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ON THE LIMIT CYCLES OF THE POLYNOMIAL DIFFERENTIAL SYSTEMS WITH A LINEAR NODE AND HOMOGENEOUS NONLINEARITIES

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ABSTRACT. We consider the class of polynomial differential equations $\dot{x} = \lambda x + P_n(x, y)$, $\dot{y} = \mu y + Q_n(x, y)$ in $\mathbb{R}^2$ where $P_n(x, y)$ and $Q_n(x, y)$ are homogeneous polynomials of degree $n > 1$ and $\lambda \neq \mu$, i.e. the class of polynomial differential systems with a linear node with different eigenvalues and homogeneous nonlinearities. For this class of polynomial differential equations we study the existence and non–existence of limit cycles surrounding the node localized at the origin of coordinates.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

A two dimensional *polynomial differential system* in $\mathbb{R}^2$ is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where the dependent variables x and y, and the independent one (the time) t are real, and $P(x, y)$ and $Q(x, y)$ are polynomials in the variables x and y with real coefficients. The *degree* of the polynomial system is $m = \max\{\deg P, \deg Q\}$. Recall that a *limit cycle* of a system (1) is an isolated periodic solution in the set of all periodic solutions of system (1).

One of the more difficult problems in the qualitative theory of the polynomial differential equations in the plane $\mathbb{R}^2$ is the study of their limit cycles. Thus a classical problem related with these polynomial differential systems is the second part of the unsolved 16–th Hilbert problem [12], which essentially consists in finding a uniform upper bound for the maximum number of limit cycles that a planar polynomial differential system of a given degree can have, see for more details the surveys [13] and [11].

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Here we want to study the existence and non–existence of limit cycles for the class of polynomial differential systems in $\mathbb{R}^2$ of the form

$$\dot{x} = P_1(x,y) + P_n(x,y), \qquad \dot{y} = Q_1(x,y) + Q_n(x,y), \tag{2}$$

where $n > 1$, and $P_k(x,y)$ and $Q_k(x,y)$ are homogeneous polynomials of degree k.

The limit cycles of the polynomial differential systems (2) of the form

$$\dot{x} = -y + P_n(x,y), \qquad \dot{y} = x + Q_n(x,y),$$

i.e. having a focus at the origin have been intensively studied, see for instance [2, 3, 4, 6, 8, 9, 10, 14, 16]. But there are almost no results about the limit cycles of the polynomial differential systems of the form

$$\dot{x} = \lambda x + P_n(x,y), \qquad \dot{y} = \mu y + Q_n(x,y), \tag{3}$$

i.e. having a node at the origin. Recently, the polynomial differential systems (3) with $\lambda = \mu$ and $n > 1$ have been studied in [1], where it is proved that if n is even then these systems have no periodic solutions, and that if n is odd then they have at most one limit cycle, and there are examples with one limit cycle. In Proposition 6.3 and Remark 6.4 of [7] the authors provide examples of systems (3) having two, one or zero limit cycles surrounding the origin.

The objective of this paper is to study the existence and non–existence of limit cycles for the class of polynomial differential systems in $\mathbb{R}^2$ of the form (3) with $\lambda \neq \mu$ having the same sign and $n > 1$. Our methods are different from those of [1] for studying limit cycles around a star node. We can assume that $\mu < \lambda$, otherwise we interchange the names of the variables x and y.

In order to state our results we write the polynomial differential system (3) in polar coordinates (r,θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then the system becomes

$$\begin{aligned} \dot{r} &= (\lambda\cos^2\theta + \mu\sin^2\theta)r + f(\theta)r^n, \\ \dot{\theta} &= (\mu - \lambda)\cos\theta\sin\theta + g(\theta)r^{n-1}, \end{aligned} \tag{4}$$

where

$$\begin{aligned} f(\theta) &= \cos\theta P_n(\cos\theta,\sin\theta) + \sin\theta Q_n(\cos\theta,\sin\theta), \\ g(\theta) &= \cos\theta Q_n(\cos\theta,\sin\theta) - \sin\theta P_n(\cos\theta,\sin\theta). \end{aligned}$$

Note that $f(\theta)$ and $g(\theta)$ are homogeneous trigonometric polynomials of degree $n+1$ in the variables $\cos\theta$ and $\sin\theta$. For simplifying the notation we define

$$\theta_0 = 0,\ \theta_1 = \frac{\pi}{2},\ \theta_2 = \pi,\ \theta_3 = \frac{3\pi}{2},\ \theta_4 = 2\pi. \tag{5}$$

Our main results on the non–existence of limit cycles for the polynomial differential systems (3) surrounding the origin are stated in the next theorem.

Theorem 1. *Consider a polynomial differential system with homogeneous nonlinearities* (3) *with* $\mu < \lambda$, $\mu\lambda > 0$ *and* $n > 1$.

(a) *If n is even, then system* (3) *has no periodic solutions surrounding the origin.*

(b) *If $g(\theta)$ has a zero θ^* such that* $\cos\theta^* \sin\theta^* = 0$, *then system* (3) *has no limit cycles surrounding the origin.*

(c) *Let θ^* be a zero of $g(\theta)$ and θ_i be one of those defined in* (5). *If $g(\theta)\cos\theta\sin\theta > 0$ for $\theta \in (\theta_i, \theta^*)$ or (θ^*, θ_i), then system* (3) *has no limit cycles surrounding the origin.*

(d) *If $g(\theta)$ has two consecutive zeros, one in the interval (θ_{i-1}, θ_i) and the other in the interval (θ_i, θ_{i+1}) for some $i \in \{1, 2, 3\}$, then system* (3) *has no limit cycles surrounding the origin.*

Theorem 1 is proved in section 2.

The curve $\dot{\theta} = 0$ along the orbits shall play a main role in the proof of Theorem 1. From (4) this curve is given by

$$r(\theta) = \left(\frac{(\lambda - \mu)\cos\theta\sin\theta}{g(\theta)}\right)^{\frac{1}{n-1}} \quad \text{when} \quad \frac{(\lambda - \mu)\cos\theta\sin\theta}{g(\theta)} > 0. \tag{6}$$

In what follows we shall denote this curve by $r = r^*(\theta)$.

Our main results on the existence of limit cycles for the polynomial differential systems (3) surrounding the origin are stated in the following theorem.

Theorem 2. *Consider a polynomial differential system with homogeneous nonlinearities* (3) *with* $\mu < \lambda$, $\mu\lambda > 0$ *and* $n > 1$ *odd.*

(a) *If system* (3) *has a limit cycle surrounding the origin, then it cannot intersect the curve* $r = r^*(\theta)$.

(b) *If $\lambda f(\theta) < 0$ for all $\theta \in [0, 2\pi)$, and the origin is the unique singularity of system* (3), *then at least one limit cycle surrounds the origin.*

(c) *If μ is sufficiently close to λ, $g(\theta) \neq 0$ and $\lambda f(\theta) < 0$ for all $\theta \in [0, 2\pi)$, and the origin is the unique singularity of system* (3), *then the system has exactly one limit cycle surrounding the origin.*

(d) *If μ is sufficiently close to λ and $g(\theta) \neq 0$ for all $\theta \in [0, 2\pi)$, then system* (3) *has at most two limit cycles surrounding the origin.*

Theorem 2 is proved in section 3. We remark that in statements (b) and (c) we have the additional assumption: the origin is the unique equilibrium point of system (3). This condition also appeared in [1, Theorem 2(c)], but there it is not necessary because it can be obtained from $g(\theta) \neq 0$ for $\theta \in [2\pi, 0)$. But for $\lambda \neq \mu$ it is not the case as the following example shows. This example also satisfies all the conditions of statement (c).

Example. Consider the polynomial differential system

$$\begin{aligned} \dot{x} &= \lambda x + x^3 - x^2 y + x y^2 - y^3, \\ \dot{y} &= \mu y + x^3 + x^2 y + x y^2 + y^3. \end{aligned} \tag{7}$$

Write it in polar coordinates we have

$$\begin{aligned} \dot{r} &= (\lambda \cos^2\theta + \mu \sin^2\theta) r + r^3, \\ \dot{\theta} &= (\mu - \lambda)\cos\theta \sin\theta + r^2. \end{aligned} \tag{8}$$

Using the notation given in (6) we have $f(\theta) = g(\theta) = 1$. So for any given $\lambda < 0$, if $|\mu - \lambda| \ll 1$ then system (8) (consequently (7)) has the origin as a unique singularity. By statement (c) of Theorem 2 system (7) has a unique limit cycle.

Of course our results on the limit cycles for the degree n odd are partial. But to control the maximum number of limit cycles for a given class of polynomial differential systems is in general a very difficult problem specially when this class depends on many parameters as in our systems (3). For instance if the polynomials of the differential system (1) are quadratic (i.e. of degree 2), there are more than one thousand of papers published on those systems and their maximum number of limit cycles is unknown.

2. Proof of Theorem 1

Recall that

$$\theta_0 = 0,\ \theta_1 = \frac{\pi}{2},\ \theta_2 = \pi,\ \theta_3 = \frac{3\pi}{2},\ \theta_4 = 2\pi.$$

Proof of statement (*b*) *of Theorem 1.* Clearly $\theta^* = \theta_i$ for some $i = 0, 1, 2, 3$. If $\theta^* \in \{\theta_0, \theta_2\}$, then Q_n has the factor y, and so $y = 0$ is an invariant straight line. Consequently system (3) has no limit cycles around the origin. If $\theta \in \{\theta_1, \theta_3\}$, then the straight line $x = 0$ is invariant. And consequently system (3) has no limit cycles surrounding the origin. □

In what follows we consider the case that $g(\theta)$ does not vanish in θ_i for $i = 0, 1, 2, 3$.

Proof of statement (c) *of Theorem 1.* We consider the case that $g(\theta)\cos\theta\sin\theta > 0$ in the interval (θ_i, θ^*). The proof for the other interval is completely similar. Since by the assumption $\mu < \lambda$, we have that the curve $r = r^*(\theta)$ of (6) has a branch γ defined in the interval (θ_i, θ^*). This branch is at the origin at $\theta = \theta_i$ and tends to infinity when $\theta \to \theta^*$.

We can assume that in the topological sector S_1 limited by the ray $\theta = \theta_i$ and the branch γ we have $\dot\theta > 0$, and in the topological sector S_2 limited by the branch γ and the ray $\theta = \theta^*$ we have $\dot\theta < 0$; otherwise we reverse the sign of the time in the differential system (3).

Let Γ be a limit cycle surrounding the origin of system (3). We separate the proof in two cases.

Case 1: Assume that Γ crosses the branch γ. In the interior of the sector S_2 we have that $\dot\theta < 0$. Just after Γ enters in S_2 its distance to the origin decreases. If Γ does not exit the sector S_2, then either its ω–limit set is the origin (see Figure 1(a)), or it is a singularity on γ (see Figure 1(b)). If Γ exits the sector S_2 (see Figure 1(c)), then the ω–limit set of Γ must be either a singularity on γ, or a periodic orbit surrounding a singularity on γ. So Γ cannot be a limit cycle.

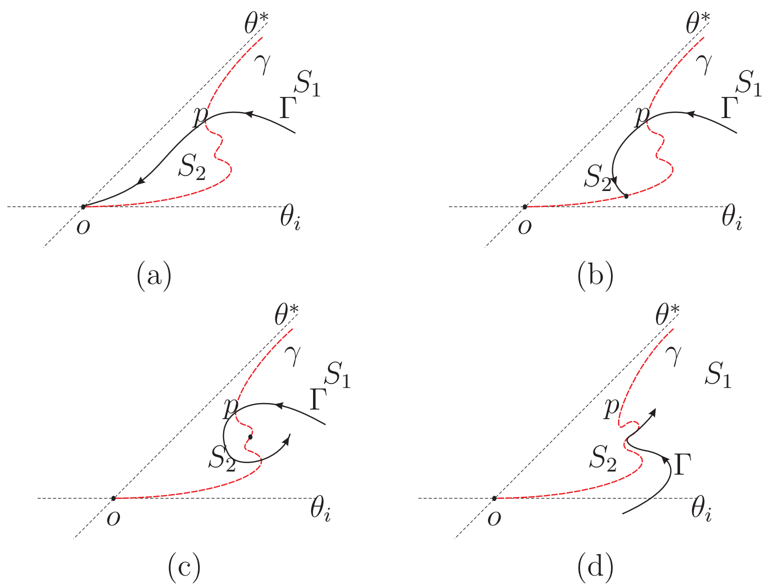


FIGURE 1. The branch having one end at the origin and the other at infinity

Case 2: Suppose that Γ is tangent to the branch γ at some points (see Figure 1(d)). In a neighborhood of a tangent regular point the orbits in the bounded component limited by Γ when enters in the sector S_2 where $\dot{\theta} < 0$, their angular coordinate θ decreases and such an orbit cannot follow the limit cycle Γ, in contradiction with the Theorem of Continuous Dependence on the Initial Conditions (see [17]). $\square$

Proof of statement (*d*) *of Theorem 1.* Let $\theta_1^* \in (\theta_{i-1}, \theta_i)$ and $\theta_2^* \in (\theta_i, \theta_{i+1})$ be two consecutive zeros of $g(\theta)$. Then we have $g(\theta)\cos\theta\sin\theta > 0$ either for $\theta \in (\theta_1^*, \theta_i)$, or for $\theta \in (\theta_i, \theta_2^*)$. So, by statement (c) of Theorem 1 system (3) cannot have limit cycles. $\square$

Lemma 3. *If $g(\theta)$ has an odd number of zeros taking into account their multiplicities in some interval (θ_i, θ_{i+1}) with $i \in \{0, 1, 2, 3\}$, then system* (3) *has no limit cycles surrounding the origin.*

Proof. Let $\theta_1^* \le \ldots \le \theta_{2l+1}^*$ be the zeros of $g(\theta)$ in (θ_i, θ_{i+1}) being l a non–negative integer. Then $g(\theta)$ has different signs in the intervals (θ_i, θ_1^*) and $(\theta_{2l+1}^*, \theta_{i+1})$. Therefore, we must have $g(\theta)\cos\theta\sin\theta > 0$ for the values of θ either in (θ_i, θ_1^*), or in $(\theta_{2l+1}^*, \theta_{i+1})$. So from statement (c) of Theorem 1 it follows that system (3) has no limit cycles around the origin. $\square$

Proof of statement (*a*) *of Theorem 1.* If n is even, $g(\theta)$ is a homogeneous trigonometric polynomial of odd degree. So $g(\theta)$ has an odd number of zeros taking into account their multiplicities. This implies that in at least one of the intervals (θ_i, θ_{i+1}), $i = 0, 1, 2, 3$, the function $g(\theta)$ has an odd number of zeros taking into account their multiplicities. Hence the proof follows Lemma 3. $\square$

3. Proof of Theorem 2

In what follows and from the proof of statement (c) of Theorem 1 we only need to consider the case that the curve $\dot{\theta} = 0$, i.e. the curve $r = r^*(\theta)$ given by (6) has all its branches having both endpoints either at the origin, or at infinity. See Figures 2 and 3. Moreover, due to statement (d) of Theorem 1 in every quadrant of the plane determined by the two axes of coordinates x and y, only can be branches of the same kind. In fact, if in a quadrant there is one branch with endpoints at the origin, then this is the unique branch of $r = r^*(\theta)$ in this quadrant.

Obviously $g(\theta) \neq 0$ on $r = r^*(\theta)$. Also if $g(\theta) \neq 0$ for all $\theta \in [0, 2\pi]$, then $r = r^*(\theta)$ has only two branches, both having their two endpoints at the origin.

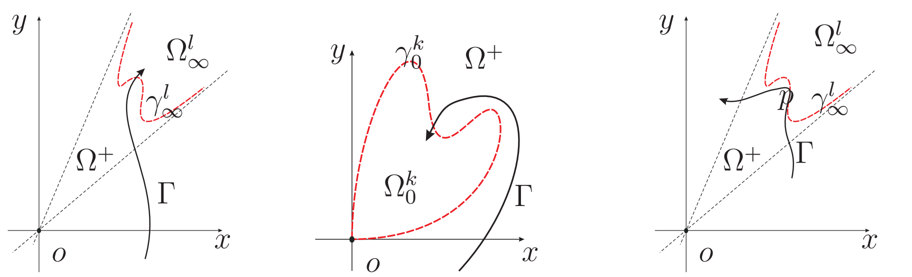
Proof of statement (a) of Theorem 2. All singularities of system (3) are on the curve $r = r^*(\theta)$, so they are located on the curve $r = r^*(\theta)$.

Since the curve $r = r^*(\theta)$ has only branches satisfying that their two endpoints are at the origin, or at the infinity. Denote by γ_0^k for $k = 1, 2$ the formers and by γ_∞^l the latters for $l = 1, \ldots, 2L$. From the comments at the beginning of this section there are only two branches γ_0^k. The number of the branches γ_∞^l is even due to the fact that if $g(\theta) = 0$ then $g(\theta + \pi) = 0$, because $g(\theta)$ is a homogeneous trigonometric polynomial in the variables $\cos\theta$ and $\sin\theta$.

We denote by Ω_0^k the closed region of the plane whose boundary is formed uniquely by the curve γ_0^k and the origin (see Figure 2(b)), and by Ω_∞^l the closed region of the plane whose boundary is formed uniquely by the curve γ_∞^l (see Figure 2(a)). Set

$$\Omega_+ = \mathbb{R}^2 \setminus \{\Omega_0^1 \cup \Omega_0^2 \cup \Omega_\infty^1 \cup \cdots \cup \Omega_\infty^{2L}\}.$$

Without loss of generality (if necessary we change the sign of the time) we assume that $\dot\theta > 0$ in Ω_+ and $\dot\theta < 0$ in $\Omega_0^1 \cup \Omega_0^2 \cup \Omega_\infty^1 \cup \cdots \cup \Omega_\infty^{2L}$.



(a) Branch γ_∞^l (b) Branch γ_0^k (c) Orbit tangent to γ_∞^l

FIGURE 2. The branches with their two ends either both at infinity or both at the origin

Let Γ be a limit cycle of system (3). Then, from the shapes of γ_0^k and γ_∞^l it follows that since Γ surrounds the origin then it cannot be completely contained in Ω_0^k or Ω_∞^l.

We separate the rest of the proof in two cases.

Case 1: Assume that Γ crosses the curve $\dot\theta = 0$. First we suppose that it crosses the branch γ_∞^l of the curve $r = r^*(\theta)$, see Figure 2(a). In the interior of the region Ω_∞^l we have that $\dot\theta < 0$. Just after Γ enters in the region Ω_∞^l its distance to the origin increases. If Γ does not exit the region Ω_∞^l, then its ω–limit set is either a singularity in γ_∞^l, or it is at infinity. If Γ exits the region Ω_∞^l, then the ω–limit set of Γ must be either a singularity on γ^l_∞, or a periodic orbit surrounding a singularity on γ^l_∞. So Γ cannot be a limit cycle.

Suppose that Γ crosses the curve γ^k_0, see Figure 2(b). Arguments completely similar to the ones of the previous paragraph show that if Γ crosses the curve γ^k_0, then it cannot be a limit cycle.

Case 2: Assume now that the limit cycle Γ is tangent to the curve γ^l_∞ in some points, see Figure 2(c). In a neighborhood of a tangent regular point the orbits in the unbounded component limited by Γ when enters in the region where $\dot\theta < 0$, their angular coordinate θ decreases and such an orbit cannot follow the limit cycle Γ, in contradiction with the Theorem of Continuous Dependence on the Initial Conditions.

If the limit cycle Γ is tangent to the curve γ^k_0, then similar arguments show that this is not possible. In short the statement (a) is proved. $\square$

Proof of statement (*b*) *of Theorem 2.* Since $\lambda f(\theta) < 0$ for all $\theta \in [0, 2\pi)$, we assume that $\lambda > 0$ and $f(\theta) < 0$ for all $\theta \in [0, 2\pi)$. The proof when $\lambda < 0$ and $f(\theta) > 0$ for all $\theta \in [0, 2\pi)$ is completely similar.

From the expression of $\dot r$ in the system (4) it follows that for the boundary of a very big disc centered at the origin the flow of the system enters because $f(\theta) < 0$ for all $\theta \in [0, 2\pi)$. Then, since the origin is an unstable node and there are no other singularities, by the Poincaré–Bendixson Theorem (see for instance Corollary 1.30 of [5]) it follows that at least exists one periodic orbit surrounding the origin. We claim that this periodic orbit is isolated in the set of all periodic orbits of system (4). Then it is a limit cycle and the statement (b) is proved. Now we prove the claim.

Suppose that all the periodic orbits surrounding the origin are forming an annulus A filled only by periodic orbits. Since the origin is a node, it cannot be in the boundary of the annulus A. Take the periodic orbit γ of the boundary of this annulus more closest to the unstable node localized at the origin. The orbit γ is the ω–limit set of the orbits which have its α–limit set in the node at the origin. When one of such orbits is sufficiently close to γ it spirals tending to γ in forward time. The return map is defined on the annulus and sufficiently near it. Since the system is analytic the return map is analytic (see for instance [17]). Clearly this return map is the identity on the annulus, since it is analytic, it is also the identity sufficiently near to the annulus, in contradiction with the fact that sufficiently near to the annulus the orbits spiral tending to the annulus. So the claim is proved. $\square$

We shall use the following result of Lloyd, see Theorems 3 and 4 of [15].

Theorem 4. *Consider the differential system in polar coordinates*

$$\dot r = R(r,\theta), \qquad \dot\theta = \Theta(r,\theta) \quad (r>0),$$

where both functions R and Θ are 2π–periodic in the variable θ. Suppose that in a topological annular region $\mathcal{A}$ surrounding the origin, the function $\Theta \neq 0$. Let $S(r,\theta) = R(r,\theta)/\Theta(r,\theta)$ in $\mathcal{A}$.

(a) *If either $\partial S/\partial r > 0$ or $\partial S/\partial r < 0$ in $\mathcal{A}$, then $\mathcal{A}$ can entirely contain no more than one limit cycle.*

(b) *If either $\partial^2 S/\partial r^2 > 0$ or $\partial^2 S/\partial r^2 < 0$ in $\mathcal{A}$, then $\mathcal{A}$ can entirely contain no more than two limit cycles.*

Proof of statement (c) *of Theorem 2.* By statement (a) of Theorem 2 the limit cycles surrounding the origin must live in the unbounded open topological annular region $\mathcal{A}$ which has as boundary all the branches of the curve $r = r^*(\theta)$, see Figure 3 for illustration (we remark that in the case $g(\theta) \neq 0$ the branches γ_∞^l do not appear). Using the notation of the proof of statement (a) of Theorem 2 we have that $\mathcal{A} = \Omega_+$. By definition in $\mathcal{A}$ either $\dot\theta > 0$, or $\dot\theta < 0$. Therefore our system in polar coordinates (4) satisfies the assumption $\Theta \neq 0$ of Theorem 2.

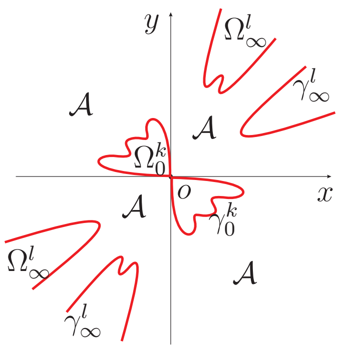


FIGURE 3. The unbounded annulus bounded by the branches of $r = r^*(\theta)$

Since $\lambda f(\theta) < 0$, we can assume without loss of generality that $f(\theta) < 0$. The assumption $g(\theta) \neq 0$ implies that the infinity is a periodic orbit, and it is unstable. Moreover the origin is an unstable node because $\lambda > 0$. So there exist a neighborhood U_0 of the origin and a neighborhood U_∞ of the infinity such that the orbits of system (3) or (4) starting from U_0 or U_∞ will positively and transversally intersect the boundary of U_0 or U_∞. In addition, a tedious but easy computation shows that the function S in the statement of Theorem 2 for our system (4) satisfies

$$\frac{\partial S}{\partial r} = \frac{r^{-n}\left(f(\theta)r^n + (2-n)\lambda r\right)}{g(\theta)} + O(\lambda-\mu).$$

Set $\mathcal{A}^* = \mathcal{A} \cap (\mathbb{R}^2 \setminus (U_0 \cup U_\infty))$. Combining the proof of the last paragraph and from statement (a) and its proof, we get that for $|\mu-\lambda|$ sufficiently small, if system (4) has a limit cycle it must be located in $\mathcal{A}^*$, because all orbits starting from Ω_0 or Ω_∞ will also positively pass through the boundary of the region and get into $\mathbb{R}^2 \setminus (\Omega_0 \cup \Omega_\infty)$ for $|\mu-\lambda| \ll 1$. Moreover, since $\mathcal{A}^*$ is compact and $\partial S/\partial r$ is either positive or negative for $\lambda=\mu$, it follows that $\partial S/\partial r$ is either positive or negative in the annulus $\mathcal{A}^*$ for $|\lambda-\mu|$ sufficiently small. Therefore, from the statement (a) of Theorem 4 we obtain that in the topological annulus $\mathcal{A}^*$ there is at most one limit cycle. Moreover, by statement (b) there is one limit cycle. So statement (c) is proved. □

Proof of statement (d) *of Theorem 2.* By statement (d) of Theorem 2 of [1] and its proof we know that system (4) with $\mu=\lambda$ has at most one periodic orbit in the finite plane. Since $g(\theta) \neq 0$, the infinity is an invariant periodic orbit. So working in a similar way as that in the proof of statement (c) we have a compact annulus $\mathcal{A}^*$ as that in the proof of statement (c) such that for $|\mu-\lambda|$ sufficiently small any periodic orbit of system (4) in the finite plane must be located in $\mathcal{A}^*$.

Some tedious but direct calculation shows that the function S in the statement of Theorem 2 for our system (4) satisfies

$$\frac{\partial^2 S}{\partial r^2} = \frac{(n-2)(n-1)r^{-n}\lambda}{g(\theta)} + O(\lambda-\mu).$$

Under the assumptions of statement (d) the derivative $\partial^2 S/\partial r^2$ always is either positive or negative when $\lambda=\mu$. Therefore, it follows from the compactness of $\mathcal{A}^*$ that $\partial^2 S/\partial r^2$ is either positive or negative in $\mathcal{A}^*$ for $|\lambda-\mu|$ sufficiently small. Then from the statement (b) of Theorem 4 we obtain that in the annulus $\mathcal{A}^*$ there is at most two limit cycles. This completes the proof of statement (d). □

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