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ENTIRE SPACELIKE H-GRAPHS IN LORENTZIAN PRODUCT SPACES

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Abstract: In this work we establish sufficient conditions to ensure that an entire spacelike graph immersed with constant mean curvature in a Lorentzian product space, whose Riemannian fiber has sectional curvature bounded from below, must be a trivial slice of the ambient space.

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1. Introduction and statements of the results

The last few decades have seen a steadily growing interest in the study of the geometry of spacelike hypersurfaces immersed in a Lorentzian space. Apart from physical motivations, from the mathematical point of view this is mostly due to the fact that such hypersurfaces exhibit nice Bernstein-type properties, and one can truly say that the first remarkable results in this branch were the rigidity theorems of Calabi in [**10**] and Cheng and Yau in [**11**], who showed (the former for $n \leq 4$, and the latter for general n) that the only maximal (that is, with zero mean curvature) complete noncompact spacelike hypersurfaces of the Lorentz–Minkowski space $\mathbb{L}^{n+1}$ are the spacelike hyperplanes. However, in the case that the mean curvature is a positive constant, Treibergs [**20**] astonishingly showed that there are many entire solutions of the corresponding constant mean curvature equation in $\mathbb{L}^{n+1}$, which he was able to classify by their projective boundary values at infinity.

On the other hand, Xin [**21**] and Aiyama [**1**], working independently, characterized spacelike hyperplanes as the only complete constant mean curvature spacelike hypersurfaces in $\mathbb{L}^{n+1}$ whose Gauss mapping image is contained in a geodesic ball of the n-dimensional hyperbolic space. Later on, Aledo and Alías [**6**], among other results, showed that a complete constant mean curvature spacelike hypersurface which lies between two parallel spacelike hyperplanes of $\mathbb{L}^{n+1}$ must be, in fact, a spacelike hyperplane.

It is also natural to treat these same questions in a wide class of Lorentzian manifolds. When the ambient space is a Lorentzian product space, Salavessa [**19**] considered spacelike graphs in $-\mathbb{R}\times M^n$ and, under the assumption that the Cheeger constant of the fiber M^n is zero and some conditions on the second fundamental form at infinity, she concluded that if the spacelike graph has parallel mean curvature then the graph must be maximal. When M^n is the hyperbolic space $\mathbb{H}^n$, for any constant $c \in \mathbb{R}$ the author described an explicit foliation of $-\mathbb{R} \times \mathbb{H}^n$ by hypersurfaces with constant mean curvature c. Meanwhile, Albujer [**2**] obtained new explicit examples of complete and non-complete entire maximal spacelike graphs in $-\mathbb{R} \times \mathbb{H}^2$.

Afterwards, Albujer and Alías [**3**] established Calabi–Bernstein results for maximal spacelike surfaces immersed into a Lorentzian product space $-\mathbb{R} \times M^2$. In particular, when M^2 is a Riemannian surface with nonnegative Gaussian curvature, they proved that any complete maximal spacelike surface in $-\mathbb{R} \times M^2$ must be totally geodesic. Besides, assuming that the fiber M^2 is non-flat, the authors concluded that it must be a slice $\{t\} \times M^2$. In [**14**], Li and Salavessa generalized such results of [**3**] to higher dimension and codimension.

In [**5**], the first author jointly with Albujer and Camargo established uniqueness results concerning complete spacelike hypersurfaces with constant mean curvature immersed in $-\mathbb{R}\times\mathbb{H}^n$. Next, Albujer and Alías [**4**] obtained some parabolicity criteria for maximal surfaces immersed into a Lorentzian product space $-\mathbb{R}\times M^2$, where M^2 is supposed to have nonnegative Gaussian curvature. As an application of their main result, they deduced that every maximal graph over a starlike domain $\Omega \subset M^2$ is parabolic. This allowed them to give an alternative proof of the nonparametric version of the Calabi–Bernstein theorem for entire maximal graphs in such ambient space. Later, the first author jointly with Parente [**13**] obtained a lower estimate of the index of relative nullity of complete maximal spacelike hypersurfaces immersed in a so-called Robertson–Walker spacetime and, in particular, we also proved a sort of weak extension of the Calabi–Bernstein theorem in Lorentzian product spaces. More recently, the authors [**12**] applied some generalized maximum principles in order to establish uniqueness results concerning complete spacelike hypersurfaces with constant mean curvature in $-\mathbb{R}\times M^n$, extending the results of [**5**].

Motivated by these works described above, in this article we deal with entire spacelike graphs $\Sigma(u) = \{(u(x), x);\, x \in M^n\}$ with constant mean curvature in a Lorentzian product space $-\mathbb{R}\times M^n$, whose Riemannian fiber M^n has sectional curvature bounded from below. According to the current literature, since the mean curvature H of $\Sigma(u)$ is supposed to be constant, we call $\Sigma(u)$ an entire spacelike H-graph. In this setting, we obtain the following Calabi–Bernstein type result:

Theorem 1. *Let $\overline{M}^{n+1}=-\mathbb{R}\times M^n$ be a Lorentzian product space, such that the sectional curvature K_M of its Riemannian fiber M^n satisfies $K_M\geq-\kappa$, for some positive constant κ. Let $\Sigma(u)$ be an entire spacelike H-graph over M^n, with u bounded and H_2 bounded from below. If*

$$|Du|_M^2\leq\frac{|A|^2}{\kappa(n-1)+|A|^2}, \tag{1.1}$$

then $u\equiv t_0$ for some $t_0\in\mathbb{R}$.

Here, $H_2=\frac{2}{n(n-1)}S_2$ is the mean value of the second elementary symmetric function S_2 on the eigenvalues of the shape operator A of $\Sigma(u)$, Du stands the gradient of the smooth function $u\colon M^n\to\mathbb{R}$ in M^n and $|Du|_M$ its norm, both with respect to the metric of M^n.

In the context of Lorentzian product spaces, we note that our restriction on the sectional curvature K_M of the fiber M^n in Theorem 1 is a weaker restriction when compared with the so-called *null (timelike) convergence condition*, which means that the Ricci curvature of the ambient space is nonnegative on null or lightlike (timelike) directions (for a thorough discussion about such convergence conditions, see for example [**7, 8, 9, 15**]). Furthermore, through the example described in Remark 3, we see that Theorem 1 is sharp in the sense that it does not hold when the function u is unbounded.

The proof of Theorem 1 is given in Section 3. From Theorem 1 jointly with Theorem 3.3 of [**3**], it is not difficult to see that we also get the following result, where 1-maximal means that H_2 vanishes identically on the graph:

Corollary 1. *Let $\overline{M}^{n+1}=-\mathbb{R}\times M^n$ be a Lorentzian product space, such that the sectional curvature K_M of its Riemannian fiber M^n is nonnegative. If $\Sigma(u)$ is an entire 1-maximal spacelike H-graph over M^n with u bounded, then $\Sigma(u)$ is totally geodesic. In addition, if $n=2$ and $K_M(p)>0$ at some point $p\in M^2$, then $u\equiv t_0$ for some $t_0\in\mathbb{R}$.*

We observe that, when the ambient space is the Lorentz–Minkowski space $\mathbb{L}^{n+1}$, Corollary 1 reads as follows:

Corollary 2. *The only bounded entire 1-maximal spacelike H-graphs over a spacelike hyperplane of $\mathbb{L}^{n+1}$ are the spacelike hyperplanes.*

2. Preliminaries

In what follows, we deal with a spacelike hypersurface Σ^n immersed into an $(n+1)$-dimensional Lorentzian product space $\overline{M}^{n+1}$ of the form $\mathbb{R}\times M^n$, where M^n is an n-dimensional connected Riemannian manifold and $\overline{M}^{n+1}$ is endowed with the Lorentzian metric

$$\langle\,,\,\rangle = -\pi_{\mathbb{R}}^*(dt^2) + \pi_M^*(\langle\,,\,\rangle_M),$$

where $\pi_{\mathbb{R}}$ and π_M denote the canonical projections from $\mathbb{R}\times M$ onto each factor, and $\langle\,,\,\rangle_M$ is the Riemannian metric on M^n.

For simplicity, we will just write $\overline{M}^{n+1} = -\mathbb{R}\times M^n$ and $\langle\,,\,\rangle = -dt^2 + \langle\,,\,\rangle_M$. In this setting, for each fixed $t_0 \in \mathbb{R}$, we say that $M^n_{t_0} = \{t_0\}\times M^n$ is a *slice*, which is a totally geodesic spacelike hypersurface of $\overline{M}^{n+1}$. We recall that a smooth immersion $\psi\colon \Sigma^n \to -\mathbb{R}\times M^n$ of an n-dimensional connected manifold Σ^n is said to be a *spacelike hypersurface* if the induced metric via ψ is a Riemannian metric on Σ^n, which, as usual, is also denoted for $\langle\,,\,\rangle$.

Since $\partial_t = (\partial/\partial_t)_{(t,x)}$, $(t,x)\in -\mathbb{R}\times M^n$, is a unitary timelike vector field globally defined on the ambient spacetime, then there exists a unique timelike unitary normal vector field N globally defined on the spacelike hypersurface Σ^n which is in the same time-orientation as ∂_t. By using Cauchy–Schwarz inequality, we get $\langle N, \partial_t\rangle \le -1$ on Σ^n. We will refer to that normal vector field N as the future-pointing Gauss map of the spacelike hypersurface Σ^n.

Let $\overline{\nabla}$ and ∇ denote the Levi–Civita connections in $-\mathbb{R}\times M^n$ and Σ^n, respectively. Then the Gauss and Weingarten formulas for the spacelike hypersurface $\psi\colon \Sigma^n \to -\mathbb{R}\times M^n$ are given by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle AX, Y\rangle N \tag{2.1}$$

and

$$AX = -\overline{\nabla}_X N, \tag{2.2}$$

for every tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. Here $A\colon \mathfrak{X}(\Sigma)\to\mathfrak{X}(\Sigma)$ stands for the shape operator (or Weingarten endomorphism) of Σ^n with respect to the future-pointing Gauss map N.

As in [**17**], the curvature tensor R of the spacelike hypersurface Σ^n is given by

$$R(X,Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z,$$

where [ ] denotes the Lie bracket and $X, Y, Z \in \mathfrak{X}(\Sigma)$. Another fact well known is that the curvature tensor R of the spacelike hypersurface Σ^n can be described in terms of the shape operator A and the curvature tensor $\overline{R}$ of the ambient spacetime $-\mathbb{R} \times M^n$ by the so-called Gauss equation given by

$$R(X,Y)Z = (\overline{R}(X,Y)Z)^\top - \langle AX, Z\rangle AY + \langle AY, Z\rangle AX, \tag{2.3}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$.

Now, we consider two particular functions naturally attached to a spacelike hypersurface Σ^n immersed into a Lorentzian product space $-\mathbb{R} \times M^n$, namely, the (vertical) height function $h = (\pi_{\mathbb{R}})|_\Sigma$ and the support function $\langle N, \partial_t\rangle$, where we recall that N denotes the future-pointing Gauss map of Σ^n and ∂_t is the coordinate vector field induced by the universal time on $-\mathbb{R} \times M^n$.

Let us denote by $\overline{\nabla}$ and ∇ the gradients with respect to the metrics of $-\mathbb{R} \times M^n$ and Σ^n, respectively. Then, a simple computation shows that the gradient of $\pi_{\mathbb{R}}$ on $-\mathbb{R} \times M^n$ is given by

$$\overline{\nabla}\pi_{\mathbb{R}} = -\langle \overline{\nabla}\pi_{\mathbb{R}}, \partial_t\rangle\partial_t = -\partial_t,$$

so that the gradient of h on Σ^n is

$$\nabla h = (\overline{\nabla}\pi_{\mathbb{R}})^\top = -\partial_t^\top = -\partial_t - \langle N, \partial_t\rangle N, \tag{2.4}$$

where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M}^{n+1})$ along Σ^n. Thus, we get

$$|\nabla h|^2 = \langle N, \partial_t\rangle^2 - 1, \tag{2.5}$$

where $|\ |$ denotes the norm of a vector field on Σ^n. Since ∂_t is parallel on $-\mathbb{R} \times M^n$, we have that

$$\overline{\nabla}_X \partial_t = 0, \tag{2.6}$$

for every tangent vector field $X \in \mathfrak{X}(\Sigma)$. Writing $\partial_t = -\nabla h - \langle N, \partial_t\rangle N$ along the hypersurface Σ^n and using formulas (2.1) and (2.2), from (2.4) and (2.6) we get that

$$\nabla_X \nabla h = \langle N, \partial_t\rangle AX, \tag{2.7}$$

for every tangent vector field $X \in \mathfrak{X}(\Sigma)$. Therefore, from (2.7) we obtain that the Laplacian on Σ^n of its height function h is given by

$$\Delta h = -nH\langle N, \partial_t\rangle, \tag{2.8}$$

where $H = -\frac{1}{n}\operatorname{tr}(A)$ denotes the mean curvature of Σ^n with respect to its future-pointing Gauss mapping N.

Moreover, from (2.4) and (2.6) we also have that

$$X(\langle N, \partial_t\rangle) = -\langle A(X), \partial_t\rangle = -\langle X, A(\partial_t^\top)\rangle = \langle X, A(\nabla h)\rangle,$$

for all $X \in \mathfrak{X}(\Sigma)$. Thus,

$$\nabla\langle N, \partial_t\rangle = -A(\partial_t^\top) = A(\nabla h). \tag{2.9}$$

Supposing that Σ^n is a constant mean curvature spacelike hypersurface of $-\mathbb{R}\times M^n$, as a particular case of Corollary 8.2 in [**7**] we also obtain that the Laplacian on Σ^n of its support function $\langle N, \partial_t\rangle$ is given by

$$\Delta\langle N, \partial_t\rangle = (\mathrm{Ric}_M(N^*, N^*) + |A|^2)\langle N, \partial_t\rangle, \tag{2.10}$$

where Ric_M is the Ricci curvature of the fiber M^n, $N^* = N + \langle N, \partial_t\rangle\partial_t$ is the projection of N onto M^n and $|A|$ stands for the Hilbert–Schmidt norm of the shape operator A of Σ^n.

3. Proof of Theorem 1

Let $-\mathbb{R}\times M^n$ be a Lorentzian product space. We recall that an entire graph over the fiber M^n is determined by a smooth function $u \in C^\infty(M)$ and it is given by

$$\Sigma(u) = \{(u(x), x); x \in M^n\} \subset -\mathbb{R}\times M^n.$$

The metric induced on M^n from the Lorentzian metric on the ambient space via $\Sigma(u)$ is

$$\langle\, ,\, \rangle = -du^2 + \langle\, ,\, \rangle_M. \tag{3.1}$$

Remark 1. It can be easily seen from (3.1) that an entire graph $\Sigma(u)$ is a spacelike hypersurface if, and only if, $|Du|_M^2 < 1$. Note that, when the fiber M^n is simply connected, every complete spacelike hypersurface in $-\mathbb{R}\times M^n$ is an entire graph in such space (see, for instance, Lemma 3.1 of [**3**]). However, according to the examples of non-complete entire maximal graphs in $-\mathbb{R}\times\mathbb{H}^2$ due to Albujer in Section 3 of [**2**], we see that an entire spacelike graph in a Lorentzian product space is not necessarily complete, in the sense that the induced Riemannian metric (3.1) is not necessarily complete.

If $\Sigma(u)$ is an entire graph over the fiber M^n, with a straightforward computation we verify that the vector field

$$N = \frac{1}{\sqrt{1-|Du|_M^2}}(\partial_t + Du) \tag{3.2}$$

defines the future-pointing Gauss map of $\Sigma(u)$.

Let us study the shape operator A of $\Sigma^n(u)$ with respect its orientation given by (3.2). For any $X \in \mathfrak{X}(\Sigma(u))$, since $X = X^* - \langle Du, X^*\rangle_M \partial_t$, we have that

$$AX = -\overline{\nabla}_X N = \langle Du, X^*\rangle_M \overline{\nabla}_{\partial_t} N - \overline{\nabla}_{X^*} N. \tag{3.3}$$

Consequently, from (3.2), (3.3), and with aid of Proposition 7.35 of [**17**], we verify that

$$AX = -\frac{1}{\sqrt{1-|Du|_M^2}} D_{X^*} Du - \frac{\langle D_{X^*} Du, Du\rangle_M}{(1-|Du|_M^2)^{3/2}} Du, \tag{3.4}$$

where D denotes the Levi–Civita connection in M^n with respect to the metric $\langle\, ,\, \rangle_M$.

From (3.4) we obtain that the mean curvature of $\Sigma(u)$ is given by

$$nH = \operatorname{Div}\left(\frac{Du}{\sqrt{1-|Du|_M^2}}\right), \tag{3.5}$$

where Div stands for the divergence operator on M^n with respect to the metric $\langle\, ,\, \rangle_M$.

In order to prove Theorem 1, we will need two key lemmas. The first one gives a suitable lower estimate for the Ricci curvature of a spacelike hypersurface immersed in $-\mathbb{R} \times M^n$.

Lemma 1. *Let Σ^n be a spacelike hypersurface immersed in a Lorentzian product space $-\mathbb{R} \times M^n$, whose sectional curvature K_M of its fiber M^n verifies $K_M \geq -\kappa$ for some positive constant κ. Then, for all $X \in \mathfrak{X}(\Sigma)$, the Ricci curvature of Σ^n satisfies the following inequality*

$$\operatorname{Ric}(X,X) \geq -\kappa(n-1)(1+|\nabla h|^2)|X|^2 - \frac{n^2H^2}{4}|X|^2. \tag{3.6}$$

Proof: let us consider $X \in \mathfrak{X}(\Sigma)$ and a local orthonormal frame $\{E_1, \dots, E_n\}$ of $\mathfrak{X}(\Sigma)$. Then, it follows from Gauss equation (2.3) that

$$\begin{aligned}\operatorname{Ric}(X,X) &= \sum_{i=1}^n \langle \overline{R}(X,E_i)X, E_i\rangle + nH\langle AX, X\rangle + \langle AX, AX\rangle \\ &= \sum_{i=1}^n \langle \overline{R}(X,E_i)X, E_i\rangle - \frac{n^2H^2}{4}|X|^2 + \left|AX + \frac{nH}{2}X\right|^2.\end{aligned}$$

Moreover, we have that

$$\begin{aligned}\langle \overline{R}(X,E_i)X, E_i\rangle &= \langle R(X^*,E_i^*)X^*, E_i^*\rangle_M \\ &= K_M(X^*,E_i^*)(\langle X^*,X^*\rangle_M\langle E_i^*,E_i^*\rangle_M - \langle X^*,E_i^*\rangle_M^2).\end{aligned} \tag{3.7}$$

On the other hand, since $X^* = X + \langle X, \partial_t \rangle \partial_t$, $E_i^* = E_i + \langle E_i, \partial_t \rangle \partial_t$, and $\nabla h = -\partial_t^\top$, with a straightforward computation we see that

$$\langle X^*, X^* \rangle_M \langle E_i^*, E_i^* \rangle_M = (1 + \langle E_i, \nabla h \rangle^2)(|X|^2 + \langle X, \nabla h \rangle^2)$$

and

$$\begin{aligned}\langle X^*, E_i^* \rangle_M^2 = \langle X, E_i \rangle^2 &+ 2\langle X, \nabla h \rangle \langle E_i, \nabla h \rangle \langle X, E_i \rangle \\ &+ \langle X, \nabla h \rangle^2 \langle E_i, \nabla h \rangle^2.\end{aligned}$$

Therefore, since we are supposing that $K_M \geq -\kappa$ for some positive constant κ, we obtain

$$\begin{aligned}\sum_{i=1}^{n} \langle \overline{R}(X, E_i)X, E_i \rangle &\geq -\kappa \left((n-1)|X|^2 + (n-2)\langle X, \nabla h \rangle^2 + |X|^2 |\nabla h|^2 \right) \\ &\geq -\kappa (n-1)(1 + |\nabla h|^2)|X|^2,\end{aligned}$$

which jointly with (3.7) yields (3.6). □

The second auxiliary lemma is the well known generalized maximum principle due to Omori [**16**] and Yau [**22**], which is quoted below.

Lemma 2. *Let Σ^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and ϑ be a smooth function on Σ^n which is bounded from below. Then, for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in \Sigma^n$ such that*

$$\inf_\Sigma \vartheta \leq \vartheta(p_\varepsilon) < \inf_\Sigma \vartheta + \varepsilon, \quad |\nabla \vartheta(p_\varepsilon)| < \varepsilon, \quad \textit{and} \quad \Delta \vartheta(p_\varepsilon) > -\varepsilon.$$

Now, we are in position to present the proof of Theorem 1.

Proof of Theorem 1: Observe first that, under the assumptions of the theorem, $\Sigma(u)$ is indeed a complete spacelike hypersurface. In fact, from (3.1) and the Cauchy–Schwarz inequality we get

$$\langle X, X \rangle = \langle X^*, X^* \rangle_M - \langle Du, X^* \rangle_M^2 \geq (1 - |Du|_M^2) \langle X^*, X^* \rangle_M, \tag{3.8}$$

for every tangent vector field X on $\Sigma(u)$.

On the other hand, we have that the Hilbert–Schmidt norm of the shape operator A of $\Sigma(u)$ satisfies the following algebraic identity

$$|A|^2 = n^2 H^2 - n(n-1)H_2. \tag{3.9}$$

Thus, since H is constant and H_2 is supposed to be bounded from below, from (3.9) it holds that $\sup_{p \in \Sigma(u)} |A_p|^2 < +\infty$. So, from (1.1) we see that there exists a constant $0 < \alpha < 1$ such that $|Du|_M \leq \alpha$. Hence, from (3.8) we get

$$\langle X, X \rangle \geq (1 - \alpha^2) \langle X^*, X^* \rangle_M.$$

This implies that $L \geq \sqrt{c}L_M$, where L and L_M denote the length of a curve on $\Sigma(u)$ with respect to the Riemannian metrics $\langle\, , \rangle$ and $\langle\, , \rangle_M$, respectively, and $c = 1 - \alpha^2$. As a consequence, since we are supposing that M^n is complete, then the induced metric on $\Sigma(u)$ from the metric of $-\mathbb{R} \times M^n$ is also complete.

Now, let us consider on $\Sigma(u)$ the functions $\eta = 1 - e^{-ku}$, with $k \in \mathbb{N}$, and $W = \sqrt{1 - |Du|_M^2}$. Since we are supposing that u is bounded, we have that the function $\vartheta = \eta W$ is bounded from below. On the other hand, since H is constant and taking into account hypothesis (1.1) jointly with (3.12), from Lemma 1 we have that the Ricci curvature of $\Sigma(u)$ is also bounded from below. Hence, we can apply Lemma 2 to the function ϑ, obtaining a sequence of points $\{p_{k,\varepsilon}\}$ in $\Sigma(u)$ such that, for each fixed $k > 0$,

$$|\nabla\vartheta|(p_{k,\varepsilon}) \leq \varepsilon, \quad \vartheta(p_{k,\varepsilon}) \leq \inf_{\Sigma(u)} \vartheta + \varepsilon, \quad \text{and} \quad \Delta\vartheta(p_{k,\varepsilon}) \geq -\varepsilon.$$

Computing $\Delta\vartheta$ we obtain

$$\Delta\vartheta = \Delta(\eta W) = W\Delta\eta + \eta\Delta W + 2\langle\nabla W, \nabla\eta\rangle. \tag{3.10}$$

Therefore, since $W\nabla\eta = \nabla\vartheta - \eta\nabla W$, from (3.10) we get

$$\Delta\vartheta = W\Delta\eta + \eta\left(\Delta W - 2\frac{|\nabla W|^2}{W}\right) + \frac{2}{W}\langle\nabla W, \nabla\vartheta\rangle. \tag{3.11}$$

On the other hand, since $N = -\langle N, \partial_t\rangle\partial_t + N^*$ where N^* denotes the projection of N onto the fiber M^n, from equation (2.4) it is not difficult to see that $N^{*\top} = -\langle N, \partial_t\rangle\nabla u$ and $|\nabla u|^2 = \langle N^*, N^*\rangle_M$. Here, we are taking into account that the height function h of $\Sigma(u)$ is nothing but the function u regarded as a function on $\Sigma(u)$. Thus, from (3.2) we obtain that

$$|\nabla u|^2 = \frac{|Du|_M^2}{1 - |Du|_M^2} = \frac{1 - W^2}{W}. \tag{3.12}$$

Consequently, from (2.5) and (3.12) we have that

$$\langle N, \partial_t\rangle = -\frac{1}{W}. \tag{3.13}$$

Hence, taking into account that

$$\Delta\left(\frac{1}{W}\right) = -\frac{1}{W^2}\left(\Delta W - \frac{2|\nabla W|^2}{W}\right),$$

we can use formula (2.10) to rewrite (3.11) as

$$\Delta\vartheta = W\Delta\eta - \eta(\mathrm{Ric}_M(N^*, N^*) + |A|^2)W + \frac{2}{W}\langle\nabla W, \nabla\vartheta\rangle. \tag{3.14}$$

Hence, along the minimizing sequence $\{p_{k,\varepsilon}\}$, from (3.14) we get

$$-\varepsilon \leq W\Delta\eta - \eta(\mathrm{Ric}_M(N^*, N^*) + |A|^2)W + \frac{2}{W}\langle \nabla W, \nabla \vartheta\rangle. \tag{3.15}$$

So, using Cauchy–Schwartz inequality in (3.15), we obtain that

$$-\varepsilon \leq W\Delta\eta - \eta(\mathrm{Ric}_M(N^*, N^*) + |A|^2)W + \frac{2\varepsilon|\nabla W|}{W}. \tag{3.16}$$

On the other hand, since we are assuming that $K_M \geq -\kappa$ for some positive constant κ, we have

$$\mathrm{Ric}_M(N^*, N^*) \geq -\kappa(n-1)|N^*|_M^2 = -\kappa(n-1)|\nabla u|^2. \tag{3.17}$$

But, from (1.1) and (3.12) it holds that

$$|\nabla u|^2 \leq \frac{|A|^2}{\kappa(n-1)}.$$

Consequently, from (3.17) we obtain

$$\mathrm{Ric}_M(N^*, N^*) + |A|^2 \geq -\kappa(n-1)|\nabla u|^2 + |A|^2 \geq 0. \tag{3.18}$$

Furthermore, up to translation, we can assume $u > 0$ and, hence, we have that $\eta > 0$ on $\Sigma(u)$. Therefore, from (3.16) and (3.18) we get

$$-\varepsilon\left(\frac{W + 2|\nabla W|}{W^2}\right) \leq \Delta\eta. \tag{3.19}$$

Using the general formula $\Delta f(u) = f'\Delta u + f''|\nabla u|$, we also have that

$$\Delta\eta = e^{-ku}(k\Delta u - k^2|\nabla u|^2). \tag{3.20}$$

Thus, from (3.19) and (3.20) we obtain

$$-\varepsilon\left(\frac{W + 2|\nabla W|}{W^2}\right) \leq e^{-ku}(k\Delta u - k^2|\nabla u|^2). \tag{3.21}$$

Hence, taking into account (2.8) and (3.5), from (3.21) we must have

$$-\varepsilon e^{ku}(W + 2|\nabla W|) \leq (-nHkW - k^2|Du|_M^2). \tag{3.22}$$

We claim that ∇W is also bounded. Indeed, from (3.13) we have that

$$\nabla W = -\frac{1}{\langle N, \partial_t\rangle^2}\nabla\langle N, \partial_t\rangle. \tag{3.23}$$

Hence, from (2.9) and (3.23) we get

$$|\nabla W| \leq W^2|A||\nabla u| \leq W^2\frac{|A|^2}{\sqrt{\kappa(n-1)}}.$$

Thus, letting $\varepsilon \to 0$ in (3.22) and taking the lim sup on ε, we obtain the following estimate

$$0 \leq -n|H| \limsup_{\varepsilon\to 0} W - k \limsup_{\varepsilon\to 0} |Du|_M^2. \tag{3.24}$$

Now, multiplying (3.24) by $\frac{1}{k}$ and making $k \to \infty$ as we take the lim sup over k we get the next

$$\limsup_{k\to\infty} \limsup_{\varepsilon\to 0} |Du|_M^2 = 0. \tag{3.25}$$

Consequently, since $W^2 = 1 - |Du|_M^2$, we have

$$\liminf_{k\to\infty} \liminf_{\varepsilon\to 0} W^2 = 1. \tag{3.26}$$

Since these sequences are minimizing, by Lemma 2 on an arbitrary point we have the ensuing

$$\eta^2 W^2(p_{k,\varepsilon}) \leq \eta^2 W^2 + \varepsilon,$$

which implies that

$$\begin{aligned} |Du|_M^2 &\leq 1 - \frac{\eta_*^2}{\eta^2} W^2(p_{k,\varepsilon}) + \frac{\varepsilon}{\eta^2} \\ &\leq 1 - (1 - e^{-ku_*})^2 W^2(p_{k,\varepsilon}) + \frac{\varepsilon}{(1 - e^{-ku_*})^2}, \end{aligned}$$

where $\eta_* = \inf_{\Sigma(u)} \eta$ and $u_* = \inf_{\Sigma(u)} u$. Without loss of generality, denoting $u^* = \sup_{\Sigma(u)} u$, we can suppose that $u^* \geq u \geq u_* > 0$. Thus,

$$|Du|_M^2 \leq 1 - (1 - e^{-ku_*})^2 W^2(p_{k,\varepsilon}) + \frac{\varepsilon}{(1 - e^{-ku^*})^2}. \tag{3.27}$$

Since ε does not appear in the left hand side of (3.27), we can take $\limsup_{\varepsilon\to 0}$ on both sides of (3.27) obtaining

$$|Du|_M^2 \leq 1 - (1 - e^{-ku_*})^2 \liminf_{\varepsilon\to 0} W^2(p_{k,\varepsilon}). \tag{3.28}$$

In an analogous way, taking $\limsup_{k\to\infty}$ on (3.28), we finally conclude that $|Du|_M^2 = 0$ on $\Sigma(u)$, that is, $u \equiv t_0$ for some $t_0 \in \mathbb{R}$. □

Remark 2. We recall that the Cheeger constant $\mathfrak{h}(M)$ of a complete Riemannian manifold M^n is given by

$$\mathfrak{h}(M) = \inf_D \frac{A(\partial D)}{V(D)},$$

where D ranges over all open submanifolds of M^n with compact closure in M^n and smooth boundary, and where $V(D)$, $A(\partial D)$ are the volume of D resp. the area of ∂D, relative to the metric of M^n.

Returning to the context of Theorem 1, assuming that there exists an entire spacelike H-graph with $H > 0$ and such that (1.1) holds, from (3.5) we can apply an argument due to Salavessa [**18**] to get

$$nHV(D) \le \int_D nH\,dV = \int_D \operatorname{Div}\left(\frac{Du}{\sqrt{1-|Du|_M^2}}\right) dV$$

$$= \oint_{\partial D} \left\langle \frac{Du}{\sqrt{1-|Du|_M^2}}, \nu \right\rangle dA \le \sqrt{\frac{n}{(n-1)\kappa}} HA(\partial D),$$

where ν is the outward unit normal of ∂D. Yielding the following lower estimate for the Cheeger constant of the fiber M^n

$$\sqrt{n(n-1)\kappa} \le \mathfrak{h}(M).$$

Furthermore, recalling the stability operator $\mathcal{J} = \Delta + \operatorname{Ric}(N,N) + |A|^2$, a spacelike H-hypersurface Σ^n is said to be stable if

$$\int_\Sigma \mathcal{J} f \cdot f \ge 0, \quad \forall\, f \in C_0^2(\Sigma). \tag{3.29}$$

We also note that, under the stated hypothesis of Theorem 1, entire spacelike H-graph is, in fact, a slice and therefore $\operatorname{Ric}(\partial_t, \partial_t) = 0$ and $|A|^2 \equiv 0$. Hence, in this case, from (3.29) we see that such graph is stable.

Remark 3. According to Example 4.4 of [**12**], taking $0 < |a| < 1$, we have that the entire vertical graph

$$\Sigma(u) = \{(a \ln y, x, y); y > 0\} \subset -\mathbb{R} \times \mathbb{H}^2$$

is such that

$$|Du|_{\mathbb{H}^2}^2 = |a|^2$$

and, hence, $\Sigma(u)$ is a complete spacelike surface in $-\mathbb{R} \times \mathbb{H}^2$. Moreover, with a straightforward computation we verify that $\Sigma^2(u)$ has constant mean curvature $H = -\frac{a}{2\sqrt{1-a^2}}$, $H_2 = 0$, and

$$|Du|_{\mathbb{H}^2}^2 = \frac{|A|^2}{1+|A|^2}. \tag{3.30}$$

Hence, we conclude that Theorem 1 does not hold when the function u is unbounded.

Furthermore, since $\langle N, \partial_t \rangle$ is constant on $\Sigma(u)$, from formula (2.10) and taking into account equations (3.12) and (3.30), we get

$$\Delta \langle N, \partial_t \rangle = (|A|^2 - |\nabla h|^2) \langle N, \partial_t \rangle = 0. \tag{3.31}$$

Consequently, according to the stability criteria given in (3.29), from equation (3.31) we also conclude that $\Sigma(u)$ constitutes a nontrivial example of stable surface in $-\mathbb{R}\times\mathbb{H}^2$. Therefore, concerning the context of Theorem 1, we see that the stability of the entire spacelike H-graph cannot alone guarantee the uniqueness result.

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