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Extending possibilistic logic over Godel logic☆

Pilar Dellunde[1,2] Lluís Godo[2] and Enrico Marchioni[2]

[1] Universitat Autonoma de Barcelona
08193 Bellaterra, Spain
pilar.dellunde@uab.cat

[2] IIIA - CSIC
08193 Bellaterra, Spain
{enrico,pilar,godo}@iiia.csic.es

Abstract

In this paper we present several fuzzy logics trying to capture different notions of necessity (in the sense of Possibility theory) for Godel logic formulas. Based on different characterizations of necessity measures on fuzzy sets, a group of logics with Kripke style semantics are built over a restricted language, namely a two level language composed of non-modal and modal formulas, the latter moreover not allowing for nested applications of the modal operator N. Completeness and some computational complexity results are shown.

Keywords: Posibilistic Logic, Necessity Measures, Godel Logic, Fuzzy Logic

1. Introduction

The handling and modelling of uncertainty is a key issue in artificial intelligence reasoning tasks. The most general notion of uncertainty is captured by monotone set functions with two natural boundary conditions. In the literature, these functions have received several names, like Sugeno measures [34] or plausibility measures [28]. Many popular uncertainty measures, like probabilities, upper and lower probabilities, Dempster-Shafer plausibility and belief functions, or possibility and necessity measures, can be seen as particular classes of Sugeno measures.

In this paper, we specially focus on possibilistic models of uncertainty. A possibility measure on a Boolean algebra of events $\mathbf{U} = (U, \wedge, \vee, \neg, \bar{0}^{U}, \bar{1}^{U})$ is a Sugeno measure $\mu^* : U \to [0,1]$ (i.e. it satisfies $\mu^*(\bar{0}^U) = 0$, $\mu^*(\bar{1}^U) = 1$ and $\mu^*(u_1) \leq \mu^*(u_2)$ whenever $u_1 \leq u_2$) such that the following $\vee$-decomposition property

$$\mu^*(u_1 \vee u_2) = \max(\mu^*(u_1), \mu^*(u_2))$$

☆This is a fully revised and extended version of the conference paper [10].

holds. A necessity measure is a Sugeno measure μ_* satisfying the $\wedge$-decomposition property

$$\mu_*(u_1 \wedge u_2) = \min(\mu_*(u_1), \mu_*(u_2)).$$

Possibility and necessity are dual classes of measures, in the sense that if μ^* is a possibility measure, then the function $\mu_*(u) = 1 - \mu^*(\neg u)$ is a necessity measure, and vice versa. If U is the power set of a set X, then any dual pair of measures (μ^*, μ_*) on U is induced by a normalized possibility distribution, i.e. a mapping $\pi : X \to [0, 1]$ such that, $\sup_{x \in X} \pi(x) = 1$, and, for any $A \subseteq X$,

$$\mu^*(A) = \sup\{\pi(x) \mid x \in A\} \text{ and } \mu_*(A) = \inf\{1 - \pi(x) \mid x \notin A\}.$$[1]

A usual approach in artificial intelligence is to integrate uncertainty models within logic-based knowledge representation formalisms by attaching some belief quantification to declarative statements. In this line, Possibilistic logic is a weighted logic that handles possibilistic uncertainty by associating certainty, or priority levels, to classical logic formulas [14]. Possibilistic logic (propositional) formulas are pairs (ϕ, r), where ϕ is a classical propositional formula and $r \in (0, 1]$ is interpreted as a lower bound for the necessity of ϕ. The semantics is given by possibility distributions on the set Ω of classical interpretations. Namely, if $\pi : \Omega \to [0, 1]$, then π satisfies a pair (ϕ, r), written $\pi \models (\phi, r)$ when $N_\pi(\phi) = \inf\{1 - \pi(w) \mid w \in \Omega, w(\phi) = 0\} \geq r$. Possibilistic logic is axiomatized by taking as axioms the axioms of Classical Propositional Calculus (CPC) with weight 1 and as inference rules:

- from (ϕ, α) and $(\phi \to \psi, \beta)$ infer $(\psi, \min(\alpha, \beta))$ (weighted modus ponens);
- from (ϕ, α) infer (ϕ, β), with $\beta \leq \alpha$ (weight weakening).

Since its introduction in the mid-eighties, multiple facets of possibilistic logic have been developed and used for a number of applications in AI, like handling exceptions in default reasoning, modeling belief revision, providing a graphical Bayesian-like network representation counterpart to a Possibilistic logic knowledge base or representing positive and negative information in a bipolar setting with applications to preferences fusion and to version space learning. Moreover, Possibilistic logic copes with inconsistency by taking advantage of the stratification of the set of formulas induced by the associated levels. It is also useful for representing preferences expressed as sets of prioritized goals, in this case, the weight is to be understood as the priority level of a goal. Several extensions of Possibilistic logic deal with time, multiple agents' mutual beliefs, a symbolic treatment of priorities for handling partial orders between levels, or learning stratified hypotheses for coping with exceptions. The reader is referred to [14] for an extensive overview on Possibilistic logic and its main applications.

[1] Indeed, $\mu_*(A) = \inf\{1 - \pi(x) \mid x \notin A\}$ holds true only when $A = X$, when $A = X$ then it must be $\mu_*(A) = 1$.

When we go beyond the classical framework of Boolean algebras of events to more general frameworks, appropriate extensions of uncertainty measures need to be considered in order to represent and reason about the uncertainty of non-classical or fuzzy events. For instance, the notion of (finitely additive) probability has been generalized in the setting of MV-algebras by means of the notion of state [29][2] and has been used in [18] to provide a logical framework for reasoning about the probability of (finitely-valued) fuzzy events.

Within the posibilistic framework, several extensions of the notions of possibility and necessity measures for fuzzy sets have been proposed in different logical systems extending the well-known Dubois-Lang-Prade's Possibilistic logic to fuzzy events, see e.g. [12, 15, 23, 3, 2, 4]. Actually, the different generalizations proposed in the literature arise from two observations. First of all, in contraposition to the classical case, $[0,1]$-valued mappings on the set of fuzzy sets in some (finite) domain X, $\Pi, N : [0,1]^X \to [0,1]$ satisfying the usual boundary conditions $\Pi(\emptyset) = N(\emptyset) = 0$ and $\Pi(X) = N(X) = 1$ and the characteristic decomposition properties

$$\Pi(A \vee B) = \max(\Pi(A), \Pi(B)), \quad N(A \wedge B) = \min(N(A), N(B)),$$

for any $A, B \in [0,1]^X$ where $\vee$ and $\wedge$ denote the pointwise maximum and minimum operations, are not univocally determined by a possibility distribution π on the set X. The second observation is that in the classical case the expressions of possibility and necessity measures on subsets of a set X in terms of a possibility distribution on X can be equivalently rewritten as

$$\Pi(A) = \sup_{x\in X} \min(\pi(x), A(x)), \quad N(A) = \inf_{x\in X} \max(1 - \pi(x), A(x))$$

where the subset $A \subseteq X$ is identified with its membership function $A : X \to \{0,1\}$. Therefore, natural generalizations of these expressions when $A : X \to [0,1]$ is a fuzzy subset of X are

$$\Pi(A) = \sup_{x\in X} \pi(x) \otimes A(x), \quad N(A) = \inf_{x\in X} \pi(x) \Rightarrow A(x) \qquad (*)$$

where $\otimes$ is a t-norm and $\Rightarrow$ is some suitable fuzzy implication function[3].

In particular, the following implication functions have been discussed in the literature as instantiations of the $\Rightarrow$ operation in (*):

(1) $u \Rightarrow_{KD} v = \max(1 - u, v)$ (Kleene-Dienes implication);

(2) $u \Rightarrow_{RG} v = \begin{cases} 1, \text{if } u \leq v \\ 1 - u, \text{otherwise} \end{cases}$ (reciprocal of Godel implication);

[2] Another generalization of the notion of probability has been recently studied in depth in [1] by defining probabilistic states over Gödel algebras.

[3] The minimum properties required to a binary operation $\Rightarrow$: $[0,1] \times [0,1] \to [0,1]$ to be considered as fuzzy counterpart of the classical $\{0,1\}$-valued implication truth-function are: $1 \Rightarrow 1 = 0 \Rightarrow 0 = 1$, $1 \Rightarrow 0 = 0$, $\Rightarrow$ is non-increasing in the first variable and non-decreasing in the second variable.

(3) $u \Rightarrow_L v = \min(1, 1 - u + v)$ (Lukasiewicz implication).

All these functions actually lead to proper extensions of the above definition of necessity over classical sets or events in the sense that if A is a crisp set, i.e. $A(x) \in \{0, 1\}$ for all $x \in X$, then (*) gives $N(A) = \inf_{x \in X}\{\max(1 - \pi(x), A(x))\}$.

In the literature different logical formalizations to reason about such extensions of the necessity of fuzzy events can be found. In [27], and later in [24], a full many-valued modal approach is developed over finitely-valued Lukasiewicz logics in order to capture the notion of necessity defined using $\Rightarrow_{KD}$. A logic programming approach over Godel logic is investigated in [3] and in [2] by relying on $\Rightarrow_{KD}$ and $\Rightarrow_{RG}$, respectively. More recently, following the approach of [18], modal-like logics to reason about the necessity of fuzzy events in the framework of MV-algebras have been defined in [19], in order to capture the notion of necessity defined by $\Rightarrow_{KD}$ and $\Rightarrow_L$.

The purpose of this paper is to explore different logical approaches to reason about the necessity of fuzzy events over Godel fuzzy logic. In more concrete terms, our aim is to study a modal-like expansion of the [0, 1]-valued Godel logic with a modality N such that the truth-value of a formula $N\phi$ (in [0, 1]) can be interpreted as the degree of necessity of ϕ. In this context, although it does not extend the classical possibilistic logic, it seems also interesting to investigate the notion of necessity definable from Godel implication, which corresponds to the standard fuzzy interpretation of the implication connective in Godel logic:

(4) $u \Rightarrow_G v = \begin{cases} 1, \text{if } u \leq v \\ v, \text{otherwise} \end{cases}$ (Godel implication).

This work is structured as follows. After this introduction and recalling some background on Godel fuzzy logic and related concepts, in Section 3 we recall a characterization of necessity measures on fuzzy sets defined by the implications $\Rightarrow_{KD}$ and $\Rightarrow_{RG}$ and provide a (new) characterization of those defined by $\Rightarrow_G$. These characterizations are the basis for the completeness results of several logics introduced in Sections 4, 5 and 6 capturing the corresponding notions of necessity for Godel logic formulas. These logics, with Kripke style semantics, are built over a two-level language composed of modal and non-modal formulas, the former not allowing nested applications of the modal operator. A common fragment of these logics akin to classical possibilistic logic is considered in Section 8. Some remarks about the computational complexity for these logics are made in Section 9. Finally, in Section 10, we mention some open problems and new research goals we plan to address in the near future.

2. Preliminaries on the Godel fuzzy logic G and its expansions $G_\Delta(C)$

All this section is devoted to preliminaries on the Godel fuzzy logic G and its expansions $G_\Delta(C)$. We present their syntax and semantics, their main logical properties and the notation we use throughout the article.

The language of Godel propositional logic is built as usual from a countable set of propositional variables V, the constant $\overline{0}$ and the binary connectives $\wedge$

and $\to$. Disjunction and negation are respectively defined as $\phi \vee \psi := ((\phi \to \psi) \to \psi) \wedge ((\psi \to \phi) \to \phi)$ and as $\neg\phi := \phi \to 0$, and the constant 1 is taken as $0 \to 0$.

As a many-valued logic, Godel logic is the axiomatic extension of Hájek's Basic Fuzzy Logic BL [24] (which is the logic of continuous t-norms and their residua) by means of the contraction axiom $\phi \to (\phi \wedge \phi)$. Since the unique idempotent continuous t-norm is the minimum, this yields that Godel logic is strongly complete with respect to its standard fuzzy semantics that interprets formulas over the structure $[0,1]_G = ([0,1], \min, \Rightarrow_G, 0, 1)$[4], i.e. semantics defined by truth-evaluations e such that $e(\phi \wedge \psi) = \min(e(\phi), e(\psi))$, $e(\phi \to \psi) = e(\phi) \Rightarrow_G e(\psi)$ and $e(0) = 0$. As a consequence $e(\phi \vee \psi) = \max(e(\phi), e(\psi))$ and $e(\neg\phi) = \neg_G e(\phi) = e(\phi) \Rightarrow_G 0$.

Godel logic can also be seen as the axiomatic extension of intuitionistic propositional logic by the prelinearity axiom $(\phi \to \psi) \vee (\psi \to \phi)$. Its algebraic semantics is therefore given by the variety of prelinear Heyting algebras, also known as Godel algebras. A Godel algebra is a structure $A = (A, *, \Rightarrow, 0, 1)$ which is a (bounded, integral, commutative) residuated lattice satisfying the contraction equation

$$x * x = x$$

and pre-linearity equation

$$(x \Rightarrow y) \vee (y \Rightarrow x) = 1,$$

where $x \vee y = ((x \Rightarrow y) \Rightarrow y) * ((y \Rightarrow x) \Rightarrow x))$. Godel algebras are locally finite, i.e. given a Godel algebra A and a finite set F of elements of A, the Godel subalgebra generated by F is finite as well.

On the other hand, Hajek also shows [24] that Godel logic can be expanded with the so-called Monteiro-Baaz's projection connective Δ. Truth-evaluations of Godel logic are extended adding the stipulation $e(\Delta\phi) = \delta(e(\phi))$ where

$$\delta(x) = \begin{cases} 1, & \text{if } x = 1 \\ 0, & \text{otherwise} \end{cases}$$

Axioms and rules for G_Δ are those of Godel logic plus the following axioms

($\Delta 1$) $\Delta\phi \vee \neg\Delta\phi$ ($\Delta 2$) $\Delta(\phi \vee \psi) \to (\Delta\phi \vee \Delta\psi)$
($\Delta 3$) $\Delta\phi \to \phi$ ($\Delta 4$) $\Delta\phi \to \Delta\Delta\phi$
($\Delta 5$) $\Delta(\phi \to \psi) \to (\Delta\phi \to \Delta\psi)$

and the Necessitation rule for Δ: from ϕ derive $\Delta\phi$. G_Δ is shown to be standard complete (i.e. wrt. the structure $[0,1]_{G_\Delta} = ([0,1], \min, \Rightarrow_G, \delta, 0, 1))$ for deductions from finite theories.

It is worth noticing that Godel logic satisfies the classical deduction theorem, $\Gamma \cup \{\phi\} \vdash_G \psi$ iff $\Gamma \vdash_G \phi \to \psi$, while the deduction theorem for G_Δ reads as follows: $\Gamma \cup \{\phi\} \vdash_{G_\Delta} \psi$ iff $\Gamma \vdash_{G_\Delta} \Delta\phi \to \psi$.

[4] Called standard Gödel algebra.

Finally, the logic G_Δ can be expanded with truth-constants $\overline{r}$ for each r belonging to a countable $C \subseteq [0,1]$, containing 0 and 1 (in particular one may take $C = [0,1] \cap Q$, the set of rational numbers in the unit real interval). Additional axioms of $G_\Delta(C)$ [17] are book-keeping axioms for the different connectives of the logics: namely, for any $r, s \in C$, the axioms $\overline{r} \wedge \overline{s} \leftrightarrow \overline{\min(r,s)}$, $\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow_G s}$, and the axioms $\Delta\overline{r} \leftrightarrow \overline{\delta(r)}$. So defined, the logic $G_\Delta(C)$ can be shown to satisfy the same deduction theorem as G_Δ and to be standard complete for deductions from finite theories wrt. the standard structure $[0,1]_{G_\Delta}$ expanded with truth constants $\overline{r}$ interpreted by their own value[5] r, for each $r \in C$. In case of expanding G_Δ with a finite set of truth-constants C, Hájek already proved that the resulting logic $G_\Delta(C)$ is strongly standard complete [24, Th. 4.2.21].

As a last remark, let us notice that the algebraic semantics for $G_\Delta(C)$ is given by the varieties of $G_\Delta(C)$-algebras, which are expansions of Godel algebras defined in the natural way, and are also locally finite.

3. Some representable classes of necessity measures over Godel algebras of fuzzy sets and their characterizations

Let $(L, \leq, 0, 1)$ be a bounded linearly ordered set. Let X be a finite set and let $F(X) = L^X$ be the set of functions $f : X \to L$, in other words, the set of L-fuzzy sets over X. $F(X)$ can be regarded as a Godel algebra equipped with the pointwise extension of the operations of the linearly ordered Godel algebra over $\underline{L}$. In the following, for each $r \in L$, we will denote by $\overline{r}$ the constant function $\overline{r}(x) = r$ for all $x \in X$.

In this section we provide characterizations of some classes of necessity measures on Godel algebras of functions of $F(X)$ which are representable in terms of possibility distributions $\pi : X \to L$ and implication functions considered in the Introduction. But we first introduce a general notion of basic necessity over particular subsets of $F(X)$ that encompasses all those representable classes, just by requiring to satisfy the corresponding extensions of the two defining properties in the Boolean case. For the sake of a simpler notation we denote by $F(X)$ both the set of L-fuzzy sets over X and the Godel algebra of functions which has as domain this set.

Definition 1. Let L be a bounded linearly ordered set and let $U \subseteq F(X)$ be closed under $\wedge$ and such that $\overline{0}, \overline{1} \in U$. A mapping $N : U \to L$ satisfying

(N1) $N(f \wedge g) = \min(N(f), N(g))$

(N2) $N(\overline{r}) = r$, for all $r \in L$ such that $\overline{r} \in U$

is called a basic necessity.

If $U \subseteq F(X)$ is a Godel subalgebra and $N : U \to L$ is a basic necessity then it is easy to check that N also satisfies the following properties:

[5] If Δ was not present in the logic, we would still have completeness but w.r.t. to algebras over [0, 1] with truth-constants not necessarily interpreted by their own values.

(i) $\min(N(f), N(\neg_G f)) = 0$
(ii) $N(f \Rightarrow_G g) \le N(f) \Rightarrow_G N(g)$

where $\neg_G f = f \Rightarrow_G \overline{0}$. The classes of necessity measures based on the Kleene-Dienes implication and the reciprocal of Godel implication have been already characterized in [3, 2], but we provide the proof below to keep the paper self-contained. We do not consider here the one based on Lukasiewicz implication, which was addressed in [19]. Notice that, for these characterizations to work one has to consider necessities over the full set of functions $F(X)$.

For each $x \in X$, let us denote by x its characteristic function, i.e. the function in $F(X)$ such that $x(y) = 1$, if $y = x$ and $x(y) = 0$, otherwise. Observe that each $f \in F(X)$ can be written as

$$f = \bigwedge_{x \in X} x \Rightarrow \overline{f(x)},$$

where $\Rightarrow$ is an implication function (as the ones introduced in Section 1) such that $1 \Rightarrow u = u$ for any $u \in L$. Therefore, if N is a basic necessity on $F(X)$, by (N1) we have

$$N(f) = \inf_{x \in X} N(x \Rightarrow \overline{f(x)}).$$

Using this then we have the following characterizations.

Proposition 2 (cf [3, 2]). Let $N : F(X) \to L$ be a basic necessity over $F(X)$, where $\{0, 1\} \subseteq L \subseteq [0, 1]$ is closed by the standard negation $n(x) = 1 - x$. Then we have:

(i) N satisfies the following property for all $f \in F(X)$ and $r \in L$:

(N$_{KD}$) $\quad N(\overline{r} \Rightarrow_{KD} f) = r \Rightarrow_{KD} N(f)$

if and only if there exists a normalized possibility distribution $\pi : X \to L$ such that, for all $f \in F(X)$, $N(f) = \inf_{x \in X}(\pi(x) \Rightarrow_{KD} f(x))$;

(ii) N satisfies the following property for all $f \in F(X)$ and $r \in L$:

(N$_{RG}$) $\quad N(\overline{r} \Rightarrow_{KD} f) = r \Rightarrow_G N(f)$

if and only if there exists a normalized possibility distribution $\pi : X \to L$ such that, for all $f \in F(X)$, $N(f) = \inf_{x \in X}(\pi(x) \Rightarrow_{RG} f(x))$.

Proof: (i) Assume N is defined as $N(f) = \inf_{x \in X}(\pi(x) \Rightarrow_{KD} f(x))$. Then, since $1 - r = 1 - r$, $N(r \Rightarrow_{KD} f) = \inf_{x \in X}((1 - \pi(x)) \vee ((1 - r) \vee f(x)) = (1 - r) \vee \inf_{x \in X}((1 - \pi(x)) \vee f(x)) = r \Rightarrow_{KD} N(f)$. As for the other direction, assume N satifies (N$_{KD}$). From the above remarks, for each $f \in F(X)$ we have $N(f) = \inf_{x \in X} N(x \Rightarrow_{KD} f(x)) = \inf_{x \in X} N(1 - f(x) \Rightarrow_{KD} (1 - x))$. Now, defining $\pi(x) = 1 - N(1 - x)$ and applying (N$_{KD}$), we get $N(f) = \inf_{x \in X}(1 - f(x)) \Rightarrow_{KD} (1 - \pi(x)) = \inf_{x \in X} \pi(x) \Rightarrow_{KD} f(x)$.

(ii) Assume N is defined as $N(f) = \inf_{x\in X}(\pi(x) \Rightarrow_{RG} f(x))$. We use the fact that, for any $u, v, t \in [0, 1]$, it is easy to check that the identity $u \Rightarrow_{RG} (v \vee t) = (1 - t) \Rightarrow_G (u \Rightarrow_{RG} v)$ holds. Then, again since $1 - r = 1 - r$, we have $N(r \Rightarrow_{KD} f) = \inf_{x\in X}(\pi(x) \Rightarrow_{RG} (r \Rightarrow_{KD} f)(x)) = \inf_{x\in X}(\pi(x) \Rightarrow_{RG} (1 - r \vee f(x))) = r \Rightarrow_G (\inf_{x\in X} \pi(x) \Rightarrow_{RG} f(x)) = r \Rightarrow_G N(f)$. Conversely, assume N satisfies (N_{RG}). Then, reasoning as above and defining $\pi(x) = 1 - N(1 - x)$, we have $N(f) = \inf_{x\in X} N(x \Rightarrow_{KD} f(x)) = \inf_{x\in X} N(1 - f(x) \Rightarrow_{KD} (1 - x)) = \inf_{x\in X}(1 - f(x)) \Rightarrow_G N(1 - x) = \inf_{x\in X} \pi(x) \Rightarrow_{RG} f(x)$. ✷

The characterization of the necessity measures based on the Godel implication function is a bit more involved since it requires the presence of an associated class of possibility measures which are not dual in the usual strong sense.

Definition 3. Let $U \subseteq F(X)$ be a Gödel subalgebra. A mapping $\Pi : U \to L$ satisfying

(Π1) $\Pi(f \vee g) = \max(\Pi(f), \Pi(g))$
(Π2) $\Pi(r) = r$, for all $r \in L$ such that $r \in U$

is called a basic possibility.

Note that if $\Pi : U \subseteq F(X) \to L$ is a basic possibility then it also satisfies $\max(\Pi(\neg_G f), \Pi(\neg_G \neg_G f)) = 1$. Now observe that each $f \in F(X)$ can be written as

$$f = \bigvee_{x\in X} x \wedge \overline{f(x)}.$$

and hence, since at the beginning of this section we have assumed that X is finite, if Π is a basic possibility on $F(X)$, by (Π1) we have

$$\Pi(f) = \sup_{x\in X} \Pi(x \wedge \overline{f(x)}).$$

Proposition 4. Let $\Pi : F(X) \to L$ be a basic possibility over $F(X)$. Π further satisfies

(Π3) $\Pi(f \wedge \bar{r}) = \min(\Pi(f), r)$, for all $r \in L$

iff there exists a normalized $\pi : X \to L$ and, for all $f \in F(X)$, $\Pi(f) = \sup_{x\in X} \min(\pi(x), f(x))$.

Proof: One direction is easy. Conversely, assume that $\Pi : F(X) \to L$ satisfies (Π1) and (Π3). Then, taking into account the above observations, we have

$$\Pi(f) = \sup_{x\in X} \Pi(x \wedge \overline{f(x)}) = \sup_{x\in X} \min(\Pi(x), f(x)).$$

Hence, the claim easily follows by defining $\pi(x) = \Pi(x)$. ✷

Proposition 5. Let $N : F(X) \to L$ be a basic necessity and $\Pi : F(X) \to L$ be a basic possibility satisfying (Π3). N and Π further satisfy

(N Π) $\quad N(f \Rightarrow_G \bar{r}) = \Pi(f) \Rightarrow_G r$, for all $r \in L$

iff there exists a normalized $\pi : X \rightarrow L$ and, for all $f \in F(X)$,

$$N(f) = \inf_{x \in X} \pi(x) \Rightarrow_G f(x) \text{ and } \Pi(f) = \sup_{x \in X} \min(\pi(x), f(x)).$$

Proof: Given a possibility distribution π on X, define $N(f) = \inf_{x\in X} \pi(x) \Rightarrow_G f(x)$ and $\Pi(f) = \sup_{x\in X} \min(\pi(x), f(x))$, for each $f \in F(X)$. It is clear that so defined N and Π are a basic necessity and a basic possibility respectively. But we also have $N(f \Rightarrow_G \bar{r}) = \inf_{x\in X}(\pi(x) \Rightarrow_G (f(x) \Rightarrow_G r)) = \inf_{x\in X}((\pi(x) \wedge f(x)) \Rightarrow_G r) = (\sup_{x\in X}\pi(x) \wedge f(x)) \Rightarrow_G r = \Pi(f) \Rightarrow_G r$. Hence, Π and N satisfy (N Π).

Conversely, suppose that N and Π are a basic necessity and possibility satisfying (N Π). Then, defining $\pi : X \rightarrow L$ by $\pi(x) = \Pi(x)$ for each $x \in X$, we have $N(f) = \inf_{x\in X} N(x \Rightarrow_G f(x)) = \inf_{x\in X} \Pi(x) \Rightarrow_G f(x) = \inf_{x\in X} \pi(x) \Rightarrow_G f(x)$. Moreover, if Π also satisfies (Π3), by Proposition 4, we have $\Pi(f) = \sup_{x\in X} \min(\pi(x), f(x))$.

✷

4. The Logic of Basic Necessity $NG^0(C)$

In this and the following three sections we define several logics to capture reasoning about the necessity of fuzzy events, understood as equivalence classes of formulas of Godel logic with a finite set of truth constants C. Following previous approaches like [24], the idea is to introduce a modal operator N such that, for every formula ϕ of the logic $G_\Delta(C)$, $N\phi$ is fuzzy proposition which reads "ϕ is certain" whose truth-value is to be interpreted as the necessity degree of ϕ. The logics we will define next are built over a two-level language: non-modal formulas to describe and reason about the events and modal formulas (without nesting of the operators N) to represent and reason about the certainty or necessity of the non-modal formulas. Since possibilistic models are basically qualitative models of uncertainty (the comparative ordering is what matters rather than absolute degrees), we choose again Godel logic (with a finite set of truth-constants) as fuzzy logic to reason about the modal $N\phi$ formulas. Due to the semantics of the logic $G_\Delta(C)$ one can express that the necessity of ϕ is at least r or at most s (with $r, s \in C$) by the formulas $\bar{r} \rightarrow N\phi$ and $N\phi \rightarrow \bar{s}$, while a formula $N\phi \rightarrow N\psi$ expresses the qualitative comparative statement "ψ is at least as certain as ϕ".

The particular semantics of the necessity operators N will vary from one section to the other. In this section we axiomatize the operators N whose semantics is given by the class of basic necessity measures, while in the following sections we will successively consider the logics for the operators N whose semantics are given by the three classes of representable measures studied in Section 3. As shown there, the basic differences between all these logics will be the way N handle truth-constants, this is a main reason to include a set C of truth-constants in the languages of both modal and non-modal formulas of our logics. A second reason is to be able to express statements of the kind "the necessity of ϕ is at least r", as explained above.

4.1. Syntax of $NG^0(C)$

Let us fix a set of propositional variables V and a countable set $C \subseteq [0, 1] \cap Q$ containing 0 and 1.

The language of $NG^0(C)$ consists of two classes of formulas:

(i) the set Fm(V, C) of non-modal formulas $\phi, \psi \ldots$, which are formulas of $G_\Delta(C)$ (Godel logic G expanded with the Monteiro-Baaz's projection connective Δ and truth constants $\overline{r}$ for each rational $r \in C \subset [0, 1]$) built from the set of propositional variables $V = \{p_1, p_2, \ldots\}$;

(ii) and the set MFm(V, C) of modal formulas $\Phi, \Psi \ldots$, built from atomic modal formulas $N\phi$, with $\phi \in Fm(V, C)$, where N denotes the_modality necessity, using the connectives from G_Δ and truth constants r for each rational $r \in C \subset [0, 1]$. Notice that nested modalities are not allowed. By "modal theory" we will simply mean a set of modal formulas.

The axioms of the logic $NG^0(C)$ of basic necessity are the axioms of $G_\Delta(C)$ for non-modal and modal formulas plus the following necessity related modal axioms:

(N1) $N(\phi \to \psi) \to (N\phi \to N\psi)$

(N2) $N(r) \leftrightarrow r$, for each $r \in C \subseteq [0, 1] \cap Q$.

The rules of inference of $NG^0(C)$ are modus ponens (for modal and non-modal formulas) and necessitation for non-modal formulas: if ϕ is a theorem of $G_\Delta(C)$ then $N\phi$ is a theorem of $NG^0(C)$. These axioms and rules define a notion of proof, denoted $\vdash_{NG^0(C)}$, in the usual way. It is worth noting that $NG^0(C)$ proves the formula $N(\phi \wedge \psi) \leftrightarrow (N\phi \wedge N\psi)$, which encodes a characteristic property of necess ity measures.

In the particular case of no additional truth-constants in the language other than $\overline{1}$ and $\overline{0}$, i.e. when $C = \{0, 1\}$, there is no need to consider the expansion with the Δ operator, and hence one should consider the logic G instead of the logic $G_\Delta(C)$.

4.2. Semantics of $NG^0(C)$

For the semantics of $NG^0(C)$ we consider several classes of possibilistic Kripke models.

A C-basic necessity Kripke model is a system $M = \langle W, e, I \rangle$ where:

- W is a non-empty set whose elements are called nodes or worlds,
- $e : W \times V \to [0, 1]$ is such that, for each $w \in W$, $e(w, \cdot) : V \to [0, 1]$ is an evaluation of propositional variables which is extended to a $G_\Delta(C)$-evaluation of non-modal formulas of Fm(V, C) in the usual way.

- For each $\phi \in Fm(V, C)$ we define its associated function $\hat{\phi} : W \to [0,1]$, where $\hat{\phi}(w) = e(w, \phi)$. Let $\widehat{Fm(V, C)} = \{\hat{\phi} \mid \phi \in Fm(V, C)\}$

- $I : \widehat{Fm(V, C)} \to [0,1]$ is a basic necessity over $\widehat{Fm(V, C)}$ (as a G-algebra), i.e. it satisfies

  (i) $I(\hat{\bar{r}}) = r$, for all $r \in C \subseteq [0,1] \cap Q$

  (ii) $I(\hat{\phi}_1 \wedge \hat{\phi}_2) = \min(I(\hat{\phi}_1), I(\hat{\phi}_2))$.

Now, given a modal formula Φ, the truth value of Φ in $M = \langle W, e, I\rangle$, denoted $\|\Phi\|_M$, is inductively defined as follows:

- If Φ is an atomic modal formula $N\phi$, then $\|N\phi\|_M = I(\hat{\phi})$

- If Φ is a non-atomic modal formula, then its truth value is computed by evaluating its atomic modal subformulas, and then by using the truth functions associated to the $G_\Delta(C)$-connectives occurring in Φ.

We will denote by $N(C)$ the class of C-basic necessity Kripke models.

4.3. Completeness of $NG^0(C)$

Taking into account that $G_\Delta(C)$-algebras are locally finite, following the same approach of [19] with the necessary modifications, one can prove the following result.

Theorem 6. Let $C \subseteq [0,1] \cap Q$ be finite. $NG^0(C)$ is sound and complete for modal theories wrt. the class $N(C)$ of C-basic necessity Kripke models.

Proof: Let Γ and Φ be a modal theory and a modal formula respectively, and asssume $\Gamma \not\vdash_{NG^0(C)} \Phi$. We will prove that there exists a basic necessity Kripke model M satisfying Γ but not Φ. We follow the strategy adopted in [24, 18] that amounts to translating theories over $NG^0(C)$ into theories over $G_\Delta(C)$. For each modal formula Ψ, let us denote by Ψ^* the corresponding $G_\Delta(C)$-formula obtained from Ψ by considering any atomic subformula of the form $N\phi$ as a new propositional variable. Define, therefore, $\Gamma^* = \{\Psi^* \mid \Psi \in \Gamma\}$ and

$$Ax^* = \{Y^* \mid Y \text{ is an instance of axiom (Ni), } i = 1,2\} \cup \{(N\phi)^* \mid \vdash_{G_\Delta(C)} \phi\}.$$

Using the same technique used in [24] it is not difficult to prove that

$$\Gamma \not\vdash_{NG^0(C)} \Phi \text{ iff } \Gamma^* \cup Ax^* \not\vdash_{G_\Delta(C)} \Phi^*. \quad (1)$$

Now, since $G_\Delta(C)$ is strongly complete for C finite [24, Theorem 4.2.21], we know that there exists a $G_\Delta(C)$-evaluation v that is a model of $\Gamma^* \cup Ax^*$ such that $v(\Phi^*) < 1$. We define then the following Kripke model: $M = (W, e, I)$, where W is the set of G-evaluations of propositional variables V, $e : W \times V \to [0,1]$ defined by

- $e(w, p_i) = w(p_i)$, for all $w \in W$ and $p_i \in V$, and extended to $G_\Delta(C)$-formulas as usual, and $I : \widehat{Fm(V, C)} \to [0, 1]$ defined by

- $I(\hat{\phi}) = v((N\,\phi)^?)$.

It is fairly easy to see that M is indeed a Kripke model equipped with a basic necessity I, since:

(i) $I(\hat{\phi} \wedge \hat{\psi}) = I(\widehat{\phi \wedge \psi}) = v((N\,(\phi \wedge \psi))^?) = v((N\,\phi)^? \wedge (N\,\psi)^?) = \min(v((N\phi)^?), v((N\,\psi)^?)) = \min(I(\hat{\phi}), I(\hat{\psi}))$.

(ii) $I(\hat{\bar{r}}) = v((N\,\bar{r})^?) = v(\bar{r}) = r$, for each $r \in C$.

Notice that the only constant functions in $\widehat{Fm(V, C)}$ are those of the form $\hat{\bar{r}}$ for $r \in C$. And that while M is a model for Γ, it holds that $||\Phi||_M < 1$. ✷

Actually, when the set of propositional variables V is assumed to be finite, by suitably modifying the above proof one can prove that $NG^0(C)$ is not only complete wrt. the class $N(C)$ of basic necessity Kripke models but also wrt. the subclass of finite structures (and hence also wrt. the subclass of rational-valued structures).

Corollary 7. Let V and C be finite. Then $NG^0(C)$ is complete for finite modal theories wrt. the class of finite C-basic necessity Kripke models.

Proof: Assume $\Gamma \not\vdash_{NG^0} \Phi$, with Γ finite, and proceed as above until there exists a $G_\Delta(C)$-evaluation v that is a model of $\Gamma^? \cup Ax^?$ such that $v(\Phi^?) < 1$. Since the variety of Godel algebras is locally finite, the set of equivalence classes of formulas of Fm(V, C) modulo provable equivalence is finite as well. Hence we may assume that $Ax^?$ is finite as well. Moreover, since the set of tautologies of G is the intersection of the tautologies of the n-valued Godel logics G_n , the above implies that there exists a sufficiently large $n \in N$ such that $\Phi^?$ does not follow from $\Gamma^? \cup Ax^?$ over $G_{n,\Delta}(C)$, the finitely-valued Godel logic G_n, with truth values $C_n = \{0, 1/n, \ldots, 1\}$, expanded with Δ and truth constants from $C \subseteq C_n$. Therefore, we have that $\Gamma^? \cup Ax^? \not\models_{G_{n,\Delta}(C)} \Phi^?$, i.e. there exists a C_n-valued Godel assignment v^0 such that $v^0(\Psi^?) = 1$ for all $\Psi^? \in \Gamma^? \cup Ax^?$ and $v^0(\Phi^?) < 1$.

We define the following Kripke model: $M = (W_n, e, I)$, where W_n is the finite set of G_n-evaluations of propositional variables V, $e : W_n \times V \to C_n$ defined by

- $e(w, p) = w(p)$, for all $w \in W_n$ and $p \in V$

and extended to $G_{n,\Delta}(C_n)$ formulas as usual using $G_{n,\Delta}(C)$ logic connectives, and $I : \widehat{Fm(V, C)} \to C_n$ is defined by

- $I(\hat{\phi}) = v^0((N\,\phi)^?)$.

In particular, since v^0 is a model of the translation of the (N2) axiom, for each $r \in C$ we have $I(\hat{\overline{r}}) = v^0((N\,\overline{r})^\star) = v^0(\overline{r}) = r$, for each $r \in C$. It is easy to check that, so defined, I is a basic necessity, and hence M is indeed a finite basic necessity Kripke model. Notice also that $I(\hat{\phi}) = v^0((N\,\phi)^\star)$ if $\phi \in Fm(V, C)$. Finally, one can check that M is a model for Γ and that $\|\Phi\|_M < 1$. ✷

Let us say that a modal formula Φ is 1-satisfiable if there exists a necessity Kripke structure $M = \langle W, e, I\rangle$ such that $\|\Phi\|_M = 1$. As a corollary of the model construction in the above proof, we have the following result.

Corollary 8. Let Φ be a modal formula of MFm(V, C) with V and C finite. If Φ is 1-satisfiable then it is satisfiable in a finite model, that is, there exists $M = \langle W, e, I\rangle$ with W being finite such that $\|\Phi\|_M = 1$.

Proof: The condition of Φ being 1-satisfiable is equivalent to the condition that $\neg\Delta\Phi$ is not a valid formula in the the class $N(C)$ of basic necessity Kripke models, and then, following the proof of the completeness result in Corollary 7, one can build a finite model M where $\|\neg\Delta\Phi\|_M < 1$, i.e. where $\|\Phi\|_M = 1$. ✷

In next sections we consider extensions of $NG^0(C)$ which faithfully capture the three different notions of necessity measures introduced before.

5. The Kleene-Dienes implication-based necessity logic $NG_{KD}(C)$

In this section we assume C to be closed by the standard negation, i.e. if $r \in C$, then $1 - r \in C$ as well. We start by considering the following additional axiom:

$$(N_{KD}) \quad N(\overline{r} \vee \phi) \leftrightarrow (\overline{r} \vee N\phi), \quad \text{for each } r \in C$$

Let $NG_{KD}(C)$ be the axiomatic extension of $NG^0(C)$ with the axiom (N_{KD}). The aim is to show completeness of $NG_{KD}(C)$ with respect to the subclass $N_{KD}(C)$ of $N(C)$ structures $M = (W, e, I)$ such that for every $\phi \in Fm(V, C)$, $I(\hat{\phi}) = \inf_{w \in W} \pi(w) \Rightarrow_{KD} \hat{\phi}(w)$ for some possibility distribution $\pi : W \to [0, 1]$ on the set of possible worlds W.

Lemma 9. Let $C \subseteq C^0$. Then $NG_{KD}(C^0)$ is a conservative extension of $NG_{KD}(C)$.

Proof: Let Γ and Φ be a modal theory and a modal formula respectively in the language of $NG_{KD}(C)$, i.e. $\Gamma \cup \{\Phi\} \subseteq MFm(V, C)$, such that $\Gamma \models_{NG_{KD}(C^0)} \Phi$. Reasoning as in Theorem 6, $\Gamma \vdash_{NG_{KD}(C^0)} \Phi$ iff $\Gamma^\star \cup Ax^\star \vdash_{G_\Delta(C^0)} \Phi^\star$, where Ax is the set of instances of the axioms using only constants from C. Now, the lemma follows using the fact that $G_\Delta(C^0)$ is a conservative extension of $G_\Delta(C)$. ✷

Theorem 10. Let $\Gamma \cup \{\Phi\}$ be a subset of MFm(V, C). Then:

(i) If $\Gamma \vdash_{NG_{KD}(C)} \Phi$ then $\Gamma \models_{NG_{KD}(C)} \Phi$;

(ii) Let V and C be finite. There exists a sufficiently large finite $C^0 \supseteq C$ such that, if $\Gamma \not\vdash_{NG_{KD}(C^0)} \Phi$, then $\Gamma \not\models_{NG_{KD}(C^0)} \Phi$.

Proof: Assume $\Gamma \not\vdash_{NG_{KD}(C)} \Phi$ for some finite theory Γ. Notice that, due to Lemma 9, this means $\Gamma \not\vdash_{NG_{KD}(C^0)} \Phi$ for each $C^0 \supseteq C$. The proof begins exactly as the proof of Corollary 7 for $NG^0(C)$. Let $n \in N$ be the natural number found there such that $\Phi^?$ does not follow from $\Gamma^? \cup Ax^?$ over $G_{n,\Delta}(C)$. If $C_n \subseteq C$ then there exists $n^0 \geq n$ such that $C_{n^0} = C$, and we take $C^0 = C_{n^0} = C$. Otherwise, if $C_n \supset C$ then take $n^0 = n$ and $C^0 = C_n$. Let v be a $G_{n^0,\Delta}(C^0)$-evaluation such that $v(\Psi^*) = 1$ for each $\Psi \in \Gamma^? \cup Ax^?$, and $v(\Phi^*) < 1$.

Now consider the model $M = (W, e, I)$ where W is the finite set of C^0-valued evaluations of propositional variables of V, $e : W \times V \to C^0$ is defined as $e(w, p) = w(p)$ and $e(w, \cdot)$ is extended to $G_{n^0,\Delta}(C^0)$-evaluations to formulas of $Fm(V, C^0)$ as usual. Finally, the necessity $I : \widehat{Fm(V, C^0)} \to C^0$ is defined as $I(\phi) = v((N \phi)^?)$. Notice that now $\widehat{Fm(V, C^0)}$ is the set of all functions C^W, including all constant functions $\mathbf{r}$ for each $r \in C^0$. But, since v is a model of $Ax^?$ containing all necessary instances of the translation of (N_{KD}), I is easily seen to satisfy also $I((\overline{1 - r} \vee \phi)^b) = I(\overline{1 - r} \vee \phi) = \max(1 - r, I(\phi))$, for every $r \in C^0$. Therefore, by (i) of Lemma 2, there exists $\pi : W \to C^0$ such that $I(\phi) = \min_{w \in W} \max(1 - \pi(w), w(\phi))$. Hence $M = (W, e, I) \in NG_{KD}(C)$ is a model such that $\|\Psi\|_M = 1$ for each $\Psi \in \Gamma$ and $\|\Phi\|_M < 1$. ✷

6. The reciprocal of Godel implication-based necessity logic $NG_{RG}(C)$

To capture RG-necessities, let us assume that C is closed under the standard negation, i.e. if $r \in C$, then $1 - r \in C$ as well. We define $NG_{RG}(C)$ as the axiomatic extension of $NG^0(G)$ over $G_\Delta(C)$ with the following axiom:

$$(N_{RG}) \quad N(\phi \vee \bar{r}) \leftrightarrow (\overline{1 - r} \to N\phi), \qquad \text{for each } r \in C$$

and define $N_{RG}(C)$ as the subclass of $N(C)$ structures $M = (W, e, I)$ such that for every $\phi \in Fm(V, C)$, $I(\phi) = \inf_{w \in W} \pi(w) \Rightarrow_{RG} \phi(w)$ for some possibility distribution $\pi : W \to [0, 1]$ on the set of possible worlds W.

Then, using again Lemma 2 and an analog of Lemma 9 for $NG_{RG}(C)$, one can prove the following result arguing as in the proof of Theorem 10.

Theorem 11. Let $\Gamma \cup \{\Phi\}$ be a subset of $MFm(V, C)$. Then:

(i) If $\Gamma \vdash_{NG_{RG}(C)} \Phi$ then $\Gamma \models_{NG_{RG}(C)} \Phi$;

(ii) Let V and C be finite. There exists a sufficiently large finite $C^0 \supseteq C$ closed by the standard negation such that, if $\Gamma \not\vdash_{NG_{RG}(C^0)} \Phi$, then $\Gamma \not\models_{NG_{RG}(C^0)} \Phi$.

It is worth pointing out that if we add the Boolean axiom $\phi \vee \neg\phi$ to the logics $N_{KD}(C)$ and $N_{RG}(C)$, both extensions would basically collapse into the classical possibilistic logic.

7. The Godel implication-based necessity logic $N\Pi_G(C)$

Finally, to define a logic capturing N_G-necessities, we need to expand the language of $NG^0(C)$ with an additional operator Π to capture the associated possibility measures according to Proposition 5. Therefore we consider the extended set $MFm(V, C)^+$ of modal formulas $\Phi, \Psi \ldots$ as those built from atomic modal formulas $N\phi$ and $\Pi\phi$, with $\phi \in Fm(V, C)$, truth-constants $\bar{r}$ for each $r \in C \subseteq [0, 1] \cap Q$ and G_Δ connectives. Then the axioms of the logic $N\Pi_G(C)$ are those of $G_\Delta(C)$ for non-modal and modal formulas, plus the following necessity related modal axioms:

(N1) $N(\phi \to \psi) \to (N\phi \to N\psi)$
(N2) $N(\bar{r}) \leftrightarrow \bar{r}$,
(Π1) $\Pi(\phi \vee \psi) \leftrightarrow (\Pi\phi \vee \Pi\psi)$
(Π2) $\Pi(\bar{r}) \leftrightarrow \bar{r}$,
(Π3) $\Pi(\phi \wedge \bar{r}) \leftrightarrow (\Pi\phi \wedge \bar{r})$
(NΠ) $N(\phi \to \bar{r}) \leftrightarrow (\Pi\phi \to \bar{r})$

where (N2), (Π2), (Π3) and (NΠ) hold for each $r \in C$. Inference rules of $N\Pi_G(C)$ are those of $G_\Delta(C)$ and necessitation for N and Π.

Now, we also need to consider expanded Kripke structures of the form $M = \langle W, e, I, P \rangle$, where W and e are as above and the mappings $I, P :\to [0, 1]$ are such that, for every $\phi \in Fm(V, C)$, $I(\phi) = \inf_{w \in W} \pi(w) \Rightarrow_G \phi(w)$ and $P(\phi) = \sup_{w \in W} \min(\pi(w), \phi(w))$, for some possibility distribution $\pi : W \to [0, 1]$. Call $NP_G(C)$ the class for such structures. Then, using Proposition 5 we get the following result.

Theorem 12.
(i) $N\Pi_G(C)$ is sound wrt. the class $NP_G(C)$ of structures.
(ii) Let V be finite and let $\Gamma \cup \{\Phi\}$ be a finite subset of $MFm(V, C)$. There exists a sufficiently large finite $C^0 \supseteq C$ such that such that if $\Gamma \not\vdash_{N\Pi_G(C^0)} \Phi$ then
$\Gamma \not\models_{N\Pi_G(C^0)} \Phi$.

Proof: Assume $\Gamma \not\vdash_{N\Pi_G(C)} \Phi$. Again, the proof proceeds very similar to the pre- vious ones, but now the model that is built $M = (W, e, I, P)$ has been expanded with a mapping $P : Fm(V, C^0) \to C^0$ defined by

- $P(\phi) = v((\Pi\phi)^?)$.

Then, it is easy to check that so defined P is a basic possibility (see Definition 3). Moreover, since $Ax^?$ contains the necessary instances of axioms (N1), (N2), (Π1), (Π2), (Π3), and (NΠ), the mappings I and P satisfy the conditions of Proposition 5, and hence there exists $\pi : W \to C^0$ such that $I(\phi) = \min_{w \in W} \pi(w) \Rightarrow_G w(\phi)$ and $P(\phi) = \max_{w \in W} \min(\pi(w), w(\phi))$. Therefore $M = (W_n, e, I, P)$ is a $NP_G(C^0)$-model satisfying all formulas in Γ but $\|\Phi\|_M < 1$. ✷

The main features of the four logics defined so far are summarized in Table 1.

Table 1: Logics capturing different notions of necessity measures.

Logic	Axioms	Semantics $M = (W, e, I)$	Implication Function
$NG^0(C)$	$N(\phi \to \psi) \to (N\phi \to N\psi)$ $N(\bar{r}) \leftrightarrow \bar{r}$	$I(\hat{r}) = r$ $I(\hat{\phi} \wedge \hat{\psi}) = \min(I(\hat{\phi}), I(\hat{\psi}))$	
$NG_{KD}(C)$	$N(\bar{r} \vee \phi) \leftrightarrow (\bar{r} \vee N\phi)$,	$\pi : W \to [0,1]$ $I(\hat{\phi}) = \inf_{w \in W} \pi(w) \Rightarrow_{KD} \hat{\phi}(w)$	Kleene-Dienes $u \Rightarrow_{KD} v = \max(1-u, v)$
$NG_{RG}(C)$	$N(\phi \vee \bar{r}) \leftrightarrow (\overline{1-r} \to N\phi)$,	$\pi : W \to [0,1]$ $I(\hat{\phi}) = \inf_{w \in W} \pi(w) \Rightarrow_{RG} \hat{\phi}(w)$	Reciprocal of Gödel $u \Rightarrow_{RG} v = \begin{cases} 1, \text{if } u \leq v \\ 1-u, \text{other.} \end{cases}$
$N\Pi_G(C)$	$N(\phi \to \bar{r}) \leftrightarrow (\Pi\phi \to \bar{r})$ $\Pi(\bar{r}) \leftrightarrow \bar{r}$ $\Pi(\phi \vee \psi) \leftrightarrow (\Pi\phi \vee \Pi\psi)$ $\Pi(\phi \wedge \bar{r}) \leftrightarrow (\Pi\phi \wedge \bar{r})$	$M = (W, e, I, P)$, $\pi : W \to [0,1]$ $I(\hat{\phi}) = \inf_{w \in W} \pi(w) \Rightarrow_G \hat{\phi}(w)$ $P(\hat{\phi}) = \sup_{w \in W} \min(\pi(w), \hat{\phi}(w))$	Gödel $u \Rightarrow_G v = \begin{cases} 1, \text{if } u \leq v \\ v, \text{other.} \end{cases}$

8. A common fragment of $NG_{KD}(C)$, $NG_{RG}(C)$ and $N\Pi_G(C)$

In this section we turn our attention to a fragment of the previous logics, that is syntactically very close to classical possibilistic logic, consisting of formulas of the kind $\bar{r} \to N\phi$, where ϕ contains neither additional truth-constants nor the Δ operator, and $r \in C \setminus \{0\}$. By analogy to classical possibilistic logic, we will simply write (ϕ, r) for $r \to N\phi$.

Let PGL be the common fragment of $NG_{KD}(C)$, $NG_{RG}(C)$ and $N\Pi_G(C)$ over the language of formulas of the kind (ϕ, r) and axiomatized by the following axioms

- $(\phi, 1)$, with ϕ being an axiom of Godel logic

and the following inference rules:

- from $(\phi, r), (\phi \to \psi, s)$ infer $(\psi, \min(r, s))$ (weighted modus ponens)
- from (ϕ, r) infer (ϕ, s), with $s \leq r$ (weight weakening)

We will denote by $\vdash_{PGL}$ the notion of deduction using the above axioms and rules. Since axiom (K) and the necessitation rule are present in all the three logics, PGL is sound with respect to the three classes of Kripke models $N_{KD}(C)$, $N_{RG}(C)$ and $N P_G(C)$. Moreover we can show that deductions in PGL and in Godel logic are very related, mimicking the relationship between deductions in (classical) possibilistic logic and CPC deductions within stratified propositional logic formulas.

In the following, for any set of PGL-formulas $T = \{(\psi_1, r_1), \ldots (\psi_m, r_m)\}$, we will denote by $T^*_s = \{\psi \mid (\psi, r) \in T \text{ and } r \geq s\}$ and by T^* the whole set $\{\psi_1, \ldots, \psi_m\}$ of the underlying set of Godel logic formulas,

Lemma 13. Let $T = \{(\psi_1, r_1), \ldots (\psi_m, r_m)\}$. Then $T \vdash_{PGL} (\phi, s)$ iff $T^*_s \vdash_G \phi$.

Proof: Easy by noticing that proofs can be easily transferred from one logic to the other and that if a formula $(\psi_i, r_i) \in T$ is used in the proof of (ϕ, s), because of the two inference rules, then necessarily $s \geq r_i$. ✷

Lemma 14. Let $L \in \{NG_{KD}(C), NG_{RG}(C), N\Pi_G(C)\}$, and let T be a finite set of PGL-formulas. If $T \models_L (\phi, r)$ for some $r > 0$, then $T^* \models_G \phi$.

Proof: Assume $T \models_L (\phi, r)$ with T^* being G-consistent, otherwise the result trivially holds. Let w be a Gödel evaluation such that w is a model of T^* and let π_w be the possibility distribution such that $\pi_w(w) = 1$ and $\pi_w(w^0) = 0$ for $w^0 = w$. Consider then the Kripke model $M_w = (W, e, I_w)$ where W is the set of G-evaluations for G-formulas, and $I_w(\hat{\psi}) = \inf_{w^0 \in W} \pi_w(w^0) \Rightarrow w^0(\psi)$, where $\Rightarrow \in \{\Rightarrow_{KD}, \Rightarrow_{RG}, \Rightarrow_G\}$.

Let $L \in \{NG_{KD}(C), N\Pi_G(C)\}$. An easy computation shows that, for any $G_\Delta(C)$-formula ψ, $\|N\psi\|_{M_w} = w(\psi)$ and hence $M_w \models (\psi, r)$ iff $w(\psi) \geq r$. Therefore, M_w is clearly a model of T. Therefore we have proved that for each G-model w of T^*, $w(\phi) \geq r$, in other words, $T^* \models_{G_\Delta(C)} \overline{r} \to \phi$, and using the deduction theorem for the logic $G_\Delta(C)$, $\models_{G_\Delta(C)} r \to (\Delta\varphi_T \to \phi)$, where φ_T is the conjunction of all the formulas of the theory T^*. Now, using results from [22][6], it holds that $\models_{G_\Delta} \Delta\varphi_T \to \phi$, i.e. $T^* \models_{G_\Delta} \phi$, and since T^* and ϕ do not contain the Δ connective, this is equivalent to $T^* \models_G \phi$.

Let $L = NG_{RG}$ $\quad$ M_w $\quad$ $N\psi\|_{M_w} = 1$ otherwise. Hence $M_w \models (\psi, r)$ with $r > 0$ iff $w(\psi) = 1$. Therefore, M_w is clearly a model of T. Therefore we have proved that for each G-model w of T^*, $w(\phi) = 1$, in other words, $T^* \models_G \phi$. ✷

The next theorem shows partial completeness results for PGL with respect to the three different representable notions of necessity measures.

Theorem 15. Let $L \in \{NG_{KD}(C), NG_{RG}(C), N\Pi_G(C)\}$. Let $T = \{(\psi_1, r_1), \ldots (\psi_m, r_m)\}$ be a finite set of PGL formulas. Then, if $T \vdash_{PGL} (\phi, r)$ then $T \models_L (\phi, r)$. Conversely, if $T \models_L (\phi, r)$ with $r > 0$, then $T \vdash_{PGL} (\phi, \min(r_1, \ldots, r_m))$.

[6] In [22] the author studies the functions from the unit hypercube $[0, 1]^n$ into $[0, 1]$ associated to formulas of Godel logic (with n being the number of variables of the formula) and from the form of these functions it follows that there is no Godel logic formula ϕ such that, given $r \in (0, 1]$, $v(\phi) \geq r$ for each valuation v. In particular this means that if, for some truth-constant $r > 0$ from C, $\overline{r} \to \phi$ is a tautology of $G_\Delta(C)$, then necessarily ϕ itself must be a tautology of Gödel logic.

Proof: One direction is soundness. As for the other direction, let $T = \{(\psi_1, r_1), \ldots (\psi_m, r_m)\}$ and assume $T \models_L (\phi, r)$. By Lemma 14, $T^* \models_G \phi$, and by completeness of Gödel logic, $T^* \vdash_G \phi$. Finally, by Lemma 13, $T \vdash_{PGL} (\phi, \min(r_1, \ldots, r_m))$.

✷

9. Some complexity issues

Let $L \in \{NG^0(C), N_{KD}(C), N_{RG}(C), N\Pi_G(C)\}$. We begin by showing that, whenever we fix the set of propositional variables, the satisfiability problem for L is decidable in polynomial time.

Theorem 16. Let Φ a formula in the L-language with n propositional variables. Then, checking the satisfiability of Φ is a problem decidable in polynomial time.

Proof: We just deal with $NG^0(C)$; the other cases are similar and left to the reader.

Following the completeness proof, satisfiability of Φ in $NG^0(C)$ is equivalent to the satisfiability of a theory in $G_\Delta(C)$ obtained as translation from Φ as follows. Let $\Phi^?$ be obtained from Φ by replacing every occurrence of an atomic subformula of the form $N\phi$ by a new propositional variable p_ϕ. Then, inductively define the mapping ? from modal formulas into $G_\Delta(C)$-formulas as follows:

- $(N(\phi))^? = p_\phi$,
- $(\bar{r})^? = \bar{r}$,
- $(\Phi \to \Psi)^? = \Phi^? \to \Psi^?$,
- $(\Delta(\Phi))^? = \Delta(\Phi^?)$

Let, therefore, Γ be the union of $\Phi^?$ and $\mathsf{F}^?$ defined as

$$\mathsf{F}^? = \{Y^? \mid Y \text{ is an instance of the axioms}\} \cup \{p_\phi \mid \vdash_{G_\Delta(C)} \phi\},$$

where all the non-modal formulas belong to $F_n(G_\Delta(C))$, the $G_\Delta(C)$-algebra of equivalence classes of formulas over n generators modulo provable equivalence. Notice that $\mathsf{F}^?$ is a finite theory over $G_\Delta(C)$ whose language has as new propositional variables $Var^0 = \{p_\phi \mid \phi \in F_n(G_\Delta(C))\}$. The cardinality of $\mathsf{F}^?$ depends only on the original number of variables n.

Now, checking the satisfiability of Φ over $NG^0(C)$ is equivalent to checking the satisfiability of $\Phi^? \cup \mathsf{F}^?$ over $G_\Delta(C)$. Since the number of variables in Var^0 is fixed once we fix n, we only need to consider a chain with $|Var^0| + 1$ elements. Then, in order to decide the computation problem for the satisfiability of $\Phi^? \cup \mathsf{F}^?$, it is enough to calculate the truth table corresponding to a formula over a $(|Var^0| + 1)$-valued G-chain, and hence it is linear in the length of the formula. ✷

Notice that the above translation works whenever the number of variables is fixed. The most interesting and general case, in which the number of variables is not specified, still is an open problem.

Finally, as for the fragment PGL, it be can easily noticed that the problem of deciding whether (ϕ, r) can be deduced from a finite theory T, i.e. whether $T \vdash_{PGL} (\phi, r)$ holds true, is equivalent to deciding whether $T_r^* \vdash_G \phi$ holds true in Godel logic, which is known to be a co-NP complete problem (see e.g. [24]).

10. Future Work

Several issues related to the logics we have introduced in this paper deserve further investigation. These logics are not proper modal logics, since the notion of well-formed formula excludes those formulas with occurrences of nested modalities. Work in progress is devoted to the study of the Possibilistic Necessity Godel logic (PNG, for short), presented in [10]. PNG is a full fuzzy modal logic for graded necessity over Godel logic, that is, a modal expansion of the [0, 1]-valued Godel logic with a modality N such that the truth-value of a formula $N\phi$ (in [0, 1]) can be interpreted as the degree of necessity of ϕ, according to some suitable semantics. The language of PNG is defined as follows: formulas of PNG are built from the set of G-formulas using G-connectives and the operator N. Axioms of PNG are those of Godel logic plus the following modal axioms:

1. $N(\phi \to \psi) \to (N\phi \to N\psi)$.
2. $N\psi \leftrightarrow NN\psi$.
3. $\neg N0$.

Deduction rules for PNG are Modus Ponens and Necessitation for N (from ψ derive $N\psi$). These axioms and rules define a notion of proof $\vdash_{PNG}$ in the usual way. It would be worth to study in depth the relation between PNG and the logic of Basic Necessity $NG^0(C)$ (and of some of its meaningful axiomatic extensions). Since the logic PNG may only capture the logic of basic necessities, additional axioms (and possibly operators as well) must be considered in order to capture more specific families of necessities, as those related to axioms (N_{KD}), (N_{RG}), ($\Pi3$) or the axiom ($N\Pi$). For instance, a candidate axiom as the PNG counterpart of (N_{KD}) is

$$N(N\phi \vee \psi) \leftrightarrow N\phi \vee N\psi$$

and a candidate counterpart of (N_{RG}) would be

$$N(N\phi \vee \psi) \leftrightarrow \sim N\phi \to N\psi$$

where ~ is an additional involutive negation, and

$$\Pi(\phi \wedge \Pi\psi) \leftrightarrow \Pi\phi \wedge \Pi\psi$$
$$N(\phi \to N\psi) \leftrightarrow \Pi\phi \to N\psi$$

would be candidate axioms for counterparts of (Π3) and (N Π) respectively.

From the algebraic study of specific logics a general theory of the algebraization of logics slowly emerged during the last century with the aim of obtaining general results relating the properties of a logic with the properties of the class of algebras (or algebra related structures) associated with it. The algebraizable logics are purported to be the logics with the strongest possible link with their natural class of algebras. A precise concept of algebraic semantics and of algebraizable logic was introduced by Blok and Pigozzi in [5]. The main point in Blok and Pigozzi's concept of algebraic semantics comes from the realization that the set of designated elements considered in the algebraic semantics of known logics is in fact the set of solutions of an equation. Then, in the study of a logic, we should find an equational way to define uniformly in every algebra a set of designated elements in order to obtain an algebraic soundness and completeness theorem. In [10] we showed that PNG is finitely algebraizable and has a strong completeness algebraic theorem. Future work will be devoted to study further the algebraic semantics of PNG, the class of NG-algebras. From an algebraic point of view, it seems interesting also to investigate faithful NG-algebras (see [10]) since they enjoy very nice properties.

As mentioned in Section 2, Godel propositional logic is an extension of the Intuitionistic logic **Int**, by the axiom of prelinearity. Hence, necessity-valued Godel logics could be regarded as extensions of the Intuitionistic Modal Logic $\mathbf{IntK}_*$, which is axiomatized by adding to **Int** axioms K, $*(p \wedge q) \leftrightarrow (*p \wedge *q)$ and $*>$ and whose deduction rules are Modus Ponens and Necessitation for $*$. The logic $\mathbf{IntK}_*$ and its extensions were introduced and investigated in different articles, among them [31], [7], [33] and [30]. Kripke-style models for intuitionistic modal logics using two accessibility relations between worlds, one of which is intuitionistic and the other one modal, were introduced in [33], and a completeness theorem for $\mathbf{IntK}_*$ using this semantics and the standard canonical model technique, can be found in [11]. In the future we would like to explore the relationship between the logics introduced in this paper and fragments of well-know many-valued modal logics in the literature (see e.g. [6]). Among them, specially relevant for our research are the Godel modal logics studied by Caicedo and Rodríguez in [9].

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