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On Elementary Equivalence in Fuzzy Predicate Logics

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Abstract Our work is a contribution to the model theory of fuzzy predicate logics. In this paper we characterize elementary equivalence between models of fuzzy predicate logic using elementary mappings. Refining the method of diagrams we give a solution to an open problem of P. Hájek and P. Cintula (Conjectures 1 and 2 of [HaCi06]). We investigate also the properties of elementary extensions in witnessed and quasi-witnessed theories, generalizing some results of Section 7 of [HaCi06] and of Section 4 of [CeEs09] to non-exhaustive models.

Keywords Mathematical Logic and Foundations · Model Theory · Fuzzy Predicate Logics · Elementary Extensions · Witnessed Models · Quasi-witnessed Models

1 Introduction

This work is a contribution to the model theory of fuzzy predicate logics. Model theory is the branch of mathematical logic that studies the construction and classification of structures. Construction means building structures or families of structures, which have some feature that interests us. Classifying a class of structures means grouping the structures into subclasses in a useful way, and then proving that every structure in the collection does belong in just one of the subclasses. The most basic classification in classical model theory is given by the relations of elementary equivalence and isomorphism. Our purpose in the present article is to characterize the relation of elementary equivalence between two structures of a fuzzy predicate language in terms of elementary extensions.

The basic notion of elementary equivalence between models is due to A. Tarski (see [Ta35]) and the fundamental results on elementary extensions and elementary

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chains were introduced by A. Tarski and R. Vaught in [TaVa57]. For a general survey on the subject and an historical overview we refer the reader to [ChKe90]. In the context of fuzzy predicate logics, elementarily equivalent structures were defined in [HaCi06] (Definition 10), our starting point is the research done in this article. There the authors presented a characterization of conservative extension theories using the elementary equivalence relation (see Theorems 6 and 11 of [HaCi06]). Future work will be devoted to analise the relationship of our investigation with other approaches, for instance the one presented in [NoPerMo99], where a notion of elementary equivalent models *in a degree d* was introduced (Definition 4.33).

P. Hájek and P. Cintula proved in Theorem 6 of [HaCi06] that, in core fuzzy logics, a theory T_2 is a conservative extension of another theory T_1 if and only if each exhaustive model of T_1 is elementarily embedded in a model of T_2. Then, they conjectured the same result to be true for arbitrary structures (Conjectures 1 and 2 of [HaCi06]). In this paper we present a counterexample to Conjectures 1 and 2, using a refinement of the method of diagrams developed in [De11].

Special attention is devoted to witnessed and quasi-witnessed elementary extensions. The interest for the study of this kind of structure has grown recently (cf. [Ha07a], [Ha07b], [HaCi06] and [CeEs09]). The aforementioned papers show the importance of witnessed and quasi-witnessed models for applications in logic-based knowledge representation in artificial intelligence. In this article we prove that these classes of models have some good model-theoretic properties, allowing us to generalize some results of Section 7 of [HaCi06] and of Section 4 of [CeEs09] to non-exhaustive models.

This paper is a revised and extended version of the contribution [DeEs10] of the authors to the IPMU'10 Conference. The article is structured as follows: Section 2 is devoted to preliminaries on fuzzy predicate logics. In Section 3 we study the basic properties of elementary extensions and we present an analog to the Tarski-Vaught Test in the fuzzy context. By using this test, in Section 4 we provide new proofs for the Löwenheim-Skolem Theorems in fuzzy predicate logics. In Section 5 we introduce some known definitions and basic facts on canonical models (see section 4 and 5 of [HaCi06]) and of the method of diagrams developed in [De11]. Later on in this section, we prove some new propositions related to canonical models and diagrams. Section 6 is devoted to the study of witnessed and quasi-witnessed extensions, and in Section 7 we present a counterexample to Conjectures 1 and 2 of [HaCi06], using the results of Section 4. Finally, in Section 8 we present a characterization theorem of elementary equivalence in fuzzy predicate logics. We conclude the article with a section of work in progress and future work.

2 Preliminaries

In his seminal book [Ha98], Hájek considered the problem of finding a common base for the most important fuzzy logics, namely Łukasiewicz, Gödel and product logics. There, he introduced a logic, named BL, and he proposed it for the role of basic fuzzy logic. Hájek's proposal was greatly supported by the proof that BL is the logic of all continuous t-norms of their residua. But in [EsGo01] the authors observed that the minimal condition for a t-norm to have a residuum, and therefore to determine a logic, is left-continuity (continuity is not necessary). There, they proposed a weaker logic, called MTL (monoidal t-norm based logic), and conjectured that MTL is the logic of left-continuous t-norms and their residua. This conjecture was proved in [JeMo02].

In the literature of t-norm based logics, one can find not only a number of axiomatic extensions of MTL but also extensions by means of expanding the language with new connectives such as the Δ connective or with an involutive negation. All of MTL extensions and most of its expansions defined elsewhere share the property of being complete with respect to a corresponding class of linearly ordered algebras. To encompass all these logics and prove general results common to all of them, Cintula introduced in [Ci05] the notion of *core fuzzy logics* (he also defines the class of Δ-*core fuzzy logics* to capture all expansions having the Δ connective).

Our study of the model theory of fuzzy predicate logics is focused on the basic fuzzy predicate logic MTL∀ and some of its expansions based on propositional (Δ-)*core fuzzy logics*. We start by introducing the definition of propositional (Δ-)core fuzzy logic:

Definition 1 *A propositional logic* L *is a* core fuzzy logic *iff* L *satisfies:*

- L *expands* MTL.
- (Cong) *For all formulas* ϕ, φ, α,

$$\varphi \leftrightarrow \phi \vdash_{\mathrm{L}} \alpha(\varphi) \leftrightarrow \alpha(\phi)$$

- ($\mathcal{LDT}$) Local Deduction Theorem: *for each theory* T *and formulas* ϕ, φ:

$$T, \varphi \vdash_{\mathrm{L}} \phi \textit{ iff there is a natural number } n \textit{ such that } T \vdash_{\mathrm{L}} \varphi^n \to \phi.$$

and we say that L *is a* Δ-core fuzzy logic *if*

- L *expands* MTL_Δ.
- (Cong) *For all formulas* ϕ, φ, α,

$$\varphi \leftrightarrow \phi \vdash_{\mathrm{L}} \alpha(\varphi) \leftrightarrow \alpha(\phi)$$

- ($\mathcal{DT}_\Delta$) Delta Deduction Theorem: *for each theory* T *and formulas* ϕ, φ:

$$T, \varphi \vdash_{\mathrm{L}} \phi \textit{ iff } T \vdash_{\mathrm{L}} \Delta\varphi \to \phi.$$

So defined (Δ-)core fuzzy logics are axiomatic extensions of MTL (of MTL_Δ, respectively). We state now, without proof, a useful property of (Δ-)core fuzzy logics that we will use later on (for a proof of this result see Theorem 1 of [CiHa10]).

Theorem 2 [Proof by Cases Property] *If* $T, \phi \vdash \chi$ *and* $T, \psi \vdash \chi$, *then* $T, \phi \vee \psi \vdash \chi$.

For a thorough treatment of (Δ-)core fuzzy logics we refer to [CiHa10], [HaCi06] and [CiEsGiGoMoNo09].

Following [Ha98] we introduce the syntax of fuzzy predicate logics. A *predicate language* Γ is a triple (**P**,**F**,**A**) where **P** is a non-empty set of predicate symbols, **F** is a set of function symbols and **A** is a mapping assigning to each predicate and function symbol a natural number called the *arity of the symbol*. The function symbols F for which $\mathbf{A}(F) = 0$ are called the *object constants*. The predicate symbols P for which $\mathbf{A}(P) = 0$ are called the *truth constants*.

Formulas of the predicate language Γ are built up from the symbols in (**P**,**F**,**A**), the connectives and truth constants of a fixed (Δ-)core fuzzy logic L, the logical symbols ∀ and ∃, variables and punctuation. From now on, the formulas of a predicate language Γ will be called Γ*-formulas*. A Γ-sentence is a Γ-formula without free variables.

Throughout the paper we consider the equality symbol as a binary predicate symbol not as a logical symbol, we work in equality-free fuzzy predicate logics. That is, the equality symbol is not necessarily present in all the languages and its interpretation is not fixed. Given a propositional (Δ-)core fuzzy logic L, we denote by L∀ the corresponding fuzzy predicate logic.

Let L be a fixed (Δ-)core fuzzy logic, we introduce now an axiomatic system for the predicate logic L∀:

(P) the axioms resulting from the axioms of L by the substitution of the propositional variables by the Γ-formulas.
(∀1) $\forall x\phi(x) \to \phi(t)$, where t is substitutable for x in ϕ.
(∃1) $\phi(t) \to \exists x\phi(x)$, where t is substitutable for x in ϕ.
(∀2) $\forall x(\phi \to \varphi) \to (\phi \to \forall x\varphi)$, where x is not free in ϕ.
(∀3) $\forall x(\phi \vee \varphi) \to (\phi \vee \forall x\varphi)$, where x is not free in ϕ.

The deduction rules are those of L (modus ponens and, in the case where L is a Δ-core fuzzy logic, necessitation for the Baaz-Monteiro connective Δ: from ϕ infer $\Delta\phi$) and *generalization*: from ϕ infer $\forall x\phi$. By $\Sigma \vdash_{\mathrm{L}\forall} \alpha$ we denote that the formula α follows from the set of formulas Σ in the axiomatic system of the fuzzy predicate logic L∀. When it is clear by the context we omit the subscript L∀.

Let L be a fixed propositional (Δ-)core fuzzy logic, we introduce now the semantics for the fuzzy predicate logic L∀:

Definition 3 *Given an* L*-algebra* $\mathbf{B}$*, a* $\mathbf{B}$-structure *for a predicate language* Γ *is a tuple*

$$\mathbf{M} = (M, (P_{\mathbf{M}})_{P\in\Gamma}, (F_{\mathbf{M}})_{F\in\Gamma})$$

where:

1. *M is a non-empty set.*
2. *For each n-ary predicate $P \in \Gamma$, if $n > 0$, $P_{\mathbf{M}}$ is a $\mathbf{B}$-fuzzy relation $P_{\mathbf{M}}\colon M^n \to B$. If $n = 0$, $P_{\mathbf{M}}$ is an element of B.*
3. *For each n-ary function symbol $F \in \Gamma$, if $n > 0$, $F_{\mathbf{M}}\colon M^n \to M$ is a crisp function. If $n = 0$, $F_{\mathbf{M}}$ is an element of M.*

Given an L-algebra $\mathbf{B}$ and a $\mathbf{B}$-structure $\mathbf{M}$, an $\mathbf{M}$-*evaluation* of the object variables is a mapping v which assigns to each variable an element from M. By $\phi(x_1, \ldots, x_k)$ we mean that all the free variables of ϕ are among $x_1, \ldots, x_k$. Let v be an $\mathbf{M}$-evaluation, x a variable, and $d \in M$, we denote by $v[x \to d]$ the $\mathbf{M}$-evaluation such that $v[x \to d](x) = d$ and for each variable y different from x, $v[x \to d](y) = v(y)$.

Let $\mathbf{B}$ be an L-algebra, $\mathbf{M}$ be a $\mathbf{B}$-structure and v be an $\mathbf{M}$-evaluation, we define the values of the terms and *truth values* of the formulas as follows:

$$\|c\|^{\mathbf{B}}_{\mathbf{M},v} = c_{\mathbf{M}},\ \|x\|^{\mathbf{B}}_{\mathbf{M},v} = v(x)$$

$$\|F(t_1, \ldots, t_n)\|^{\mathbf{B}}_{\mathbf{M},v} = F_{\mathbf{M}}(\|t_1\|^{\mathbf{B}}_{\mathbf{M},v}, \ldots, \|t_n\|^{\mathbf{B}}_{\mathbf{M},v})$$

for each variable x, each object constant $c \in \Gamma$, each n-ary function symbol $F \in \Gamma$ for $n > 0$ and Γ-terms $t_1, \ldots, t_n$, respectively.

$$\|P(t_1, \ldots, t_n)\|^{\mathbf{B}}_{\mathbf{M},v} = P_{\mathbf{M}}(\|t_1\|^{\mathbf{B}}_{\mathbf{M},v}, \ldots, \|t_n\|^{\mathbf{B}}_{\mathbf{M},v})$$

for each n-ary predicate $P \in \Gamma$,

$$\|\delta(\phi_1, \dots, \phi_n)\|^{\mathbf{B}}_{\mathbf{M},v} = \delta_{\mathbf{B}}(\|\phi_1\|^{\mathbf{B}}_{\mathbf{M},v}, \dots, \|\phi_n\|^{\mathbf{B}}_{\mathbf{M},v})$$

for each n-ary connective $\delta \in$L and Γ-formulas $\phi_1, \dots, \phi_n$. Finally, for the quantifiers,

$$\|\forall x\phi\|^{\mathbf{B}}_{\mathbf{M},v} = \inf\{\|\phi\|^{\mathbf{B}}_{\mathbf{M},v[x\to d]} : d \in M\}$$

$$\|\exists x\phi\|^{\mathbf{B}}_{\mathbf{M},v} = \sup\{\|\phi\|^{\mathbf{B}}_{\mathbf{M},v[x\to d]} : d \in M\}$$

provided the infimum/supremum exists; otherwise the truth value of the formula in question is undefined. Remark that, since the L-algebras we work with are not necessarily complete, the above suprema and infima could be not defined in some cases. It is said that a **B**-structure is *safe* if such suprema and infima are defined for all the formulas. From now on we assume that all our structures are safe. Safe models were introduced in [Ra74] under the name of *interpretations* and in [DiNoGe86] as *completely valued models*.

B-structures were introduced in [RaSi63] under the name of *realizations* and in [DiNoGe86] under the name of *fuzzy models*. The formalization of this concept that we use at present is due to [Ha98]. If v is an evaluation such that for each $0 < i \leq n$, $v(x_i) = d_i$, and λ is either a Γ-term or a Γ-formula, we abbreviate by $\|\lambda(d_1, \dots, d_n)\|^{(\mathbf{B},\mathbf{M})}$ the expression $\|\lambda(x_1, \dots, x_n)\|^{\mathbf{B}}_{\mathbf{M},v}$.

Definition 4 *Let ϕ be a Γ-sentence, given an* L*-algebra* **B** *and a* **B**-structure **M***, it is said that* **M** *is a* model *of ϕ iff $\|\phi\|^{(\mathbf{B},\mathbf{M})} = 1$. And it is said that* **M** *is a model of a set of Γ-sentences Σ iff for all $\phi \in \Sigma$,* **M** *is a model of ϕ.*

Definition 5 *Let $T \cup \{\phi\}$ be a set of Γ-sentences. We say that ϕ is a* semantical consequence *of T (denoted by $T \models \phi$) iff for every* L*-algebra* **B** *and every* **B**-structure **M***, if* **M** *is a model of T, then* **M** *is also a model of ϕ.*

From now on, given an L-algebra **B**, we say that $(\mathbf{B}, \mathbf{M})$ is a *Γ-structure* instead of saying that **M** is a **B**-structure for a predicate language Γ.

Definition 6 *Let $(\mathbf{B}, \mathbf{M})$ be a Γ-structure, by $Alg(\mathbf{B}, \mathbf{M})$ we denote the subalgebra of* **B** *whose domain is the set*

$$\{\|\phi(d_1, \dots, d_n)\|^{(\mathbf{B},\mathbf{M})} : d_1, \dots, d_n \in M \textit{ and } \phi(x_1, \dots, x_n) \textit{ is a } \Gamma\textit{-formula}\}$$

Then, it is said that $(\mathbf{B}, \mathbf{M})$ is exhaustive *iff $Alg(\mathbf{B}, \mathbf{M}) = \mathbf{B}$.*

Given two Γ-structures $(\mathbf{B}_1, \mathbf{M}_1)$ and $(\mathbf{B}_2, \mathbf{M}_2)$, we denote by $(\mathbf{B}_1, \mathbf{M}_1) \equiv (\mathbf{B}_2, \mathbf{M}_2)$ the fact that $(\mathbf{B}_1, \mathbf{M}_1)$ and $(\mathbf{B}_2, \mathbf{M}_2)$ are *elementarily equivalent*, that is, that they are models of exactly the same Γ-sentences. The definition of elementarily equivalent classical structures is due to A. Tarski (see [Ta35]). In the fuzzy setting was introduced in [HaCi06] (Definition 10).

From now on and throughout the article, we will assume that **B** is always an L-chain. In this section we have presented only a few definitions and notation, a detailed introduction to the syntax and semantics of fuzzy predicate logics can be found in [Ha98] and [CiHa10].

2.1 Weak Homomorphisms, σ-Mappings and Homomorphisms

There are several ways to extend classical model theory to fuzzy logic. The concepts of embedding and elementary embedding can be extended in several ways to fuzzy logic. Our choice for the counterparts of these notions in fuzzy logic (weak homomorphisms and elementary mappings) has been motivated, not only for the desire to find extensions of the corresponding notions of classical predicate logics but also to try to encompass the most commonly used definitions in the literature of predicate fuzzy logic. Different definitions have been used so far for basic model-theoretic operations on structures. For instance, the notion of *elementary submodel, morphism* (see [DiNoGe86]), *elementary embeddings and submodels*(see [HaCi06]), *fuzzy submodel, elementary fuzzy submodel* and *isomorphism* of structures of first-order fuzzy logic with graded syntax (see [NoPerMo99]), *complete morphism* in languages with a similarity predicate (see [Be02]) and the notion of σ-embedding (see [CiEsGiGoMoNo09]). Taking as our starting point all these works, we have defined these model-theoretic notions as general as possible. We want to contemplate the possibility that our languages contain function symbols and also the equality symbol but not as a logical symbol (for instance, interpreted as a similarity). For some mathematical purposes, such as algebraic applications, this could seem to be useless. But in other scientific disciplines, such as computer science or artificial intelligence, it is important to have fuzzy predicate logics able to deal with similarities, and at the same time with functions. In this section we recall the notions of weak homomorphism and of homomorphism as introduced in [De11].

In fuzzy predicate languages, since we often do not work with crisp equalities, we can find mappings that preserve all quantifier-free formulas but are not homomorphisms (in the classical sense) between the algebraic reducts of the models (that is, between the interpretations of the function symbols). Since these two notions do not coincide, unlike in classical first-order logic, we define both notions, recall their differential properties and their relationship to basic constructions on model theory.

Definition 7 *Let* $(\mathbf{B}_1, \mathbf{M}_1)$ *be a* Γ_1*-structure and* $(\mathbf{B}_2, \mathbf{M}_2)$ *be a* Γ_2*-structure, with* $\Gamma_1 \subseteq \Gamma_2$*. We say that the pair* (g, f) *is a* weak homomorphism *iff*

1. $g \colon \mathbf{B}_1 \to \mathbf{B}_2$ *is an* L*-algebra homomorphism of* $\mathbf{B}_1$ *into* $\mathbf{B}_2$*.*
2. $f \colon M_1 \to M_2$ *is a mapping of* M_1 *into* M_2*.*
3. *For each quantifier-free* Γ_1*-formula* $\phi(x_1, \ldots, x_n)$ *and elements* $d_1, \ldots, d_n \in M_1$*,*

$$g(\|\phi(d_1, \ldots, d_n)\|^{(\mathbf{B}_1,\mathbf{M}_1)}) = \|\phi(f(d_1), \ldots, f(d_n))\|^{(\mathbf{B}_2,\mathbf{M}_2)}$$

Moreover, if in addition:

- Condition 3 *above holds for every* Γ_1*-formula,* (g, f) *is said to be an* elementary mapping.
- g *preserves the existing infima and suprema,* (g, f) *is said to be a* σ-mapping.
- *For each n-ary function symbol* $F \in \Gamma_1$ *and elements* $d_1, \ldots, d_n \in M_1$*,*

$$f(F_{\mathbf{M}_1}(d_1, \ldots, d_n)) = F_{\mathbf{M}_2}(f(d_1), \ldots, f(d_n))$$

  . (g, f) *is said to be a* homomorphism.

If (g, f) is both a σ-mapping and a homomorphism we would say that (g, f) is a *σ-homomorphism.* Moreover, we would say that (g, f) is an *embedding* when (g, f) is a homomorphism and both g and f are one-to-one, and we denote by $(\mathbf{B}, \mathbf{M}) \cong$

$(\mathbf{A}, \mathbf{N})$ when these two structures are *isomorphic* (that is, there is an embedding (g, f) from $(\mathbf{B}, \mathbf{M})$ into $(\mathbf{A}, \mathbf{N})$ with g and f onto). Homomorphisms that in addition are elementary mappings will be called *elementary homomorphisms*. Remark that, unlike [Be02], homomorphisms are crisp when restricted to the algebraic reducts of the models. Observe also that, working with predicate languages without function symbols, the notions of weak homomorphism and homomorphism coincide, but it is not the case for arbitrary structures.

In the recent development of model theory for fuzzy predicate languages (see for instance [HaCi06] and [CiHa10]), elementary mappings have been used to study fundamental notions such as elementary equivalence or the notion of conservative extension of a theory. However, it is customary in classical model theory to use the notion of isomorphism instead. The main interest for using a weaker notion comes from the fact that, working in equality-free languages, it is possible to find mappings that preserve all the formulas of the language, yet not capturing the structural properties of the models (and this suffices for some of our purposes). Remark that this does not happen in classical predicate logic, being the isomorphisms the only mappings satisfying both conditions (preserving all the formulas and, at the same time, preserving the structure of the models).

Note that, by definition, weak homomorphisms are not always σ-mappings and homomorphisms are not always σ-homomorphisms (as are in [DiNoGe86] or [Be02]). We will see now that σ-mappings enjoy some good properties. The following proposition is a reformulation of Propositions 6.1 and 6.2 in [DiNoGe86]:

Proposition 8 *Let (g, f) be a weak homomorphism of the Γ-structure $(\mathbf{B}_1, \mathbf{M}_1)$ into the Γ-structure $(\mathbf{B}_2, \mathbf{M}_2)$. If (g, f) is a σ-mapping with f onto, then (g, f) is an elementary mapping and $(\mathbf{B}_1, \mathbf{M}_1) \equiv (\mathbf{B}_2, \mathbf{M}_2)$.*

The previous statement shows us that there is a close relationship between elementary mappings and σ-mappings (σ-homomorphisms in particular) with f onto. However, as shown in [De11], these two notions do not coincide.

3 Elementary Extensions

In this section we introduce the notion of elementary substructure and we present an analog to the Tarski-Vaught Test for fuzzy predicate logics. Later on in this section we provide new proofs for the Löwenheim-Skolem Theorems.

Definition 9 *Let $(\mathbf{A}, \mathbf{N})$ be a Γ-structure, we say that a Γ-structure $(\mathbf{B}, \mathbf{M})$ is a* substructure *of $(\mathbf{A}, \mathbf{N})$ iff*

1. *For every object constant $c \in \Gamma$, $c_M = c_N$.*
2. *For each n-ary function symbol $F \in \Gamma$, and elements $d_1, \ldots, d_n \in M$, $F_M(d_1, \ldots, d_n) = F_N(d_1, \ldots, d_n)$.*
3. *$\mathbf{B}$ is an* L*-subalgebra of $\mathbf{A}$.*
4. *For each n-ary predicate $P \in \Gamma$, $P_{\mathbf{M}}$ is the restriction of $P_{\mathbf{N}}$ to M.*

Moreover, it is said that $(\mathbf{B}, \mathbf{M})$ is an elementary substructure *iff it is a substructure such that, for every Γ-formula $\phi(x_1, \ldots, x_n)$ and elements $d_1, \ldots, d_n \in M$,*

$$\|\phi(d_1, \ldots, d_n)\|^{(\mathbf{B},\mathbf{M})} = \|\phi(d_1, \ldots, d_n)\|^{(\mathbf{A},\mathbf{N})} \quad (\diamond)$$

When $(\mathbf{B}, \mathbf{M})$ is an elementary substructure of $(\mathbf{A}, \mathbf{N})$ we say also that $(\mathbf{A}, \mathbf{N})$ is an *elementary extension* of $(\mathbf{B}, \mathbf{M})$. Remember that we have assumed that all our structures are safe. It is easy to check, by induction on the complexity of the formulas, that, for every substructure, condition $(\diamond)$ holds restricted to quantifier-free formulas. The transitivity of the notion of elementary mapping was stated in [HaCi06]. Now we introduce the notion of *definable set* of elements of an L-algebra.

Definition 10 *Let* $(\mathbf{A}, \mathbf{N})$ *be a* Γ*-structure,* $K \subseteq N$*,* $e_1, \ldots, e_n \in K$ *and* $\phi(x, y_1, \ldots, y_n)$ *be a* Γ*-formula. We denote by* $X^{(\mathbf{A},\mathbf{N})}_{\phi, e_1, \ldots, e_n, K}$ *the following subset of* A*:*

$$\{\|\phi(d, e_1, \ldots, e_n)\|^{(\mathbf{A},\mathbf{N})} : d \in K\}$$

It is said that a subset Y *of* A *is* definable with parameters *in* $(\mathbf{A}, \mathbf{N})$ *if there are* $K \subseteq N$*,* $e_1, \ldots, e_n \in N$ *and a* Γ*-formula* $\phi(x, y_1, \ldots, y_n)$ *such that* $Y = X^{(\mathbf{A},\mathbf{N})}_{\phi, e_1, \ldots, e_n, K}$

The following result is a fuzzy version of the Tarski-Vaught Test for elementary extensions. The Tarski-Vaught criterion is a necessary and sufficient condition for a substructure to be an elementary substructure. It can be useful for constructing an elementary substructure of a large structure. The proof of this result follows easily by induction on the complexity of the formulas by using the definition introduced above.

Proposition 11 [Tarski-Vaught Test] *Let* $(\mathbf{A}, \mathbf{N})$ *and* $(\mathbf{B}, \mathbf{M})$ *be two* Γ*-structures. Then the following are equivalent:*

1. $(\mathbf{B}, \mathbf{M})$ *is an elementary substructure of* $(\mathbf{A}, \mathbf{N})$*.*
2. $(\mathbf{B}, \mathbf{M})$ *is a substructure of* $(\mathbf{A}, \mathbf{N})$ *and for every* Γ*-formula* $\phi(x, y_1, \ldots, y_n)$ *and* $e_1, \ldots, e_n \in M$*,* $X^{(\mathbf{A},\mathbf{N})}_{\phi, e_1, \ldots, e_n, M}$ *and* $X^{(\mathbf{A},\mathbf{N})}_{\phi, e_1, \ldots, e_n, N}$ *have the same infimum and supremum in* $\mathbf{A}$*.*

4 Löwenheim-Skolem Theorems

Löwenheim-Skolem Theorems are one of the pillars of classical model theory. A great deal of the work done in the third quarter of the twentieth century was devoted to working out the consequences of these theorems. In 1915, L. Löwenheim proved that if a first-order sentence has a model, then it has a model whose domain is countable. In 1922, T. Skolem generalized this result to whole sets of sentences. He proved that if a countable collection of first-order sentences has an infinite model, then it has a model whose domain is only countable. Now we show that similar theorems hold for fuzzy predicate languages.

Di Nola and G. Gerla introduced in [DiNoGe86] the notions of valuation structure and fuzzy model of a given first-order language in a categorial setting. G. Gerla introduced in [Ge86] the notions of d-filter, of reduced product and of ultraproduct of a family of fuzzy models with definable quantifiers. That is, models such that for each quantifier there is a formula of the classical first-order language with equality with a unique monadic predicate P that defines it (for a reference see Definition 8.1 of [Ge86]). By using these constructions he showed analogues to the Löwenheim-Skolem-Tarski Theorems for fuzzy models. Here we present new proofs for these theorems without making use of the ultraproduct construction and, in the case of the Upward Löwenheim-Skolem Theorem, we improve the theorem obtaining a model over the same

L-algebra. Our proofs use, on the one hand, the Tarski-Vaught Test introduced before (downward), and on the other hand, a terms construction (upwards).

Some notation is required before we proceed with the statement of the theorems. We denote by $|\Gamma|$ the cardinality of the predicate language Γ. Given a model $(\mathbf{B}, \mathbf{M})$, we say that its *cardinality* is the cardinality of the domain M, denoted by $|M|$. In the fuzzy case, it is interesting also to consider another cardinality related to the algebra of truth values $\mathbf{B}$. Assume that every subset of B definable with parameters in $(\mathbf{B}, \mathbf{M})$ has infimum and supremum. We denote by $p(\mathbf{B})$ the minimum cardinal γ such that for every $X \subseteq B$, definable with parameters in $(\mathbf{B}, \mathbf{M})$, there is $Y \subseteq X$ of cardinality $\leq \gamma$ and such that $\inf X = \inf Y$ and $\sup X = \sup Y$. A similar notion was introduced in [Ge86] under the name *quantifier cardinal*. For instance, $p([0,1]_{\text{Ł}}) = \omega$.

Theorem 12 **[Downward Löwenheim-Skolem Theorem]** *Let* $(\mathbf{B}, \mathbf{M})$ *be an infinite* Γ*-structure. Assume that every subset of* B *definable with parameters in* $(\mathbf{B}, \mathbf{M})$ *has infimum and supremum. For every cardinal* κ *with* $\max(p(\mathbf{B}), |\Gamma|, \omega) \leq \kappa \leq |M|$ *and every* $Z \subseteq M$ *with* $|Z| \leq \kappa$*, there is a model* $(\mathbf{B}, \mathbf{O})$ *which is an elementary substructure of* $(\mathbf{B}, \mathbf{M})$ *of cardinality* $\leq \kappa$ *and* $Z \subseteq O$*.*

Proof We define inductively a chain $(Z_n : n \in \omega)$ of subsets of M with $Z \subseteq Z_0$ and for each $n \in \omega$, $|Z_n| \leq \kappa$. We start by choosing Z_0 to be any subset of M containing Z and of cardinality $\leq \kappa$. Now given Z_n, Z_{n+1} is obtained in the following form: for every Γ-formula $\phi(x, y_1, \dots, y_n)$ and $e_1, \dots, e_n \in Z_n$, we take a subset of $X^{(\mathbf{B},\mathbf{M})}_{\phi, e_1, \dots, e_n, Z_n}$, say $Y_{\phi, e_1, \dots, e_n, Z_n}$ (see Definition 10) such that $p(\mathbf{B}) \geq \left|Y_{\phi, e_1, \dots, e_n, Z_n}\right|$, with

$$\sup Y_{\phi, e_1, \dots, e_n, Z_n} = \sup X^{(\mathbf{B},\mathbf{M})}_{\phi, e_1, \dots, e_n, Z_n}$$

and

$$\inf Y_{\phi, e_1, \dots, e_n, Z_n} = \inf X^{(\mathbf{B},\mathbf{M})}_{\phi, e_1, \dots, e_n, Z_n}$$

Now, for every $b \in Y_{\phi, e_1, \dots, e_n, Z_n}$ choose an arbitrary $d_b \in M$ such that

$$\|\phi(d_b, e_1, \dots, e_n)\|^{(\mathbf{B},\mathbf{M})} = b$$

and then take Z_{n+1} to be the domain of the substructure of $(\mathbf{B}, \mathbf{M})$ generated by the set $Z_n \cup \{d_b : b \in Y_{\phi, e_1, \dots, e_n, Z_n}\}$. Since $\max(p(\mathbf{B}), |\Gamma|, \omega) \leq \kappa$, Z_{n+1} has also cardinality $\leq \kappa$. Finally let $(\mathbf{B}, \mathbf{O})$ be the substructure of $(\mathbf{B}, \mathbf{M})$ that has as domain the union $\bigcup_{n \in \omega} Z_n$. Clearly $|O| \leq \kappa$. Using the Tarski-Vaught Test stated before it is easy to check that $(\mathbf{B}, \mathbf{O})$ is the desired elementary substructure of $(\mathbf{B}, \mathbf{M})$.

Theorem 13 **[Upward Löwenheim-Skolem Theorem]** *For every* Γ*-structure* $(\mathbf{B}, \mathbf{M})$ *and every cardinal* κ *with* $\max(|M|, |\Gamma|) \leq \kappa$*, there is a* Γ*-structure* $(\mathbf{B}, \mathbf{O})$ *of cardinality* κ *such that* $(\mathbf{B}, \mathbf{M})$ *is elementarily mapped in* $(\mathbf{B}, \mathbf{O})$*.*

Proof First we define a Γ-structure $(\mathbf{B}, \mathbf{O})$ and a σ-homomorphism $(Id_{\mathbf{B}}, f)$ from $(\mathbf{B}, \mathbf{O})$ onto $(\mathbf{B}, \mathbf{M})$. Later on we will show that $(\mathbf{B}, \mathbf{M})$ is elementarily mapped in $(\mathbf{B}, \mathbf{O})$.

We fix an enumeration (possibly with repetitions) $(d_j : j \in \kappa)$ of M. The first-order part, the algebraic reduct of $(\mathbf{B}, \mathbf{O})$ (that is the interpretation of the function symbols in Γ) will be the algebra Ter_κ of Γ-terms generated by a set of new variables $V_\kappa = \{v_j : j \in \kappa\}$. We define the function $f_0 : V_\kappa \to M$ by: for every $j \in \kappa$, $f_0(v_j) = d_j$. Then we extend f_0 in the usual way to a homomorphism f from Ter_κ onto M. Finally, we define the interpretation of the predicate symbols in $(\mathbf{B}, \mathbf{O})$: for each n-ary predicate $P \in \Gamma$ and elements $t_1, \ldots, t_n \in O$, $P_O(t_1, \ldots, t_n) = P_M(f(t_1), \ldots, f(t_n))$. So defined, $(Id_{\mathbf{B}}, f)$ is clearly a σ-homomorphism onto $(\mathbf{B}, \mathbf{M})$, because the identity mapping preserves existing infima and suprema. Remark that, by Proposition 8, since f is onto and $(Id_{\mathbf{B}}, f)$ is a σ-homomorphism, $(Id_{\mathbf{B}}, f)$ is also an elementary mapping.

Now we define another mapping $(Id_{\mathbf{B}}, h)$ in the other direction, that is, from $(\mathbf{B}, \mathbf{M})$ into $(\mathbf{B}, \mathbf{O})$ as follows: for every $d \in M$ we choose an arbitrary variable $v_d \in V_\kappa$ such that $f(v_d) = d$ and then we take $h(d) = v_d$. Since $(Id_{\mathbf{B}}, f)$ is an elementary mapping, it is easy to check that $(Id_{\mathbf{B}}, h)$ is also an elementary mapping. Moreover, by definition, h is clearly one-to-one. Without loss of generality we can identify each v_d with d, being then $(\mathbf{B}, \mathbf{O})$ our desired model.

Notice that the mapping $(Id_{\mathbf{B}}, h)$ of the previous theorem it is not necessarily an embedding. In the case where Γ is a predicate language without function symbols, by the construction of the terms structure, $(\mathbf{B}, \mathbf{O})$ is an elementary extension of $(\mathbf{B}, \mathbf{M})$.

5 Diagrams and Canonical Models

The method of diagrams, due to L. A. Henkin and A. Robinson, has proved to be a useful tool for model theory. During the 1950's the maps between structures came to play a deeper role in model theory, not just as possible research fields but as essential tools of the subject. For a general reference on the method of diagrams in classical predicate logic see [ChKe90].

For the special case of structures of first-order fuzzy logic with graded syntax and languages with the equality symbol, diagrams are presented in [NoPerMo99]. For fuzzy predicate logics in general, diagrams are used in the proofs of Lemma 4 in [HaCi06], and full diagrams can be found in the proof of Theorem 5.9 in [CiEsGiGoMoNo09]. Finally, the diagram technique was developed for arbitrary predicate core fuzzy logics in [De11].

5.1 The Method of Diagrams in Fuzzy Predicate Logics

Certain classes of mappings preserve all formulas with certain syntactic forms. Conversely, we can classify mappings by means of the formulas they preserve. In this subsection we recall some definitions and characterizations of mappings in fuzzy predicate logics developed in [De11]. Later on we prove some new propositions related to canonical models and diagrams.

Definition 14 *Let Γ be a predicate language and $(\mathbf{B}, \mathbf{M})$ be a Γ-structure, we define:*

1. *$\Gamma_{\mathbf{M}}$ is the expansion of Γ by adding a constant symbol c_d, for each $d \in M$.*
2. *$(\mathbf{B}, \mathbf{M}')$ is the expansion of $(\mathbf{B}, \mathbf{M})$ to the language $\Gamma_{\mathbf{M}}$, by interpreting for each $d \in M$, the constant c_d by d.*
3. The Basic Elementary Diagram of $(\mathbf{B}, \mathbf{M})$, *denoted by* $\mathrm{EDIAG}_0(\mathbf{B}, \mathbf{M})$*, is the set of all sentences of $\Gamma_{\mathbf{M}}$ true in $(\mathbf{B}, \mathbf{M}')$.*

Definition 15 *Let $(\mathbf{B}, \mathbf{M})$ be a Γ-structure, we expand the language further adding new symbols to $\Gamma_{\mathbf{M}}$ and we define:*

1. $\Gamma_{(\mathbf{B},\mathbf{M})}$ *is the expansion of* $\Gamma_{\mathbf{M}}$ *by adding a truth constant* P_b*, for each* $b \in B$.
2. $(\mathbf{B},\mathbf{M}^\sharp)$ *is the expansion of* $(\mathbf{B},\mathbf{M}')$ *to the language* $\Gamma_{(\mathbf{B},\mathbf{M})}$*, by interpreting for each* $b \in B$*, the truth constant symbol* P_b *by* b.
3. The Full Diagram *of* $(\mathbf{B},\mathbf{M})$*, denoted by* $\mathrm{FDIAG}(\mathbf{B},\mathbf{M})$*, is the set of all* $\Gamma_{(\mathbf{B},\mathbf{M})}$*-sentences true in* $(\mathbf{B},\mathbf{M}^\sharp)$.
4. $\mathrm{EQ}(\mathbf{B})$ *is the set of* $\Gamma_{(\mathbf{B},\mathbf{M})}$*-sentences of the form* $\delta(P_{b_1},\ldots,P_{b_n}) \leftrightarrow \epsilon(P_{a_1},\ldots,P_{a_k})$ *such that* $\mathbf{B} \models \delta(b_1,\ldots,b_n) = \epsilon(a_1,\ldots,a_k)$*, where* δ,ϵ *are* L*-terms and* $a_1,\ldots,a_k$, $b_1,\ldots,b_n \in B$
5. $\mathrm{NEQ}(\mathbf{B})$ *is the set of* $\Gamma_{(\mathbf{B},\mathbf{M})}$*-sentences of the form* $\delta(P_{b_1},\ldots,P_{b_n}) \leftrightarrow \epsilon(P_{a_1},\ldots,P_{a_k})$ *such that* $\mathbf{B} \models \delta(b_1,\ldots,b_n) \neq \epsilon(a_1,\ldots,a_k)$*, where* δ,ϵ *are* L*-terms and* $a_1,\ldots,a_k$, $b_1,\ldots,b_n \in B$
6. The Elementary Diagram *of* $(\mathbf{B},\mathbf{M})$*, denoted by* $\mathrm{EDIAG}(\mathbf{B},\mathbf{M})$*, is the set*

$$\mathrm{EQ}(\mathbf{B}) \cup \{\phi \leftrightarrow P_b : \phi \text{ is a } \Gamma_{\mathrm{M}}\text{-sentence and } \|\phi\|^{(\mathbf{B},\mathbf{M}')} = b\}$$

Proposition 16 [Proposition 31 of [De11]] *Let* $(\mathbf{B},\mathbf{M})$ *and* $(\mathbf{A},\mathbf{N})$ *be two* Γ*-structures. The following are equivalent:*

1. *There is an expansion of* $(\mathbf{A},\mathbf{N})$ *that is a model of* $\mathrm{EDIAG}(\mathbf{B},\mathbf{M})$.
2. *There is an elementary mapping* (g,f) *from* $(\mathbf{B},\mathbf{M})$ *into* $(\mathbf{A},\mathbf{N})$.

Moreover, g *is one-to-one iff for every sentence* $\psi \in \mathrm{NEQ}(\mathbf{B})$*, the expansion of* $(\mathbf{A},\mathbf{N})$ *in condition 1. is not a model of* ψ.

Corollary 17 [Remark after Proposition 31 of [De11]] *Let* $(\mathbf{B},\mathbf{M})$ *and* $(\mathbf{A},\mathbf{N})$ *be two* Γ*-structures such that* $(\mathbf{B},\mathbf{M})$ *is exhaustive. The following are equivalent:*

1. *There is an expansion of* $(\mathbf{A},\mathbf{N})$ *that is a model of* $\mathrm{EDIAG}_0(\mathbf{B},\mathbf{M})$.
2. *There is an elementary mapping* (g,f) *from* $(\mathbf{B},\mathbf{M})$ *into* $(\mathbf{A},\mathbf{N})$.

Moreover, g *is one-to-one iff for every sentence of* $\Gamma_{\mathbf{M}}$*,* $\psi \notin \mathrm{EDIAG}_0(\mathbf{B},\mathbf{M})$*, the expansion of* $(\mathbf{A},\mathbf{N})$ *in condition 1. is not a model of* ψ.

Remark that, as pointed out in [De11], the mapping f of Proposition 16 and of Corollary 17 is not necessarily one-to-one, because we do not work with a crisp equality.

5.2 Canonical Models

Now we recall some definitions and basic facts on canonical models of fuzzy predicate logics (cf. section 4 and 5 of [HaCi06]).

Definition 18 *A* Γ*-theory* T *is* linear *iff for each pair of* Γ*-sentences* $\phi,\psi \in \Gamma$*,* $T \vdash \phi \to \psi$ *or* $T \vdash \psi \to \phi$.

Definition 19 *A* Γ*-theory* Ψ *is* directed *iff for each pair of* Γ*-sentences* $\phi,\psi \in \Psi$*, there is a* Γ*-sentence* $\chi \in \Psi$ *such that both* $\phi \to \chi$ *and* $\psi \to \chi$ *are probable.*

Definition 20 *Let* Γ *and* Γ' *be predicate languages such that* $\Gamma \subseteq \Gamma'$ *and let* T *be a* Γ'*-theory. We say that* T *is* Γ-Henkin *if for each formula* $\psi(x) \in \Gamma$ *such that* $T \nvdash \forall x\psi$*, there is a constant* $c \in \Gamma'$ *such that* $T \nvdash \psi(c)$*. And we say that* T *is* $\exists$-Γ–Henkin *if for each formula* $\psi(x) \in \Gamma$ *such that* $T \vdash \exists x\psi$*, there is a constant* $c \in \Gamma'$ *such that* $T \vdash \psi(c)$*. Finally, a* Γ*-theory is called* doubly-Γ-Henkin *if it is both* Γ*-Henkin and* $\exists$-Γ*-Henkin. In case that* $\Gamma = \Gamma'$*, we say that* T *is* Henkin *(*$\exists$*-Henkin, doubly Henkin, respectively).*

Theorem 21 [Theorem 2.20 of [CiHa10]] *Let T_0 be a Γ-theory and Ψ a directed set of Γ-sentences such that $T_0 \not\vdash \Psi$. Then, there is a linear doubly Henkin theory $T \supseteq T_0$ in a predicate language $\Gamma' \supseteq \Gamma$ such that $T \not\vdash \Psi$.*

Definition 22 *Let T be a Γ-theory. The* canonical model *of T, denoted by $(\mathbf{Lind}_T, \mathbf{CM}(T))$, where $\mathbf{Lind}_T$ is the Lindenbaum algebra of T (that is, the L-algebra of classes of T-equivalent Γ–sentences) is defined as follows: the domain of $\mathbf{CM}(T)$ is the set of closed Γ-terms, for every n-ary function symbol $F \in \Gamma$,*

$$F_{(\mathbf{Lind}_T, \mathbf{CM}(T))}(t_1 \dots t_n) = F(t_1 \dots t_n)$$

and for each n-ary predicate symbol $P \in \Gamma$, $P_{(\mathbf{Lind}_T, \mathbf{CM}(T))}(t_1 \dots t_n) = [P(t_1 \dots t_n)]_T$.

For now on we will use a shorter notation and write $\mathbf{CM}(T)$ instead of $(\mathbf{Lind}_T, \mathbf{CM}(T))$.

Lemma 23 [Lemma 2.24 of [CiHa10]] *Let T be a Henkin Γ-theory. Then,*

- $\mathbf{Lind}_T$ *is an* L*-chain iff T is linear*
- *For every sentence $\phi \in \Gamma$, $\|\phi\|^{(\mathbf{Lind}_T, \mathbf{CM}(T))} = [\phi]_T$*
- *For every sentence $\phi \in \Gamma$, $T \vdash \phi$ iff $\mathbf{CM}(T) \models \phi$*
- $\mathbf{CM}(T)$ *is exhaustive*

5.3 Refinements of the Method of Diagrams

Now we prove some new facts on diagrams and elementary extensions, using canonical models. The following proposition can be regarded as an improvement of Proposition 32 of [De11] in two main aspects. On the one hand, the technique obtains now an elementary extension (a canonical model) which is well-known for us and has some good model-theoretic properties. On the other hand, we have an elementary mapping (g, f) with g and f one-to-one, fact that does not hold in general for arbitrary structures.

Proposition 24 *Let $(\mathbf{B}, \mathbf{M})$ be a Σ-structure and $T_0 \supseteq \mathrm{EDIAG}(\mathbf{B}, \mathbf{M})$ a consistent theory in a predicate language $\Gamma \supseteq \Sigma$. If $\Psi \supseteq \mathrm{NEQ}(\mathbf{B})$ is a directed set of Γ-sentences such that $T_0 \not\vdash \Psi$, then there is a linear doubly Henkin theory $T \supseteq T_0$ in a predicate language $\Gamma' \supseteq \Gamma$ such that $T \not\vdash \Psi$ and an elementary mapping (g, f) from $(\mathbf{B}, \mathbf{M})$ into $\mathbf{CM}(T)$, with g and f one-to-one.*

Proof By Theorem 21, there is a linear doubly Henkin theory $T \supseteq T_0$ in a predicate language $\Gamma' \supseteq \Gamma$ such that $T \not\vdash \Psi$. By Lemma 23, $\mathbf{CM}(T)$ is a model of $\mathrm{EDIAG}(\mathbf{B}, \mathbf{M})$. Then, by Proposition 16 (Proposition 32 of [De11]), there is an elementary mapping (g, f) from $(\mathbf{B}, \mathbf{M})$ into $\mathbf{CM}(T)$, defined as follows: for each $d \in M$, $f(d) = c_d$ and for each $b \in B$, $g(b) = [P_b]_T$. Moreover, since $T \not\vdash \mathrm{NEQ}(\mathbf{B})$, for every sentence $\psi \in \mathrm{NEQ}(\mathbf{B})$, $\mathbf{CM}(T)$ is not a model of ψ and thus, g is one-to-one: indeed, if $b \neq b'$, then $P_b \leftrightarrow P_{b'} \in \mathrm{NEQ}(\mathbf{B})$ and, by assumption, it is not true in $\mathbf{CM}(T)$ and consequently, $[P_b]_T \neq [P_{b'}]_T$ and thus $g(b) \neq g(b')$. Finally, by definition of $\mathbf{CM}(T)$, f is also one-to-one.

Observe that the elementary mapping defined in Proposition 24 is not necessarily a homomorphism: in general we have for every closed term t, $f(t_M) = c_{t_M}$ and $c_{t_M} \neq t$.

Observe that Theorem 21 does not hold for Δ-core fuzzy logics (for an explanation see [HaCi06]).

Theorem 25 [Theorem 2.21 of [CiHa10]] *Let L be a Δ-core fuzzy logic, T_0 a Γ-theory and Ψ a directed set of Γ-sentences such that $T_0 \not\vdash \Psi$. Then, there is a linear Henkin theory $T \supseteq T_0$ in a predicate language $\Gamma' \supseteq \Gamma$ such that $T \not\vdash \Psi$.*

Nevertheless, imitating the proof of Proposition 24 and using Theorem 25, it is possible to obtain a version of the previous result for Δ-core fuzzy logics:

Proposition 26 *Let L be a Δ-core fuzzy logic, $(\mathbf{B}, \mathbf{M})$ be a Σ-structure and $T_0 \supseteq \mathrm{EDIAG}(\mathbf{B}, \mathbf{M})$ a consistent theory in a predicate language $\Gamma \supseteq \Sigma$. If $\Psi \supseteq \mathrm{NEQ}(\mathbf{B})$ is a directed set of Γ-sentences such that $T_0 \not\vdash \Psi$, then there is a linear Henkin theory $T \supseteq T_0$ in a predicate language $\Gamma' \supseteq \Gamma$ such that $T \not\vdash \Psi$ and an elementary mapping (g, f) from $(\mathbf{B}, \mathbf{M})$ into $\mathbf{CM}(T)$, with f and g one-to-one.*

Now as a Corollary of Propositions 17 and 24 we obtain the following result for exhaustive structures (an analogous result could be obtained for Δ-core fuzzy logics using Propositions 17 and 26). We denote by $\overline{\mathrm{EDIAG}_0}(\mathbf{B}, \mathbf{M})$ the set of all $\Gamma_{\mathbf{M}}$-sentences that do not belong to $\mathrm{EDIAG}_0(\mathbf{B}, \mathbf{M})$.

Corollary 27 *Let $(\mathbf{B}, \mathbf{M})$ be an exhaustive Σ-structure and $T_0 \supseteq \mathrm{EDIAG}_0(\mathbf{B}, \mathbf{M})$ a consistent theory in a predicate language $\Gamma \supseteq \Sigma$. If $\Psi \supseteq \overline{\mathrm{EDIAG}_0}(\mathbf{B}, \mathbf{M})$ is a directed set of formulas of Γ such that $T_0 \not\vdash \Psi$, then there is a linear doubly Henkin theory $T \supseteq T_0$ in a predicate language $\Gamma' \supseteq \Gamma$ such that $T \not\vdash \Psi$ and an elementary mapping (g, f) from $(\mathbf{B}, \mathbf{M})$ into $\mathbf{CM}(T)$, with g and f one-to-one.*

6 Witnessed and Quasi-witnessed Models

In this section we focus on extensions of witnessed and quasi-witnessed models. We will show a direct application of Proposition 24, giving a generalization of Lemmas 3.3 of [CeEs09] and 5 of [HaCi06] for non-exhaustive models.

Intuitively speaking, a *witnessed model* is one in which every quantified formula has an element of the model which witnesses it. More precisely:

Definition 28 *Let $(\mathbf{B}, \mathbf{M})$ be a Γ-structure. A Γ-formula $\phi(y, x_1, \ldots, x_n)$ is* witnessed *in $(\mathbf{B}, \mathbf{M})$ iff for every $d_1, \ldots, d_n \in M$ the following two conditions hold:*

1. *there is $d \in M$ such that $\|\exists y \phi(d_1, \ldots, d_n)\|^{(\mathbf{B},\mathbf{M})} = \|\phi(d, d_1, \ldots, d_n)\|^{(\mathbf{B},\mathbf{M})}$*
2. *there is $e \in M$ such that $\|\forall y \phi(d_1, \ldots, d_n)\|^{(\mathbf{B},\mathbf{M})} = \|\phi(e, d_1, \ldots, d_n)\|^{(\mathbf{B},\mathbf{M})}$*

Moreover, it is said that $(\mathbf{B}, \mathbf{M})$ is a witnessed model *iff every Γ-formula $\phi(y, x_1, \ldots, x_n)$ is witnessed in $(\mathbf{B}, \mathbf{M})$.*

Notice that every model of classical predicate logic is witnessed, as well as all the models of finitely-valued logics. However, when we move to infinitely valued logics, this is not always the case. The infimum or supremum of a set of truth values might not be included in this set, and thus we can find models in which some quantified formula has no witness. Following these ideas, Hájek introduced in [Ha07a], [Ha07b] the aforementioned notion of witnessed model and proved that this is an important property because it implies a limited form of finite model property for certain fragments of predicate fuzzy logic (see [Ha05]). In [HaCi06] the following axiom schemes, originally introduced by Baaz, are discussed: (C$\forall$) $\exists x(\phi(x) \to \forall y \phi(y))$ and (C$\exists$) $\exists x(\exists y \phi(y) \to \phi(x))$. Cintula and Hájek showed in [HaCi06] that adding the witnessed axioms (C$\forall$)

and (C∃) to any first order core fuzzy logic, we obtain a logic complete with respect this kind of models. Subsequently, they proved that these axioms are derivable in Łukasiewicz First Order Logic, showing that Ł∀ is complete with respect to witnessed models (we will say that Ł∀ has the *witnessed model property*). Nevertheless, they proved that neither G∀ nor Π∀ share this property (the witnessed axioms are not theorems of these logics). In fact, no other first order logic of a continuous t-norm enjoys this property. This characteristic is strongly related to the continuity of the truth functions, a property that only Łukasiewicz logic has.

Now we show a direct application of Proposition 24 to obtain witnessed models, giving a generalization of Lemma 5 of [HaCi06] to non-exhaustive structures.

Proposition 29 *Let T be a Γ-theory and T' its extension with axioms* C∀ *and* C∃. *Then every Γ-structure model of T' is elementarily one-to one mapped into a witnessed model of T.*

Proof Let $(\mathbf{B},\mathbf{M})$ be a Γ-structure model of T'. We consider the theory $T_0 = \mathrm{FDIAG}(\mathbf{M},\mathbf{B})$. Now let Ψ be the closure of NEQ($\mathbf{B}$) under disjunctions. Clearly Ψ is a directed set. We show that $T_0 \not\vdash \Psi$: it is enough to prove that for every $\alpha,\beta \in$NEQ($\mathbf{B}$), $\alpha \vee \beta \notin T_0$. Assume the contrary, since $\mathbf{B}$ is an L-chain, we have that either $\alpha \to \beta \in T_0$ or $\beta \to \alpha \in T_0$. Then, since L is a core fuzzy logic, by Theorem 2, we will have either that $\alpha \in T_0$ or $\beta \in T_0$, which is absurd, by definition of NEQ($\mathbf{B}$).

Then, by Proposition 24, since $T_0 \supseteq$ EDIAG($\mathbf{B},\mathbf{M}$) and $\Psi \supseteq$ NEQ($\mathbf{B}$), there is a linear doubly Henkin theory $T^* \supseteq T_0$ such that $T^* \not\vdash \Psi$ and $(\mathbf{B},\mathbf{M})$ elementarily one-to one mapped into $\mathbf{CM}(T^*)$.

Now we see that $\mathbf{CM}(T)$ is witnessed: let $\phi(y, x_1, \dots, x_n)$ be a formula and $t_1, \dots, t_n$ closed terms elements of CM(T), assume that the constants that occur in these terms are $c_1, \dots, c_k$. Consider now the formula $\phi'(y, c_1, \dots, c_k)$ obtained from $\phi(y, x_1, \dots, x_n)$ by substituting the variables $x_1, \dots, x_n$ for the terms $t_1, \dots, t_n$. By assumption, $T \vdash (C\exists)$, we have then that $T \vdash \exists z(\exists y\phi'(y, c_1, \dots, c_k) \to \phi'(z, c_1, \dots, c_k))$ and thus, since T is ∃-Henkin, there is a constant d such that $T \vdash \exists y\phi'(y, c_1, \dots, c_k) \to \phi'(d, c_1, \dots, c_k)$. Then, by definition of $\mathbf{CM}(T)$, we have that

$$\|\exists y\phi(y, t_1, \dots, t_n)\|^{(\mathbf{Lind}_T,\mathbf{CM}(T))} = \|\exists y\phi'(y, c_1, \dots, c_k)\|^{(\mathbf{Lind}_T,\mathbf{CM}(T))} =$$

$$= \|\phi'(d, c_1, \dots, c_k)\|^{(\mathbf{Lind}_T,\mathbf{CM}(T))} = \|\phi(d, t_1, \dots, t_n)\|^{(\mathbf{Lind}_T,\mathbf{CM}(T))}$$

The proof for the universal step is analogous by using axiom (C∀).

In [LaMa07] it is proved that Product Predicate Logic Π∀ enjoys a property weaker than the witnessed model property, the so-called *quasi-witnessed model property.* Quasi-witnessed models are models in which universally quantified formulas taking truth-values greater than 0 have witnesses, while existentially quantified formulas are always witnessed. In [CeEs09] the authors introduced the so-called *quasi-witnessed axioms*:

$$(\Pi C\forall)\ \neg\neg\forall x\phi(x) \to (\exists x(\phi(x) \to \forall y\phi(y))$$
$$(C\exists)\ \exists x(\exists y\phi(y) \to \phi(x))$$

and they proved that the axiomatic extension of any strict core fuzzy logic[1] with the quasi-witnessed axioms is complete w.r.t. quasi-witnessed models. In particular they proved that in Π∀ these axioms are deducible, and thus Π∀ is complete w.r.t.

[1] A core fuzzy logic is *strict* iff it expands SMTL (for the details see Definition 3 of [CeEs09]).

quasi-witnessed models (result already proved directly in [LaMa07]). Finally it is also proved that no other logic of a continuous t-norm except Ł∀ and Π∀ satisfy the quasi-witnessed axioms. Notice that we have taken the name coined in [CeEs09] for this kind of models instead of the original one of [LaMa07], which was *strictly closed models*. The main reason is that the new name seems more informative about the properties of the described model. Moreover, with this notation we avoid possible confusions with other usages of the name *closed* in mathematics and logic. Imitating the proof of Proposition 29 and using the method of diagrams we can generalize Lemma 19 of [CeEs09] to non-exhaustive models.

Proposition 30 *Let T be a Γ-theory and T' its extension with axioms* ΠC∀ *and* C∃. *Then every Γ-structure model of T' is elementary one-to one mapped into a quasi-witnessed model of T.*

7 Counterexample to Conjectures 1 and 2 of [HaCi06]

Given two theories $T_1 \subseteq T_2$ in the respective predicate languages $\Gamma_1 \subseteq \Gamma_2$, it is said that T_2 is a *conservative extension* of T_1 if and only if each Γ_1-formula provable in T_2 is also provable in T_1. P. Hájek and P. Cintula proved in Theorem 6 of [HaCi06] that, in core fuzzy logics, a theory T_2 is a conservative extension of another theory T_1 if and only if each exhaustive model of T_1 can be elementarily one-to one mapped into some model of T_2. In Theorem 7 of [HaCi06], they conjectured the same result to be true for arbitrary structures, showing that the following two conjectures were equivalent:

Conjecture 1 of [HaCi06]: Let P be a truth constant symbol and for $i \in \{1, 2\}$, T_i be a Γ_i-theory, and T_i^+ be a $\Gamma_i \cup \{P\}$-theory such that $T_i^+ = T_i$ (i.e. P is added to the language but no new axioms are added). If T_2 is a conservative extension of T_1, then T_2^+ is a conservative extension of T_1^+.

Conjecture 2 of [HaCi06]: A theory T_2 is a conservative extension of another theory T_1 if and only if each model of T_1 can be elementarily one-to one mapped into some model of T_2.

We present here a counterexample to Conjecture 2 (and thus to Conjecture 1). Let L be the logic that has as equivalent algebraic semantics the variety generated by the union of the classes of Łukasiewicz and Product chains. L is an axiomatic extension of BL, to find an axiomatization we refer to [CigEsGoTo00] where it is proved also that the set of chains of the variety coincide with the union of the sets of Łukasiewicz and Product chains). Let now $(\{0, 1\}, \mathbf{M})$ be a classical first-order structure in a predicate language Γ, and let $\mathbf{B}_1 = [0, 1]_\Pi$ and $\mathbf{B}_2 = [0, 1]_Ł$ be the canonical Product and Łukasiewicz chains, respectively.

Remark that the structure $(\{0, 1\}, \mathbf{M})$ can also be regarded as a Γ-structure over both $\mathbf{B}_1$ and $\mathbf{B}_2$ chains, since for every two-valued n-ary predicate $P_\mathbf{M} : M^n \to \{0, 1\}$, $P_\mathbf{M}$ is also a fuzzy relation $P_\mathbf{M} : M^n \to [0, 1]_\Pi$ and $P_\mathbf{M} : M^n \to [0, 1]_Ł$. Thus, we have $(\mathbf{B}_1, \mathbf{M}) \equiv (\mathbf{B}_2, \mathbf{M})$ (in fact we have that $(\mathbf{B}_1, \mathbf{M}') \equiv (\mathbf{B}_2, \mathbf{M}')$, where $\mathbf{M}'$ is as in Definition 14).

Let T_1=EDIAG$_0(\mathbf{B}_1, \mathbf{M})$ and T_2=FDIAG$(\mathbf{B}_2, \mathbf{M})$. We have that T_2 is a conservative extension of T_1: for every $\Gamma_\mathbf{M}$-formula ϕ, if $T_2 \vdash \phi$, then since $\|\phi\|^{(\mathbf{B}_2, \mathbf{M}')}$=1 and $(\mathbf{B}_1, \mathbf{M}') \equiv (\mathbf{B}_2, \mathbf{M}')$ and then $\phi \in T_1$. Now we show that there is a model of T_1 that can not be elementarily one-to one mapped into some model of T_2, this model is $([0, 1]_\Pi, \mathbf{M})$. Suppose, contrary to our claim, that there is a model of T_2, say $(\mathbf{A}, \mathbf{N})$, in which $([0,1]_\Pi, \mathbf{M})$ is elementarily one-to one mapped. By Proposition 16, since $(\mathbf{A}, \mathbf{N})$ is a model of T_2, there is an elementary mapping from $([0,1]_{\text{Ł}}, \mathbf{M})$ into $(\mathbf{A}, \mathbf{N})$. Consequently, there is an L-embedding k from $[0,1]_\Pi$ into $\mathbf{A}$ and at the same time there is an L-homomorphism h from $[0,1]_{\text{Ł}}$ into $\mathbf{A}$ (not necessarily one-to-one). If $\mathbf{A}$ is an L-chain, it is clear that this is not possible. We show now that, for any arbitrary L-algebra $\mathbf{A}$, this fact leads to a contradiction[2]. If such embeddings k and h exist, and c and b are the images of $1/2$ under h and k respectively, we have $b = \neg b$ (because h is an L-homomorphism), $c < 1$ and $\neg c = 0$ (because k is an L-embedding and the negation in $[0,1]_\Pi$ is the Gödel negation). If we decompose $\mathbf{A}$ as a subdirect product of an indexed family of subdirectly irreducible BL-chains, say $(\mathbf{A}_i : i \in I)$, every such $\mathbf{A}_i$ is either a Łukasiewicz, or a Product chain (for a reference see [Ha98] and [CigEsGoTo00]). Therefore, if we take an index i such that the i-component, c_i, satisfies $0 < c_i < 1$, we will have at the same time $\neg c_i = 0$ and for the i-component b_i, $b_i = \neg b_i$, which is absurd, because $\mathbf{A}_i$ can not be, at the same time, a Łukasiewicz and a Product chain.

8 A Characterization Theorem of Elementary Equivalence

In this section we characterize when two exhaustive structures are elementarily equivalent using elementary mappings. We provide an example showing that the result can not be extended to arbitrary models.

Theorem 31 *Let $(\mathbf{B}_1, \mathbf{M}_1)$ and $(\mathbf{B}_2, \mathbf{M}_2)$ be two exhaustive Γ-structures. The following are equivalent:*

1. $(\mathbf{B}_1, \mathbf{M}_1) \equiv (\mathbf{B}_2, \mathbf{M}_2)$.
2. *There is a Γ-structure $(\mathbf{A}, \mathbf{N})$, such that $(\mathbf{B}_1, \mathbf{M}_1)$ and $(\mathbf{B}_2, \mathbf{M}_2)$ are elementarily mapped into $(\mathbf{A}, \mathbf{N})$.*

Proof 2. $\Rightarrow$ 1. is clear.

1. $\Rightarrow$ 2. First we expand the language introducing two disjoint sets of new constants, C_{M_1} and C_{M_2} for the elements of M_1 and M_2, respectively, that are not interpretations of the constant symbols in Γ.

Now consider the theory $T_0 = \text{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1) \cup \text{EDIAG}_0(\mathbf{B}_2, \mathbf{M}_2)$ in the language expanded with the set of constants C_{M_1} and C_{M_2} respectively. Let us show that T_0 is consistent: If $T_0 \vdash \bot$, since $\text{EDIAG}_0(\mathbf{B}_2, \mathbf{M}_2)$ is closed under conjunction and the proof is finitary, there is $\psi \in \text{EDIAG}_0(\mathbf{B}_2, \mathbf{M}_2)$ such that $\text{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1), \psi \vdash \bot$. Then, by the Local Deduction Theorem (see Definition 1), there is a natural number n such that $\text{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1) \vdash (\psi)^n \to \bot$. Let $\widehat{\psi}$ be the formula obtained by replacing each constant $c \in C_{M_2}$ by a new variable x_c. Thus we have $\text{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1) \vdash (\widehat{\psi})^n \to \bot$ and by generalization over the new variables we obtain $\text{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1) \vdash (\forall ...)((\widehat{\psi})^n \to \bot)$, thus $(\forall ...)((\widehat{\psi})^n \to \bot) \in \text{Th}(\mathbf{B}_1, \mathbf{M}_1) = \text{Th}(\mathbf{B}_2, \mathbf{M}_2)$ (because $(\mathbf{B}_1, \mathbf{M}_1) \equiv (\mathbf{B}_2, \mathbf{M}_2)$) and consequently, $\bot \in \text{Th}(\mathbf{B}_2, \mathbf{M}_2)$, which is absurd.

Now let $\Psi = \overline{\text{EDIAG}_0}(\mathbf{B}_1, \mathbf{M}_1)$. It is easy to check that Ψ is a directed set: given $\alpha, \beta \in \Psi$, we show that $\alpha \vee \beta \in \Psi$. Assume the contrary, if $\alpha \vee \beta \in \text{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1)$, using the fact that $\mathbf{B}_1$ is an L-chain, we have that either $\alpha \to \beta \in \text{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1)$ or $\beta \to \alpha \in \text{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1)$. Then, since L is a core fuzzy logic, by Theorem 2, we

[2] Our example is inspired in one used by F. Montagna in [Mo06].

will have either that $\alpha \in \mathrm{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1)$ or $\beta \in \mathrm{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1)$, which is absurd because $\alpha, \beta \in \Psi$.

We show now that $T_0 \nvdash \Psi$. Otherwise, if for some $\alpha \in \Psi$, $T_0 \vdash \alpha$, since the set $\mathrm{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1)$ is closed under conjunction and the proof is finitary, there is $\psi \in \mathrm{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1)$ such that $\mathrm{EDIAG}_0(\mathbf{B}_2, \mathbf{M}_2), \psi \vdash \alpha$. Then, by the same kind of argument we used to show that T_0 is consistent, we would obtain that $\alpha \in \mathrm{EDIAG}_0(\mathbf{B}_1, \mathbf{M}_1)$, which is absurd.

Then, by Corollary 27, there is a linear doubly Henkin theory $T \supseteq T_0$ in a predicate language $\Gamma' \supseteq \Gamma$ such that $T \nvdash \Psi$ and an elementary mapping (g, f) from $(\mathbf{B}_1, \mathbf{M}_1)$ into $\mathbf{CM}(T)$, with g and f one-to-one. Moreover, since $\mathbf{CM}(T)$ is also a model of $\mathrm{EDIAG}_0(\mathbf{B}_2, \mathbf{M}_2)$, by Corollary 17, $(\mathbf{B}_2, \mathbf{M}_2)$ is elementarily mapped into $\mathbf{CM}(T)$. Finally, by Lemma 23, $\mathbf{Lind}_T$ is an L-chain.

By the analogue of Corollary 27 for Δ-core fuzzy logics, if we substitute in the first line of the last paragraph of the previous proof the expression 'linear doubly Henkin theory' by 'linear Henkin theory', then Theorem 31 holds also for Δ-core fuzzy logics.

Remark that Theorem 31 can not be generalized to arbitrary structures. If we take the structures of the counterexample to Conjectures 1 and 2 of Section 7, we have $([0,1]_\Pi, \mathbf{M}) \equiv ([0,1]_{\text{Ł}}, \mathbf{M})$, but there is not a Γ-structure $(\mathbf{A}, \mathbf{N})$ in which both are elementary mapped.

9 Future Work

Work in progress is devoted to find characterizations of the notion of elementary equivalence using other model-theoretic constructions such as ultraproducts. In our future research we plan to analise the relationship of our study with other approaches, for instance the one presented in [NoPerMo99], where a notion of elementary equivalent models *in a degree d* was introduced (cf. Definition 4.33).

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References

[Be02] Belohlávek R (2002) Fuzzy relational systems: foundations and principles. Springer

[CeEs09] Cerami M, Esteva F (2011) Strict core fuzzy logics and quasi-witnessed models. Archive for Mathematical Logic, 50(5/6):625–641

[ChKe90] Chang CC, Keisler HJ (1990) Model theory, Studies in Logic, 73, 3rd edn. North-Holland, Amsterdam

[CigEsGoTo00] Cignoli R, Esteva F, Godo L, Torrens A (2000) Basic fuzzy logic is the logic of continuous t-norms and their residua. Soft Computing 4(2):106–112

[Ci05] Cintula P (2005) From Fuzzy Logic to Fuzzy Mathematics. Ph.D. dissertation, Czech Technical University, Prague (Czech Republic).

[CiHa10] Cintula P, Hájek P (2010) Triangular norm based predicate fuzzy logics. Fuzzy Sets and Systems 161:311–346

[CiEsGiGoMoNo09] Cintula P, Esteva F, Gispert J, Godo L, Montagna F, Noguera C (2009) Distinguished algebraic semantics for t-norm based fuzzy logics: Methods and algebraic equivalencies. Annals of Pure and Applied Logic 160(1):53–81

[De11] Dellunde P (2011) Preserving mappings in fuzzy predicate logics. Journal of Logic and Computation, in press.

[DeEs10] Dellunde P, Esteva F (2010) On elementary extensions for fuzzy predicate logics. In: Proceedings of IPMU 2010, 6178:747–756

[EsGo01] Esteva F, Godo L (2001) Monoidal t-norm based logic: towards a logic for left-continuous t-norms. Fuzzy Sets and Systems 124(3):271–288

[Ge86] Gerla G (1986) The category of the fuzzy models and Löwenheim-Skolem theorem. Mathematics of Fuzzy Systems 121–141

[Ha98] Hájek P (1998) Metamathematics of fuzzy logic, Trends in Logic—Studia Logica Library, 4, Kluwer Academic Publishers, Dordrecht

[Ha05] Hájek P (2005) Making fuzzy description logic more general. Fuzzy Sets and Systems 154(1):1–15

[Ha07a] Hájek P (2007) On witnessed models in fuzzy logic. Mathematical Logic Quarterly 53(1):66–77

[Ha07b] Hájek P (2007) On witnessed models in fuzzy logic II. Mathematical Logic Quarterly 53(6):610–615

[HaCi06] Hájek P, Cintula P (2006) On theories and models in fuzzy predicate logics. The Journal of Symbolic Logic 71(3):863–880

[JeMo02] Jenei S, Montagna F (2002) A proof of standard completeness for Esteva and Godo's logic MTL. Studia Logica 70(2):183–192

[LaMa07] Laskowski MC, Malekpour S (2007) Provability in predicate product logic. Archive for Mathematical Logic 46(5):365–378

[Mo06] Montagna F (2006) Interpolation and Beth's property in propositional many-valued logics: A semantic investigation. Annals of Pure and Applied Logic 141(1-2):148–179

[DiNoGe86] Di Nola, Gerla G (1986) Fuzzy models of first order languages. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 32:331–340

[NoPerMo99] Novák V, Perfilieva I, Močkoř J (1999) Mathematical principles of fuzzy logic, The Kluwer International Series in Engineering and Computer Science, 517. Kluwer Academic Publishers, Boston, MA

[Ra74] Rasiowa H (1974) An algebraic approach to non-classical logics. North-Holland

[RaSi63] Rasiowa H, Sikorski R (1963) The mathematics of the metamathematics. P.W.N., Warszawa

[Ta35] Tarski A (1935) Der wahrheitsbegriff in den formalisierten sprachen. Studia Philosophica 1:261–405

[TaVa57] Tarski A, Vaught RL (1957) Arithmetical extensions of relational systems. Compositio Math 13:81–102