x)}{r} + 3\tau_1;$$

(12.73) follows, and now we are ready to apply an almost-constant density result from [**D5**].

Now let r satisfy (12.64). We want to apply Proposition 30.3 of [**D5**] to the set E, with the radii $r_1 = r/2$, $r_0 = r_2 = \frac{3Ar}{2}$, and the small constant $\tau_8 = (3A/2)^{-d}\tau$. So we check the various assumptions.

We take $U = B(y, r_0)$, a single L_j equal to L, and we do not need to straighten things here, i.e., we can take the bilipschitz mapping ξ of (30.2) in [**D5**] to be the identity. Since we took $r \leq (2A)^{-1}$ in (12.64), we get that $2r_0 \leq \frac{3}{2}$ and E is almost minimal in $B(y, 2r_0)$, with $h(2r_0)$ as small as we want (by (1.42)). This takes care of (30.5) in [**D5**] (where

the small ε depends on τ_8, but this is all right); similarly, (12.73) says that

$$(12.76) \qquad |\theta_y(r_2) - \theta_y(r_1)| \le C r_1^{-1} \delta(x) + 6\tau_1 \le C N^{-1} + 6\tau_1$$

because $2\delta(x) < r_1 < r_2 < \frac{18}{10}$ and by (12.64). If N^{-1} and τ_1 are small enough, this implies (30.6) in [**D5**], and we can apply Proposition 30.3 there. We get the existence of a coral minimal cone X (with a boundary condition coming from L), which satisfies (12.65) and (12.66) (by (30.7) and (30.8) there), and (12.67) and (12.68) (by (30.9) and (30.10) there). Finally, (12.69) follows from (12.68) (applied with $t = r$) and the fact that $|\theta_y(r) - \omega_d| \le \tau$, by (12.73) (and if N^{-1} and τ_1 are small enough). This completes our proof of Proposition 12.7. □

Remark 12.8. When $d = 2$, the author believes that any sliding minimal cone X that satisfies (12.69) must lie in $\mathbb{V}(L)$. The proof would follow, for instance, the proof of the description of the minimal cones that was given in [**D2**], and show that $X \cap \partial B(y, 1)$ is composed of arcs of great circles with constraints on how they meet L or each other. But he did not check the details. Possibly this is also true in some higher dimensions too. Then the description in Proposition 12.7 becomes a little better. The author also expects that when $d = 2$, we should be able to get a better local description of E, possibly even with a local C^1 parameterization.

Remark 12.9. We did not try to state Corollary 1.8 or the results of this section when L is a smooth $(d-1)$-dimensional surface. The main ingredients, namely the almost monotonicity formula (Theorem 7.1), and the approximation results (Corollary 9.3 and its earlier analogues in [**D2**] and [**D5**]) are valid in this context, but the author did not check that the rest of the proofs in this section goes through too. Again the author hopes to return to this issue and prove more precise statements in the near future.

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